Zhang, *Cogent Mathematics* (2017), 4: 1339369
https://doi.org/10.1080/23311835.2017.1339369

cogent mathematics

PURE MATHEMATICS | RESEARCH ARTICLE

The growth theorems for subclasses of biholomorphic mappings in several complex variables

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Received: 06 April 2017
Accepted: 01 June 2017
First Published: 12 June 2017

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Reviewing editor:
Hari M. Srivastava, University of Victoria, Canada

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Abstract: In this article, the growth theorems for some subclasses of biholomorphic mappings are obtained using the method of parametric representation. As the application, some well-known results can be got when special functions are taken on the unit disk in the complex plane.

Subjects: Analysis - Mathematics; Complex Variables; Pure Mathematics

Keywords: growth theorems; *g*-almost star-like mappings of order α; *g*-spiral-like mappings of type β

AMS subject classifications: Primary 32A30; Secondary 30C45

1. Introduction

In geometric theory of one complex variable, the following growth theorem for biholomorphic functions was well known.

THEOREM 1.1 *(Duren, 1983) Let f be a biholomorphic function on the unit disk $U = \{\zeta \in \mathbb{C}:|\zeta| < 1\}$, and $f(0) = f'(0) - 1 = 0$. Then*

$$\frac{|\zeta|}{(1+|\zeta|)^2} \le |f(\zeta)| \le \frac{|\zeta|}{(1-|\zeta|)^2}.$$

It is natural to extend the above beautiful results to higher dimensions. However, Cartan (1933) pointed out the above theorem for normalized biholomorphic mappings would not hold in several complex variables. And he also suggested to study the star-like mappings and convex mappings as appropriate topics for generalization. Untill 1991, Barnard, Fitzgerald, and Gong (1991) firstly established the growth and $\frac{1}{4}$-theorems for normalized biholomorphic star-like mappings on the unit ball $B^n = \{z \in \mathbb{C}^n:\|z\| = \left(\sum_{j=1}^{n} |z_j|^2\right)^{\frac{1}{2}} < 1\}$. And after that, a lot of researchers came to study the

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PUBLIC INTEREST STATEMENT

It is an important work to establish the growth theorems for subclasses of normalized biholomorphic mappings in geometric function theory of several complex variables. As well-known results, many researchers have studied the growth theorems for star-like mappings, covex mappings, and their subclasses. The aim of this article is to discuss growth theorems on the Euclidean unit ball for the generalized star-like mappings and the generalized spiral-like mappings that are defined by Loewner chains.

growth theorem for star-like mappings and the subclasses of star-like mappings on different domains; the reader can consult the references (Feng, Liu, & Ren, 2007; Hamada, Honda, & Kohr, 2006; Liu & Ren, 1998).

The subject of Loewner chains in higher dimensions was initiated by Pfaltzgraff (1974). He generalized to higher dimensions the Loewner differential equation and developed existence and uniqueness theorems for its solutions on the Euclidean unit ball B^n in $\mathbb{C}^n$. Poreda (1987a, 1987b) obtained some applications of parametric representation to growth theorems and coefficient estimates on the unit polydisk in $\mathbb{C}^n$. Poreda (1989) also deduced certain generalizations on the unit ball of finite dimensional complex Banach space. The existence and regularity of the theory of Loewner chains in higher dimensions were considered by Duren, Graham, Hamada, and Kohr (2010), Graham, Hamada, and Kohr (2002), Hamada and Kohr (2000), etc. Many details and applications of the theory of Loewner chains in several complex variables may be found in the monograph of Graham and Kohr (2003).

Chirilă (2014a) used the method of Loewner chains to generate certain subfamilies of normalized biholomorphic mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$, which have interesting geometric characterizations. In this paper, we will continue to study these biholomorphic mappings introduced by Chirilă. Furthermore, we will obtain the growth theorems for g-almost star-like mapping of order α $(0 \le \alpha < 1)$ and g-spiral-like mapping of type $\beta\left(-\frac{\pi}{2} < \beta < \frac{\pi}{2}\right)$ on the unit ball B^n using the method of parametric representation. As the application, some well-known results can be got when special functions g are taken on the unit disk in the complex plane.

