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Research Article

Another Representation for the Maximal Lie Algebra of $sl(n+2,\mathbb{R})$ in Terms of Operators

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We provide an alternate representation to the result that the Lie algebra of generators of the system of n differential equations, $(y^a)'' = 0$, is isomorphic to the Lie algebra of the special linear group of order $(n+2)$, over the real numbers, $sl(n+2,\mathbb{R})$. In this paper, we provide an alternate representation of the symmetry algebra by simple relabelling of indices. This provides one more proof of the result that the symmetry algebra of $(y^a)'' = 0$ is $sl(n+2,\mathbb{R})$.

1. Introduction

The classification of all scalar second-order ordinary differential equations, according to the Lie algebra of generators they admit, is complete [1]; for example, the free particle equation $y'' = 0$ admits eight Lie symmetries [2], which is the maximum number of symmetries admitted by any second-order differential equation defined on a domain in the plane [1]. This Lie algebra is isomorphic to $sl(3,\mathbb{R})$ [3]. The maximality of the Lie algebra of $y'' = 0$ was proved by Lie, using a geometric argument [4]. The fact that the Lie algebra of the n-dimensional vector equation $(y^a)'' = 0$ is $sl(n+2,\mathbb{R})$ will be demonstrated here by an algebraic method of relabelling indices. This result had apparently been published earlier by Aminova in a relatively inaccessible journal [5, 6]. Before her, Leach [7] showed that an n-dimensional, uncoupled, undamped, and unforced linear system has the complete symmetry group $sl(n+2,\mathbb{R})$, and Prince and Eliezer [8] studied that the full symmetry group of the time-dependent oscillator in n-dimensions is $sl(n+2,\mathbb{R})$ of n^2+4n+3 operators. The number of symmetries of the equation has also been discussed in [9].

2. The Algebra of $(y^a)'' = 0$

Ibragimov [10] lists symmetry generators of the 3-dimensional vector $(y^a)'' = 0$:

$$X_0 = \frac{\partial}{\partial x}, \qquad X_a = \frac{\partial}{\partial y^a}, \qquad S = x\frac{\partial}{\partial x}, \qquad P^a = y^a\frac{\partial}{\partial x}, \qquad Q_a = x\frac{\partial}{\partial y^a},$$
$$Y^a_b = y^a\frac{\partial}{\partial y^b}, \qquad Z^0 = x^2\frac{\partial}{\partial x} + xy^b\frac{\partial}{\partial y^b}, \qquad Z^a = xy^a\frac{\partial}{\partial x} + y^a y^b\frac{\partial}{\partial y^b}, \tag{2.1}$$

where we have used the Einstein summation convention that repeated indices are summed over and rewritten so that they balance. These are twenty four generators in all.

As the Lie algebra is eight dimensional for the scalar equation and twenty four dimensional for the 3-dimensional equation, one may have supposed that the number of symmetry generators for the 2-dimensional equation would be sixteen. However, it turns out that the number of infinitesimal generators is fifteen. This equals the number for $sl(4,\mathbb{R})$. Infact, for $n = 3$, the number of generators in (2.1) equals that for $sl(5,\mathbb{R})$. We may, therefore, guess the following.

Theorem 2.1. *The Lie algebra, for the second-order n-dimensional vector equation, is $sl(n+2,\mathbb{R})$.*

Proof. The generators of the Lie algebra $gl(n,\mathbb{R})$ are [11, 12]

$$Y^\mu_\nu = \eta^{\gamma\mu}\frac{\partial}{\partial \eta^{\gamma\nu}}, \quad \mu,\nu = 1,2,3,\ldots,n, \tag{2.2}$$

which satisfy the commutation relation

$$\left[Y^\mu_\nu, Y^\rho_\tau\right] = \delta^\rho_\nu Y^\mu_\tau - \delta^\mu_\tau Y^\rho_\nu, \tag{2.3}$$

where δ^μ_ν is the usual Kronecker delta. Further, setting $Y^\alpha_\alpha = 0$ gives the Lie algebra of $sl(n,\mathbb{R})$. $\square$

It can be easily verified that the algebra for n dependent variables is (2.1) with $a = 1,2,3,\ldots,n$. Now define $y^\alpha = x$ for $\alpha = 0$ and y^a for $\alpha = a$. Then the generators can be rewritten as

$$X_\alpha = \frac{\partial}{\partial y^\alpha}, \quad Y^\alpha_\beta = y^\alpha\frac{\partial}{\partial y^\beta}, \quad Z^\alpha = y^\alpha y^\beta\frac{\partial}{\partial y^\beta}, \quad \alpha,\beta = 0,1,2,\ldots,n, \tag{2.4}$$

where $Y^0_0 = S$ and $Y^a_0 = P^a$. Now, further putting

$$Y^{n+1}_\alpha = X_\alpha, \qquad Y^\alpha_{n+1} = -Z^\alpha, \tag{2.5}$$

we only need to define Y^{n+1}_{n+1}. This may be defined by setting $Y^\mu_\mu = 0$, where $\mu,\nu = 0,1,2,\ldots,$ $n+1$. Then the generators given by (2.4) and (2.5) satisfy (2.3). The negative sign in (2.5) is introduced, so that the generators satisfy the required algebra. It is allowable to introduce the negative sign as $-V^{\alpha}$ will be a generator if V^{α} is. Hence the maximal symmetry algebra of the second-order n-dimensional vector differential equation *is* $sl(n+2,\mathbb{R})$.

3. Remarks

This representation of the symmetry algebra $(y^a)'' = 0$, $a = 1,2,\ldots,n$, has been obtained by merely relabelling the symmetry generators, as such we feel that it is especially simple and elegant.

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