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AN ACCELERATING SCHEME OF CONVERGENCE TO SOLVE FUZZY NON-LINEAR EQUATIONS

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Abstract. In this paper, we propose an accelerating scheme of convergence of numerical solutions of fuzzy non-linear equations. Numerical experiments show that the new method has significant acceleration of convergence of solutions of fuzzy non-linear equation. Three-dimensional graphical representation of fuzzy solutions is also provided as a reference of visual convergence of the solution sequence.

1. Introduction

Fuzzy logic presented by Zadeh [11] in 1965 is applicable in many fields including decision making, production planning and scheduling, location, transportation, microprocess, information processing, automatic control, physics, engineering, medical science, artificial intelligence, etc. Some systems depend on the roots of fuzzy non-linear equations. One can use analytical methods to compute the roots of certain types of fuzzy non-linear equations, but there are many problems so that one may struggle with getting the exact root or even may not get the exact one. In such cases, numerical methods may be useful to solve those problems.

There are many numerical methods to solve fuzzy non-linear equations. Abbasbandy and Asady [1] proposed a kind of Newton's method, Shokri [9] suggested a new two-step iterative method, Khorasani and Aghcheghloo [3] presented a kind of secand method, Saha and Shirin [6, 7] introduced methods using fixed point iteration algorithm and using Bisection algorithm, Waziri and Majid [10] proposed a method using Broyden's and Newton's methods, Paripour *et al.* [4] presented a method based on homotopy, and Senthilkumar and Ganesan [8] suggested a method using harmonic mean. However, there is no literature so far dealing with acceleration of

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convergence of numerical solutions of fuzzy non-linear equations. In this paper, we propose an accelerating method of convergence to solve fuzzy non-linear equations.

The paper is organized as follows. In Section 2, we provide some preliminary definitions in fuzzy real numbers. In Section 3, an accelerating algorithm is presented to solve fuzzy non-linear equations. Numerical example is provided in Section 4. Lastly, we will make concluding remarks in Section 5.

2. Preliminaries

In this section, we discuss some important definitions and properties of linear fuzzy real numbers [5, 6, 7]. When we consider the set of all real numbers R, one way to associate a fuzzy number with a fuzzy subset of real numbers is as a function $\mu : R \to [0,1]$, where the value $\mu(x)$ is to represent a degree of belonging to the subset of R.

Definition 2.1 (Linear fuzzy real number)**.** Let R be the set of all real numbers and $\mu : R \to [0,1]$ be a function defined by

$$\mu(x) = \begin{cases} 0, & \text{if } x < a \text{ or } x > c, \\ \frac{x-a}{b-a}, & \text{if } a \le x < b, \\ 1, & \text{if } x = b, \\ \frac{c-x}{c-b}, & \text{if } b < x \le c. \end{cases}$$

Then $\mu(a,b,c)$ is called a *linear fuzzy real number* with associated triple of real numbers (a,b,c) where $a \le b \le c$ shown in Figure 1.

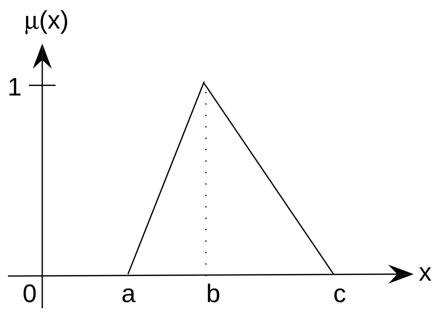

Figure 1. Linear fuzzy real number $\mu(a,b,c)$

Let LFR be the set of all linear fuzzy real numbers. Then we note that any real number $b \in R$ can be written as a linear fuzzy real number $r(b) \in LFR$, where $r(b) = \mu(b,b,b)$, and hence $R \subseteq LFR$. As a linear fuzzy real number, we consider

$r(b)$ to represent the real number b itself. Operations, square root, function, non-linear equation on LFR [5, 6, 7] are defined as the followings.

