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Some generalized results related to Fibonacci sequence

Neeraj Kumar Paul[1] orcid.org/0000-0003-2804-6564
Helen K. Saikia[2] orcid.org/0000-0003-1971-9472

Gauhati University, Dept. of Mathematics, Guwahati, AS, India.
[1] neeraj_dmj@yahoo.co.in ; [2] hsaikia@yahoo.com

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Abstract:

Cassini's identity states that for the n^{th} Fibonacci number of the Fibonacci sequence $\{F_n\}$, $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. We generalize Fibonacci sequence in terms of the number of sequences. Fibonacci sequence is the particular case of generating only one sequence. This generalization is used to generalize Cassini's identity. Moreover we prove few more results which can be seen as generalized form of the results which hold for Fibonacci sequence.

Keywords: Fibonacci sequence; Cassini's identity.

MSC (2020): 11B39.

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1. Introduction

Fibonacci sequence (F_k) is generated by the recursive formula $F_k = F_{k-1} + F_{k-2}$ for $k \geq 3$ with $F_1 = 1, F_2 = 1$ [2,3]. That is, each term in the sequence (third term onwards) is the sum of the two terms that immediately precede it. The Fibonacci sequence is the first known recursive sequence in mathematical work. First few terms of the sequence [4] are 1,1,2,3,5,8,13,21,.... Many generalizations of the sequence and hence its properties are available in the literature.

In [1], we generated m number of sequences following certain recursive rules as follows. When the number of sequence is one, i.e. $m = 1$, these rules coincide with those generating Fibonacci numbers and we get the Fibonacci Sequence.

We consider m interconnected sequences

$$(S_{1,k}), (S_{2,k}), (S_{3,k}), \dots, (S_{m,k})$$

which can be generated according to the following rule

$$\begin{aligned} &S_{1,1} = S_{2,1} = S_{3,1} = \ldots = S_{m,1} = 1,\ S_{1,2} = 1, && \\ &S_{i,k} = S_{i-1,k-1} + S_{i-1,k}, && 1 < i \leq m, k \geq 2 \\ &S_{1,k} = S_{m,k-1} + S_{m,k-2}, && k \geq 3 \end{aligned}$$

k	$S_{1,k}$	$S_{2,k}$	$S_{3,k}$	$S_{4,k}$	$S_{5,k}$	$S_{6,k}$
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	7	8	10	13	17	22
4	28	35	43	53	66	83
5	105	133	168	211	264	330
6	413	518	651	819	1030	1294
7	1624	2037	2555	3206	4025	5055
8	6349	7973	10010	12565	15771	19796
...	...	...	...	...	...	...

Columns show the terms in six sequences (i.e. for $m = 6$).

For $m = 1$, we identify the only sequence $(S_{1,k})$ as the Fibonacci sequence (F_k).

Notation: $S_{p,q}$ denotes the q^{th} term of the p^{th} sequence.

With this definition we generalized, in [1], the following results for Fibonacci sequence, along with others.

- $F_k + F_{k+1} = F_{k+2}$
  Generalization: Sum of k^{th} terms in all the sequences and $(k+1)^{th}$ term in the first sequence equals the $(k+2)^{th}$ term in the first sequence.

- $\sum_{k=1}^{n} F_k = F_{n+2} - 1$
  Generalization: Sum of first n terms of all the sequences is one less than the $(n+2)^{th}$ term of the first sequence.

- $\sum_{k=0}^{n-1} F_{2k+1} = F_{2n}$
  Generalization: Sum of all the terms in odd positions (upto $(2n+1)^{th}$ position) in all the sequences equals the $(2n)^{th}$ term of the last sequence.

- $\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1$
  Generalization: Sum of all the terms in even positions (upto $(2n)^{th}$ position) in all the sequences is one less than the $(2n+1)^{th}$ term of the last sequence.

- $\lim_{k\to\infty} \frac{S_{i,k+1}}{S_{i,k}}$ is a zero of the polynomial $x^{m+1} - (x+1)^m$ for $i = 1, 2, \ldots, m$, which is the generalization of the fact that the golden ratio is a zero of the polynomial $x^2 - x - 1$.

