Matrix orthogonality in the plane versus scalar orthogonality in a Riemann surface

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Abstract

We consider a non-Hermitian matrix orthogonality on a contour in the complex plane. Given a diagonalizable and rational matrix valued weight, we show that the Christoffel–Darboux (CD) kernel, which is built in terms of matrix orthogonal polynomials, is equivalent to a scalar valued reproducing kernel of meromorphic functions in a Riemann surface. If this Riemann surface has genus 0, then the matrix valued CD kernel is equivalent to a scalar reproducing kernel of polynomials in the plane. We find that, interestingly, this scalar reproducing kernel is not necessarily a scalar CD kernel.

We provide several applications of our result to the theory of tiling models with doubly periodic weightings. In particular, we show that the correlation kernel of any lozenge tiling model with 2×1 or 2×2 periodic weightings admits a double contour integral representation involving only a scalar CD kernel. This simplifies a formula of Duits and Kuijlaars.

1 Introduction and statement of results

We say that P is an $r \times r$ matrix polynomial of degree $N-1$ if it can be written in the form

$$P(z) = \sum_{k=0}^{N-1} C_k z^k \quad \text{for some } C_0, \ldots, C_{N-1} \in \mathbb{C}^{r\times r} \text{ with } C_{N-1} \neq 0_r,$$

where 0_r denotes the $r \times r$ zero matrix.

Consider the following bilinear pairing between $r \times r$ matrix polynomials

$$\langle P, Q \rangle = \int_\gamma P(z)W(z)Q(z)dz, \tag{1.1}$$

where $\gamma \subset \mathbb{C}$ is a finite union of piecewise smooth, oriented curves and W is a continuous $r \times r$ matrix weight. Because the matrix product does not commute, in general one has $\langle P, Q \rangle \neq \langle Q, P \rangle$, and therefore the pairing (1.1) gives rise to two families of matrix orthogonal polynomials (MOPs): the left MOPs and the right MOPs (see e.g. [54, 20] and below).

The study of MOPs has been initiated by Krein [46, 47] in the 1940's, motivated by a moment problem arising in operator theory. MOPs have then been studied sporadically, until a resurgence in the 1980's. They have found applications in scattering theory [33, 4], matrix valued spherical functions [45, 37, 42, 43, 51, 2, 44], system theory [32], Gaussian quadrature for matrix functions [53], the analysis of sequences of polynomials satisfying higher order recurrence relations [26, 29], integrable systems [50, 12, 5, 13, 38], Toda lattices [3, 21], among others. For a survey on MOPs up to 2008, we refer to [20]. The two standard settings that one often encounters in the literature are the following:

- Matrix orthogonality on the real line: $\gamma \subset \mathbb{R}$, and $W(z)$ is real valued, symmetric and positive definite for every $z \in \gamma$.
- Matrix orthogonality on the unit circle: $\gamma = \{z \in \mathbb{C} : |z| = 1\}$ is oriented positively, and $W(z)\frac{dz}{|dz|} = izW(z)$ is Hermitian and positive definite for every $z \in \gamma$.

In each of these two cases, the positive definiteness property ensures the existence of the MOPs, which are built from the left and right matrix inner products $\langle P, Q\rangle_L := \langle P, Q^*\rangle$ and $\langle P, Q\rangle_R := \langle P^*, Q\rangle$, where $P(z)^*$ denote the transpose conjugate of $P(z)$, see also [54, 20] for more details.

In this work, we deviate from the standard set-ups in several aspects. We consider a *non-Hermitian* matrix orthogonality on a general contour $\gamma \subset \mathbb{C}$. This means that no assumptions are made on W that would ensure the presence of certain inner products associated to (1.1). In our setting, there is no guarantee of existence and uniqueness for the MOPs, see also Section 1.1. We will restrict our attention to *rational* matrix weights, that is, each entry of W is a rational function with no pole on γ. This type of matrix orthogonality arises in the theory of tiling models [25, 10, 16], see also Section 2. Our main result will be valid under the following assumption on the weight W.

Assumption 1.1. The rational $r \times r$ matrix valued function $z \mapsto W(z)$ has no pole on γ and is diagonalizable for all but finitely many $z \in \mathbb{C}$.

Since W is rational, its eigenvalues $\lambda_1, \dots, \lambda_r$ are (branches of) meromorphic functions on $\mathbb{C}$. Assumption 1.1 implies that for all but finitely many $z \in \mathbb{C}$, there exists an invertible $r \times r$ matrix $E(z)$ such that

$$W(z) = E(z)\Lambda(z)E(z)^{-1}, \tag{1.2}$$

where $\Lambda(z) = \operatorname{diag}(\lambda_1(z), \dots, \lambda_r(z))$. Note that it does not matter for us if W fails to be diagonalizable for finitely many points on the contour γ itself. Assumption 1.1 essentially rules out the weights with a Jordan block structure. For example, the weight

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

is only diagonalizable at $z = 0$, and therefore does not satisfy Assumption 1.1. Our main result is stated in Theorem 1.5 and is described in terms of the Christoffel-Darboux (CD) kernel. It can roughly be summarized as *if W satisfies Assumption 1.1, the non-Hermitian matrix orthogonality induced by* (1.1) *is equivalent to a scalar orthogonality in a Riemann surface.* If this Riemann surface has genus 0, it can be mapped to the plane and the matrix orthogonality is equivalent to a scalar orthogonality in the plane. For instance, we show in Example 1.18 below that the orthogonality on the unit circle associated to the $r \times r$ matrix weight

$$W(z) = z^{-R} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ z & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}^L, \qquad L, R \in \mathbb{N}_{>0}, \tag{1.3}$$

is equivalent to the scalar orthogonality, also on the unit circle, associated to the well-studied [49] Jacobi weight $\zeta^{-rR}(1+\zeta)^L$ with non-standard parameters. We describe some applications of our results to the theory of tiling models in Section 2.

This work is inspired from [16], in which it was established that the matrix orthogonality on the unit circle associated to

$$W(z) = \frac{1}{z^n} \begin{pmatrix} \alpha^2 + z & 1 + \alpha \\ (1+\alpha^3)z & 1 + \alpha^2 z \end{pmatrix}^n, \qquad \alpha \in (0,1),\ n \in \mathbb{N}_{>0} \tag{1.4}$$

is equivalent to the scalar orthogonality associated to

$$\mathcal{W}(\zeta) = \left(\frac{(\zeta - \alpha c)(\zeta - \alpha c^{-1})}{\zeta(\zeta - c)(\zeta - c^{-1})} \right)^n, \qquad \text{where} \qquad c = \sqrt{\frac{\alpha}{1 - \alpha + \alpha^2}},$$

on a suitable contour which we do not describe here. Several steps in the proof of [16] rely on the exact expression (1.4) of W. Here, we generalize these ideas to handle any W satisfying Assumption 1.1, and we feel our proof is also simpler.

Remark 1.2. *In the context of matrix orthogonality on the real line, it was already observed by several authors, see e.g. [28, 14], that if the matrix weight is diagonalizable with* ***constant*** *matrices, then the matrix orthogonality is nothing really different from a scalar orthogonality on the real line. The major difference with our situation is that $z \mapsto E(z)$ is obviously not necessarily constant.*

We now introduce the necessary material to state our results.

CD kernel. Given a contour γ, a matrix weight W and $N \in \mathbb{N}_{>0}$, the associated CD kernel $\mathcal{R}_N^W(w,z)$ is defined as the unique bivariate $r \times r$ matrix polynomial of degree $\leq N-1$ in both w and z that satisfies either

$$\int_\gamma P(w)W(w)\mathcal{R}_N^W(w,z)dw = P(z), \qquad \text{for every } P \in \mathcal{P}_{N-1}^{r\times r} \text{ and } z \in \mathbb{C}, \tag{1.5}$$

$$\text{or} \quad \int_\gamma \mathcal{R}_N^W(w,z)W(z)P(z)dz = P(w), \qquad \text{for every } P \in \mathcal{P}_{N-1}^{r\times r} \text{ and } w \in \mathbb{C}, \tag{1.6}$$

where for $j,k \in \mathbb{N}_{>0}$, $\mathcal{P}_{N-1}^{j\times k} = \{\sum_{\ell=0}^{N-1} C_\ell z^\ell : C_0, \ldots, C_{N-1} \in \mathbb{C}^{j\times k}\}$ is the vector space of all $j \times k$ matrix polynomials of degree $\leq N-1$. Because of (1.5)–(1.6), $\mathcal{R}_N^W$ is also called the *reproducing kernel* for $\mathcal{P}_{N-1}^{r\times r}$, and we refer to [24, Proposition 2.4] (see also [25, Lemma 4.6]) for a proof that $\mathcal{R}_N^W$ is indeed unique. There also exists an explicit expression for $\mathcal{R}_N^W$ in terms of the left and right MOPs, but since this expression is not needed to state our results, we defer it to Section 1.1. For non-Hermitian matrix orthogonality, there is in general no guarantee of existence for $\mathcal{R}_N^W$, although in concrete situations one can sometimes prove it, see [25, Lemma 4.8]. If γ is unbounded, then a necessary condition for the existence of $\mathcal{R}_N^W$ is

$$W(z) = \mathcal{O}(z^{-2N}), \qquad z \to \infty.$$

This condition ensures the convergence of the integrals in (1.5)–(1.6).

Let I_r be the $r \times r$ identity matrix. Since W is rational, we can write $\det(W(z) - \lambda I_r) = P_W(z,\lambda)/Q_W(z)$ for some polynomials P_W and Q_W. Let us consider the zero set of P_W, namely

$$\{(z,\lambda) \in \mathbb{C}^2 : P_W(z,\lambda) = 0\}. \tag{1.7}$$

It is well-known, see e.g. [52, Example 6 of Section 4.2 and Chapter 5], that any zero set of a polynomial in two variables can be completed to an algebraic curve (=compact Riemann surface). Since P_W is of degree r in the variable λ, the Riemann surface $\mathcal{M}$ associated to (1.7) can be represented as an r-sheeted covering of $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ (and $\mathcal{M}$ is connected if and only if P_W is irreducible). We will use the notation z, w to denote points of $\mathcal{M}$, and z, w for points of $\mathbb{C}$. If z and z appear in the same equation, then z denotes the projection of z on $\widehat{\mathbb{C}}$ (and similarly for w and w). We choose the numbering of the sheets such that the function

$$\mathrm{z} \mapsto \lambda(\mathrm{z}) := \lambda_k(z), \qquad \text{if } \mathrm{z} \text{ is on the } k\text{-th sheet of } \mathcal{M}, \tag{1.8}$$

is meromorphic on $\mathcal{M}$. Assumption 1.1 implies, see Appendix A for the details, that we can (and do) choose the matrix of eigenvectors $E(z)$ such that the functions

$$z \mapsto \mathfrak{e}(z) := E(z)e_k, \qquad \text{if z is on the } k\text{-th sheet of } \mathcal{M}, \tag{1.9}$$

$$z \mapsto \mathfrak{e}^{-1}(z) := e_k^T E(z)^{-1}, \qquad \text{if z is on the } k\text{-th sheet of } \mathcal{M}, \tag{1.10}$$

are also meromorphic on $\mathcal{M}$, where e_k is the k-th column of the identity matrix and ^{T} denotes the transpose operation.

The Riemann surface $\mathcal{M}$ associated with the zero set (1.7) is just one example of a Riemann surface for which one can define meromorphic functions λ, $\mathfrak{e}$ and $\mathfrak{e}^{-1}$ as in (1.8)–(1.10). However, in certain cases it is more convenient to work with a slightly different $\mathcal{M}$. Assume for example that the weight W is of the form $W(z) = A(z)^L$ for a certain rational matrix A and a certain $L \in \mathbb{N}_{>0}$, and assume that

$$A(z) = E(z)\widetilde{\Lambda}(z)E(z)^{-1}, \qquad \widetilde{\Lambda}(z) = \operatorname{diag}(\widetilde{\lambda}_1(z), \ldots, \widetilde{\lambda}_r(z)), \qquad \text{for all but finitely many } z \in \mathbb{C}.$$

The eigenvalues of W are obviously $\lambda_k(z) = \widetilde{\lambda}_k(z)^L$, $k = 1, \ldots, r$. Therefore, in this case, instead of using the Riemann surface associated to (1.7), $\mathcal{M}$ can be defined as the (simpler) algebraic curve constructed from the zero set $\{(z, \widetilde{\lambda}) \in \mathbb{C}^2 : P_A(z, \widetilde{\lambda}) = 0\}$.

From now, $\mathcal{M}$ denotes an arbitrary r-sheeted Riemann surface such that the functions λ, $\mathfrak{e}$ and $\mathfrak{e}^{-1}$ defined in (1.8)–(1.10) are meromorphic. Assumption 1.1 implies the existence of such a $\mathcal{M}$, but the exact choice of the zero set from which $\mathcal{M}$ is constructed does not matter for our results.

Remark 1.3. *There is a slight abuse of notation in the definitions (1.9)–(1.10). Since $\mathfrak{e}$ is of size $r \times 1$ and $\mathfrak{e}^{-1}$ is of size $1 \times r$, there are not the inverse of each other in the usual matrix sense. However, from the relations $E(z)E(z)^{-1} = I = E(z)^{-1}E(z)$, we deduce that they satisfy*[1]

$$\mathfrak{e}^{-1}(z^{(j)})\mathfrak{e}(z^{(k)}) = \delta_{j,k}, \qquad \text{for all } z \in \mathbb{C} \text{ and } 1 \leq j, k \leq r, \tag{1.11}$$

$$\sum_{j=1}^{r} \mathfrak{e}(z^{(j)})\mathfrak{e}^{-1}(z^{(j)}) = I_r, \qquad \text{for all } z \in \mathbb{C}, \tag{1.12}$$

where for a given $z \in \mathbb{C}$, $z^{(k)}$ denotes the point on the k-th sheet of $\mathcal{M}$ whose projection on $\mathbb{C}$ is z.

Let us illustrate with an example how to compute in practice the functions λ, $\mathfrak{e}$ and $\mathfrak{e}^{-1}$.

Example 1.4. Consider the weight

$$W(z) = \begin{pmatrix} 1 & 1 \\ z^k & 1 \end{pmatrix}, \qquad k \in \mathbb{Z}.$$

For all $z \in \mathbb{C} \setminus \{0\}$, we can write $W(z) = E(z)\Lambda(z)E(z)^{-1}$ with

$$E(z) = \begin{pmatrix} 1 & 1 \\ z^{\frac{k}{2}} & -z^{\frac{k}{2}} \end{pmatrix}, \qquad \Lambda(z) = \begin{pmatrix} 1 + z^{\frac{k}{2}} & 0 \\ 0 & 1 - z^{\frac{k}{2}} \end{pmatrix}, \qquad E(z)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & z^{-\frac{k}{2}} \\ 1 & -z^{-\frac{k}{2}} \end{pmatrix}, \tag{1.13}$$

where the principal branch is chosen for $z^{\frac{k}{2}}$ if $k \bmod 2 = 1$. We let $\mathcal{M}$ be the Riemann surface associated to the zero set $\{(z, \eta) \in \mathbb{C}^2 : \eta^2 = z^k\}$. We choose the numbering of the sheets such that

[1] Strictly speaking, (1.11)-(1.12) hold for each $z \in \mathbb{C}$ that is not a pole of E or E^{-1}, but by continuity they hold for all $z \in \mathbb{C}$.

$\eta = z^{\frac{k}{2}}$ on the first sheet and $\eta = -z^{\frac{k}{2}}$ on the second sheet (if $k \bmod 2 = 0$, $\mathcal{M}$ is simply the disjoint union of two copies of $\widehat{\mathbb{C}}$). The functions λ, $\mathfrak{e}$ and $\mathfrak{e}^{-1}$, defined by (1.8) and (1.9)–(1.10), are explicitly given by

$$\lambda((z,\eta)) = 1+\eta, \qquad \mathfrak{e}((z,\eta)) = \begin{pmatrix} 1 & \eta \end{pmatrix}^T, \qquad \mathfrak{e}^{-1}((z,\eta)) = \frac{1}{2}\begin{pmatrix} 1 & \eta^{-1} \end{pmatrix},$$

where $\mathrm{z} = (z,\eta)$ denotes a point of $\mathcal{M}$. These functions are meromorphic on $\mathcal{M}$, as required. Note that the use of (z,η) for a point of $\mathcal{M}$ is a slight abuse of notation, because e.g. if $k = 2$, $\mathcal{M}$ is the disjoint union of two copies of $\widehat{\mathbb{C}}$, and this notation does not distinguish the points $0^{(1)}$ and $0^{(2)}$, which are both denoted $(0,0)$. We will use again this notation several times for convenience, but if this can lead to confusion we will clarify it.