In the following, we will give some notations and definitions. Let $\mathbb{C}$ be the complex plane and $U_r = \{\zeta \in \mathbb{C}:|\zeta| < r\}$. The unit disk in $\mathbb{C}$ is denoted by U. Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, \dots, z_n)'$ with the Euclidean inner product $\langle z, w\rangle = \sum_{i=1}^n z_i\overline{w_i}$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{\frac{1}{2}}$, where $z, w \in \mathbb{C}^n$ and the symbol "′" means transpose. The unit ball $B^n = \{z \in \mathbb{C}^n:\|z\| < 1\}$. Let X denote the complex Banach space with norm $\|\cdot\|$, $\mathcal{B} = \{x \in X:\|x\| < 1\}$ be the unit ball in X. Let Ω be a domain in X, $f:\Omega \to X$. If for any $x \in \Omega$, there is a linear mapping $Df(x)$ from X to X such that

$$\lim_{h\to 0}\frac{\|f(x+h) - f(x) - Df(x)h\|}{\|h\|} = 0,$$

then f is said to be holomorphic on Ω. The linear map $Df(x)$ is called the Fréchet derivative of f at x. In $\mathbb{C}^n$, $Df(x)$ is the Jacobian, always written by $J_f(z)$. We denote by $H(\Omega)$ the set of holomorphic mappings from Ω into $\mathbb{C}^n$. Let $f:\Omega \to X$ be a holomorphic mapping; if its Fréchet derivative $Df(x)$ is nonsingular at each $x \in \Omega$, then f is said to be locally biholomorphic on Ω. If f^{-1} means the inverse of f exists and it is holomorphic on the open set $f(\Omega)$, then f is said to be biholomorphic. If $f(0) = 0$ and $Df(0) = I$, then f is called normalized, where I is the identity operator.

The following families play a key role in our discussion:

$$\mathcal{P} = \{p \in H(U): p(0) = 1,\ Rep(\zeta) < 0,\ \zeta \in U\};$$

$$\mathcal{N} = \left\{h \in H(B^n): h(0) = 0, Re\left\langle h(z), \frac{z}{\|z\|^2}\right\rangle \ge 0, z \in B^n\backslash\{0\}\right\};$$

$$\mathcal{M} = \{h \in \mathcal{N}: Dh(0) = I\}.$$

2. Definition and lemmas

Definition 2.1 (Hamada et al., 2006) Let $g \in H(U)$ be a biholomorphic function such that $g(0) = 1,\ g(\overline{\zeta}) = \overline{g(\zeta)}$ for $\zeta \in U$ (i.e. g has real coefficients in its power series expansion), $Reg(\zeta) \ge 0$ on U and assume that g satisfies the following conditions for $r \in (0, 1)$

$$\begin{cases} \min\limits_{|\zeta|=r} Reg(\zeta) = \min\{g(r), g(-r)\}; \\ \max\limits_{|\zeta|=r} Reg(\zeta) = \max\{g(r), g(-r)\}. \end{cases} \tag{2.1}$$

Definition 2.2 (Chirilă, 2014a) Let $\alpha \in [0, 1)$. A normalized biholomorphic mapping $f:B^n \to \mathbb{C}^n$ is said to be g-almost star-like mapping of order α if

$$\frac{1}{1-\alpha}\left\langle J_f^{-1}(z)f(z), \frac{z}{\|z\|^2}\right\rangle - \frac{\alpha}{1-\alpha} \in g(U), \ z \in B^n\backslash\{0\},$$

where g satisfies the requirements of Definition 2.1.

Remark 1

(a) Since $Reg(\zeta) \geq 0$, we have $Re\left\langle J_f^{-1}(z)f(z), z\right\rangle \geq \alpha\|z\|^2$. Hence, the g-almost star-like mapping of order α is a subclass of almost star-like mapping of order α on B^n. And hence biholomorphic on B^n.

