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The criterion of the best approximant for the multivariable functions in the space $L_{p_1,\ldots,p_{i-1},1,p_{i+1}\ldots,p_n}$

Досліджено питання характеризації елемента найкращого наближення для функцій багатьох змінних у просторі зі змішаною інтегральною метрикою. Одержано критерій елемента найкращого наближення у просторі $L_{p_1,\ldots,p_{i-1},1,p_{i+1}\ldots,p_n}$.

Ключові слова: змішана інтегральна метрика, елемент найкращого наближення.

Исследованы вопросы характеризации элемента наилучшего приближения для функций многих переменных в пространстве со смешанной интегральной метрикой. Получен критерий элемента наилучшего приближения в пространстве $L_{p_1,\ldots,p_{i-1},1,p_{i+1}\ldots,p_n}$.

Ключевые слова: смешанная интегральная метрика, элемент наилучшего приближения.

The questions of the characterization of the best approximant for the multivariable functions in the space with mixed integral metric were considered in this article. The criterion of the best approximant in space $L_{p_1,\ldots,p_{i-1},1,p_{i+1}\ldots,p_n}$ is obtained.

Key words: mixed integral metric, the best approximant.

MSC2010: Pri 41A45, Sec 46E30, 41A10

Let $L_{p_1,\ldots,p_n}(1 \le p_i < \infty, 1 \le i \le n)$ be the space of all real-valued summable on $K = I_1 \times I_2 \times \ldots \times I_n$, $(I_i = [a_i, b_i], 1 \le i \le n)$ functions $f : K \to \mathbb{R}$ such that

$$\|f\|_{p_1,\ldots,p_n} = \left[\int\limits_{I_n} \ldots \left[\int\limits_{I_2} \left[\int\limits_{I_1} |f(x)|^{p_1} dx_1 \right]^{\frac{p_2}{p_1}} dx_2 \right]^{\frac{p_3}{p_2}} \ldots dx_n \right]^{\frac{1}{p_n}}.$$

We set

$$|f|_{p_k,\ldots,p_i} = \left[\int\limits_{I_i} \ldots \left[\int\limits_{I_{k+1}} \left[\int\limits_{I_k} |f(x)|^{p_k} dx_k \right]^{\frac{p_{k+1}}{p_k}} dx_{k+1} \right]^{\frac{p_{k+2}}{p_{k+1}}} \ldots dx_i \right]^{\frac{1}{p_i}},$$

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where $1 \le k < i \le n$.

Consider also the classes $L_{p_1,\dots,p_n}$ (where at least one $p_i = \infty$) of functions f, respectively with norms

$$\|f\|_{p_1,\dots,p_{n-1},\infty} = \operatorname{ess}\sup_{x_n \in I_n} |f|_{p_1,\dots,p_{n-1}} < \infty,$$

$$\|f\|_{p_1,\dots,p_{i-1},\infty,p_{i+1},\dots,p_n} =$$

$$= \left[\int\limits_{I_n} \dots \left[\int\limits_{I_{i+1}} \left(\operatorname{ess}\sup_{x_i \in I_i} |f|_{p_1,\dots,p_{i-1}} \right)^{p_{i+1}} dx_{i+1} \right]^{\frac{p_{i+2}}{p_{i+1}}} \dots dx_n \right]^{\frac{1}{p_n}} < \infty,$$

where $1 \le i < n$.

For a system of linearly independent functions $\{\varphi_1, \dots, \varphi_m\} \subset L_{p_1,\dots,p_n}$ we denote $H_m = \operatorname{span}\{\varphi_1, \dots, \varphi_m\}$. Then the elements of the set H_m (polynomials P_m) are representable in the form

$$P_m = \sum_{k=1}^{m} \lambda_k \varphi_k.$$

For $f \in L_{p_1,\dots,p_n}$ the quantity

$$E_m(f)_{p_1,\dots,p_n} = \inf_{P_m \in H_m} \|f - P_m\|_{p_1,\dots,p_n} \tag{1}$$

is called the best approximation of f in the space $L_{p_1,\dots,p_n}$ by the set H_m. Any polynomial P_m^* which realizes the "inf" on the right-hand side of (1) is said to be a best approximant to f by H_m.