Main results. Let $\mathcal{M}_*$ be the set $\mathcal{M}$ with all points at infinity removed, and let $\mathcal{Q}$ and $\widehat{\mathcal{Q}}$ be the finite sets of all poles of $\mathfrak{e}$ and $\mathfrak{e}^{-1}$, respectively.[2] For $\mathrm{z} \in \mathcal{M}_* \setminus \mathcal{Q}$ and $\mathrm{w} \in \mathcal{M}_* \setminus \widehat{\mathcal{Q}}$, we define

$$R_N^{\lambda}(\mathrm{w},\mathrm{z}) = \mathfrak{e}^{-1}(\mathrm{w})\mathcal{R}_N^W(w,z)\mathfrak{e}(\mathrm{z}), \tag{1.14}$$

where we recall that z and w denote the projections on the complex plane of z and w, respectively. Note that R_N^{λ} is scalar valued, but is equivalent to $\mathcal{R}_N^W$ in the sense that we can completely recover $\mathcal{R}_N^W$ from R_N^{λ} by

$$\left[R_N^{\lambda}(w^{(j)}, z^{(k)})\right]_{j,k=1}^r = E(w)^{-1}\mathcal{R}_N^W(w,z)E(z). \tag{1.15}$$

Let $\gamma_{\mathcal{M}} = \cup_{j=1}^r \gamma^{(j)}$ be the closed contour on $\mathcal{M}$ that consists of r copies of γ, one on each sheet, and let L_N and L_N^* be the vector spaces of scalar meromorphic functions on $\mathcal{M}$ given by

$$L_N = \{\mathrm{z} \mapsto P(z)\mathfrak{e}(\mathrm{z}) : P \in \mathcal{P}_{N-1}^{1\times r}\}, \qquad L_N^* = \{\mathrm{z} \mapsto \mathfrak{e}^{-1}(\mathrm{z})P(z) : P \in \mathcal{P}_{N-1}^{r\times 1}\}. \tag{1.16}$$

Our first main result states that R_N^{λ} satisfies some reproducing properties for L_N and L_N^* on the contour $\gamma_{\mathcal{M}}$.

Theorem 1.5. *Let $\gamma \subset \mathbb{C}$ be a finite union of piecewise smooth, oriented curves, let W be a rational $r \times r$ matrix weight, and let $N \in \mathbb{N}_{>0}$. Suppose that W satisfies Assumption 1.1 and that $\mathcal{R}_N^W$ exists. Let $\mathcal{M}$ be an r-sheeted Riemann surface such that the functions λ, $\mathfrak{e}$ and $\mathfrak{e}^{-1}$ defined in (1.8)–(1.10) are meromorphic, and define R_N^{λ} as in (1.14). Then R_N^{λ} exists,* $\dim L_N = \dim L_N^* = rN$, *and we have the following:*

(a) $\mathrm{z} \mapsto R_N^{\lambda}(\mathrm{w},\mathrm{z}) \in L_N$ *for every* $\mathrm{w} \in \mathcal{M}_* \setminus \widehat{\mathcal{Q}}$,

(b) $\mathrm{w} \mapsto R_N^{\lambda}(\mathrm{w},\mathrm{z}) \in L_N^*$ *for every* $\mathrm{z} \in \mathcal{M}_* \setminus \mathcal{Q}$,

(c) R_N^{λ} *satisfies the following reproducing property for* L_N*:*

$$\int_{\gamma_{\mathcal{M}}} f(\mathrm{w})\lambda(\mathrm{w})R_N^{\lambda}(\mathrm{w},\mathrm{z})dw = f(\mathrm{z}), \qquad \textit{for every } f \in L_N \textit{ and } \mathrm{z} \in \mathcal{M}_* \setminus \mathcal{Q}. \tag{1.17}$$

(d) R_N^{λ} *satisfies the following reproducing property for* L_N^**:*

$$\int_{\gamma_{\mathcal{M}}} R_N^{\lambda}(\mathrm{w},\mathrm{z})\lambda(\mathrm{z})f(\mathrm{z})dz = f(\mathrm{w}), \qquad \textit{for every } f \in L_N^* \textit{ and } \mathrm{w} \in \mathcal{M}_* \setminus \widehat{\mathcal{Q}}. \tag{1.18}$$

[2] z is a pole of $\mathfrak{e}$ (resp. of $\mathfrak{e}^{-1}$) if z is a pole for at least one entry of $\mathfrak{e}$ (resp. of $\mathfrak{e}^{-1}$).

Proof. See Section 3.1. $\square$

L_N and L_N^* are contained in certain spaces of meromorphic functions with prescribed zeros and allowed poles. To describe this relation, we briefly introduce some notation and definitions.

Definition 1.6. (divisors) A divisor D is a formal sum of the form $D = \sum_{j=1}^{n} \ell_j \mathrm{z}_j$, where $n \in \mathbb{N}_{>0}$, $\ell_j \in \mathbb{Z}$ and $\mathrm{z}_j \in \mathcal{M}$. The divisor of a non-zero scalar valued meromorphic function f on $\mathcal{M}$ is defined as $\mathrm{div}(f) = \sum_{\mathrm{z}\in\mathcal{Z}_f} \ell_\mathrm{z}\mathrm{z} - \sum_{\mathrm{z}\in\mathcal{Q}_f} \ell_\mathrm{z}\mathrm{z}$, where $\mathcal{Z}_f$ and $\mathcal{Q}_f$ are the finite sets of all zeros and poles of f, respectively, and $\ell_\mathrm{z} \in \mathbb{N}_{>0}$ is the order of z.

Definition 1.7. (poles and zeros) Given a matrix valued meromorphic function F on $\mathcal{M}$, we say that z is a zero of F (of order m) if z is a common zero of all entries of F (of order $\geq m$ for each entry, and of order exactly m for at least one entry). Similarly, z is a pole of F (of order m) if z is a pole (of order m) for at least one entry of F (and the other entries of F have either no pole at z, or a pole of order $\leq m$). Of course, if F is scalar valued, the number of its poles equals the number of its zeros (counting multiplicities), but this not true in general if F is matrix valued. For instance, the function $\mathfrak{e}$ of Example 1.4 with $k \bmod 2 = 1$ and $k > 0$ has a pole of order k at $\infty^{(1)} = \infty^{(2)}$ and no zero.

Definition 1.8. (n_z and $\widehat{n}_\mathrm{z}$) Let $\mathcal{Z}$ and $\widehat{\mathcal{Z}}$ be the finite sets of all zeros of $\mathfrak{e}$ and $\mathfrak{e}^{-1}$, respectively, and recall that $\mathcal{Q}$ and $\widehat{\mathcal{Q}}$ are the finite sets of all poles of $\mathfrak{e}$ and $\mathfrak{e}^{-1}$, respectively. The order of a zero $\mathrm{z} \in \mathcal{Z}$ of $\mathfrak{e}$ is denoted by n_z, and the order of a pole $\mathrm{z} \in \mathcal{Q}$ of $\mathfrak{e}$ is denoted by $-n_\mathrm{z}$. That is, $n_\mathrm{z} > 0$ if $\mathrm{z} \in \mathcal{Z}$ and $n_\mathrm{z} < 0$ if $\mathrm{z} \in \mathcal{Q}$. Similarly, for each $\mathrm{z} \in \widehat{\mathcal{Z}} \cup \widehat{\mathcal{Q}}$, we associate an integer $\widehat{n}_\mathrm{z} \in \mathbb{Z}\setminus\{0\}$, which represents the order of the zero, or the opposite of the order of the pole of $\mathfrak{e}^{-1}$ at z.

Definition 1.9. ($\infty^{(j)}$) Let $\infty^{(j)}$ be the point at infinity on j-th sheet of $\mathcal{M}$. If on each sheet $\mathcal{M}$ has no branch point at infinity, then $\#\{\infty^{(1)}, \dots, \infty^{(r)}\} = r$; otherwise $\#\{\infty^{(1)}, \dots, \infty^{(r)}\} < r$.

Given a scalar polynomial $P \in \mathcal{P}_{N-1} := \mathcal{P}_{N-1}^{1\times 1}$, the function

$$\mathrm{z} \mapsto P(z), \qquad \text{where } z \in \widehat{\mathbb{C}} \text{ is the projection of z},$$

is meromorphic on $\mathcal{M}$ with a pole of order $(N-1)m_j$ at $\infty^{(j)}$, $j = 1, \dots, r$, where

$$m_j = \#\{\ell \in \{1, \dots, r\} : \infty^{(\ell)} = \infty^{(j)}\}.$$

Therefore, it is immediate to see from (1.16) that

$$L_N \subseteq \{f : \mathrm{div}(f) \geq -\sum_{j=1}^{r}(N-1)\cdot\infty^{(j)} + \sum_{\mathrm{z}\in\mathcal{Z}\cup\mathcal{Q}} n_\mathrm{z}\mathrm{z}\}, \tag{1.19}$$

$$L_N^* \subseteq \{f : \mathrm{div}(f) \geq -\sum_{j=1}^{r}(N-1)\cdot\infty^{(j)} + \sum_{\mathrm{z}\in\widehat{\mathcal{Z}}\cup\widehat{\mathcal{Q}}} \widehat{n}_\mathrm{z}\mathrm{z}\}. \tag{1.20}$$

Remark 1.10. *The opposite inclusion $\supseteq$ does not hold in general. To see this, consider Example 1.4 with $k \bmod 2 = 1$, $k > 1$. Since η has a pole of order k at $\infty^{(1)} = \infty^{(2)}$, one has*

$$L_N = \{(z,\eta) \mapsto P_1(z) + \eta P_2(z) : P_1, P_2 \in \mathcal{P}_{N-1}\} \subseteq \{f : \mathrm{div}(f) \geq -(2(N-1)+k)\infty^{(1)}\},$$

which is consistent with (1.19). However, the function

$$(z,\eta) \mapsto \sqrt{z} = \frac{\eta}{z^{\frac{k-1}{2}}}$$

has only a simple pole at $\infty^{(1)}$, but does not belong to L_N. In particular,

$$L_N \subsetneq \{f : \mathrm{div}(f) \geq -(2(N-1)+k)\infty^{(1)}\}.$$

Genus 0 situation. Our second main result, Theorem 1.16, states that if $\mathcal{M}$ is a connected Riemann surface of genus 0, $\mathcal{R}_N^W$ is in fact completely equivalent to the scalar reproducing kernel $\mathfrak{R}_{rN}^{\mathcal{W}}$ of certain polynomial spaces $\mathcal{V}$ and $\mathcal{V}^*$. This scalar kernel is associated to a scalar weight $\mathcal{W}$ on a contour $\gamma_{\mathbb{C}} \subset \mathbb{C}$ that are described below. In certain situations, it holds that $\mathcal{V} = \mathcal{V}^* = \mathcal{P}_{rN-1}$, in which case $\mathfrak{R}_{rN}^{\mathcal{W}}$ is exactly the CD kernel $\mathfrak{R}_{rN}^{\mathcal{W}} = \mathcal{R}_{rN}^{\mathcal{W}}$. However, we emphasize that in general $\mathfrak{R}_{rN}^{\mathcal{W}}$ is not a CD kernel. We now define the relevant quantities that will appear in the statement of Theorem 1.16.

We assume from now that $\mathcal{M}$ is a connected Riemann surface of genus 0.

Definition 1.11. (the maps φ and ϕ) Since $\mathcal{M}$ is of genus 0, there is a one-to-one map $\zeta \mapsto \varphi(\zeta)$ from $\widehat{\mathbb{C}}$ to $\mathcal{M}$. The projection of $\varphi(\zeta)$ to the complex plane will be denoted by $\phi(\zeta)$.

Definition 1.12. (the scalar kernel $\mathfrak{R}_{rN}^{\mathcal{W}}$) We define $\mathfrak{R}_{rN}^{\mathcal{W}}$ by

$$\begin{aligned}\mathfrak{R}_{rN}^{\mathcal{W}}(\omega,\zeta) &= \widehat{h}(\omega) R_N^{\lambda}(\varphi(\omega),\varphi(\zeta)) h(\zeta), \\ &= \widehat{h}(\omega)\mathfrak{e}^{-1}(\varphi(\omega))\mathcal{R}_N^W(\phi(\omega),\phi(\zeta))\mathfrak{e}(\varphi(\zeta))h(\zeta), \qquad \zeta,\omega\in\mathbb{C},\end{aligned} \tag{1.21}$$

where h and $\widehat{h}$ are the two scalar valued rational functions given by

$$h(\zeta) = \prod_{\substack{j=1 \\ \infty^{(j)}\neq\varphi(\infty)}}^{r} (\zeta-\varphi^{-1}(\infty^{(j)}))^{N-1} \prod_{\mathrm{z}\in(\mathcal{Z}\cup\mathcal{Q})\setminus\{\varphi(\infty)\}} (\zeta-\varphi^{-1}(\mathrm{z}))^{-n_{\mathrm{z}}}, \tag{1.22}$$

$$\widehat{h}(\zeta) = \prod_{\substack{j=1 \\ \infty^{(j)}\neq\varphi(\infty)}}^{r} (\zeta-\varphi^{-1}(\infty^{(k_j)}))^{N-1} \prod_{\mathrm{z}\in(\widehat{\mathcal{Z}}\cup\widehat{\mathcal{Q}})\setminus\{\varphi(\infty)\}} (\zeta-\varphi^{-1}(\mathrm{z}))^{-\widehat{n}_{\mathrm{z}}}. \tag{1.23}$$

The role of h and $\widehat{h}$ is to remove all the poles of

$$\zeta \mapsto \mathfrak{e}^{-1}(\varphi(\omega))\mathcal{R}_N^W(\phi(\omega),\phi(\zeta))\mathfrak{e}(\varphi(\zeta)) \qquad \text{and} \qquad \omega \mapsto \mathfrak{e}^{-1}(\varphi(\omega))\mathcal{R}_N^W(\phi(\omega),\phi(\zeta))\mathfrak{e}(\varphi(\zeta)),$$

and also to reduce the order of some the zeros of these functions.

Definition 1.13. ($\mathcal{W}$ and $\gamma_{\mathbb{C}}$) The contour $\gamma_{\mathbb{C}}$ is defined by $\gamma_{\mathbb{C}} = \varphi^{-1}(\gamma_{\mathcal{M}}) = \varphi^{-1}(\cup_{j=1}^r \gamma^{(j)})$, and the scalar weight $\mathcal{W}$ by

$$\mathcal{W}(\omega) = \frac{\lambda(\varphi(\omega))}{h(\omega)\widehat{h}(\omega)}\phi'(\omega). \tag{1.24}$$

Definition 1.14. (The polynomial spaces $\mathcal{V}$ and $\mathcal{V}^*$) We define

$$\mathcal{V} = \{\zeta \mapsto p(\zeta) = f(\varphi(\zeta))h(\zeta) : f \in L_N\}, \qquad \mathcal{V}^* = \{\zeta \mapsto p(\zeta) = f(\varphi(\zeta))\widehat{h}(\zeta) : f \in L_N^*\}. \tag{1.25}$$

Remark 1.15. *If $P \in \mathcal{P}_{N-1}$, then $\zeta \mapsto P(\varphi(\zeta))$ is a meromorphic function on $\widehat{\mathbb{C}}$ with poles of order $(N-1)m_j$ at $\varphi^{-1}(\infty^{(j)})$, $j = 1, ..., r$. Hence, by (1.16) and (1.22)–(1.23), the elements of $\mathcal{V}$ and $\mathcal{V}^*$ are polynomials of degree*

$$\leq r(N-1) - \sum_{\mathrm{z}\in\mathcal{Z}\cup\mathcal{Q}} n_{\mathrm{z}} \qquad \textit{and} \qquad \leq r(N-1) - \sum_{\mathrm{z}\in\widehat{\mathcal{Z}}\cup\widehat{\mathcal{Q}}} \widehat{n}_{\mathrm{z}}, \qquad \textit{respectively.}$$

Also, since $\mathcal{V}$ is in bijection with L_N via the obvious map $[\mathrm{z} \mapsto f(\mathrm{z})] \mapsto [\zeta \mapsto f(\varphi(\zeta))h(\zeta)]$, we have $\dim \mathcal{V} = rN$. Similarly, $\mathcal{V}^*$ is in bijection with L_N^* and $\dim \mathcal{V}^* = rN$. In particular, we always have

$$-\sum_{\mathrm{z}\in\mathcal{Z}\cup\mathcal{Q}} n_{\mathrm{z}} \geq r-1 \qquad \textit{and} \qquad -\sum_{\mathrm{z}\in\widehat{\mathcal{Z}}\cup\widehat{\mathcal{Q}}} \widehat{n}_{\mathrm{z}} \geq r-1, \tag{1.26}$$

and $\mathcal{V} = \mathcal{P}_{rN-1}$ (resp. $\mathcal{V}^* = \mathcal{P}_{rN-1}$) if and only if $-\sum_{\mathrm{z}\in\mathcal{Z}\cup\mathcal{Q}} n_{\mathrm{z}} = r-1$ (resp. $-\sum_{\mathrm{z}\in\widehat{\mathcal{Z}}\cup\widehat{\mathcal{Q}}} \widehat{n}_{\mathrm{z}} = r-1$).