(b) If $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$, this class reduces to the class of almost star-like mappings of order α on B^n. If $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $\zeta \in U$, $\gamma \in (0, 1)$, the almost star-like mappings of order α and type γ on B^n can be got (see Chirilă, 2014b).

Definition 2.3 (Zhang & Feng, 2013) Let $\rho \in [0, 1)$. A normalized biholomorphic mapping $f:B^n \to \mathbb{C}^n$ is said to be parabolic star-like mappings of order ρ if

$$\left|\left\langle J_f^{-1}(z)f(z), \frac{z}{\|z\|^2}\right\rangle - 1\right| \leq (1-2\rho) + Re\left\langle J_f^{-1}(z)f(z), \frac{z}{\|z\|^2}\right\rangle, \ z \in B^n \setminus \{0\}.$$

This shows that if f is a parabolic star-like of order ρ, then $\left\langle J_f^{-1}(z)f(z), \frac{z}{\|z\|^2}\right\rangle$ is a mapping from B^n onto the parabolic region in the right half-plane Ω_ρ, where
$\Omega_\rho = \{w = u + iv: v^2 \leq 4(1-\rho)(u-\rho)\} = \{w: |w-1| \leq (1-2\rho) + Rew\}$.

Definition 2.4 (Chirilă, 2014a) Let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. A normalized biholomorphic mapping $f:B^n \to \mathbb{C}^n$ is said to be g-spiral-like mapping of type β if

$$i\frac{\sin\beta}{\cos\beta} + \frac{e^{-i\beta}}{\cos\beta}\left\langle J_f^{-1}(z)f(z), \frac{z}{\|z\|^2}\right\rangle \in g(U), \ z \in B^n\backslash\{0\},$$

where g satisfies the requirements of Definition 2.1.

Remark 2

(1) Obviously, if f is g-spiral-like of type β, then

$$Re\left\{e^{-i\beta}\left\langle J_f^{-1}(z)f(z), \frac{z}{\|z\|^2}\right\rangle\right\} \geq 0.$$

Hence, f is also a normalized biholomorphic spiral-like mapping of type β on B^n.

(2) If $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$, this class becomes the class of spiral-like mappings of type β on B^n. If $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $\zeta \in U$, $\gamma \in (0, 1)$, we obtain the class of spiral-like mappings of type β and order γ on B^n. If $g(\zeta) = \frac{1+(1-2\gamma)\zeta}{1-\zeta}$, $\zeta \in U$, $\gamma \in (0, 1)$, we obtain the class of almost spiral-like mappings of type β and order γ on B^n.

LEMMA 2.5 *(Gurganus, 1975) Let $h \in \mathcal{N}$. Then, for each $z \in B^n$, the initial value problem*

$$\begin{cases} \frac{\partial v}{\partial t}(z, t) = -h(v(z, t)), \\ \qquad v(z, 0) = z \end{cases} \tag{2.2}$$

has a unique solution $v(t) = v(z, t)$ defined for all $t > 0$, and $v(z, t) \to 0$ as $t \to +\infty$. For fixed t, $v(\cdot, t)$ is a biholomorphic Schwarz function on B^n.

LEMMA 2.6 *(Liu & Lu, 2002) Let $f{:}B^n \to \mathbb{C}^n$ be a normalized biholomorphic star-like mapping. Then,*

$$f(z) = \lim_{t\to+\infty} \{e^t v(z, t)\},$$

where $v(z, t)$ is the solution of the initial value problem (2.2), and the corresponding $h(z) = J_f^{-1}(z)f(z)$.

LEMMA 2.7 *(Liu, Zhang, & Lu, 2006) Let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and let $f{:}B^n \to \mathbb{C}^n$ be a normalized biholomorphic spiral-like mapping of type β. Then,*

$$f(z) = \lim_{t\to+\infty} \left\{\exp(te^{-i\beta})v(z, t)\right\},$$

where $v(z, t)$ is the solution of the initial value problem (2.2), and the corresponding $h(z) = e^{-i\beta}J_f^{-1}(z)f(z)$.