In 1973 G.S. Smirnov [2] proved the criterion of the best approximant in the spaces with mixed integral metric for the functions of two variables. V,M. Traktynska [4] extended this result to the multivariable functions. If at least one $p_i = 1$, these criteroins are true under the condition $f - P_m^* \ne 0$ almost every where on K. This restrictionis removed by G.S. Smirnov [3] for the functions of two variables in the spaces $L_{p,1}(I_1 \times I_2)$ i $L_{1,q}(I_1 \times I_2)$). O.D. Kostyuk, V.M. Traktynska and M.Ye. Tkachenko [1] extended this result to the multivariable functions in the spaces $L_{1,p_2,\dots,p_n}$ and $L_{p_1,\dots,p_{n-1},1}$.

The purpose of this article is getting the criterion in the space $L_{p_1,\dots,p_{i-1},1,p_{i+1}\dots,p_n}$.

Let $f \in L_{p_1,\dots,p_{i-1},1,p_{i+1}\dots,p_n}$ with $||f||_{p_1,\dots,p_{i-1},1,p_{i+1}\dots,p_n} > 0$ and $g \in L_{q_1,\dots,q_{i-1},\infty,q_{i+1}\dots,q_n}$ with $||g||_{q_1,\dots,q_{i-1},\infty,q_{i+1}\dots,q_n} \le 1$, $\frac{1}{p_j} + \frac{1}{q_j} = 1$, $1 < p_j < \infty$, $j = 1, 2, \dots, i-1, i+1, \dots, n$ be given.

Note that

$$\|f\|_{p_1,\dots,p_{i-1},1,p_{i+1}\dots,p_n} = \left[\int\limits_{I_n} \dots \left[\int\limits_{I_{i+1}} \left[\int\limits_{I_i} |f(x)|_{p_1,\dots,p_{i-1}} dx_i \right]^{p_{i+1}} dx_{i+1} \right]^{\frac{p_{i+2}}{p_{i+1}}} \dots dx_n \right]^{\frac{1}{p_n}}.$$

Applying Gelder's inequality and Fubini's Theorem, we obtain

$$\int\limits_K f(x)g(x)dx_1\dots dx_n \le \int\limits_{I_n}\dots\int\limits_{I_1} |f||g|dx_1\dots dx_n \le$$

$$\le \int\limits_{I_n}\dots\int\limits_{I_i}\left[\int\limits_{I_{i-1}}\dots\left[\int\limits_{I_2}\left[\int\limits_{I_1}|f|^{p_1}dx_1\right]^{\frac{p_2}{p_1}}dx_2\right]^{\frac{p_3}{p_2}}\dots dx_{i-1}\right]^{\frac{1}{p_{i-1}}}\times$$

$$\times\left[\int\limits_{I_{i-1}}\dots\left[\int\limits_{I_2}\left[\int\limits_{I_1}|g|^{q_1}dx_1\right]^{\frac{q_2}{q_1}}dx_2\right]^{\frac{q_3}{q_2}}\dots dx_{i-1}\right]^{\frac{1}{q_{i-1}}}dx_i\dots dx_n=$$

$$=\int\limits_{I_n}\dots\int\limits_{I_i}\left(|f|_{p_1,\dots,p_{i-1}}|g|_{q_1,\dots,q_{i-1}}\right)dx_i\dots dx_n\le$$

$$\le\int\limits_{I_n}\dots\int\limits_{I_{i+1}}\left(\left(\int\limits_{I_i}|f|_{p_1,\dots,p_{i-1}}dx_i\right)\times\operatorname*{ess\,sup}_{x_i\in I_i}|g|_{q_1,\dots,q_{i-1}}\right)dx_{i+1}\dots dx_n\le$$

$$\le\left(\int\limits_{I_n}\dots\left(\int\limits_{I_{i+1}}\left(\int\limits_{I_i}|f|_{p_1,\dots,p_{i-1}}dx_i\right)^{p_{i+1}}dx_{i+1}\right)^{\frac{p_{i+2}}{p_{i+1}}}\dots dx_n\right)^{\frac{1}{p_n}}\times$$