Theorem 1.16. *Let $\gamma \subset \mathbb{C}$ be a finite union of piecewise smooth, oriented curves, let W be a rational $r \times r$ matrix weight, and let $N \in \mathbb{N}_{>0}$. Suppose that W satisfies Assumption 1.1 and that $\mathcal{R}_N^W$ exists. Let $\mathcal{M}$ be an r-sheeted Riemann surface such that the functions λ, $\mathfrak{e}$ and $\mathfrak{e}^{-1}$ defined in (1.8)–(1.10) are meromorphic, and assume that $\mathcal{M}$ is of genus 0. Define $\mathfrak{R}_{rN}^{\mathcal{W}}$, φ, ϕ, $\mathcal{W}$, $\gamma_{\mathbb{C}}$, $\mathcal{V}$ and $\mathcal{V}^*$ as in Definitions 1.11–1.14. The scalar kernel $\mathfrak{R}_{rN}^{\mathcal{W}}$ exists, $\dim \mathcal{V} = \dim \mathcal{V}^* = rN$ and we have:*

(a) $\zeta \mapsto \mathfrak{R}_{rN}^{\mathcal{W}}(\omega, \zeta) \in \mathcal{V}$ for every $\omega \in \mathbb{C}$,

(b) $\omega \mapsto \mathfrak{R}_{rN}^{\mathcal{W}}(\omega, \zeta) \in \mathcal{V}^*$ for every $\zeta \in \mathbb{C}$,

(c) $\mathfrak{R}_{rN}^{\mathcal{W}}$ satisfies the following reproducing property for $\mathcal{V}$:

$$\int_{\gamma_{\mathbb{C}}} p(\omega)\mathcal{W}(\omega)\mathfrak{R}_{rN}^{\mathcal{W}}(\omega,\zeta)d\omega = p(\zeta), \qquad \text{for every } p \in \mathcal{V} \text{ and } \zeta \in \mathbb{C}. \tag{1.27}$$

(d) $\mathfrak{R}_{rN}^{\mathcal{W}}$ satisfies the following reproducing property for $\mathcal{V}^*$:

$$\int_{\gamma_{\mathbb{C}}} \mathfrak{R}_{rN}^{\mathcal{W}}(\omega,\zeta)\mathcal{W}(\zeta)p(\zeta)d\zeta = p(\omega), \qquad \text{for every } p \in \mathcal{V}^* \text{ and } \omega \in \mathbb{C}. \tag{1.28}$$

(e) We have the following equivalences

$$-\sum_{\mathrm{z} \in \mathcal{Z} \cup \mathcal{Q}} n_{\mathrm{z}} = r-1 \;\Leftrightarrow\; -\sum_{\mathrm{z} \in \widehat{\mathcal{Z}} \cup \widehat{\mathcal{Q}}} \widehat{n}_{\mathrm{z}} = r-1 \;\Leftrightarrow\; \mathcal{V} = \mathcal{P}_{rN-1} \;\Leftrightarrow\; \mathcal{V}^* = \mathcal{P}_{rN-1} \;\Leftrightarrow\; \mathfrak{R}_{rN}^{\mathcal{W}} = \mathcal{R}_{rN}^{\mathcal{W}},$$

where $\mathcal{R}_{rN}^{\mathcal{W}}$ denotes the scalar CD kernel associated to $\mathcal{W}$.

Proof. See Section 3.2. ☐

We now illustrate how to apply Theorem 1.16 in some concrete cases. The following example considers a situation where $\mathcal{M}$ is of genus 0, but the scalar kernel $\mathfrak{R}_{rN}^{\mathcal{W}}$ is not equivalent to a CD kernel (except for a particular choice of the parameters).

Example 1.17. Consider the matrix weight

$$W(z) = \frac{1}{z^M}\begin{pmatrix} 1 & 1 \\ z^k & 1 \end{pmatrix}^L, \qquad L, M \in \mathbb{N}_{>0},$$

and $k \in \mathbb{N}_{>0}$ is odd, and let $\gamma \subset \mathbb{C}$ be such that $0 \notin \gamma$. This weight is similar but slightly more complicated than in Example 1.4. For each $z \in \mathbb{C} \setminus \{0\}$, we can write $W(z) = E(z)\Lambda(z)E(z)^{-1}$ with

$$E(z) = \begin{pmatrix} 1 & 1 \\ z^{\frac{k}{2}} & -z^{\frac{k}{2}} \end{pmatrix}, \qquad \Lambda(z) = z^{-M}\begin{pmatrix} (1+z^{\frac{k}{2}})^L & 0 \\ 0 & (1-z^{\frac{k}{2}})^L \end{pmatrix}, \qquad E(z)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & z^{-\frac{k}{2}} \\ 1 & -z^{-\frac{k}{2}} \end{pmatrix},$$

where the principal branch is chosen for $z^{\frac{k}{2}}$. In particular, W satisfies Assumption 1.1. The Riemann surface $\mathcal{M}$ associated to

$$\{(z,\eta) \in \mathbb{C}^2 : \eta^2 = z^k\}$$

is of genus 0, and the functions

$$\lambda((z,\eta)) = z^{-M}(1+\eta)^L, \qquad \mathfrak{e}((z,\eta)) = \begin{pmatrix} 1 & \eta \end{pmatrix}^T, \qquad \mathfrak{e}^{-1}((z,\eta)) = \frac{1}{2}\begin{pmatrix} 1 & \eta^{-1} \end{pmatrix} \tag{1.29}$$

are meromorphic on $\mathcal{M}$. Therefore we know by Theorem 1.16 that the matrix CD kernel $\mathcal{R}_N^W$ (assuming it exists) is equivalent to a scalar reproducing kernel $\mathfrak{R}_{rN}^{\mathcal{W}}$ which can be explicitly described as follows. First, we note that $\mathfrak{e}$ has no zero and a pole of order k at $\infty^{(1)} = \infty^{(2)}$, and $\mathfrak{e}^{-1}$ has no zero and a pole of order k at $0^{(1)} = 0^{(2)}$. Therefore, following Definition 1.8, we have

$$\mathcal{Z} = \emptyset, \qquad \mathcal{Q} = \{\infty^{(1)}\}, \qquad \widehat{\mathcal{Z}} = \emptyset, \qquad \widehat{\mathcal{Q}} = \{0^{(1)}\}, \qquad n_{\infty^{(1)}} = -k, \qquad \widehat{n}_{0^{(1)}} = -k.$$

Define the bijections $\varphi : \widehat{\mathbb{C}} \to \mathcal{M}$ and $\varphi^{-1} : \mathcal{M} \to \widehat{\mathbb{C}}$ by

$$\varphi(\zeta) = (\zeta^2, \zeta^k), \qquad\qquad \varphi^{-1}((z,\eta)) = \eta z^{-\frac{k-1}{2}} = \sqrt{z}. \tag{1.30}$$

The functions $\lambda \circ \varphi$, $\mathfrak{e} \circ \varphi$ and $\mathfrak{e}^{-1} \circ \varphi$ can be computed from (1.29) and (1.30), and we obtain

$$\lambda(\varphi(\zeta)) = \zeta^{-2M}(1+\zeta^k)^L, \qquad \mathfrak{e}(\varphi(\zeta)) = \begin{pmatrix} 1 & \zeta^k \end{pmatrix}^T, \qquad \mathfrak{e}^{-1}(\varphi(\zeta)) = \frac{1}{2}\begin{pmatrix} 1 & \zeta^{-k} \end{pmatrix}.$$

Since $\varphi(\infty) = \infty^{(1)}$ and $\varphi(0) = 0^{(1)}$, by (1.22)-(1.23) we have $h(\zeta) = 1$ and $\widehat{h}(\zeta) = \zeta^k$. Therefore, from (1.25) and (1.16), we have

$$\mathcal{V} = \mathcal{V}^* = \{P_1(\zeta^2) + \zeta^k P_2(\zeta^2) : P_1, P_2 \in \mathcal{P}_{N-1}\} \qquad \begin{matrix} = \mathcal{P}_{2N-1}, & \text{if } k = 1, \\ \neq \mathcal{P}_{2N-1}, & \text{if } k > 1. \end{matrix} \tag{1.31}$$

Also, since $\phi(\zeta) = \zeta^2$, we infer from (1.24) that

$$\mathcal{W}(\zeta) = 2\zeta^{-2M-k+1}(1+\zeta^k)^L, \qquad\qquad \gamma_{\mathbb{C}} = \varphi^{-1}(\gamma_{\mathcal{M}}) = \varphi^{-1}(\cup_{j=1}^r \gamma^{(j)}).$$

It is interesting to note that $\mathcal{W}$ above is nothing else than the Jacobi weight with non-standard parameters. It is easy to see that if γ is the unit circle, then $\gamma_{\mathbb{C}} = \gamma$, and in this case the associated scalar orthogonal polynomials have been studied in [49]. On the other hand, if $\gamma = (a,b)$ for certain $b > a > 0$, then $\gamma_{\mathbb{C}} = (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b})$; this provides an example of a "one-cut" matrix orthogonality that leads to a "two-cuts" scalar orthogonality. Finally, we note from (1.31) and criteria (e) of Theorem 1.16 that $\mathfrak{R}_{2N}^{\mathcal{W}}$ is a CD kernel if and only if $k = 1$, in which case we have $\mathfrak{R}_{2N}^{\mathcal{W}} = \mathcal{R}_{2N}^{\mathcal{W}}$.

The above example only deals with $r = 2$. Example 1.18 below provides an application of Theorem 1.16 in a situation where the matrix weight is of size $r \times r$, where $r \in \mathbb{N}_{>0}$ is arbitrary.

Example 1.18. Consider the weight W given by (1.3), and let $\gamma \subset \mathbb{C}$ be such that $0 \notin \gamma$. A simple computation shows that

$$W(z) = E(z)\frac{\widehat{\Lambda}(z)^L}{z^R}E(z)^{-1}, \qquad z \in \mathbb{C} \setminus \{0\},$$

with

$$E(z) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z^{\frac{1}{r}} & \rho_r z^{\frac{1}{r}} & \cdots & \rho_r^{r-1} z^{\frac{1}{r}} \\ \vdots & \vdots & \ddots & \vdots \\ z^{\frac{r-1}{r}} & \rho_r^{r-1} z^{\frac{r-1}{r}} & \cdots & \rho_r^{(r-1)^2} z^{\frac{r-1}{r}} \end{pmatrix}, \quad E(z)^{-1} = \frac{1}{r}\begin{pmatrix} 1 & z^{-\frac{1}{r}} & \cdots & z^{-\frac{r-1}{r}} \\ 1 & \rho_r z^{-\frac{1}{r}} & \cdots & \rho_r^{r-1} z^{-\frac{r-1}{r}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_r^{r-1} z^{-\frac{1}{r}} & \cdots & \rho_r^{(r-1)^2} z^{-\frac{r-1}{r}} \end{pmatrix},$$

$\widehat{\Lambda}(z) = \operatorname{diag}(1 + z^{\frac{1}{r}}, 1 + \rho_r z^{\frac{1}{r}}, \ldots, 1 + \rho_r^{r-1} z^{\frac{1}{r}})$ and $\rho_r = e^{\frac{2\pi i}{r}}$, and where the principal branches are taken for the roots. These computations show in particular that W satisfies Assumption 1.1. The

r-sheeted Riemann surface $\mathcal{M}$ associated to $\{(z,\eta)\in\mathbb{C}^2 : \eta^r = z\}$ is of genus 0. For convenience, the sheets are numbered such that $\eta \sim \rho_r^{k-1} z^{\frac{1}{r}}$ as $z\to+\infty$ on the k-th sheet, $k=1,\ldots,r$. The functions $\mathfrak{e}$, $\mathfrak{e}^{-1}$ and λ, defined in (1.8)–(1.10), are given by

$$\mathfrak{e}((z,\eta))^T = \begin{pmatrix}1 & \eta & \eta^2 & \cdots & \eta^{r-1}\end{pmatrix}, \quad \mathfrak{e}^{-1}((z,\eta)) = \frac{1}{r}\begin{pmatrix}1 & \eta^{-1} & \ldots & \eta^{-(r-1)}\end{pmatrix}, \quad \lambda((z,\eta)) = \frac{(1+\eta)^L}{z^R}$$

and are meromorphic on $\mathcal{M}$. Since $\mathcal{M}$ is of genus 0, Theorem 1.16 implies that $\mathcal{R}_N^W$ is equivalent to a scalar kernel $\mathfrak{R}_{rN}^{\mathcal{W}}$ which can be described as follows. Consider the natural bijections $\varphi:\widehat{\mathbb{C}}\to\mathcal{M}$, $\varphi^{-1}:\mathcal{M}\to\widehat{\mathbb{C}}$ given by

$$\varphi(\zeta) = (\zeta^r,\zeta), \qquad\qquad \varphi^{-1}((z,\eta)) = \eta.$$

Since $\mathfrak{e}$ has no zero and a pole of order $r-1$ at $\infty^{(1)} = \ldots = \infty^{(r)} = (\infty,\infty)$, by (1.22) and Definition 1.8, we have

$$\mathcal{Z}=\emptyset, \qquad \mathcal{Q}=\{\infty^{(1)}\}, \qquad n_{\infty^{(1)}} = -(r-1), \qquad h(\zeta)=1.$$

Similarly, $\mathfrak{e}^{-1}$ has no zero and a pole of order $r-1$ at $0^{(1)} = \ldots = 0^{(r)} = (0,0)$, and thus from (1.23) we get

$$\widehat{\mathcal{Z}}=\emptyset, \qquad \widehat{\mathcal{Q}}=\{0^{(1)}\}, \qquad \widehat{n}_{0^{(1)}} = -(r-1), \qquad \widehat{h}(\zeta) = (\zeta-\varphi^{-1}(0^{(1)}))^{r-1} = \zeta^{r-1}.$$

Since $-\sum_{\mathrm{z}\in\mathcal{Z}\cup\mathcal{Q}} n_{\mathrm{z}} = r-1$, we known from criteria (e) of Theorem 1.16 that $\mathcal{V}=\mathcal{V}^*=\mathcal{P}_{rN-1}$. This fact can also be verified directly from the definition (1.25):

$$\mathcal{V}=\mathcal{V}^* = \{P_1(\zeta^r)+\zeta P_2(\zeta^r)+\ldots+\zeta^{r-1}P_r(\zeta^r) : P_1,\ldots,P_r\in\mathcal{P}_{N-1}\} = \mathcal{P}_{rN-1}.$$

Therefore $\mathfrak{R}_{rN}^{\mathcal{W}} = \mathcal{R}_{rN}^{\mathcal{W}}$ is the scalar CD kernel associated to the scalar weight

$$\mathcal{W}(\zeta) = \frac{\lambda(\varphi(\zeta))}{h(\zeta)\widehat{h}(\zeta)}\phi'(\zeta) = r\zeta^{-rR}(1+\zeta)^L, \qquad \gamma_{\mathbb{C}} = \varphi^{-1}(\gamma_{\mathcal{M}}) = \varphi^{-1}(\cup_{j=1}^r \gamma^{(j)}). \tag{1.32}$$

If γ is the unit circle, we mention that the existence of $\mathcal{R}_N^W$ is guaranteed from the general result [25, Lemma 4.8] (see also Section 2). In this case, it is easy to see that $\gamma_{\mathbb{C}}=\gamma$. On the other hand, if $\gamma=(a,b)$ for certain $b>a>0$, then $\gamma_{\mathbb{C}} = \cup_{j=1}^r \rho_r^{j-1}(a^{\frac{1}{r}}, b^{\frac{1}{r}})$ is a disjoint union of r intervals in the complex plane.

We provide other applications of Theorem 1.16 in Section 2.

It is well-known that matrix valued CD kernels can be expressed in terms of MOPs and are related to certain Riemann-Hilbert (RH) problems of large size [27, 20, 24, 36, 25]. Theorem 1.16 (e) establishes a link between matrix and scalar CD kernels, and therefore it also admits two reformulations – one involving MOPs, the other one involving RH problems – which we believe are of interest.

1.1 Theorem 1.16 (e) in terms of MOPs

In the context of matrix orthogonality on the real line or on the unit circle, it is well-known that the reproducing kernel can be expressed in terms of MOPs by mean of a so-called CD formula, see e.g. [27, 20, 24, 36]. The adaptation of this formula to our setting only requires minor modifications, which we present here.