2. Recurrence relation and few generalizations

Theorem 2.1. *Each sequence can be generated independently by a recurrence relation given by*

$$S_{i,k} = {}^mC_0 S_{i,k-1} + {}^mC_1 S_{i,k-2} + {}^mC_2 S_{i,k-3} + \ldots + {}^mC_m S_{i,k-m-1}$$

with initial terms being for $1 \leq i \leq m$ *and* $-(m+1) \leq k \leq -1$

$$S_{i,k} = (-1)^{i+k+m+1} \left({}^{-k-1}C_{m-i}\right)$$

Proof. We restate the rules for generating the sequences.
We set

$$\begin{array}{ll} S_{1,1}=S_{2,1}=S_{3,1}=\ldots=S_{m,1}=1,\ S_{1,2}=1, & \\ S_{i,k}=S_{i-1,k-1}+S_{i-1,k}, & 1<i\leq m, k\geq 2 \\ S_{1,k}=S_{m,k-1}+S_{m,k-2}, & k\geq 3 \end{array}$$

Extending the sequences backward, we can easily see that
for $1 \leq i \leq m$,

$$S_{i,0}=0,$$

and for $-(m+1) \leq k \leq -1$

$$S_{i,k}=(-1)^{i+k+m+1}\left({}^{-k-1}C_{m-i}\right) \tag{2.1}$$

Clearly, $S_{i,k}=0$ for $i-m \leq k \leq -1$ and $S_{m,-1}=1$.

We now make a claim for the recurrence relation. A term in a sequence can be obtained using the following recurring relation involving the previously obtained $(m+1)$ terms of the same sequence.

$$S_{i,k}={}^{m}C_0S_{i,k-1}+{}^{m}C_1S_{i,k-2}+{}^{m}C_2S_{i,k-3}+\ldots+{}^{m}C_mS_{i,k-m-1} \tag{2.2}$$

with initial terms being $S_{i,k}$, as defined in (2.1), where $1 \leq i \leq m$ and $-(m+1) \leq k \leq -1$.

Now,

$$\begin{aligned} S_{i,0} &= {}^{m}C_0S_{i,-1}+{}^{m}C_1S_{i,-2}+{}^{m}C_2S_{i,-3}+\ldots+{}^{m}C_mS_{i,-m-1} \\ &=0 \end{aligned}$$

Thus $S_{i,k}=S_{i-1,k-1}+S_{i-1,k}$, for $1<i\leq m$, $-m \leq k \leq 0$
and $S_{1,k}=S_{m,k-1}+S_{m,k-2}$, for $-m+1 \leq k \leq 0$
are satisfied.

These are also satisfied for $k=1$ as

$$\begin{aligned} S_{i,1} &= {}^{m}C_0S_{i,0}+{}^{m}C_1S_{i,-1}+{}^{m}C_2S_{i,-2}+\ldots+{}^{m}C_mS_{i,-m} \\ &=1 \end{aligned}$$

We now establish the recurrence relation (2.2) using induction on k. Suppose the definition of the sequences are satisfied for $k \leq l$. Then

$$\begin{aligned}
S_{m,l-1} + S_{m,l} &= {}^mC_0S_{m,l-2} + {}^mC_1S_{m,l-3} + {}^mC_2S_{m,l-4} + \\
&\quad \ldots + {}^mC_mS_{m,l-m-2} + \\
&\quad {}^mC_0S_{m,l-1} + {}^mC_1S_{m,l-2} + {}^mC_2S_{m,l-3} + \\
&\quad \ldots + {}^mC_mS_{m,l-m-1} \\
&= {}^mC_0\left(S_{m,l-2} + S_{m,l-1}\right) + {}^mC_1\left(S_{m,l-3} + S_{m,l-2}\right) \\
&\quad + {}^mC_2\left(S_{m,l-4} + S_{m,l-3}\right) + \ldots \\
&\quad + {}^mC_m\left(S_{m,l-m-2} + S_{m,l-m-1}\right) \\
&= {}^mC_0S_{1,l} + {}^mC_1S_{1,l-1} + {}^mC_2S_{1,l-2} + \\
&\quad \ldots + {}^mC_mS_{1,l-m} \\
&= S_{1,l+1}
\end{aligned}$$

and for $1 < i \leq m$

$$\begin{aligned}
S_{i-1,l-1} + S_{i-1,l} &= {}^mC_0S_{i-1,l-2} + {}^mC_1S_{i-1,l-3} + {}^mC_2S_{i-1,l-4} + \\
&\quad \ldots + {}^mC_mS_{i-1,l-m-2} \\
&\quad + {}^mC_0S_{i-1,l-1} + {}^mC_1S_{i-1,l-2} + {}^mC_2S_{i-1,l-3} + \\
&\quad \ldots + {}^mC_mS_{i-1,l-m-1} \\
&= S_{i,l}
\end{aligned}$$