$$\times\left(\int\limits_{I_n}\dots\left(\int\limits_{I_{i+1}}\left(\operatorname*{ess\,sup}_{x_i\in I_i}|g|_{q_1,\dots,q_{i-1}}\right)^{q_{i+1}}dx_{i+1}\right)^{\frac{q_{i+2}}{q_{i+1}}}\dots dx_n\right)^{\frac{1}{q_n}}=$$

$$=||f||_{p_1,\dots,p_{i-1},1,p_{i+1},\dots,p_n}\cdot||g||_{q_1,\dots,q_{i-1},\infty,q_{i+1},\dots,q_n}\le||f||_{p_1,\dots,p_{i-1},1,p_{i+1},\dots,p_n},$$

or in other words

$$\int\limits_K f(x)\cdot g(x)dx_1\dots dx_n\le||f||_{p_1,\dots,p_{i-1},1,p_{i+1},\dots,p_n}. \tag{2}$$

It is easy to verify that the inequality (2) becomes equality for the functions of form

$g_0=||f||^{1-p_n}_{p_1,\dots,p_{i-1},1,p_{i+1},\dots,p_n}|f|^{p_2-p_1}_{p_1}|f|^{p_3-p_2}_{p_1,p_2}\cdot\ldots\cdot|f|^{p_{i+1}-1}_{p_1,\dots,p_{i-1},1}\cdot\ldots\cdot|f|^{p_n-p_{n-1}}_{p_1,\dots,p_{i-1},1,p_{i+1},\dots,p_n}\mathrm{sgn}f$

All another functions that give the equality in (2), coincide with g_0 almost everywhere on $I_1\times\ldots\times(I_i\backslash E(x_{i+1},\dots,x_n))\times\ldots\times I_n$, where $E(x_{i+1},\dots,x_n)=\{x_i\in I_i:|f|_{p_i,\dots,p_{i-1}}=0\}$, where in $\operatorname*{ess\,sup}_{x_i\in I_i}|g(x)|_{q_1,\dots,q_{i-1}}=\operatorname*{ess\,sup}_{x_i\in I_i}|g_0(x)|_{q_1,\dots,q_{i-1}}$ almost everywhere on $I_{i+1}\times\ldots\times I_n$.

The following theorem is formulated by V.M. Traktynska [4].

Theorem 1. *For any function $f \in L_{p_1,\dots,p_n}$ $(1 \le p_i < \infty)$ the best approximation in the norm $L_{p_1,\dots,p_n}$*

$$E_m(f)_{p_1,\dots,p_n} = \sup \int\limits_K f(x)g(x)dx_1...dx_n,$$

where sup *applies to the functions $g \in L_{q_1,\dots,q_n}$ such as $||g||_{q_1,\dots,q_n} \le 1$, and $\int\limits_K P_m(x) \cdot g(x)dx_1...dx_n = 0$ for any above-defined polynomials P_m and* sup *is realized by some functions $\varphi \in L_{q_1,\dots,q_n}$ with the norm $||\varphi||_{q_1,\dots,q_n} = 1$.*

We set

$$P_m = \sum_{k=1}^{m} c_k \varphi_k, \tag{3}$$

where $\{\varphi_k\}_{k=1}^m$ is the system of linearly independent functions from $L_{p_1,,\dots,p_{i-1},1,p_{i+1},\dots,p_n}$ $(1 < p_j < \infty,\ j = 1,\dots,i-1,i+1\dots,n)$, and c_k are real numbers.

Assume that the function $f \in L_{p_1,,\dots,p_{i-1},1,p_{i+1},\dots,p_n}$ such as $||f - P_m||_{p_1,,\dots,p_{i-1},1,p_{i+1},\dots,p_n} > 0$ holds true for each P_m.

Introduce the functions:

$$F_0 = |f-P^*_m|_1^{p_1-1}|f-P^*_m|_{p_1}^{p_2-p_1}|f-P^*_m|_{p_1,p_2}^{p_3-p_2}\cdot\ldots\cdot|f-P^*_m|_{p_1,\dots,p_{i-1}}^{1-p_{i-1}}|f-P^*_m|_{p_1,\dots,p_{i-1},1,p_{i+1}}^{p_{i+1}-1}\cdot\ldots\cdot$$

$$\cdot|f-P^*_m|_{p_1,,\dots,p_{i-1},1,p_{i+1},\dots,p_{n-1}}^{p_n-p_{n-1}}\operatorname{sgn}(f-P^*_m),$$

$$F^*_0 = \frac{F_0}{||f-P^*_m||_{p_1,,\dots,p_{i-1},1,p_{i+1},\dots,p_n}^{p_n-1}},$$

where P^*_m is some polynomial of form (3).