As mentioned in the introduction, the pairing (1.1) induces two families of MOPs.

We denote $P_j^L(z) = z^j I_r + ...$ and $P_j^R(z) = z^j I_r + ...$ for the two degree j MOPs that satisfy

$$\langle P_j^L, z^k I_r \rangle = 0_r, \qquad \langle z^k I_r, P_j^R \rangle = 0_r, \qquad k = 0, \dots, j-1, \tag{1.33}$$

and we let Q_{j-1}^L and Q_{j-1}^R denote the MOPs of degree $\leq j-1$ characterized by

$$\langle Q_{j-1}^L, z^k I_r \rangle = \delta_{k,j-1} I_r, \qquad \langle z^k I_r, Q_{j-1}^R \rangle = \delta_{k,j-1} I_r, \qquad k = 0, \dots, j-1. \tag{1.34}$$

Note that each of the four systems given in (1.33) and (1.34) gives $r^2 j$ equations for the $r^2 j$ unknown scalar coefficients of P_j^L, P_j^R, Q_{j-1}^L or Q_{j-1}^R. Furthermore, the square matrices associated to these four linear systems are identical to one another. Therefore, if we have existence and uniqueness for one polynomial among P_j^L, P_j^R, Q_{j-1}^L and Q_{j-1}^R, this implies existence and uniqueness for the other three polynomials.

We mention that in certain special situations, some of these four families of MOPs can be directly related to each other. For example:

- If $W = W^T$, then $P_j^L = (P_j^R)^T$ and $Q_{j-1}^L = (Q_{j-1}^R)^T$.
- If $Q_j^L(z) = \kappa_j^L z^j + \dots$ exists, and if $\kappa_j^L \in \mathbb{C}^{r\times r}$ is invertible, then P_j^L exists and is given by $(\kappa_j^L)^{-1} Q_j^L$. Similarly, if $Q_j^R(z) = \kappa_j^R z^j + ...$ exists with κ_j^R invertible, then P_j^R exists as well and is given by $Q_j^R(\kappa_j^R)^{-1}$. However, there is no guarantee in general that κ_j^L and κ_j^R are invertible.

If the polynomials $Q_0^R, Q_1^R, \dots, Q_{N-1}^R$ exist, then $\mathcal{R}_N^W$ exists as well and is given by

$$\mathcal{R}_N^W(w,z) = \sum_{j=0}^{N-1} Q_j^R(w) P_j^L(z) = \sum_{j=0}^{N-1} P_j^R(w) Q_j^L(z). \tag{1.35}$$

To verify the validity of the above formula, first note that the sequences of MOPs $\{P_j^L(z)\}_{j\geq 0}$ and $\{Q_j^R(z)\}_{j\geq 0}$ are biorthogonal with respect to (1.1):

$$\langle P_j^L, Q_k^R \rangle = \delta_{k,j} I_r, \qquad \text{for all } j,k \geq 0.$$

Since any polynomial $P \in \mathcal{P}_{N-1}^{r\times r}$ can be represented as $P(z) = \sum_{j=1}^{N-1} C_j P_j^L(z)$ for certain $C_j \in \mathbb{C}^{r\times r}$, one has

$$\langle P, \mathcal{R}_N^W(\cdot, z) \rangle = \int_\gamma \left(\sum_{j=1}^{m} C_j P_j^L(w) \right) W(w) \left(\sum_{k=0}^{N-1} Q_k^R(w) P_k^L(z) \right) dw = P(z).$$

The above reproducing property is equivalent to (1.5); hence the first formula in (1.35) holds by uniqueness of the CD kernel. The second formula in (1.35) can be proved in a similar way. As mentioned, the formulas (1.35) holds if and only if all polynomials $Q_0^R, Q_1^R, \dots, Q_{N-1}^R$ exist. In fact, the existence of Q_{N-1}^R alone ensures the existence of $\mathcal{R}_W$; this follows from the following CD formula:

$$\mathcal{R}_N^W(w,z) = \frac{1}{z-w} \Big(Q_{N-1}^R(w) P_N^L(z) - P_N^R(w) Q_{N-1}^L(z) \Big). \tag{1.36}$$

The above equation was obtained in [25, eq (4.33)], but there it is written in terms of the solution to a certain RH problem (see also (1.38) below) and not directly in the form (1.36). Therefore, for the convenience of the reader, we sketch a proof of (1.36) in Appendix B.

We now present a reformulation of Theorem 1.16 (e) in terms of MOPs.

Theorem 1.19. *Let $\gamma \subset \mathbb{C}$ be a finite union of piecewise smooth, oriented curves, let W be a rational $r \times r$ matrix weight, and let $N \in \mathbb{N}_{>0}$. Suppose that W satisfies Assumption 1.1 and that $\mathcal{R}_N^W$ exists. Let $\mathcal{M}$ be an r-sheeted Riemann surface such that the functions λ, $\mathfrak{e}$ and $\mathfrak{e}^{-1}$ defined in (1.8)–(1.10) are meromorphic, and assume that $\mathcal{M}$ is of genus 0. Define $\mathfrak{R}_{rN}^{\mathcal{W}}$, φ, ϕ, $\mathcal{W}$, $\gamma_{\mathbb{C}}$, $\mathcal{V}$ and $\mathcal{V}^*$ as in Definitions 1.11–1.14. If $\mathcal{V} = \mathcal{P}_{rN-1}$, we have*

$$\begin{aligned}\mathfrak{R}_{rN}^{\mathcal{W}}(\omega,\zeta) &= \widehat{h}(\omega)\mathfrak{e}^{-1}(\varphi(\omega))\frac{Q_{N-1}^R(\phi(\omega))P_N^L(\phi(\zeta)) - P_N^R(\phi(\omega))Q_{N-1}^L(\phi(\zeta))}{\phi(\zeta)-\phi(\omega)}\mathfrak{e}(\varphi(\zeta))h(\zeta) \\ &= \frac{1}{\zeta-\omega}\Big(q_{rN-1}(\omega)p_{rN}(\zeta) - p_{rN}(\omega)q_{rN-1}(\zeta)\Big) = \mathcal{R}_{rN}^{\mathcal{W}}(\omega,\zeta), \end{aligned} \tag{1.37}$$

where $p_{rN}(\zeta) = \zeta^{rN} + \ldots$ is the monic scalar orthogonal polynomial defined by

$$\int_{\gamma_{\mathbb{C}}} p_{rN}(\zeta)\mathcal{W}(\zeta)\zeta^k d\zeta = 0, \qquad k = 0,\ldots,rN-1,$$

and q_{rN-1} is the degree $\leq rN-1$ scalar orthogonal polynomial satisfying

$$\int_{\gamma_{\mathbb{C}}} q_{rN-1}(\zeta)\mathcal{W}(\zeta)\zeta^k d\zeta = \delta_{k,rN-1}, \qquad k = 0,\ldots,rN-1.$$

1.2 Theorem 1.16 (e) in terms of solutions to RH problems

RH problems are boundary value problems for analytic functions; we refer to [22] for an introduction, and to [11] for a recent historical review. It is well-known that scalar orthogonal polynomials can be characterized in terms of 2×2 RH problems [31]. The generalization of this result for MOPs has been studied in great details in [24, 15, 36, 25]. Let γ_0 be the contour γ with all endpoints and points of self-intersection removed. The RH problem that is relevant for our setting is as follows.

RH problem for Y

(a) $Y : \mathbb{C}\setminus\gamma \to \mathbb{C}^{2r\times 2r}$ is analytic.

(b) The limits of $Y(z)$ as z approaches γ_0 from left and right exist, are continuous on γ_0, and are denoted by Y_+ and Y_-, respectively (here "left" and "right" refer to the orientation of γ_0). Furthermore, they are related by

$$Y_+(z) = Y_-(z)\begin{pmatrix} I_r & W(z) \\ 0_r & I_r \end{pmatrix}, \qquad \text{for } z \in \gamma_0.$$

(c) As $z \to \infty$, we have $Y(z) = \big(I_{2r} + \mathcal{O}(z^{-1})\big)\begin{pmatrix} z^N I_r & 0_r \\ 0_r & z^{-N}I_r\end{pmatrix}$.

As $z \to z_\star \in \gamma\setminus\gamma_0$, we have $Y(z) = \mathcal{O}(\log(z-z_\star))$.

The unique solution $Y(\cdot) = Y(\cdot; W,\gamma,N)$ to the above RH problem can be explicitly written in terms of MOPs [25] (see also Appendix B), and exists if and only if P_N^L exists and is unique. Furthermore, Y satisfies $\det Y \equiv 1$, and therefore Y^{-1} exists if and only if Y exists. It follows from [25, eq (4.33)] that the CD kernel $\mathcal{R}_N^W$ can be written as

$$\mathcal{R}_N^W(w,z) = \frac{1}{2\pi i(z-w)}\begin{pmatrix} 0_r & I_r\end{pmatrix} Y^{-1}(w)Y(z)\begin{pmatrix} I_r \\ 0_r\end{pmatrix}. \tag{1.38}$$

Formula (1.38) is in fact equivalent to the CD formula (1.36), see Appendix B for details.

Theorem 1.16 (e) can also be reformulated as follows.

Theorem 1.20. *Let $\gamma \subset \mathbb{C}$ be a finite union of piecewise smooth, oriented curves, let W be a rational $r \times r$ matrix weight, and let $N \in \mathbb{N}_{>0}$. Suppose that W satisfies Assumption 1.1 and that $\mathcal{R}_N^W$ exists. Let $\mathcal{M}$ be an r-sheeted Riemann surface such that the functions λ, $\mathfrak{e}$ and $\mathfrak{e}^{-1}$ defined in (1.8)–(1.10) are meromorphic, and assume that $\mathcal{M}$ is of genus 0. Define $\mathfrak{R}_{rN}^{\mathcal{W}}$, φ, ϕ, $\mathcal{W}$, $\gamma_{\mathbb{C}}$, $\mathcal{V}$ and $\mathcal{V}^*$ as in Definitions 1.11–1.14. If $\mathcal{V} = \mathcal{P}_{rN-1}$, we have*

$$\begin{aligned}\mathfrak{R}_{rN}^{\mathcal{W}}(\omega,\zeta) &= \widehat{h}(\omega)\mathfrak{e}^{-1}(\varphi(\omega))\frac{\begin{pmatrix}0_r & I_r\end{pmatrix} Y^{-1}(\phi(\omega))Y(\phi(\zeta))\begin{pmatrix}I_r\\ 0_r\end{pmatrix}}{2\pi i(\phi(\zeta)-\phi(\omega))}\mathfrak{e}(\varphi(\zeta))h(\zeta)\\ &= \frac{1}{2\pi i(\zeta-\omega)}\begin{pmatrix}0 & 1\end{pmatrix}\mathcal{Y}^{-1}(\omega)\mathcal{Y}(\zeta)\begin{pmatrix}1\\0\end{pmatrix} = \mathcal{R}_{rN}^{\mathcal{W}}(\omega,\zeta),\end{aligned} \tag{1.39}$$

where $\mathcal{Y}$ is the solution to the following RH problem.

RH problem for $\mathcal{Y}$

(a) $\mathcal{Y}: \mathbb{C}\setminus\gamma_{\mathbb{C}} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The limits of $\mathcal{Y}(\zeta)$ as ζ approaches $\gamma_{\mathbb{C},0} := \varphi^{-1}(\cup_{j=1}^r \gamma_0^{(j)})$ from left and right exist, are continuous on $\gamma_{\mathbb{C},0}$, and are denoted by $\mathcal{Y}_+$ and $\mathcal{Y}_-$, respectively. Furthermore, they are related by

$$\mathcal{Y}_+(\zeta) = \mathcal{Y}_-(\zeta)\begin{pmatrix}1 & \mathcal{W}(\zeta)\\ 0 & 1\end{pmatrix}, \qquad \text{for } \zeta \in \gamma_{\mathbb{C},0}.$$

(c) As $\zeta \to \infty$, we have $\mathcal{Y}(\zeta) = \left(I_2 + \mathcal{O}(\zeta^{-1})\right)\begin{pmatrix}\zeta^{rN} & 0\\ 0 & \zeta^{-rN}\end{pmatrix}$.

As $\zeta \to \zeta_\star \in \gamma_{\mathbb{C}}\setminus\gamma_{\mathbb{C},0}$, we have $\mathcal{Y}(\zeta) = \mathcal{O}(\log(\zeta-\zeta_\star))$.

Remark 1.21. *The Deift–Zhou [23] steepest descent method is a powerful tool for asymptotic analysis of RH problems, and is particularly well-developed for RH problems of size 2×2. It is also possible to implement this method on RH problems of larger sizes, but it represents in general a much more complicated task. Theorem 1.20 implies that, in a situation where $\mathcal{M}$ has genus 0 and $\mathcal{V} = \mathcal{P}_{rN-1}$ (such as in Example 1.18), one can study asymptotic properties of $\mathcal{R}_N^W$ by means of a 2×2 RH problem instead of a $2r \times 2r$ RH problem. This fact has already been proved useful in [16] in a situation where $r = 2$.*

2 Applications to tiling models

Tiling models with doubly periodic weightings form a class of determinantal point processes with new interesting features [18, 19, 7, 25, 10, 17, 9, 16, 8]. As it turns out, the correlation structure of these models can be studied by means of certain double contour integrals [25] which involve a matrix valued CD kernel. In this section, we use Theorems 1.5 and 1.16 to simplify this formula. We first present the necessary material to invoke the formula from [25]. To simplify the presentation, we are going to focus on *lozenge tilings of a hexagon*. We mention however that the main theorem of [25], and therefore also Theorem 2.4 below, can be applied to various other tiling models.

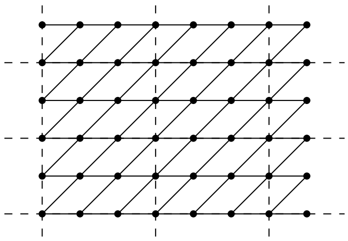
Lozenge tilings of a hexagon. Consider the infinite graph $\mathcal{G}_H$ whose vertex set is $\mathbb{Z}\times\mathbb{Z}$, and whose edges are of the form $\big((x,y),(x+1,y)\big)$ or $\big((x,y),(x+1,y+1)\big)$. A weighting on $\mathcal{G}_H$ consists of assigning to each edge a positive number. Here, we consider weightings that are $r\times q$ periodic, which means periodic of period r in the vertical direction, and periodic of period q in the horizontal direction. More precisely, an $r\times q$ periodic weighting depends on $2rq$ edge weights, denoted by $a_{\ell,j}$, $b_{\ell,j}$, $0\le \ell\le q-1$, $0\le j\le r-1$, which we assign as follows:

$$\text{weight of } \big((\ell,j),(\ell+1,j+\delta)\big)=\begin{cases} b_{\ell,j}, & \text{if } \delta=0,\\ a_{\ell,j}, & \text{if } \delta=1,\end{cases} \qquad 0\le \ell\le q-1,\ 0\le j\le r-1. \tag{2.1}$$

Then, the weighting is extended over all edges of $\mathcal{G}_H$ by

$$\text{weight of } \big((\ell+m_1q,j+m_2r),(\ell+m_1q+1,j+m_2r+\delta)\big)=\text{weight of } \big((\ell,j),(\ell+1,j+\delta)\big), \tag{2.2}$$

for all $m_1,m_2\in\mathbb{Z}$, and $\delta\in\{0,1\}$. The situation is illustrated in Figure 1 for $r=2$ and $q=3$.



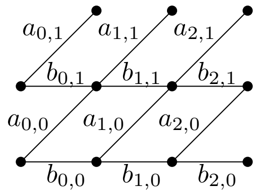


Figure 1: Left: the graph $\mathcal{G}_H$. The dashed lines emphasize the 2×3 periodicity, but are not part of $\mathcal{G}_H$. Right: the assignment of the weights on a 2×3 block of $\mathcal{G}_H$.