Hence the claim. □

Theorem 2.2. *For each of the m sequences, gcd of $(m+1)$ consecutive terms is 1. This is generalization of the fact that pair of consecutive Fibonacci numbers are relatively prime.*

Proof. We first write $m+1$ consecutive terms of a particular sequence as below.

$$\begin{array}{lllll}
S_{i,k} & = {}^mC_0S_{i,k-1} & + {}^mC_1S_{i,k-2} & + \ldots + & {}^mC_mS_{i,k-m-1} \\
S_{i,k+1} & = {}^mC_0S_{i,k} & + {}^mC_1S_{i,k-1} & + \ldots + & {}^mC_mS_{i,k-m} \\
S_{i,k+2} & = {}^mC_0S_{i,k+1} & + {}^mC_1S_{i,k} & + \ldots + & {}^mC_mS_{i,k-m+1} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
S_{i,k+m-1} & = {}^mC_0S_{i,k+m-2} & + {}^mC_1S_{i,k+m-3} & + \ldots + & {}^mC_mS_{i,k-2} \\
S_{i,k+m} & = {}^mC_0S_{i,k+m-1} & + {}^mC_1S_{i,k+m-2} & + \ldots + & {}^mC_mS_{i,k-1}
\end{array}$$

Suppose g divides all the above terms. Then from the last expression, we can write

$${}^mC_mS_{i,k-1} = S_{i,k+m} - {}^mC_0S_{i,k+m-1} - {}^mC_1S_{i,k+m-2} - \ldots - {}^mC_{m-1}S_{i,k}$$

which implies that g divides $S_{i,k-1}$.

Now considering the fact that g divides the $m+1$ consecutive terms $S_{i,k-1},\ S_{i,k},\ S_{i,k+1},\ S_{i,k+2},\ \ldots,\ S_{i,k+m-1}$, we proceed as above to get g

divides $S_{i,k-2}$. Continuing in similar fashion, we obtain that g divides $S_{i,1}$. This implies $g = 1$. □

Theorem 2.3. *Generalization of Cassini's Identity* $F_{k+1}F_{k-1} - F_k^2 = (-1)^k$

For i^{th} sequence,

$$\begin{vmatrix} S_{i,k+1} & S_{i,k} & S_{i,k-1} & \dots & S_{i,k-(m-1)} \\ S_{i,k} & S_{i,k-1} & S_{i,k-2} & \dots & S_{i,k-m} \\ S_{i,k-1} & S_{i,k-2} & S_{i,k-3} & \dots & S_{i,k-(m+1)} \\ \dots & \dots & \dots & \dots & \dots \\ S_{i,k-(m-2)} & S_{i,k-(m-1)} & S_{i,k-m} & \dots & S_{i,k-(2m-2)} \\ S_{i,k-(m-1)} & S_{i,k-m} & S_{i,k-(m+1)} & \dots & S_{i,k-(2m-1)} \end{vmatrix}$$

$$= (-1)^{m(i-1)} \begin{cases} (-1)^{\frac{m}{2}} \textit{ when } m \textit{ is even,} \\ \\ (-1)^{\frac{m-1}{2}} \textit{ when } m \textit{ is odd, and } k \textit{ is even,} \\ \\ (-1)^{\frac{m+1}{2}} \textit{ when both } m \textit{ and } k \textit{ are odd.} \end{cases}$$

Proof. We write

$$\begin{array}{lllllll} S_{i,k+m} & = & {}^mC_0 S_{i,k+m-1} + & {}^mC_1 S_{i,k+m-2} + \dots + & {}^mC_{m-1} S_{i,k} + & {}^mC_m S_{i,k-1} \\ S_{i,k+m-1} & = & 1\ S_{i,k+m-1} + & 0\ S_{i,k+m-2} + \dots + & 0\ S_{i,k} + & 0\ S_{i,k-1} \\ S_{i,k+m-2} & = & 0\ S_{i,k+m-1} + & 1\ S_{i,k+m-2} + \dots + & 0\ S_{i,k} + & 0\ S_{i,k-1} \\ \dots & \dots & \dots & \dots\ \dots & & \dots \\ \dots & \dots & \dots & \dots\ \dots & & \dots \\ S_{i,k} & = & 0\ S_{i,k+m-1} + & 0\ S_{i,k+m-2} + \dots + & 1\ S_{i,k} + & 0\ S_{i,k-1} \end{array}$$