It is easy to see that F^*_0 has the form of the function that give the equality in (2): $F^*_0 \in L_{q_1,\dots,q_{i-1},\infty,q_{i+1},\dots,q_n}$, $||F^*_0||_{q_1,,\dots,q_{i-1},\infty,q_{i+1},\dots,q_n} = 1$ and

$$\int\limits_K (f-P^*_m)F^*_0 dx_1,,,dx_n = ||f-P^*_m||_{p_1,,\dots,p_{i-1},1,p_{i+1},\dots,p_n}.$$

For any $(x_{i+1},\dots,x_n) \in I_{i+1} \times \ldots \times I_n$, define

$$e(x_{i+1},\dots,x_n) = \{x_i \in I_i : |f-P^*_m|_{p_1,\dots,p_{i-1}} = 0\}.$$

Theorem 2. *For the polynomial P^*_m to be the best approximant for the function f in $L_{p_1,,\dots,p_{i-1},1,p_{i+1},\dots,p_n}$ $(1 < p_j < \infty,\ j = 1,\dots,i-1,i+1,\dots,n)$, it is necessary and sufficient to have*

$$\left|\int\limits_K P_m \cdot F^*_0 dx_1...dx_n\right| \le \int\limits_{I_n}\ldots\int\limits_{I_{i+1}}\left[\int\limits_{e(x_{i+1},\dots,x_n)} |P_m|_{p_1,\dots,p_{i-1}}dx_i\right]\cdot$$

$$\cdot \left[\operatorname*{ess\,sup}_{x_i \in I_i} |F_0^*|_{q_1,\dots,q_{i-1}} \right] dx_{i+1}...dx_n \tag{4}$$

for any P_m by form (3).

Proof. At first prove the necessity. Let P_m^* be the best approximant for the function f in $L_{p_1,,\dots,p_{i-1},1,p_{i+1},\dots,p_n}$. According to Theorem 1, there exists a function $F_1 \in L_{q_1,,\dots,q_{i-1},\infty,q_{i+1},\dots,q_n}$ such as:

1) $||F_1||_{q_1,,\dots,q_{i-1},\infty,q_{i+1},\dots,q_n} = 1$;
2) $\int\limits_K P_m \cdot F_1 dx_1...dx_n = 0$ for any P_m by form (3);
3) $\int\limits_K f \cdot F_1 dx_1...dx_n = \int\limits_K (f - P_m^*) F_1 dx_1...dx_n = ||f - P_m^*||_{p_1,,\dots,p_{i-1},1,p_{i+1},\dots,p_n}$.

The latter equality holds true iff F_1 satisfies the condition of equality in the inequality (2). Thus $F_0^*(x) = F_1(x)$ for any points x such as $x_i \notin e(x_{i+1},\dots,x_n)$ and $\operatorname*{ess\,sup}_{x_i \in I_i} |F_0^*|_{q_1,\dots,q_{i-1}} = \operatorname*{ess\,sup}_{x_i \in I_i} |F_1|_{q_1,\dots,q_{i-1}}$ almost every where on $I_{i+1} \times \ldots \times I_n$.

For any polynomial P_m we have

$$\left| \int\limits_K P_m \cdot F_0^* dx_1...dx_n \right| = \left| \int\limits_K P_m \cdot (F_0^* - F_1) dx_1...dx_n \right| \le \int\limits_K |P_m| \cdot |F_0^* - F_1| dx_1...dx_n =$$

$$= \int\limits_{I_n} \cdots \int\limits_{I_{i+1}} \int\limits_{e(x_{i+1},\dots,x_n)} |P_m|_{p_1,\dots,p_{i-1}} \cdot |F_0^* - F_1|_{q_1,\dots,q_{i-1}} dx_1...dx_n =$$