We define q matrices $A_0,\ldots,A_{q-1}$, each of them of size $r\times r$, by

$$A_\ell(z)=\begin{pmatrix} b_{\ell,0} & a_{\ell,0} & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & b_{\ell,1} & a_{\ell,1} & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{\ell,r-2} & a_{\ell,r-2}\\ za_{\ell,r-1} & 0 & 0 & 0 & \cdots & 0 & 0 & b_{\ell,r-1}\end{pmatrix}, \qquad \ell=0,\ldots,q-1, \tag{2.3}$$

and for all $m_1\in\mathbb{Z}$, we define $A_{\ell+m_1q}=A_\ell$. We can retrieve the weighting from these matrices by

$$\text{weight of } \big((\ell,ry_1+j-1),(\ell+1,ry_2+k-1)\big)=\frac{1}{2\pi i}\oint_\gamma (A_\ell(z))_{j,k}z^{y_1-y_2}\frac{dz}{z}, \tag{2.4}$$

where $\ell,y_1,y_2\in\mathbb{Z}$, $1\le j,k\le r$, and γ is the unit circle oriented positively. Given three positive integers L, N and M, we consider the subgraph $\widehat{\mathcal{G}}_H$ that consists of the vertices and edges of $\mathcal{G}_H$ that lie entirely in the hexagon

$$\mathcal{H}:=\{(x,y)\in\mathbb{R}^2: 0\le x\le L,\ -\tfrac12\le y\le N+M-\tfrac12,\ -(L-M)-\tfrac12\le y-x\le N-\tfrac12\},$$

see also Figure 2 (left). We say that $\mathfrak{p}:\{0,1,\ldots,L\}\to\mathbb{Z}^2\cap\mathcal{H}$ is a path of $\widehat{\mathcal{G}}_H$ if $\mathfrak{p}(\ell+1)-\mathfrak{p}(\ell)\in\{0,1\}$ for each $\ell\in\{0,\ldots,L-1\}$. The set of all systems of N non-intersecting paths of $\widehat{\mathcal{G}}_H$ is in bijection with the set of all lozenge tilings of $\mathcal{H}$. This one-to-one correspondence is well-known, and is illustrated in Figure 2 (middle and right) for a particular example. The weighting on the edges of $\widehat{\mathcal{G}}_H$ naturally induces a weighting on the paths $\{\mathfrak{p}\}$, and on the tilings $\{\mathcal{T}\}$, by

$$\text{weight of } \mathfrak{p}=\prod_{\ell=0}^{L-1}\text{weight of } \big((\ell,\mathfrak{p}(\ell)),(\ell+1,\mathfrak{p}(\ell+1))\big) \quad\text{and}\quad \text{weight of } \mathcal{T}=\prod_{\mathfrak{p}\in\mathcal{T}}\text{weight of } \mathfrak{p}.$$

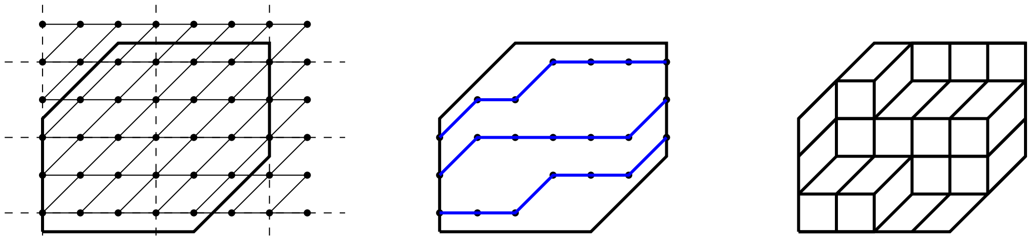


Figure 2: Left: a hexagon with 2×3 periodic weightings, and $N=3$, $M=2$, and $L=6$. Middle: a system of N non-intersecting paths, and the associated points. Right: The corresponding lozenge tiling of the hexagon.

Since the weights on the edges are positive, this defines a probability measure over the set $\{\mathcal{T}\}$ by

$$\mathbb{P}(\mathcal{T}) = \frac{\text{weight of } \mathcal{T}}{\sum_{\mathcal{T}'} \text{weight of } \mathcal{T}'}, \tag{2.5}$$

where the sum is taken over all tilings. By placing points on the paths as shown in Figure 2 (middle), each tiling $\mathcal{T}$ generates a point configuration. Hence, the probability measure (2.5) can be seen as a discrete point process. As it turns out [34, 48, 30], this point process is *determinantal*, which implies that all the information about (2.5) is encoded in a *correlation kernel* $K : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$. Let $k \geq 1$ be an integer. The determinantal property of the process means that, for any integers $x_1, \ldots, x_k, y_1, \ldots, y_k$ satisfying $0 \leq x_1, \ldots, x_k \leq L$ and $(x_j, y_j) \neq (x_\ell, y_\ell)$ if $j \neq \ell$, we have

$$\mathbb{P}\begin{bmatrix} N \text{ non-intersecting paths go through} \\ \text{each of the points } (x_1,y_1),\ldots,(x_k,y_k) \end{bmatrix} = \det\left[K(x_i,y_i,x_j,y_j)\right]_{i,j=1}^{k}. \tag{2.6}$$

The main result of [25] is the following integral formula for K involving a matrix valued CD kernel.[3]

Theorem 2.1. *[25, Theorem 4.7]. Assume that L is a multiple of q, and that N and M are multiples of r. For $x_1, x_2 \in \{1, \ldots, L-1\}$ and $y_1, y_2 \in \mathbb{Z}$, we have*

$$\begin{aligned}[K(x_1, ry_1+j, x_2, ry_2+i)]_{i,j=0}^{r-1} = &-\frac{\chi_{x_1>x_2}}{2\pi i}\oint_\gamma B_4(z)A(z)^{L_3}B_3(z)dz \\ &+ \frac{1}{2\pi i}\oint_\gamma\oint_\gamma B_2(w)A(w)^{L_2}\mathcal{R}^W_{\frac{N}{r}}(w,z)A(z)^{L_1}B_1(z)dzdw, \end{aligned} \tag{2.7}$$

where γ is the unit circle oriented positively,

$$L_1 = \lfloor \tfrac{x_1}{q} \rfloor, \qquad L_2 = \tfrac{L}{q} - \lceil \tfrac{x_2}{q} \rceil, \qquad L_3 = \max\left\{\lfloor \tfrac{x_1}{q} \rfloor - \lceil \tfrac{x_2}{q} \rceil, 0\right\}, \qquad \chi_{x_1>x_2} = \begin{cases} 1, & \text{if } x_1 > x_2, \\ 0, & \text{otherwise,} \end{cases}$$

the matrix weight is given by

$$W(z) = z^{-\frac{M+N}{r}} A(z)^{\frac{L}{q}}, \qquad A(z) = \prod_{\ell=0}^{q-1} A_\ell(z), \tag{2.8}$$

[3] R_N in [25] corresponds to $2\pi i \mathcal{R}^W_N$ in this paper.

and the matrices B_1, B_2, B_3, B_4 are given by

$$B_1(z) = z^{-y_1-1} \prod_{\ell=q\lfloor \frac{x_1}{q} \rfloor}^{x_1-1} A_\ell(z), \qquad B_2(z) = \frac{z^{y_2}}{z^{\frac{M+N}{r}}} \prod_{\ell=x_2}^{q\lceil \frac{x_2}{q} \rceil -1} A_\ell(z),$$

$$B_3(z) = z^{-y_1-1} \begin{cases} \displaystyle\prod_{\ell=q\lfloor \frac{x_1}{q} \rfloor}^{x_1-1} A_\ell(z), & \text{if } \lfloor \frac{x_1}{q} \rfloor \geq \lceil \frac{x_2}{q} \rceil, \\ I_r, & \text{otherwise}, \end{cases} \qquad B_4(z) = z^{y_2} \begin{cases} \displaystyle\prod_{\ell=x_2}^{q\lceil \frac{x_2}{q} \rceil -1} A_\ell(z), & \text{if } \lfloor \frac{x_1}{q} \rfloor \geq \lceil \frac{x_2}{q} \rceil, \\ \displaystyle\prod_{\ell=x_2}^{x_1} A_\ell(z), & \text{otherwise}. \end{cases}$$

Remark 2.2. *In this set-up, the existence of $\mathcal{R}^W_{N/r}$ is ensured from [25, Lemma 4.8].*

If one desires to understand the fine asymptotic structure of the lozenge tiling model under consideration, one usually needs to obtain asymptotics for $K(x_1, y_1, x_2, y_2)$ as $L, M, N \to +\infty$, and simultaneously $x_1, y_1, x_2, y_2 \to +\infty$ at certain critical speeds. Since the matrix products that appear in $B_1, \ldots, B_4$ always involve at most $q-1$ matrices, the B_j's are not an obstacle to an asymptotic analysis. Slightly more problematic is the fact that $L_1, L_2, L_3 \to +\infty$ in this regime, and that these quantities are exponents of A in (2.7). But this, in itself, is also not a serious obstacle if A is diagonalizable. Since W depends on the large parameters L, M and N, see (2.8), the real challenge is to obtain asymptotics of $\mathcal{R}^W_{N/r}$, especially when W is matrix-valued. Theorems 1.5 and 1.16 allows to circumvent (or simplify) this serious technical obstacle, provided A satisfies the following assumption.

Assumption 2.3. The rational $r \times r$ matrix valued function $z \mapsto A(z)$ has no pole on γ and is diagonalizable for all but finitely many $z \in \mathbb{C}$.

Since A is rational, its eigenvalues $\widehat{\lambda}_1(z), \ldots, \widehat{\lambda}_r(z)$ are (branches of) meromorphic functions on $\mathbb{C}$, which together define a meromorphic function $\widehat{\lambda}$ on an r-sheeted compect Riemann surface $\mathcal{M}$. For definiteness, we choose the numbering of the sheets such that

$$\mathrm{z} \mapsto \widehat{\lambda}(\mathrm{z}) = \widehat{\lambda}_k(z), \qquad \text{if z is on the } k\text{-th sheet of } \mathcal{M}. \tag{2.9}$$

Assumption 2.3 implies that for all but finitely many $z \in \mathbb{C}$, we can write

$$A(z) = E(z)\widehat{\Lambda}(z)E(z)^{-1}, \qquad \widehat{\Lambda}(z) = \mathrm{diag}(\widehat{\lambda}_1(z), \ldots, \widehat{\lambda}_r(z)), \tag{2.10}$$

and as in Section 3, we can (and do) choose the matrix of eigenvectors E such that the functions $\mathfrak{e}$ and $\mathfrak{e}^{-1}$ defined in (1.8)–(1.10) are meromorphic on $\mathcal{M}$. Since

$$W(z) = z^{-\frac{M+N}{r}} A(z)^{\frac{L}{q}}, \qquad \text{we have} \qquad \lambda_k(z) = z^{-\frac{M+N}{r}} \widehat{\lambda}_k(z)^{\frac{L}{q}}, \quad k = 1, \ldots, r,$$

and therefore the function λ defined in (1.8) is also meromorphic on $\mathcal{M}$.

Theorem 2.4 below is our general result for tiling models with doubly periodic weightings. If $\mathcal{M}$ is a connected Riemann surface of genus 0, then formula (2.12) provides a new double contour formula representation of K. The main advantage of this formula is that all the quantities involving the (potentially large) parameters L, M and N are now scalar. This formula can be viewed as a generalization of [16, Theorem 3.2], which was valid for the very particular 2×2 matrix weight (1.4). If $\mathcal{M}$ has either no genus (this is the case if $\mathcal{M}$ is not a connected Riemann surface), or if $\mathcal{M}$ has genus 1 or more, then formula (2.12) does not apply. In this case we instead give a new representation of the kernel which involve a scalar kernel in a Riemann surface, see formula (2.11) below. Here too, all the quantities involving the parameters L, M and N are scalar. Therefore, we expect that this formula also leads to a simpler asymptotic analysis than (2.7), although we admit that this still remains to be shown in a concrete situation.

Theorem 2.4. *Suppose that A satisfies Assumption 2.3. Let $\mathcal{M}$ be an r-sheeted Riemann surface such that the functions $\widehat{\lambda}$, $\mathfrak{e}$ and $\mathfrak{e}^{-1}$ defined in (2.9), (1.9), (1.10) are meromorphic, and define $R^{\lambda}_{N/r}$ as in (1.14). The formula (2.7) can be simplified as follows:*

$$[K(x_1, ry_1+j, x_2, ry_2+i)]_{i,j=0}^{r-1} = \frac{1}{2\pi i}\oint_{\gamma_{\mathcal{M}}}\oint_{\gamma_{\mathcal{M}}} \widehat{\lambda}(\mathrm{w})^{L_2}\widehat{\lambda}(\mathrm{z})^{L_1} B_2(w)\mathfrak{e}(\mathrm{w})R^{\lambda}_{N/r}(\mathrm{w},\mathrm{z})\mathfrak{e}^{-1}(\mathrm{z})B_1(z)dzdw$$
$$- \frac{\chi}{2\pi i}\oint_{\gamma_{\mathcal{M}}} \widehat{\lambda}(\mathrm{z})^{L_3} B_4(z)\mathfrak{e}(\mathrm{z})\mathfrak{e}^{-1}(\mathrm{z})B_3(z)dz. \tag{2.11}$$

If furthermore $\mathcal{M}$ is of genus 0, then the right-hand-side of (2.11) can be further simplified into

$$\frac{1}{2\pi i}\oint_{\gamma_{\mathbb{C}}}\oint_{\gamma_{\mathbb{C}}} \frac{\mathfrak{R}^{\mathcal{W}}_N(\omega,\zeta)}{\widehat{h}(\omega)h(\zeta)}\widehat{\lambda}(\varphi(\omega))^{L_2}\widehat{\lambda}(\varphi(\zeta))^{L_1}\phi'(\omega)\phi'(\zeta)B_2(\phi(\omega))\mathfrak{e}(\varphi(\omega))\mathfrak{e}^{-1}(\varphi(\zeta))B_1(\phi(\zeta))d\zeta d\omega$$
$$- \frac{\chi}{2\pi i}\oint_{\gamma_{\mathbb{C}}} \widehat{\lambda}(\varphi(\zeta))^{L_3} B_4(\phi(\zeta))\mathfrak{e}(\varphi(\zeta))\mathfrak{e}^{-1}(\varphi(\zeta))B_3(\phi(\zeta))\phi'(\zeta)d\zeta, \tag{2.12}$$

where $\mathfrak{R}^{\mathcal{W}}_N$, φ, ϕ, $\mathcal{W}$, $\gamma_{\mathbb{C}}$ are defined in Definitions 1.11–1.13.

Proof. Substituting $A(z) = E(z)\widehat{\Lambda}(z)E(z)^{-1}$ in (2.7), we get

$$[K(x_1, ry_1+j, x_2, ry_2+i)]_{i,j=0}^{r-1} = -\frac{\chi}{2\pi i}\oint_{\gamma} B_4(z)E(z)\widehat{\Lambda}(z)^{L_3}E(z)^{-1}B_3(z)dz$$
$$+ \frac{1}{2\pi i}\oint_{\gamma}\oint_{\gamma} B_2(w)E(w)\widehat{\Lambda}(w)^{L_2}E(w)^{-1}\mathcal{R}^{W}_{N/r}(w,z)E(z)\widehat{\Lambda}(z)^{L_1}E(z)^{-1}B_1(z)dzdw.$$

Note that for any integer x, we can write

$$\widehat{\Lambda}(z)^x = \sum_{j=1}^{r} e_j e_j^T \widehat{\lambda}_j(z)^x = \sum_{j=1}^{r} e_j e_j^T \widehat{\lambda}(z^{(j)})^x.$$

Hence, using (1.14), we obtain

$$[K(x_1, ry_1+j, x_2, ry_2+i)]_{i,j=0}^{r-1} = -\frac{\chi}{2\pi i}\sum_{j=1}^{r}\oint_{\gamma} B_4(z)\mathfrak{e}(z^{(j)})\widehat{\lambda}(z^{(j)})^{L_3}\mathfrak{e}^{-1}(z^{(j)})B_3(z)dz$$
$$+ \frac{1}{2\pi i}\sum_{j=1}^{r}\sum_{k=1}^{r}\oint_{\gamma}\oint_{\gamma} B_2(w)\mathfrak{e}(w^{(k)})\widehat{\lambda}(w^{(k)})^{L_2}R^{\lambda}_{N/r}(w^{(k)}, z^{(j)})\widehat{\lambda}(z^{(j)})^{L_1}\mathfrak{e}^{-1}(z^{(j)})B_1(z)dzdw,$$

which is equivalent to (2.11) (recal that $\gamma_{\mathcal{M}} = \cup_{j=1}^{r}\gamma^{(j)}$). The formula (2.12) then directly follows from (1.21), the change of variables $\mathrm{z} = \varphi(\zeta)$ and $\mathrm{w} = \varphi(\omega)$, and the definition $\gamma_{\mathbb{C}} = \varphi^{-1}(\gamma_{\mathcal{M}})$. $\square$

In the rest of this section, we give three applications of Theorem 2.4.