In matrix form,

$$\begin{bmatrix} S_{i,k+m} \\ S_{i,k+m-1} \\ S_{i,k+m-2} \\ S_{i,k+m-3} \\ \dots \\ S_{i,k+1} \\ S_{i,k} \end{bmatrix} = \begin{bmatrix} {}^mC_0 & {}^mC_1 & {}^mC_2 & \dots & {}^mC_{m-1} & {}^mC_m \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \begin{bmatrix} S_{i,k+m-1} \\ S_{i,k+m-2} \\ S_{i,k+m-3} \\ S_{i,k+m-4} \\ \dots \\ S_{i,k} \\ S_{i,k-1} \end{bmatrix}$$

We define

$$A = \begin{bmatrix} {}^mC_0 & {}^mC_1 & {}^mC_2 & \dots & {}^mC_{m-1} & {}^mC_m \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

Then A^k equals

$$\begin{bmatrix} \sum_{j=0}^{m} {}^mC_j S_{1,k-j} & \sum_{j=1}^{m} {}^mC_j S_{1,k+1-j} & \dots & \sum_{j=m-1}^{m} {}^mC_j S_{1,k+(m-1)-j} & S_{1,k} \\ \sum_{j=0}^{m} {}^mC_j S_{1,k-1-j} & \sum_{j=1}^{m} {}^mC_j S_{1,k-j} & \dots & \sum_{j=m-1}^{m} {}^mC_j S_{1,(k-1)+(m-1)-j} & S_{1,k-1} \\ \sum_{j=0}^{m} {}^mC_j S_{1,k-2-j} & \sum_{j=1}^{m} {}^mC_j S_{1,k-1-j} & \dots & \sum_{j=m-1}^{m} {}^mC_j S_{1,(k-2)+(m-1)-j} & S_{1,k-2} \\ \dots & \dots & \dots & \dots & \dots \\ \sum_{j=0}^{m} {}^mC_j S_{1,k-(m-1)-j} & \sum_{j=1}^{m} {}^mC_j S_{1,k-(m-2)-j} & \dots & \sum_{j=m-1}^{m} {}^mC_j S_{1,k-j} & S_{1,k-(m-1)} \\ \sum_{j=0}^{m} {}^mC_j S_{1,k-m-j} & \sum_{j=1}^{m} {}^mC_j S_{1,k-(m-1)-j} & \dots & \sum_{j=m-1}^{m} {}^mC_j S_{1,k-m+(m-1)-j} & S_{1,k-m} \end{bmatrix}$$

(p, q)th entry in the above $(m+1)\times(m+1)$ matrix is $\displaystyle\sum_{j=q-1}^{m} {}^mC_j S_{1,k-(p-q)-j}$.

or, A^k equals

$$\begin{bmatrix} S_{1,k+1} & \sum_{j=1}^{m} {}^mC_j S_{1,k+1-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j S_{1,k+(m-1)-j} & S_{1,k} \\ S_{1,k} & \sum_{j=1}^{m} {}^mC_j S_{1,k-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j S_{1,(k-1)+(m-1)-j} & S_{1,k-1} \\ S_{1,k-1} & \sum_{j=1}^{m} {}^mC_j S_{1,k-1-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j S_{1,(k-2)+(m-1)-j} & S_{1,k-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{1,k-(m-2)} & \sum_{j=1}^{m} {}^mC_j S_{1,k-(m-2)-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j S_{1,k-j} & S_{1,k-(m-1)} \\ S_{1,k-(m-1)} & \sum_{j=1}^{m} {}^mC_j S_{1,k-(m-1)-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j S_{1,k-m+(m-1)-j} & S_{1,k-m} \end{bmatrix}$$