$$= \int\limits_{I_n} \cdots \int\limits_{I_{i+1}} \int\limits_{e(x_{i+1},\dots,x_n)} |P_m|_{p_1,\dots,p_{i-1}} \cdot |F_1|_{q_1,\dots,q_{i-1}} dx_1...dx_n \le$$

$$\le \int\limits_{I_n} \cdots \int\limits_{I_{i+1}} \left[\left(\int\limits_{e(x_{i+1},\dots,x_n)} |P_m|_{p_1,\dots,p_{i-1}} dx_i \right) \cdot \left(\operatorname*{ess\,sup}_{x_i \in I_i} |F_0^*|_{q_1,\dots,q_{i-1}} \right) \right] dx_{i+1}...dx_n,$$

which was to be shown.

Prove the sufficiency. Suppose the condition (4) holds true for any polynomial P_m by form (3).

Therefore, for any polynomial $Q_m = P_m^* + P_m$ using (4) in the end, we get

$$||f - Q_m||_{p_1,,\dots,p_{i-1},1,p_{i+1},\dots,p_n} =$$

$$= ||f - P_m^* - P_m||_{p_1,,\dots,p_{i-1},1,p_{i+1},\dots,p_n} \cdot ||F_0^*||_{q_1,,\dots,q_{i-1},\infty,q_{i+1},\dots,q_n} \ge$$

$$\ge \int\limits_{I_n} \cdots \int\limits_{I_{i+1}} \left(\int\limits_{I_i} |f - P_m^* - P_m|_{p_1,,\dots,p_{i-1}} dx_i \right) \left(\operatorname*{ess\,sup}_{x_i \in I_i} |F_0^*|_{q_1,,\dots,q_{i-1}} \right) dx_{i+1}...dx_n =$$

$$= \int\limits_{I_n} ... \int\limits_{I_{i+1}} \left(\int\limits_{I_i \setminus e(x_{i+1},...,x_n)} |f - P_m^* - P_m|_{p_1,,...,p_{i-1}} dx_i \right) \times$$

$$\times \left(\operatorname{ess\,sup}_{x_i \in I_i} |F_0^*|_{q_1,,...,q_{i-1}} \right) dx_{i+1}...dx_n +$$

$$+ \int\limits_{I_n} ... \int\limits_{I_{i+1}} \left(\int\limits_{e(x_{i+1},...,x_n)} |P_m|_{p_1,,...,p_{i-1}} dx_i \right) \cdot \left(\operatorname{ess\,sup}_{x_i \in I_i} |F_0^*|_{q_1,...,q_{i-1}} \right) dx_{i+1}...dx_n \geq$$

$$\geq \int\limits_{I_n} ... \int\limits_{I_{i+1}} \int\limits_{I_i \setminus e(x_{i+1},...,x_n)} \int\limits_{I_{i-1}} \ldots \int\limits_{I_1} (f - P_m^* - P_m) \cdot F_0^* dx_1...dx_n +$$

$$+ \int\limits_{I_n} ... \int\limits_{I_{i+1}} \left(\int\limits_{e(x_{i+1},...,x_n)} |P_m|_{p_1,...,p_{i-1}} dx_i \right) \cdot \left(\operatorname{ess\,sup}_{x_i \in I_i} |F_0^*|_{q_1,...,q_{i-1}} \right) dx_{i+1}...dx_n =$$

$$= \int\limits_K (f - P_m^*) F_0^* dx_1...dx_n - \int\limits_K P_m F_0^* dx_1...dx_n +$$

$$+ \int\limits_{I_n} ... \int\limits_{I_{i+1}} \left(\int\limits_{e(x_{i+1},...,x_n)} |P_m|_{p_1,...,p_{i-1}} dx_i \right) \cdot \left(\operatorname{ess\,sup}_{x_i \in I_i} |F_0^*|_{q_1,...,q_{i-1}} \right) dx_{i+1}...dx_n \geq$$

$$\geq \int\limits_K (f - P_m^*) F_0^* dx_1...dx_n = ||f - P_m^*||_{p_1,,...,p_{i-1},1,p_{i+1},...,p_n}.$$

It means that P_m^* is the best approximant for f in $L_{p_1,,...,p_{i-1},1,p_{i+1},...,p_n}$.
This completes the proof of Theorem 2.

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