2.1 The uniform measure viewed as a $r \times 1$ periodic weighting

The uniform measure over the lozenge tilings of the hexagon has already been extensively studied [39, 6, 35, 1]. It corresponds to the case where all edges of $\mathcal{G}_H$ are assigned the same positive number, say 1. In the terminology introduce at the beginning of this section, this is a model with a 1×1

periodic weighting. Therefore, the correlation kernel of this model is given by Theorem 2.1 with $r = q = 1$ and $A(z) = A_0(z) = 1 + z$:

$$K(x_1, y_1, x_2, y_2) = \frac{1}{2\pi i} \oint_\gamma \oint_\gamma \frac{(1+w)^{L-x_2}}{w^{M+N-y_2}} \mathcal{R}_N^{\widetilde{W}}(w,z) \frac{(1+z)^{x_1}}{z^{y_1+1}} dz dw \\ - \frac{\chi_{x_1>x_2}}{2\pi i} \oint_\gamma (1+z)^{x_1-x_2} z^{y_2-y_1-1} dz, \tag{2.13}$$

where $\widetilde{W}(z) = (1+z)^L z^{-M-N}$ and γ is the unit circle. In this subsection, we do not provide new results, but we provide an example which we believe illustrates well how to use Theorem 2.4. Since the uniform measure is 1×1 periodic, it can in particular be viewed as a model with an $r \times 1$ periodic weighting, $r \geq 2$. Assume that M and N are multiple of r. In this case, by (2.3) the transition matrix $A(z) = A_0(z)$ is given by

$$A(z) = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ z & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{2.14}$$

Applying Theorem 2.1 to this case, for any $x_1, x_2 \in \{1, \ldots, L-1\}$ and $y_1, y_2 \in \mathbb{Z}$ we obtain

$$[K(x_1, ry_1 + j, x_2, ry_2 + i)]_{i,j=0}^{r-1} = \frac{1}{2\pi i} \oint_\gamma \oint_\gamma \frac{A(w)^{L-x_2}}{w^{\frac{M+N}{r} - y_2}} \mathcal{R}_{N/r}^W(w,z) \frac{A(z)^{x_1}}{z^{y_1+1}} dz dw \\ - \frac{\chi_{x_1>x_2}}{2\pi i} \oint_\gamma A(z)^{x_1-x_2} z^{y_2-y_1-1} dz,$$

or equivalently,

$$K(x_1, y_1, x_2, y_2) = \frac{1}{2\pi i} \oint_\gamma \oint_\gamma e^T_{y_2 - r\lfloor \frac{y_2}{r} \rfloor + 1} \frac{A(w)^{L-x_2}}{w^{\frac{M+N}{r} - \lfloor \frac{y_2}{r} \rfloor}} \mathcal{R}_{N/r}^W(w,z) \frac{A(z)^{x_1}}{z^{\lfloor \frac{y_1}{r} \rfloor + 1}} e_{y_1 - r\lfloor \frac{y_1}{r} \rfloor + 1} dz dw \\ - \frac{\chi_{x_1>x_2}}{2\pi i} \oint_\gamma e^T_{y_2 - r\lfloor \frac{y_2}{r} \rfloor + 1} A(z)^{x_1-x_2} e_{y_1 - r\lfloor \frac{y_1}{r} \rfloor + 1} z^{\lfloor \frac{y_2}{r} \rfloor - \lfloor \frac{y_1}{r} \rfloor - 1} dz, \tag{2.15}$$

where $W(z) = z^{-\frac{M+N}{r}} A(z)^L$ and we recall that e_j denotes the j-th column of I_r. Note that W is exactly the matrix weight written in (1.3) (see also Example 1.18) after replacing R by $\frac{M+N}{r}$. Since the right-hand-sides of (2.13) and (2.15) represent the same quantity (they are correlation kernels of one, single model), they must obviously be related (though a priori not necessarily equal[4]). However, this is fairly non-trivial task to verify this directly from the formulas. Here, we provide a direct proof for this fact using Theorem 2.4.

Proposition 2.5. *The right-hand-sides of* (2.13) *and* (2.15) *are equal.*

Proof. This proof relies on the computations done in Example 1.18. Since $\mathcal{M}$ is of genus 0, we infer from Theorem 2.4 that the right-hand-side of (2.15) can be rewritten as

$$\frac{1}{2\pi i} \oint_\gamma \oint_\gamma \frac{\mathcal{R}_N^{\mathcal{W}}(\omega, \zeta)}{\widehat{h}(\omega) h(\zeta)} \frac{\widehat{\lambda}(\varphi(\omega))^{L-x_2}}{\phi(\omega)^{\frac{M+N}{r} - \lfloor \frac{y_2}{r} \rfloor}} \frac{\widehat{\lambda}(\varphi(\zeta))^{x_1}}{\phi(\zeta)^{\lfloor \frac{y_1}{r} \rfloor + 1}} e^T_{y_2 - r\lfloor \frac{y_2}{r} \rfloor + 1} \mathfrak{e}(\varphi(\omega)) \mathfrak{e}^{-1}(\varphi(\zeta)) e_{y_1 - r\lfloor \frac{y_1}{r} \rfloor + 1} \phi'(\omega) \phi'(\zeta) d\zeta d\omega \\ - \frac{\chi_{x_1>x_2}}{2\pi i} \oint_\gamma \widehat{\lambda}(\varphi(\zeta))^{x_1-x_2} e^T_{y_2 - r\lfloor \frac{y_2}{r} \rfloor + 1} \mathfrak{e}(\varphi(\zeta)) \mathfrak{e}^{-1}(\varphi(\zeta)) e_{y_1 - r\lfloor \frac{y_1}{r} \rfloor + 1} \phi(\zeta)^{\lfloor \frac{y_2}{r} \rfloor - \lfloor \frac{y_1}{r} \rfloor - 1} \phi'(\zeta) d\zeta,$$

[4]The correlation kernel of a given point process is not unique. For example, the kernels K and $\tilde{K}(x_1, y_1, x_2, y_2) := K(x_1, y_1, x_2, y_2) \frac{f_1(x_1)}{f_1(x_2)} \frac{f_2(y_1)}{f_2(y_2)}$, where f_1 and f_2 are arbitrary non-zero functions, define the same point process, because the determinants in (2.6) remain unchanged.

where we have also used that $\mathfrak{R}_N^W = \mathcal{R}_N^{\mathcal{W}}$ and $\gamma_{\mathbb{C}} = \gamma$. Using the explicit expressions for $\mathfrak{e}$, $\mathfrak{e}^{-1}$, λ, φ, ϕ, h and $\widehat{h}$ from Example 1.18, and noting that $\widehat{\lambda}((z,\eta)) = 1+\eta$, this gives

$$\begin{aligned}&\frac{1}{2\pi i}\oint_\gamma\oint_\gamma \frac{\mathcal{R}_N^{\mathcal{W}}(\omega,\zeta)}{\omega^{r-1}}\frac{(1+\omega)^{L-x_2}}{\omega^{M+N-r\lfloor\frac{y_2}{r}\rfloor}}\frac{(1+\zeta)^{x_1}}{\zeta^{r\lfloor\frac{y_1}{r}\rfloor+r}}\omega^{y_2-r\lfloor\frac{y_2}{r}\rfloor}\zeta^{-y_1+r\lfloor\frac{y_1}{r}\rfloor}\frac{1}{r}r\omega^{r-1}r\zeta^{r-1}d\zeta d\omega\\ &-\frac{\chi_{x_1>x_2}}{2\pi i}\oint_\gamma(1+\zeta)^{x_1-x_2}\zeta^{y_2-r\lfloor\frac{y_2}{r}\rfloor}\frac{\zeta^{-y_1+r\lfloor\frac{y_1}{r}\rfloor}}{r}\zeta^{r\lfloor\frac{y_2}{r}\rfloor-r\lfloor\frac{y_1}{r}\rfloor-r}r\zeta^{r-1}d\zeta. \qquad (2.16)\end{aligned}$$

Note that the functions $\widetilde{W}$ and $\mathcal{W}$, defined respectively below (2.13) and in (1.32), are related by $\widetilde{W}(z) = \frac{1}{r}\mathcal{W}(z)$. Therefore, it is straightforward to deduce from (1.5) that $\mathcal{R}_N^{\mathcal{W}}(\omega,\zeta) = \frac{1}{r}\mathcal{R}_N^{\widetilde{W}}(\omega,\zeta)$. Substituting this relation into (2.16) and simplifying, we find (2.13). $\square$

2.2 Lozenge tiling models with a 2×1 periodic weighting

The main result of this subsection is an explicit new double contour integral formula for the kernel of an arbitrary lozenge tiling model with a 2×1 periodic weighting. In this formula, the integrand only involves a scalar CD kernel.

Let us set $r = 2$ and $q = 1$ in (2.1)–(2.2). In the most general setting, the transition matrix $A(z) = A_0(z)$ is given by

$$A(z) = \begin{pmatrix} b_0 & a_0 \\ a_1 z & b_1\end{pmatrix}, \qquad a_0, a_1, b_0, b_1 > 0, \qquad (2.17)$$

where we have written a_j, b_j for the edge weights instead of $a_{0,j}, b_{0,j}$ to lighten the notation. Let us assume that M and N are multiples of 2. We know from Theorem 2.1 that the kernel K associated to (2.5) is given by

$$\begin{aligned}[K(x_1, ry_1+j, x_2, ry_2+i)]_{i,j=0}^1 = &-\frac{\chi_{x_1>x_2}}{2\pi i}\oint_\gamma A(z)^{x_1-x_2}z^{y_2-y_1-1}dz\\ &+\frac{1}{2\pi i}\oint_\gamma\oint_\gamma A(w)^{L-x_2}\mathcal{R}_{N/2}^W(w,z)A(z)^{x_1}\frac{dzdw}{w^{\frac{M+N}{2}-y_2}z^{y_1+1}}, \qquad (2.18)\end{aligned}$$

for $x_1, x_2 \in \{1,...,L-1\}$ and $y_1, y_2 \in \mathbb{Z}$, with $W(z) = z^{-\frac{M+N}{2}}A(z)^L$. The above formula involves a CD kernel of size 2×2. Theorem 2.4 allows to simplify (2.18) as follows.

Theorem 2.6. *For $x_1, x_2 \in \{1,\dots,L-1\}$ and $y_1, y_2 \in \mathbb{Z}$, we have*

$$\begin{aligned}[K(x_1, 2y_1+j, x_2, 2y_2+i)]_{i,j=0}^1 = &-\frac{\chi_{x_1>x_2}}{2\pi i}\oint_{\gamma_{\mathbb{C}}}\frac{\left(\frac{b_0+b_1+\zeta}{2}\right)^{x_1-x_2}}{\left(\frac{\zeta^2-(b_0-b_1)^2}{4a_0a_1}\right)^{y_1-y_2+1}}\begin{pmatrix}\frac{\frac{\zeta-b_1+b_0}{2}}{\frac{\zeta^2-(b_1-b_0)^2}{4a_0}} & \frac{a_0}{\frac{\zeta+b_1-b_0}{2}}\end{pmatrix}\frac{d\zeta}{2a_0a_1}\\ &+\frac{1}{2\pi i}\oint_{\gamma_{\mathbb{C}}}\oint_{\gamma_{\mathbb{C}}}\frac{\mathcal{R}_N^{\mathcal{W}}(\omega,\zeta)\left(\frac{b_0+b_1+\zeta}{2}\right)^{x_1}\left(\frac{b_0+b_1+\omega}{2}\right)^{L-x_2}}{\left(\frac{\zeta^2-(b_1-b_0)^2}{4a_1a_0}\right)^{y_1+1}\left(\frac{\omega^2-(b_1-b_0)^2}{4a_1a_0}\right)^{\frac{M+N}{2}-y_2}}\begin{pmatrix}\frac{\frac{\zeta-(b_1-b_0)}{2}}{\frac{(\zeta-b_1+b_0)(\omega+b_1-b_0)}{4a_0}} & \frac{a_0}{\frac{\omega+b_1-b_0}{2}}\end{pmatrix}\frac{d\zeta d\omega}{4a_0^2a_1^2},\end{aligned}$$

where $\mathcal{W}(\omega) = \frac{1}{2a_0a_1}\left(\frac{b_0+b_1+\zeta}{2}\right)^L\left(\frac{4a_0a_1}{\zeta^2-(b_0-b_1)^2}\right)^{\frac{M+N}{2}}$. The contour $\gamma_{\mathbb{C}}$ is a circle oriented positively and surrounding both b_0-b_1 and b_1-b_0.

Remark 2.7. *Note that none of the (possibly large) parameters $L, M, N, x_1, y_1, x_2, y_2$ appear in the remaining matrices in the integrands. Therefore, this formula is expected to lead to a much simpler asymptotic analysis than (2.7), see also the discussion above Theorem 2.4.*

Proof. It is a simple computation to verify that A in (2.17) can be written in the form (2.10) with

$$E(z)=\begin{pmatrix}1 & 1\\ \frac{b_1-b_0+\sqrt{\Delta}}{2a_0} & \frac{b_1-b_0-\sqrt{\Delta}}{2a_0}\end{pmatrix},\qquad E(z)^{-1}=\begin{pmatrix}-\frac{b_1-b_0-\sqrt{\Delta}}{2\sqrt{\Delta}} & \frac{a_0}{\sqrt{\Delta}}\\ -\frac{b_1-b_0+\sqrt{\Delta}}{-2\sqrt{\Delta}} & \frac{a_0}{-\sqrt{\Delta}}\end{pmatrix}, \tag{2.19}$$

and

$$\widehat{\Lambda}(z)=\operatorname{diag}\left(\frac{b_0+b_1+\sqrt{\Delta}}{2},\frac{b_0+b_1-\sqrt{\Delta}}{2}\right),\qquad \Delta=\Delta(z)=4a_0a_1(z-z_1),\quad z_1=-\frac{(b_0-b_1)^2}{4a_0a_1},$$

and the principal branch is chosen for $\sqrt{\Delta}$. In particular A satisfies Assumption 2.3. Let $\mathcal{M}$ be the genus 0 Riemann surface associated to $\{(z,\eta)\in\mathbb{C}^2:\eta^2=\Delta(z)\}$. We view $\mathcal{M}$ as two copies of $\widehat{\mathbb{C}}$ that are glued along $(-\infty,z_1)$, with $\eta=\sqrt{\Delta(z)}$ on the first sheet, and $\eta=-\sqrt{\Delta(z)}$ on the second sheet. The function

$$\varphi(\zeta)=(\phi(\zeta),\zeta)\qquad\text{with}\qquad \phi(\zeta)=z_1+\frac{\zeta^2}{4a_0a_1}$$

is a bijection from $\widehat{\mathbb{C}}$ to $\mathcal{M}$ whose inverse is $\varphi^{-1}((z,\eta))=\eta$. Substituting (2.19) in (1.8)–(1.10), we obtain the following expressions

$$\mathfrak{e}(\varphi(\zeta))^T=\begin{pmatrix}1 & \frac{b_1-b_0+\zeta}{2a_0}\end{pmatrix},\qquad \mathfrak{e}^{-1}(\varphi(\zeta))=\begin{pmatrix}\frac{\zeta+b_0-b_1}{2\zeta} & \frac{a_0}{\zeta}\end{pmatrix},\qquad \lambda(\varphi(\zeta))=\frac{\widehat{\lambda}(\varphi(\zeta))^L}{\phi(\zeta)^{\frac{M+N}{2}}}, \tag{2.20}$$

with $\widehat{\lambda}(\varphi(\zeta))=\frac{b_1+b_0+\zeta}{2}$. The function $\mathfrak{e}$ has no zero and a simple pole at $\infty^{(1)}$, while $\mathfrak{e}^{-1}$ has no zero and a simple pole at $(z_1,0)=z_1^{(1)}=z_1^{(2)}$. Hence, from Definition 1.8 we have

$$\mathcal{Z}=\emptyset,\qquad \mathcal{Q}=\{\infty^{(1)}\},\qquad n_{\infty^{(1)}}=-1,\qquad \widehat{\mathcal{Z}}=\emptyset,\qquad \widehat{\mathcal{Q}}=\{z_1^{(1)}\},\qquad \widehat{n}_{z_1^{(1)}}=-1.$$

Since $\varphi(\infty)=\infty^{(1)}=\infty^{(2)}$, the functions h and $\widehat{h}$ defined in (1.22)-(1.23) reduce here to $h(\zeta)=1$ and $\widehat{h}(\zeta)=\zeta$, and the scalar weight (1.24) is given by

$$\mathcal{W}(\zeta)=\frac{\lambda(\varphi(\zeta))}{h(\zeta)\widehat{h}(\zeta)}\phi'(\zeta)=\frac{1}{2a_0a_1}\Big(\frac{b_0+b_1+\zeta}{2}\Big)^L\Big(\frac{4a_0a_1}{\zeta^2-(b_0-b_1)^2}\Big)^{\frac{M+N}{2}}.$$

Since

$$-\sum_{\mathrm{z}\in\mathcal{Z}\cup\mathcal{Q}}n_{\mathrm{z}}=1,$$

it follows from Theorem 1.16 (e) that $\mathfrak{R}_N^{\mathcal{W}}=\mathcal{R}_N^{\mathcal{W}}$. The simplified formula for K is now obtained from (2.12), (2.20) and some direct computations. It only remains to determine the shape of $\gamma_{\mathbb{C}}$. Note that γ in (2.18) can be deformed into any closed curve surrounding 0, and in particular into a small circle surrounding 0. Recalling the definition $\gamma_{\mathbb{C}}=\varphi^{-1}(\gamma_{\mathcal{M}})=\varphi^{-1}(\cup_{j=1}^2\gamma^{(j)})$, this shows that $\gamma_{\mathbb{C}}$ can be determined from a local analysis of $\varphi^{-1}(\mathrm{z})$ around $0^{(1)}=(0,\sqrt{\Delta(0)})$ and $0^{(2)}=(0,-\sqrt{\Delta(0)})$. Let us discuss first the case $b_0\neq b_1$, for which we have $0^{(1)}\neq 0^{(2)}$. Since $\varphi^{-1}(0^{(1)})=|b_0-b_1|$ and $\varphi^{-1}(0^{(2)})=-|b_0-b_1|$, it follows that the contour $\gamma_{\mathbb{C}}$ can be chosen as the union of two small circles oriented positively: one circle surrounds b_0-b_1, and the other one surrounds b_1-b_0. On the other hand, if $b_0=b_1$, then $0^{(1)}=0^{(2)}$ is a branch point of $\mathcal{M}$, and φ^{-1} maps $\gamma^{(1)}\cup\gamma^{(2)}$ into a small circle surrounding 0. We conclude that in all cases, γ can be chosen to be a single circle oriented positively and enclosing $\pm(b_0-b_1)$. $\square$

2.3 Lozenge tiling models with a 2×2 periodic weighting

In this subsection, we show that the kernel of any lozenge tiling model with a 2×2 periodic weighting admits a double contour formula representation involving a scalar CD kernel. This result is a generalization of [16, Theorem 3.2], which was valid for a particular weight.