Definition 2.2 (Operations on LFR)**.** For given two linear fuzzy real numbers $\mu_1 = \mu(a_1, b_1, c_1)$ and $\mu_2 = \mu(a_2, b_2, c_2)$, we define addition, subtraction, multiplication, and division by

(1) $\mu_1 + \mu_2 = \mu(a_1 + a_2, b_1 + b_2, c_1 + c_2)$
(2) $\mu_1 - \mu_2 = \mu(a_1 - c_2, b_1 - b_2, c_1 - a_2)$
(3) $\mu_1 \cdot \mu_2 = \mu(\min\{a_1a_2, a_1c_2, a_2c_1, c_1c_2\}, b_1b_2, \max\{a_1a_2, a_1c_2, a_2c_1, c_1c_2\})$
(4) $\frac{\mu_1}{\mu_2} = \mu_1 \cdot \frac{1}{\mu_2}$ where $\frac{1}{\mu_2} = \mu(\min\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\}, \text{median}\,\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\}, \max\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\})$.

Definition 2.3 (Square root of an LFR)**.** Square root of a $\mu(a, b, c) \in LFR$ is defined by

$$\sqrt{\mu(a, b, c)} = \mu(\sqrt{a}, \sqrt{b}, \sqrt{c})$$

where $a, b, c \geq 0$.

Definition 2.4 (Function on LFR)**.** Let $f : R \to R$ be a real-valued function and $\mu(a, b, c) \in LFR$. Let $\bar{a} = \min\{f(a), f(b), f(c)\}, \bar{b}$ =median$\{f(a), f(b), f(c)\}, \bar{c} = \max\{f(a), f(b), f(c)\}$. Then the function $\bar{f} : LFR \to LFR$ defined by

$$\bar{f}(\mu(a, b, c)) = \mu(\bar{a}, \bar{b}, \bar{c}),$$

is called the *LFR-valued function* associated with f.

We note that if $a = b = c$ then $\bar{a} = \bar{b} = \bar{c}$, i.e., $\bar{f}(r(b)) = r(f(b))$. Hence $\bar{f}$ is an extension of the function f. Furthermore, an LFR-valued function $\bar{f}$ is said to be non-linear if the associated function f is non-linear.

Definition 2.5 (Non-linear equation on LFR)**.** Let $\bar{f} : LFR \to LFR$ be a non-linear LFR-valued function. Then $\bar{f}(\mu_x) = 0$ is called a *non-linear equation* in LFR with the unknown μ_x. For example, $\bar{f}(\mu_x) = \mu_x^3 + \mu_x^2 - 3 = 0$ is a fuzzy non-linear equation in LFR.

Definition 2.6 (Sequence on LFR)**.** Let $\{\mu_n\}_{n=0}^{\infty}$ be a sequence of LFR where $\mu_n = \mu(a_n, b_n, c_n)$. The LFR-sequence $\{\mu_n\}$ has the limit $\mu^* = \mu(a^*, b^*, c^*)$ and we write $\lim_{n\to\infty} \mu_n = \mu^*$, if the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ have the limit a^*, b^*, and c^*, respectively. If $\lim_{n\to\infty} \mu_n$ exists, we say the LFR-sequence $\{\mu_n\}$ is *convergent*. Otherwise, we say the sequence is *divergent*.

3. Solving Fuzzy Non-linear Equations

As mentioned in Section 1, there are many numerical ways to solve fuzzy non-linear equations of the form $\bar{f}(\mu_x) = 0$ in LFR, such as Bisection method, Newton's method, fixed-point iteration, etc. In this paper, we focus on how to accelerate the convergence of a sequence of solutions from one initial sequence of numerical solutions of fuzzy non-linear equation. In order to get an initial sequence of solutions, we simply employ the fixed-point iterative method [6] on fuzzy non-linear equation. We point out that any other numerical schemes can be used as our initial step which will then be accelerated in our scheme. The fuzzy fixed-point iterative method begins with an initial approximation $\mu_0 = \mu(a_0, b_0, c_0) \in LFR$ and generates a sequence $\{\mu_n\}_{n=0}^{\infty}$ with $\mu_n = \mu(a_n, b_n, c_n)$ defined by

$$\mu_n = \bar{g}(\mu_{n-1}) \text{ for } n \geq 1, \tag{3.1}$$

where $\bar{f}(\mu_x) = \mu_x - \bar{g}(\mu_x) = 0$.

3.1. LFR accelerating scheme In this section, we discuss a technique to accelerate the convergence of a sequence of solutions with a modification of Aitken's method.