so that $|A^k|$ equals

$$\begin{vmatrix} S_{1,k+1} & \sum_{j=1}^{m} {}^mC_j S_{1,k+1-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j S_{1,k+(m-1)-j} & S_{1,k} \\ S_{1,k} & \sum_{j=1}^{m} {}^mC_j S_{1,k-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j S_{1,(k-1)+(m-1)-j} & S_{1,k-1} \\ S_{1,k-1} & \sum_{j=1}^{m} {}^mC_j S_{1,k-1-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j S_{1,(k-2)+(m-1)-j} & S_{1,k-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{1,k-(m-2)} & \sum_{j=1}^{m} {}^mC_j S_{1,k-(m-2)-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j S_{1,k-j} & S_{1,k-(m-1)} \\ S_{1,k-(m-1)} & \sum_{j=1}^{m} {}^mC_j S_{1,k-(m-1)-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j S_{1,k-m+(m-1)-j} & S_{1,k-m} \end{vmatrix}$$

Applying column operations, we get

$$\begin{vmatrix} S_{1,k+1} & S_{1,k-(m-1)} & S_{1,k-(m-2)} & \cdots & S_{1,k-1} & S_{1,k} \\ S_{1,k} & S_{1,k-m} & S_{1,k-(m-1)} & \cdots & S_{1,k-2} & S_{1,k-1} \\ S_{1,k-1} & S_{1,k-(m+1)} & S_{1,k-m} & \cdots & S_{1,k-3} & S_{1,k-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{1,k-(m-2)} & S_{1,k-(2m-2)} & S_{1,k-(2m-3)} & \cdots & S_{1,k-m} & S_{1,k-(m-1)} \\ S_{1,k-(m-1)} & S_{1,k-(2m-1)} & S_{1,k-(2m-2)} & \cdots & S_{1,k-(m+1)} & S_{1,k-m} \end{vmatrix}$$

$$= |A|^k = (-1)^{mk}$$

Rearranging the columns, we obtain

$$\begin{vmatrix} S_{1,k+1} & S_{1,k} & S_{1,k-1} & \cdots & S_{1,k-(m-1)} \\ S_{1,k} & S_{1,k-1} & S_{1,k-2} & \cdots & S_{1,k-m} \\ S_{1,k-1} & S_{1,k-2} & S_{1,k-3} & \cdots & S_{1,k-(m+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{1,k-(m-2)} & S_{1,k-(m-1)} & S_{1,k-m} & \cdots & S_{1,k-(2m-2)} \\ S_{1,k-(m-1)} & S_{1,k-m} & S_{1,k-(m+1)} & \cdots & S_{1,k-(2m-1)} \end{vmatrix}$$

$$= \begin{cases} (-1)^{\frac{m}{2}}(-1)^{mk} & \text{when } m \text{ is even,} \\ (-1)^{\frac{m-1}{2}}(-1)^{mk} & \text{when } m \text{ is odd.} \end{cases}$$

Thus,

$$\begin{vmatrix} S_{1,k+1} & S_{1,k} & S_{1,k-1} & \cdots & S_{1,k-(m-1)} \\ S_{1,k} & S_{1,k-1} & S_{1,k-2} & \cdots & S_{1,k-m} \\ S_{1,k-1} & S_{1,k-2} & S_{1,k-3} & \cdots & S_{1,k-(m+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{1,k-(m-2)} & S_{1,k-(m-1)} & S_{1,k-m} & \cdots & S_{1,k-(2m-2)} \\ S_{1,k-(m-1)} & S_{1,k-m} & S_{1,k-(m+1)} & \cdots & S_{1,k-(2m-1)} \end{vmatrix}$$

$$= \begin{cases} (-1)^{\frac{m}{2}} & \text{when } m \text{ is even,} \\ (-1)^{\frac{m-1}{2}} & \text{when } m \text{ is odd, } k \text{ is even,} \\ (-1)^{\frac{m+1}{2}} & \text{when } m \text{ and } k \text{ are odd.} \end{cases}$$