Lozenge tiling models with a 2×2 periodic weighting are defined by (2.1)–(2.2) and (2.5) with $r=2$ and $q=2$. Assume that M, N and L are multiples of 2. The transition matrices A_0 and A_1, defined in (2.3), are given by

$$A_0(z) = \begin{pmatrix} b_{0,0} & a_{0,0} \\ a_{0,1}z & b_{0,1} \end{pmatrix}, \qquad\qquad A_1(z) = \begin{pmatrix} b_{1,0} & a_{1,0} \\ a_{1,1}z & b_{1,1} \end{pmatrix},$$

where $a_{\ell,j}, b_{\ell,j} > 0$ for all $\ell, j \in \{0,1\}$. The quantities $\{L_j\}_{j=1}^3$ and $\{B_j\}_{j=1}^4$ of Theorem 2.1 depend on the parity of x_1 and x_2. Therefore, to ease the notation, we now invoke Theorem 2.1 with x_1 and x_2 replaced by $2x_1+\epsilon_1$ and $2x_2-\epsilon_2$, respectively, where $\epsilon_1,\epsilon_2\in\{0,1\}$.

For $\epsilon_1,\epsilon_2 \in \{0,1\}$, $y_1,y_2\in\mathbb{Z}$, and for integers x_1,x_2 such that $2x_1+\epsilon_1, 2x_2-\epsilon_2 \in \{1,\dots,L-1\}$, we have

$$\begin{aligned}[K(2x_1+\epsilon_1, 2y_1+j, 2x_2-\epsilon_2, 2y_2+i)]_{i,j=0}^1 &= -\frac{\chi_{2x_1+\epsilon_1>2x_2-\epsilon_2}}{2\pi i}\oint_\gamma A_1(z)^{\epsilon_2}A(z)^{x_1-x_2}A_0(z)^{\epsilon_1}z^{y_2-y_1-1}dz \\ &+ \frac{1}{2\pi i}\oint_\gamma\oint_\gamma \frac{A_1(w)^{\epsilon_2}}{w^{\frac{M+N}{2}-y_2}}A(w)^{\frac{L}{2}-x_2}\mathcal{R}^W_{N/2}(w,z)A(z)^{x_1}\frac{A_0(z)^{\epsilon_1}}{z^{y_1+1}}dzdw, \quad (2.21)\end{aligned}$$

where γ is the unit circle oriented positively, and

$$W(z) = z^{-\frac{M+N}{2}}A(z)^{\frac{L}{2}}, \qquad \text{with} \qquad A(z) = A_0(z)A_1(z).$$

Theorem 2.8. *The right-hand-side of (2.21) can be simplified to*

$$\begin{aligned}&-\frac{\chi_{2x_1+\epsilon_1>2x_2-\epsilon_2}}{2\pi i}\oint_{\gamma_{\mathbb{C}}}\widehat{\lambda}(\varphi(\zeta))^{x_1-x_2}\phi(\zeta)^{y_2-y_1-1}\phi'(\zeta)A_1(\phi(\zeta))^{\epsilon_2}\mathfrak{e}(\varphi(\zeta))\mathfrak{e}^{-1}(\varphi(\zeta))A_0(\phi(\zeta))^{\epsilon_1}d\zeta \\ &+\frac{1}{2\pi i}\oint_{\gamma_{\mathbb{C}}}\oint_{\gamma_{\mathbb{C}}}\frac{\mathcal{R}^{\mathcal{W}}_N(\omega,\zeta)}{\widehat{h}(\omega)h(\zeta)}\frac{\widehat{\lambda}(\varphi(\omega))^{\frac{L}{2}-x_2}}{\phi(\omega)^{\frac{M+N}{2}-y_2}}\frac{\widehat{\lambda}(\varphi(\zeta))^{x_1}}{\phi(\zeta)^{y_1+1}}\phi'(\omega)\phi'(\zeta)A_1(\phi(\omega))^{\epsilon_2}\mathfrak{e}(\varphi(\omega))\mathfrak{e}^{-1}(\varphi(\zeta))A_0(\phi(\zeta))^{\epsilon_1}d\zeta d\omega,\end{aligned}$$

(2.22)

where the various quantities that appear in the integrands depend on

$$\begin{aligned}&a_\pm = a_{1,1}a_{0,0} \pm a_{0,1}a_{1,0}, && b_\pm = b_{0,1}b_{1,1}\pm b_{0,0}b_{1,0}, \\ &c_0 = (a_{0,0}b_{1,1}+a_{1,0}b_{0,0})(a_{1,1}b_{0,1}+a_{0,1}b_{1,0}), && c_1 = (a_{0,1}b_{1,1}+a_{1,1}b_{0,0})(a_{1,0}b_{0,1}+a_{0,0}b_{1,0}), \\ &d = a_{0,0}b_{1,1}+a_{1,0}b_{0,0},\end{aligned}$$

and are given as follows.

(a) If $a_-=0$, then

$$\begin{aligned}&\phi(\zeta) = \tfrac{\zeta^2-b_-^2}{2(c_0+c_1)}, && h(\zeta)=1, && \widehat{h}(\zeta)=\zeta, \\ &\mathfrak{e}(\varphi(\zeta))^T = \begin{pmatrix} 1 & \frac{\zeta+b_-}{2d}\end{pmatrix}, && \mathfrak{e}^{-1}(\varphi(\zeta)) = \begin{pmatrix}\frac{\zeta-b_-}{2\zeta} & \frac{d}{\zeta}\end{pmatrix}, && \widehat{\lambda}(\varphi(\zeta)) = \tfrac{a_+(\zeta-\zeta_1)(\zeta-\zeta_2)}{4(c_0+c_1)}, \\ &\zeta_1 = -b_+ - \tfrac{2a_{1,0}b_{0,0}b_{0,1}}{a_{0,0}}, && \zeta_2 = -b_+ - \tfrac{2a_{0,0}b_{1,0}b_{1,1}}{a_{1,0}}, && \mathcal{W}(\omega) = \frac{\widehat{\lambda}(\varphi(\omega))^L\phi'(\omega)}{\phi(\omega)^{\frac{M+N}{2}}h(\omega)\widehat{h}(\omega)}.\end{aligned}$$

The contour $\gamma_{\mathbb{C}}$ is a circle oriented positively and surrounding both b_- and $-b_-$.

(b) If $a_- \neq 0$, then we define

$$z_\pm = \frac{1}{a_-^2}\Big(-(c_0+c_1) \pm \sqrt{(c_0+c_1)^2 - a_-^2 b_-^2}\Big), \qquad c = \frac{\sqrt{|z_-|}-\sqrt{|z_+|}}{\sqrt{|z_-|}+\sqrt{|z_+|}}. \tag{2.23}$$

They satisfy $z_- < z_+ < 0$, $c \in (0,1)$, and we have

$$\begin{aligned}
&\phi(\zeta) = \frac{z_+ - z_-}{4\zeta}(\zeta - c)(\zeta - c^{-1}), \quad h(\zeta) = \zeta^N, \quad \widehat{h}(\zeta) = \zeta^{N-2}(\zeta-1)(\zeta+1),\\
&\mathfrak{e}(\varphi(\zeta))^T = \begin{pmatrix} 1 & \frac{b_- - a_-\phi(\zeta)+\eta(\zeta)}{2d}\end{pmatrix}, \quad \mathfrak{e}^{-1}(\varphi(\zeta)) = \begin{pmatrix} \frac{a_-\phi(\zeta)+\eta(\zeta)-b_-}{2\eta(\zeta)} & \frac{d}{\eta(\zeta)}\end{pmatrix},\\
&\eta(\zeta) = a_- \frac{z_+ - z_-}{4}\left(\zeta - \zeta^{-1}\right), \qquad \widehat{\lambda}(\varphi(\zeta)) = \tfrac{a_+\phi(\zeta)+b_+ + \eta(\zeta)}{2}, \quad \mathcal{W}(\omega) = \frac{\widehat{\lambda}(\varphi(\omega))^L \phi'(\omega)}{\phi(\omega)^{\frac{M+N}{2}} h(\omega)\widehat{h}(\omega)}.
\end{aligned} \tag{2.24}$$

The contour $\gamma_{\mathbb{C}}$ is a closed curve surrounding both c and c^{-1} in the positive direction, but not surrounding 0.

Remark 2.9. *If $a_- \neq 0$, the function $\mathfrak{e}^{-1}$ has simple poles at 1 and -1. However, because $\phi'(\zeta)$ has two simple zeros at 1 and -1, the only poles of the integrand in (2.22) are 0, c and c^{-1}.*

Proof. Define

$$\Delta = \Delta(z) = a_-^2 z^2 + 2(c_0+c_1)z + b_-^2 = \begin{cases} a_-^2(z-z_+)(z-z_-), & \text{if } a_- \neq 0,\\ 2(c_0+c_1)(z-z_1), & \text{if } a_- = 0,\end{cases}$$

where z_-, z_+ are given by (2.23) and $z_1 = -\frac{b_-^2}{2(c_0+c_1)} < 0$. The eigenvalues of $A(z) = A_0(z)A_1(z)$ are given by

$$\widehat{\lambda}_1(z) = \frac{1}{2}\Big(a_+z + b_+ + \sqrt{\Delta(z)}\Big), \qquad \widehat{\lambda}_2(z) = \frac{1}{2}\Big(a_+z + b_+ - \sqrt{\Delta(z)}\Big),$$

where the branch for $\sqrt{\Delta(z)}$ is taken as follows:

$$\begin{aligned}
&\text{if } a_- \neq 0,\ \sqrt{\Delta(z)} \text{ is analytic in } \mathbb{C}\setminus[z_-,z_+] \text{ and } \sqrt{\Delta(z)} \sim a_- z \text{ as } z \to \infty,\\
&\text{if } a_- = 0,\ \sqrt{\Delta(z)} \text{ is analytic in } \mathbb{C}\setminus(-\infty,z_1] \text{ and } \sqrt{\Delta(z)} > 0 \text{ for } z > z_1.
\end{aligned}$$

A simple computation shows that $A(z)$ can be diagonalized as in (2.10) with $\widehat{\Lambda}(z) = \operatorname{diag}(\widehat{\lambda}_1(z), \widehat{\lambda}_2(z))$ and

$$E(z) = \begin{pmatrix} 1 & 1\\ \frac{b_- - a_- z + \sqrt{\Delta}}{2d} & \frac{b_- - a_- z - \sqrt{\Delta}}{2d}\end{pmatrix}, \qquad E(z)^{-1} = \begin{pmatrix} \frac{a_- z + \sqrt{\Delta} - b_-}{2\sqrt{\Delta}} & \frac{d}{\sqrt{\Delta}}\\ \frac{a_- z - \sqrt{\Delta} - b_-}{-2\sqrt{\Delta}} & \frac{d}{-\sqrt{\Delta}}\end{pmatrix}. \tag{2.25}$$

In particular, A satisfies Assumption 2.3. Let $\mathcal{M}$ be the genus 0 Riemann surface associated to $\{(z,\eta) \in \mathbb{C}^2 : \eta^2 = \Delta(z)\}$. We choose the numbering of the sheets such that $\eta = \sqrt{\Delta(z)}$ on the first sheet, and $\eta = -\sqrt{\Delta(z)}$ on the second sheet. If $a_- = 0$, we note that the maps

$$\varphi(\zeta) = \left(\frac{\zeta^2 - b_-^2}{2(c_0+c_1)}, \zeta\right) \qquad \text{and} \qquad \varphi^{-1}((z,\eta)) = \eta$$

are bijections from $\widehat{\mathbb{C}}$ to $\mathcal{M}$ and from $\mathcal{M}$ to $\widehat{\mathbb{C}}$, respectively, and therefore the claim is obtained in a similar way as in the proof of Theorem 2.6 (we omit further details). We now consider the case $a_- \neq 0$. The function

$$\varphi^{-1}((z,\eta)) = \frac{2z + 2a_-^{-1}\eta - (z_+ + z_-)}{z_+ - z_-}$$

maps the upper sheet to $\{\zeta : |\zeta| > 1\}$ and the lower sheet to $\{\zeta : |\zeta| < 1\}$. The inverse map is given by $\varphi(\zeta) = (\phi(\zeta), \eta(\zeta))$ with

$$\phi(\zeta) = \frac{z_+ + z_-}{2} + \frac{z_+ - z_-}{4}(\zeta + \zeta^{-1}) \quad \text{and} \quad \eta(\zeta) = a_- \frac{z_+ - z_-}{4}\left(\zeta - \zeta^{-1}\right). \tag{2.26}$$

The function φ satisfies

$$\begin{aligned} &\varphi(1) = (z_+, 0), \qquad \varphi(-1) = (z_-, 0), \qquad \varphi(\infty) = \infty^{(1)}, \qquad \varphi(0) = \infty^{(2)}, \\ &\varphi^{-1}(0^{(2)}) = \varphi^{-1}((0, -a_-\sqrt{z_+z_-})) = c, \qquad \varphi^{-1}(0^{(1)}) = \varphi^{-1}((0, a_-\sqrt{z_+z_-})) = c^{-1}, \end{aligned}$$

where c is defined in (2.23). Also, by definition, $\phi(\zeta)$ vanishes at $\varphi^{-1}(0^{(1)})$ and $\varphi^{-1}(0^{(2)})$, and it has simple poles at $\zeta = 0$ and $\zeta = \infty$. Hence, it can be rewritten as

$$\phi(\zeta) = \frac{z_+ - z_-}{4\zeta}(\zeta - c)(\zeta - c^{-1}).$$

The expressions (2.24) for $\mathfrak{e}$ and $\mathfrak{e}^{-1}$ follow directly from (1.9)-(1.10), (2.25) and (2.26). Since $\mathfrak{e}$ has no zero and a simple pole at $\infty^{(2)}$, by (1.22) and Definition 1.8, we have

$$\mathcal{Z} = \emptyset, \qquad \mathcal{Q} = \{\infty^{(2)}\}, \qquad n_{\infty^{(2)}} = -1, \qquad h(\zeta) = (\zeta - \varphi^{-1}(\infty^{(2)}))^{N-1}(\zeta - \varphi^{-1}(\infty^{(2)})) = \zeta^N.$$

Similarly, since $\mathfrak{e}^{-1}$ has a simple zero at $\infty^{(2)}$ and simple poles at z_+ and z_-, we have

$$\widehat{\mathcal{Z}} = \{\infty^{(2)}\}, \qquad \widehat{\mathcal{Q}} = \{z_+, z_-\}, \qquad \widehat{n}_{\infty^{(2)}} = 1, \qquad \widehat{n}_{z_+} = \widehat{n}_{z_-} = -1$$

and therefore, using (1.23) we obtain

$$\widehat{h}(\zeta) = (\zeta - \varphi^{-1}(\infty^{(2)}))^{N-1} \frac{(\zeta - \varphi^{-1}(z_+))(\zeta - \varphi^{-1}(z_-))}{\zeta - \varphi^{-1}(\infty^{(2)})} = \zeta^{N-2}(\zeta - 1)(\zeta + 1).$$