Suppose that $\{\mu_n\}_{n=0}^{\infty}$ is a convergent sequence with limit μ^*. Then define a new sequence $\{\hat{\mu}_n\}_{n=0}^{\infty}$ using the initial sequence $\{\mu_n\}_{n=0}^{\infty}$ by

$$\hat{\mu}_n = \mu_n - \frac{(\mu_{n+1} - \mu_n)^2}{\mu_{n+2} - 2\mu_{n+1} + \mu_n} \text{ for } n \geq 0. \tag{3.2}$$

Based on the next theorem, the sequence $\{\hat{\mu}_n\}_{n=0}^{\infty}$ obtained by the modified Aitken's method is convergent more rapidly to μ^* than the original sequence $\{\mu_n\}_{n=0}^{\infty}$.

Theorem 3.1. *Suppose that $\{\mu_n\}_{n=0}^{\infty}$ is a sequence that converges linearly to the limit μ^* and that $\lim_{n\to\infty} \frac{\mu_{n+1}-\mu^*}{\mu_n-\mu^*} < 1$. Then the sequence $\{\hat{\mu}_n\}_{n=0}^{\infty}$ converges to μ^* faster than $\{\mu_n\}_{n=0}^{\infty}$ in the sense that $\lim_{n\to\infty} \frac{\hat{\mu}_n-\mu^*}{\mu_n-\mu^*} = 0$.*

Proof. See in [2]. □

Now we provide the algorithm of the new scheme, referred to as LFR Aitken's algorithm, to solve fuzzy non-linear equations over LFR.

Algorithm 3.2. (LFR Aitken's algorithm)

INPUT: fuzzy function $\bar{g}(\mu_x)$, initial approx. μ_0, max. number of iter. N
OUTPUT: approximate solution μ_n and accelerated solution $\hat{\mu}_n$

Step 1: Set $i = 1$ and $j = 0$.
Step 2: While $i \leq N$ do Step 3.
Step 3: Set $\mu_i = \bar{g}(\mu_{i-1})$ and $i = i + 1$.
Step 4: While $j \leq (N - 2)$ do Step 5.
Step 5: Set $\hat{\mu}_j = \mu_j - (\mu_{j+1} - \mu_j)^2/(\mu_{j+2} - 2\mu_{j+1} + \mu_j)$ and $j = j + 1$.
Step 6: OUTPUT(μ_n, $\hat{\mu}_n$) and STOP.

Table 1. Approximate solutions μ_n using fuzzy fixed-point iteration

n	μ_n
0	$\mu_0 = \mu(1, 1.4, 1.8)$
1	$\mu_1 = \mu(1.0350983, 1.1180340, 1.2247449)$
2	$\mu_2 = \mu(1.1612361, 1.1901293, 1.2141377)$
3	$\mu_3 = \mu(1.1640143, 1.1703769, 1.1781742)$
4	$\mu_4 = \mu(1.1735844, 1.1756906, 1.1774177)$
5	$\mu_5 = \mu(1.1737882, 1.1742540, 1.1748228)$
6	$\mu_6 = \mu(1.1744883, 1.1746419, 1.1747677)$
7	$\mu_7 = \mu(1.1745032, 1.1745371, 1.1745786)$
8	$\mu_8 = \mu(1.1745542, 1.1745654, 1.1745746)$
9	$\mu_9 = \mu(1.1745553, 1.1745578, 1.1745608)$
10	$\mu_{10} = \mu(1.1745590, 1.1745598, 1.1745605)$
11	$\mu_{11} = \mu(1.1745591, 1.1745593, 1.1745595)$
12	$\boldsymbol{\mu_{12}} = \mu(1.1745594, \mathbf{1.1745594}, 1.1745595)$

Table 2. Accelerated solutions $\hat{\mu}_n$ using LFR Aitken's method

n	$\hat{\mu}_n$
0	$\hat{\mu}_0 = \mu(0.9864686, 1.1754490, 1.2139384)$
1	$\hat{\mu}_1 = \mu(1.1640769, 1.1746248, 1.2291822)$
2	$\hat{\mu}_2 = \mu(1.1600996, 1.1745642, 1.1774015)$
3	$\hat{\mu}_3 = \mu(1.1737927, 1.1745598, 1.1784856)$
4	$\boldsymbol{\hat{\mu}_4} = \mu(1.1735006, \mathbf{1.1745594}, 1.1747665)$