Suppose

$$\begin{vmatrix} S_{i,k+1} & S_{i,k} & S_{i,k-1} & \dots & S_{i,k-(m-2)} & S_{i,k-(m-1)} \\ S_{i,k} & S_{i,k-1} & S_{i,k-2} & \dots & S_{i,k-(m-1)} & S_{i,k-m} \\ S_{i,k-1} & S_{i,k-2} & S_{i,k-3} & \dots & S_{i,k-m} & S_{i,k-(m+1)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ S_{i,k-(m-2)} & S_{i,k-(m-1)} & S_{i,k-m} & \dots & S_{i,k-(2m-3)} & S_{i,k-(2m-2)} \\ S_{i,k-(m-1)} & S_{i,k-m} & S_{i,k-(m+1)} & \dots & S_{i,k-(2m-2)} & S_{i,k-(2m-1)} \end{vmatrix} = \Delta$$

By $C_j \to C_j + C_{j+1}$, for $j = 1, 2, \dots, m$

$$\begin{vmatrix} S_{i+1,k+1} & S_{i+1,k} & S_{i+1,k-1} & \dots & S_{i+1,k-(m-2)} & S_{i,k-(m-1)} \\ S_{i+1,k} & S_{i+1,k-1} & S_{i+1,k-2} & \dots & S_{i+1,k-(m-1)} & S_{i,k-m} \\ S_{i+1,k-1} & S_{i+1,k-2} & S_{i+1,k-3} & \dots & S_{i+1,k-m} & S_{i,k-(m+1)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ S_{i+1,k-(m-2)} & S_{i+1,k-(m-1)} & S_{i+1,k-m} & \dots & S_{i+1,k-(2m-3)} & S_{i,k-(2m-2)} \\ S_{i+1,k-(m-1)} & S_{i+1,k-m} & S_{i+1,k-(m+1)} & \dots & S_{i+1,k-(2m-2)} & S_{i,k-(2m-1)} \end{vmatrix} = \Delta$$

Also

$$\begin{aligned} S_{i+1,k} = \quad & S_{i+1,k+m} \\ & -(1 +^m C_0) S_{i+1,k+m-1} \\ & +(1 +^m C_0 -^m C_1) S_{i+1,k+m-2} \\ & -(1 +^m C_0 -^m C_1 +^m C_2) S_{i+1,k+m-3} \\ & + \dots \\ & +(-1)^{m-1}(1 +^m C_0 -^m C_1 +^m C_2 - \dots +^m C_{m-2}) S_{i+1,k+1} \\ & +(-1)^m S_{i,k} \end{aligned}$$

Applying column operation on the last column as per the above formula, we get

$$\begin{vmatrix} S_{i+1,k+1} & S_{i+1,k} & S_{i+1,k-1} & \dots & S_{i+1,k-(m-2)} & S_{i+1,k-(m-1)} \\ S_{i+1,k} & S_{i+1,k-1} & S_{i+1,k-2} & \dots & S_{i+1,k-(m-1)} & S_{i+1,k-m} \\ S_{i+1,k-1} & S_{i+1,k-2} & S_{i+1,k-3} & \dots & S_{i+1,k-m} & S_{i+1,k-(m+1)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ S_{i+1,k-(m-2)} & S_{i+1,k-(m-1)} & S_{i+1,k-m} & \dots & S_{i+1,k-(2m-3)} & S_{i+1,k-(2m-2)} \\ S_{i+1,k-(m-1)} & S_{i+1,k-m} & S_{i+1,k-(m+1)} & \dots & S_{i+1,k-(2m-2)} & S_{i+1,k-(2m-1)} \end{vmatrix}$$

$$= (-1)^m \Delta$$

Thus we generalize Cassini's identity as

$$\begin{vmatrix} S_{i,k+1} & S_{i,k} & S_{i,k-1} & \dots & S_{i,k-(m-1)} \\ S_{i,k} & S_{i,k-1} & S_{i,k-2} & \dots & S_{i,k-m} \\ S_{i,k-1} & S_{i,k-2} & S_{i,k-3} & \dots & S_{i,k-(m+1)} \\ \dots & \dots & \dots & \dots & \dots \\ S_{i,k-(m-2)} & S_{i,k-(m-1)} & S_{i,k-m} & \dots & S_{i,k-(2m-2)} \\ S_{i,k-(m-1)} & S_{i,k-m} & S_{i,k-(m+1)} & \dots & S_{i,k-(2m-1)} \end{vmatrix}$$

$$= (-1)^{m(i-1)} \begin{cases} (-1)^{\frac{m}{2}} & \text{when } m \text{ is even,} \\ (-1)^{\frac{m-1}{2}} & \text{when } m \text{ is odd,} \\ & \text{and } k \text{ is even,} \\ (-1)^{\frac{m+1}{2}} & \text{when both} \\ & m \text{ and } k \text{ are odd.} \end{cases}$$