Since $-\sum_{z \in \mathcal{Z} \cup \mathcal{Q}} n_z = 1$, Theorem 1.16 (e) implies that $\mathfrak{R}_N^{\mathcal{W}} = \mathcal{R}_N^{\mathcal{W}}$. Finally, since γ can be deformed into any closed curve surrounding 0, and since $\varphi^{-1}(0^{(1)}) = c^{-1}$ and $\varphi^{-1}(0^{(2)}) = c$, it follows that $\gamma_{\mathbb{C}}$ can be chosen as the union of two small circles; one surrounds c and the other one surrounds c^{-1}, but none of them surround 0. The formula (2.22) now follows from a direct application of Theorem 2.4. $\square$

3 Proofs of Theorems 1.5 and 1.16

3.1 Proof of Theorem 1.5

The existence of R_N^λ follows directly from (1.14) and the assumption that $\mathcal{R}_N^W$ exists. Given $z \in \mathbb{C}$, recall that $z^{(k)}$ denotes the point on the k-th sheet of $\mathcal{M}$ whose projection on $\mathbb{C}$ is z. Because $E(z)$ is invertible for all but finitely many $z \in \mathbb{C}$, for any $P \in \mathcal{P}_{N-1}^{1\times r}$ we have

$$P(z)\mathfrak{e}(z) \equiv 0 \;\Leftrightarrow\; P(z)\mathfrak{e}(z^{(j)}) \equiv 0 \quad \forall j \in \{1,\ldots,r\} \;\Leftrightarrow\; P(z)E(z) \equiv 0 \;\Leftrightarrow\; P(z) \equiv 0$$

from which we conclude that $\dim L_N = rN$. Similarly, for any $P \in \mathcal{P}_{N-1}^{r\times 1}$, we have

$$\mathfrak{e}^{-1}(\mathrm{z})P(z) \equiv 0 \;\Leftrightarrow\; \mathfrak{e}^{-1}(z^{(j)})P(z) \equiv 0 \quad \forall j \in \{1,\dots,r\} \;\Leftrightarrow\; E(z)^{-1}P(z) \equiv 0 \;\Leftrightarrow\; P(z) \equiv 0,$$

and therefore $\dim L_N^* = rN$. Since $\mathcal{R}_N^W(w,z)$ is a bivariate $r\times r$ matrix polynomial of degree $\leq N-1$ in both w and z, the statements (a) and (b) follow directly from (1.14) and (1.16). Next, we start from (1.5), and use (1.2) and (1.8)–(1.10) to note the following equivalences

$$\begin{aligned}
&\int_\gamma P(w)W(w)\mathcal{R}_N^W(w,z)dw = P(z), && \forall P \in \mathcal{P}_{N-1}^{r\times r},\ z \in \mathbb{C},\\
\Leftrightarrow &\int_\gamma P(w)W(w)\mathcal{R}_N^W(w,z)dw = P(z), && \forall P \in \mathcal{P}_{N-1}^{1\times r},\ z \in \mathbb{C},\\
\Leftrightarrow &\int_\gamma P(w)E(w)\Big(\sum_{j=1}^r \lambda(w^{(j)})e_je_j^T\Big)E^{-1}(w)\mathcal{R}_N^W(w,z)dw = P(z), && \forall P \in \mathcal{P}_{N-1}^{1\times r},\ z \in \mathbb{C},\\
\Leftrightarrow &\int_{\gamma_{\mathcal{M}}} P(w)\mathfrak{e}(\mathrm{w})\lambda(\mathrm{w})\mathfrak{e}^{-1}(\mathrm{w})\mathcal{R}_N^W(w,z)dw = P(z), && \forall P \in \mathcal{P}_{N-1}^{1\times r},\ z \in \mathbb{C},\\
\Leftrightarrow &\int_{\gamma_{\mathcal{M}}} P(w)\mathfrak{e}(\mathrm{w})\lambda(\mathrm{w})R_N^\lambda(\mathrm{w},\mathrm{z})dw = P(z)\mathfrak{e}(\mathrm{z}), && \forall P \in \mathcal{P}_{N-1}^{1\times r},\ \mathrm{z} \in \mathcal{M}_* \setminus \mathcal{Q},
\end{aligned}$$

where in the last two equations, w denotes the projection of w on the complex plane, and similarly for z and z in the last equation. By definition (1.16) of L_N, this last property is equivalent to (1.17), which proves (c). The proof of (d) is similar, and we omit it.

3.2 Proof of Theorem 1.16

Existence of $\mathfrak{R}_{rN}^{\mathcal{W}}$ is obvious from (1.14), (1.21) and the assumption that $\mathcal{R}_N^W$ exists. The identities $\dim\mathcal{V} = \dim\mathcal{V}^* = rN$ have already been proved in Remark 1.15. The statements (a) and (b) follow directly from the definition (1.25) and Theorem 1.5 (a)-(b). We now turn to the proof of (c). Since the sets $\mathcal{Q}$ and $\mathcal{Z}$ of Definition 1.8 are finite, we have the following equivalences

$$\begin{aligned}
&\int_{\gamma_{\mathcal{M}}} f(\mathrm{w})\lambda(\mathrm{w})R_N^\lambda(\mathrm{w},\mathrm{z})dw = f(\mathrm{z}), && \forall f \in L_N,\ \mathrm{z} \in \mathcal{M}_* \setminus \mathcal{Q},\\
\Leftrightarrow &\int_{\gamma_{\mathcal{M}}} f(\mathrm{w})\lambda(\mathrm{w})R_N^\lambda(\mathrm{w},\varphi(\zeta))dw = f(\varphi(\zeta)), && \forall f \in L_N,\ \zeta \in \mathbb{C}\setminus\varphi^{-1}(\mathcal{Q}),\\
\Leftrightarrow &\int_{\gamma_{\mathcal{M}}} f(\mathrm{w})\lambda(\mathrm{w})R_N^\lambda(\mathrm{w},\varphi(\zeta))h(\zeta)dw = f(\varphi(\zeta))h(\zeta), && \forall f \in L_N,\ \zeta \in \mathbb{C}\setminus\varphi^{-1}(\mathcal{Q}\cup\mathcal{Z}),\\
\Leftrightarrow &\int_{\gamma_{\mathbb{C}}} p(\omega)\frac{\lambda(\varphi(\omega))}{h(\omega)\widehat{h}(\omega)}\widehat{h}(\omega)R_N^\lambda(\varphi(\omega),\varphi(\zeta))h(\zeta)\phi'(\omega)d\omega = p(\zeta), && \forall p \in \mathcal{V},\ \zeta \in \mathbb{C},\\
\Leftrightarrow &\int_{\gamma_{\mathbb{C}}} p(\omega)\mathcal{W}(\omega)\mathfrak{R}_{rN}^{\mathcal{W}}(\omega,\zeta)d\omega = p(\zeta), && \forall p \in \mathcal{V},\ \zeta \in \mathbb{C},
\end{aligned}$$

which prove (c). The statement (d) follows in a similar way, and we omit the proof. Finally, recall that only one property among (1.5)-(1.6) is sufficient to uniquely determine a CD kernel. Hence, Remark 1.15 implies that

$$\begin{aligned}
-\sum_{\mathrm{z}\in\mathcal{Z}\cup\mathcal{Q}} n_{\mathrm{z}} = r-1 \;\Leftrightarrow\; \mathcal{V} = \mathcal{P}_{rN-1} \;\Leftrightarrow\; \mathfrak{R}_{rN}^{\mathcal{W}} = \mathcal{R}_{rN}^{\mathcal{W}},\\
-\sum_{\mathrm{z}\in\widehat{\mathcal{Z}}\cup\widehat{\mathcal{Q}}} \widehat{n}_{\mathrm{z}} = r-1 \;\Leftrightarrow\; \mathcal{V}^* = \mathcal{P}_{rN-1} \;\Leftrightarrow\; \mathfrak{R}_{rN}^{\mathcal{W}} = \mathcal{R}_{rN}^{\mathcal{W}},
\end{aligned}$$

which, taken together, are of course equivalent to (e).

A On the eigenvalues and eigenvectors of W

In this appendix we discuss some analytical properties of the eigenvalues and eigenvectors of W. Since W is rational, there exists a scalar polynomial p such that $W = p^{-1}T$, where T is a polynomial matrix. For each z that is not a pole of W, the eigenvalues $\lambda_1(z), \dots, \lambda_r(z)$ of $W(z)$ and the eigenvalues $\theta_1(z), \dots, \theta_r(z)$ of $T(z)$ are straightforwardly related by $\lambda_j(z) = p(z)^{-1}\theta_j(z)$, $j = 1, \dots, m$, and each eigenvector $v(z)$ of $W(z)$ satisfying $W(z)v(z) = \lambda_k(z)v(z)$ also satisfies $T(z)v(z) = \theta_k(z)v(z)$. Therefore, we restrict from now our discussion on the eigenvalues and eigenvectors of T. Most of the facts listed below are rather direct consequences of Kato's classical book [40].

On the eigenvalues of T. Let $\mathcal{M}$ be the Riemann surface constructed from the zero set (1.7). The eigenvalues of T are (branches of) meromorphic functions on $\mathbb{C} \cup \{\infty\}$, and together they define a meromorphic function on $\mathcal{M}$. Let $\theta_1(z), \dots, \theta_s(z)$ denote the *distinct* eigenvalues of $T(z)$, and let $m_1, \dots, m_s$ be their multiplicities, with $m_1 + \ldots + m_s = r$. Since $\mathcal{M}$ is compact, $s, m_1, \dots, m_s$ are constant for all $z \in \mathbb{C} \setminus \mathcal{E}$, where $\mathcal{E}$ consists of at most finitely many exceptional points.[5] Any point that is a branch point for some of the θ_j's belongs to $\mathcal{E}$; however $\mathcal{E}$ may also contain other points, see [40, page 64, Example 1.1]. Also, because T is analytic in $\mathbb{C}$, the functions $z \mapsto \theta_j(z)$ are *continuous* at any point $z \in \mathbb{C}$, also at a branch point. This fact is essentially a consequence of Rouché's theorem, see [41, p. 122]. In particular the θ_j's have no pole in $\mathbb{C}$, although they can have a pole at ∞.

On the eigenvectors of T. Eigenprojections are standard tools in analytic perturbation theory. They are closely related to the eigenvectors (see below), but allow for a simplified analysis and their properties have been studied in great depth in [40]. The eigenprojection $\mathsf{P}_k(z)$ associated to $\theta_k(z)$ is defined for $z \in \mathbb{C} \setminus \mathcal{E}$ by

$$\mathsf{P}_k(z) = -\frac{1}{2\pi i} \oint_{\Gamma_{k,z}} (T(z) - \lambda)^{-1} d\lambda, \qquad k = 1, \dots, s,$$

where $\Gamma_{k,z}$ is a small contour in the complex plane which surrounds $\theta_k(z)$, but does not surround $\theta_j(z)$, $j \neq k$. Given $z \in \widehat{\mathbb{C}}$, we let $z^{(k)}$ denotes the point on the k-th sheet of $\mathcal{M}$ whose projection on $\widehat{\mathbb{C}}$ is z. An individual P_k has the same branch cut as θ_k, and taken together the P_k's naturally define a meromorphic function on $\mathcal{M}$, which can only have poles at

$$\{\infty^{(1)}, \dots, \infty^{(r)}\} \cup \bigcup_{j=1}^{r} \mathcal{E}^{(j)},$$

see [40, Chapter II, Sections 4–6]. As its name suggests, the operator $\mathsf{P}_k(z)$ is a projection, and it satisfies [40, pages 40]

$$(T(z) - \theta_k(z))^{m_k} \mathsf{P}_k(z) = 0_r.$$

Furthermore, the images of $\mathsf{P}_k(z)$ and of $\mathsf{P}_k(z')$ are isomorphic for any $z, z' \in \mathbb{C} \setminus \mathcal{E}$, and $\dim \operatorname{Im} \mathsf{P}_k(z) = m_k$ for $z \in \mathbb{C} \setminus \mathcal{E}$, see [40, page 68].

[5] Without the compactness of $\mathcal{M}$, we would only have that $\mathcal{E}$ is locally finite. Also, the terminology *exceptional points* to denote points of $\mathcal{E}$ is standard [40].

By Assumption 1.1, $T(z)$ is diagonalizable for all $z \in \mathbb{C} \setminus \mathcal{D}$ where $\mathcal{D}$ is a finite set, and we assume without loss of generality that $\mathcal{E} \subset \mathcal{D}$. This implies in particular that the algebraic eigenspaces $\operatorname{Im} \mathsf{P}_k(z)$, $k = 1, \ldots, s$ coincide with the geometric eigenspaces, i.e. we have

$$(T(z) - \theta_k(z))\mathsf{P}_k(z) = 0_r, \qquad \text{or equivalently} \qquad T(z)\mathsf{P}_k(z) = \theta_k(z)\mathsf{P}_k(z),$$

for all $k = 1, \ldots, s$. Fix $z \in \mathbb{C} \setminus \mathcal{D}$. For each $k = 1, \ldots, s$, we take m_k linearly independent columns of $\mathsf{P}_k(z)$. Together these columns form a matrix of eigenvectors $E(z)$, and we choose the numbering of the columns such that

$$T(z)E(z) = E(z) \operatorname{diag}(\theta_1(z), \ldots, \theta_r(z)).$$

Since the columns of E are (branches of) meromorphic functions, they remain linearly independent for all but a finite number of points. Redefining $\mathcal{D}$ is necessary, we can assume that $E(z)$ is invertible for all values of $\mathbb{C} \setminus \mathcal{D}$. This finishes the construction of a matrix of eigenvectors E whose columns define a meromorphic function $\mathfrak{e}$ on $\mathcal{M}$ as in (1.9).

The rows of E^{-1} also clearly define a meromorphic function $\mathfrak{e}^{-1}$ on $\mathcal{M}$ as in (1.10), as can be seen from Cramer's formula.

B On the CD formula (1.36) for $\mathcal{R}_N^W$

The goal of this appendix is to rewrite (1.38) in the form (1.36). Assume that P_N^L exists and is unique. Then the solution to the RH problem for Y exists, is also unique, and can be explicitly written in term of MOPs as follows [25, eq (4.31)]:

$$Y(z) = \begin{pmatrix} P_N^L(z) & \dfrac{1}{2\pi i} \displaystyle\int_\gamma P_N^L(s) W(s) \frac{ds}{s-z} \\ -2\pi i Q_{N-1}^L(z) & -\displaystyle\int_\gamma Q_{N-1}^L(s) W(s) \frac{ds}{s-z} \end{pmatrix}, \qquad z \in \mathbb{C} \setminus \gamma. \tag{B.1}$$

Since Y satisfies $\det Y \equiv 1$, the existence of Y^{-1} follows from that of Y. An explicit expression for Y^{-1} is not clear from (B.1), but can be easily obtained by considering the RH problem for Y^{-1}, which is as follows.

RH problem for Y^{-1}

(a) $Y^{-1} : \mathbb{C} \setminus \gamma \to \mathbb{C}^{2r \times 2r}$ is analytic.

(b) The limits of $Y^{-1}(z)$ as z approaches γ_0 from left and right exist, are continuous on γ_0, and are denoted by Y_+^{-1} and Y_-^{-1}, respectively. Furthermore, they are related by

$$Y_+^{-1}(z) = \begin{pmatrix} I_r & -W(z) \\ 0_r & I_r \end{pmatrix} Y_-^{-1}(z), \qquad \text{for } z \in \gamma_0.$$

(c) As $z \to \infty$, we have $Y^{-1}(z) = \begin{pmatrix} z^{-N} I_r & 0_r \\ 0_r & z^N I_r \end{pmatrix} \left(I_{2r} + \mathcal{O}(z^{-1})\right)$.

As $z \to z_\star \in \gamma \setminus \gamma_0$, we have $Y^{-1}(z) = \mathcal{O}(\log(z - z_\star))$.

It is easily verified from (1.33)–(1.34) that the unique solution to the above RH problem is given by

$$Y^{-1}(z) = \begin{pmatrix} -\displaystyle\int_\gamma W(s) Q_{N-1}^R(s) \frac{ds}{s-z} & -\dfrac{1}{2\pi i} \displaystyle\int_\gamma W(s) P_N^R(s) \frac{ds}{s-z} \\ 2\pi i Q_{N-1}^R(z) & P_N^R(z) \end{pmatrix}, \qquad z \in \mathbb{C} \setminus \gamma. \tag{B.2}$$

The CD formula (1.36) is now simply obtained by substituting (B.1) and (B.2) in (1.38).

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