4. Numerical Example

Consider a fuzzy non-linear equation $\mu_x^3 + \mu_x^2 - 3 = 0$ on the interval $[0, 2]$. Let $\bar{f}(\mu_x) = \mu_x^3 + \mu_x^2 - 3$ and rewrite $\bar{f}(\mu_x) = 0$ in the form of $\mu_x = \bar{g}(\mu_x)$. If we set $\bar{g}(\mu_x) = \sqrt{\frac{3}{\mu_x+1}}$ and an initial approximation $\mu_0 = \mu(1, 1.4, 1.8) \in LFR$, then we can generate an initial sequence $\{\mu_n\}_{n=0}^{\infty}$, where $\mu_n = \mu(a_n, b_n, c_n)$, using Equation

(3.1). For example,

$$\mu_1 = \sqrt{\frac{3}{\mu_0 + 1}} = \sqrt{\frac{3}{\mu(1, 1.4, 1.8) + 1}} = \mu(1.0350983, 1.1180340, 1.2247449).$$

First few terms of the sequence are listed in Table 1. The actual root of the crisp equation $x^3 + x^2 - 3 = 0$ over R is $1.174559410292980\cdots$. Note that b_{12} in the approximate solution μ_{12} is accurate to seven decimal places. This fuzzy fixed-point iteration method produces a sequence that converges to the exact solution like in Table 1. However, we would like to accelerate the convergence of the sequence.

Once we have the initial sequence of solutions $\{\mu_n\}_{n=0}^{\infty}$, we generate a new sequence $\{\hat{\mu}_n\}_{n=0}^{\infty}$ of LFR Aitken's method using Equation (3.2) listed in Table 2. For example,

$$\hat{\mu}_0 = \mu_0 - \frac{(\mu_1 - \mu_0)^2}{\mu_2 - 2\mu_1 + \mu_0} = \mu(0.9864686, 1.1754490, 1.2139384).$$

We should point out that the later sequence $\{\hat{\mu}_n\}_{n=0}^{\infty}$ is convergent to the exact solution much faster than the initial sequence $\{\mu_n\}_{n=0}^{\infty}$ within 4 iterations versus 12. We also provide three-dimensional graphical representation of accelerated solutions $\{\hat{\mu}_n\}_{n=0}^{\infty}$ in Figure 2, in which we are able to see the fast convergence of the solution sequence.

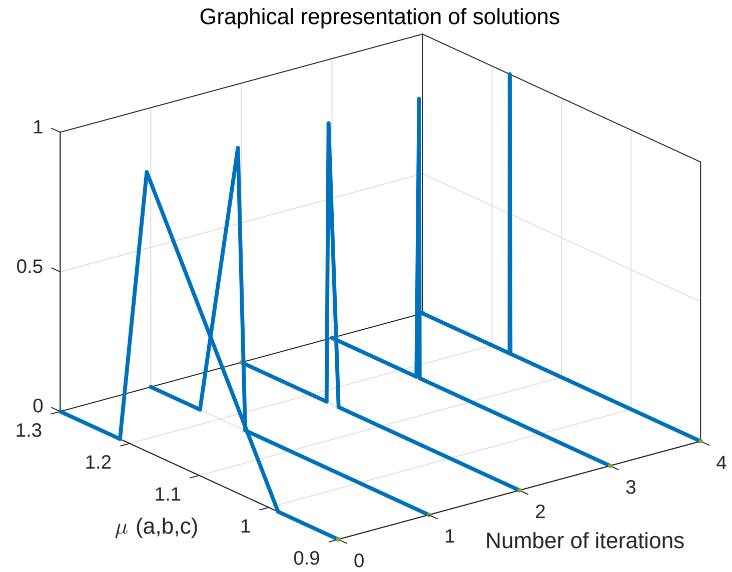

Figure 2. Graphical representation of accelerated solutions $\hat{\mu}_n$ by LFR Aitken's method

5. Conclusion

A non-linear equation over linear fuzzy real numbers is called a fuzzy non-linear equation. In this paper, a new method has been introduced to solve fuzzy non-linear equations with a modification of Aitken's method. By using this method, we could accelerate the convergence of a sequence of numerical solutions of the equation. We have also presented three-dimensional graphical representation of those solutions as a reference of visual convergence of the solution sequence.

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