For $m = 1$, this reduces to a 2×2 determinant with $i = 1$.
That is

$$\begin{vmatrix} S_{1,k+1} & S_{1,k} \\ S_{1,k} & S_{1,k-1} \end{vmatrix} = \begin{cases} 1 & \text{when } k \text{ is even,} \\ -1 & \text{when } k \text{ is odd.} \end{cases}$$

which is the Cassini's identity $F_{k+1}F_{k-1} - F_k^2 = (-1)^k$ for Fibonacci numbers. Note that for $m = 1$, $S_{1,k} = F_k$. □

Theorem 2.4. *Generalization of* $F_{p+q} = F_{p+1}F_q + F_pF_{q-1}$

$$\begin{aligned} S_{1,p+q} = \ & \left(\sum_{j=0}^{m} {}^mC_j S_{1,p-j}\right) S_{1,q} + \left(\sum_{j=1}^{m} {}^mC_j S_{1,p-j+1}\right) S_{1,q-1} \\ & + \left(\sum_{j=2}^{m} {}^mC_j S_{1,p-j+2}\right) S_{1,q-2} \\ & + \dots + \left(\sum_{j=m-1}^{m} {}^mC_j S_{1,p-j+(m-1)}\right) S_{1,q-(m-1)} + S_{1,p}S_{1,q-m} \end{aligned}$$

Proof. $A^pA^q = A^{p+q}$ gives

$$\begin{aligned} S_{1,p+q+1} = & \left(\sum_{j=0}^{m} {}^mC_jS_{1,p-j}\right)\left(\sum_{j=0}^{m} {}^mC_jS_{1,q-j}\right) \\ & + \left(\sum_{j=1}^{m} {}^mC_jS_{1,p-j+1}\right)\left(\sum_{j=0}^{m} {}^mC_jS_{1,q-j-1}\right) \\ & + \left(\sum_{j=2}^{m} {}^mC_jS_{1,p-j+2}\right)\left(\sum_{j=0}^{m} {}^mC_jS_{1,q-j-2}\right) \\ & + \ldots + \left(\sum_{j=m-1}^{m} {}^mC_jS_{1,p-j+(m-1)}\right)\left(\sum_{j=0}^{m} {}^mC_jS_{1,q-j-(m-1)}\right) \\ & + S_{1,p}\left(\sum_{j=0}^{m} {}^mC_jS_{1,q-j-m}\right) \end{aligned}$$

Note: $S_{1,q} = \sum_{j=0}^{m} {}^mC_jS_{1,(q-1)-j}$.

Therefore

$$\begin{aligned} S_{1,p+q+1} = & \left(\sum_{j=0}^{m} {}^mC_jS_{1,p-j}\right)S_{1,q+1} + \left(\sum_{j=1}^{m} {}^mC_jS_{1,p-j+1}\right)S_{1,q} \\ & + \left(\sum_{j=2}^{m} {}^mC_jS_{1,p-j+2}\right)S_{1,q-1} \\ & + \ldots + \left(\sum_{j=m-1}^{m} {}^mC_jS_{1,p-j+(m-1)}\right)S_{1,q-(m-2)} + S_{1,p}S_{1,q-(m-1)} \end{aligned}$$

Writing $q-1$ for q, we get

$$\begin{aligned} S_{1,p+q} = & \left(\sum_{j=0}^{m} {}^mC_jS_{1,p-j}\right)S_{1,q} + \left(\sum_{j=1}^{m} {}^mC_jS_{1,p-j+1}\right)S_{1,q-1} \\ & + \left(\sum_{j=2}^{m} {}^mC_jS_{1,p-j+2}\right)S_{1,q-2} \\ & + \ldots + \left(\sum_{j=m-1}^{m} {}^mC_jS_{1,p-j+(m-1)}\right)S_{1,q-(m-1)} + S_{1,p}S_{1,q-m} \end{aligned}$$

□

Conclusion

The generalization of Fibonacci sequence as discussed in this paper generalizes Cassini's identity in matrix form. Binet's Formula, Catalan's identities etc. can be explored to be generalized accordingly.

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