LEMMA 2.8 *Let $\alpha \in [0, 1)$ and let $f{:}B^n \to \mathbb{C}^n$ be a g-almost star-like of order α. Then,*

$$\begin{aligned} &\alpha\|z\|^2 + (1-\alpha)\|z\|^2 \min\{g(\|z\|), g(-\|z\|)\} \\ &\quad \le Re\left\{\left\langle J_f^{-1}(z)f(z),\ z\right\rangle\right\} \\ &\quad \le \alpha\|z\|^2 + (1-\alpha)\|z\|^2 \max\{g(\|z\|), g(-\|z\|)\}. \end{aligned}$$

Proof For $z \in B^n\backslash\{0\}$, let $z_0 = \frac{z}{\|z\|}$. Then,

$$p(\zeta) = \begin{cases} \frac{1}{1-\alpha}\frac{1}{\zeta}\left\langle J_f^{-1}(\zeta z_0)f(\zeta z_0), z_0\right\rangle - \frac{\alpha}{1-\alpha}, & \zeta \in U\backslash\{0\}, \\ 1, & \zeta = 0 \end{cases}$$

is well defined on the unit disk U, and p is biholomorphic on U. By Definition 2.2, we know that

$$p(\zeta) = \frac{1}{1-\alpha}\left\langle J_f^{-1}(\zeta z_0)f(\zeta z_0), \frac{\zeta z_0}{\|\zeta z_0\|^2}\right\rangle - \frac{\alpha}{1-\alpha} \in g(U).$$

Since $p(0) = g(0) = 1$, we have $p \prec g$. By the maximum and minimum principles for harmonic functions, we have

$$\min\{g(|\zeta|),\ g(-|\zeta|)\} \le Rep(\zeta) \le \max\{g(|\zeta|), g(-|\zeta|)\},\ \zeta \in U.$$

Let $\zeta = \|z\|$. Then,

$$\begin{aligned} &\alpha\|z\|^2 + (1-\alpha)\|z\|^2 \min\{g(\|z\|), g(-\|z\|)\} \\ &\quad \le Re\left\{\left\langle J_f^{-1}(z)f(z), z\right\rangle\right\} \\ &\quad \le \alpha\|z\|^2 + (1-\alpha)\|z\|^2 \max\{g(\|z\|), g(-\|z\|)\}. \end{aligned}$$

□

LEMMA 2.9 *Let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and let $f: B^n \to \mathbb{C}^n$ be a g-spiral-like of type β. Then,*

$$\cos\beta\|z\|^2 \min\{g(\|z\|), g(-\|z\|)\} \le Re\left\{e^{-i\beta}\left\langle J_f^{-1}(z)f(z), z\right\rangle\right\}$$
$$\le \cos\beta\|z\|^2 \max\{g(\|z\|), g(-\|z\|)\}.$$

Proof For $z \in B^n \setminus \{0\}$, let $z_0 = \frac{z}{\|z\|}$. Then,

$$p(\zeta) = \begin{cases} i\frac{\sin\beta}{\cos\beta} + \frac{e^{-i\beta}}{\cos\beta}\frac{1}{\zeta}\left\langle J_f^{-1}(\zeta z_0)f(\zeta z_0), z_0\right\rangle, & \zeta \in U\setminus\{0\}, \\ 1, & \zeta = 0 \end{cases}$$

is well defined on the unit disk U, and p is biholomorphic on U. By Definition 2.4, we know that

$$p(\zeta) = i\frac{\sin\beta}{\cos\beta} + \frac{e^{-i\beta}}{\cos\beta}\left\langle J_f^{-1}(\zeta z_0)f(\zeta z_0), \frac{\zeta z_0}{\|\zeta z_0\|^2}\right\rangle \in g(U).$$

Since $p(0) = g(0) = 1$, we have $p \prec g$. By the maximum and minimum principles for harmonic functions, we have

$$\min\{g(|\zeta|), g(-|\zeta|)\} \le Rep(\zeta) \le \max\{g(|\zeta|), g(-|\zeta|)\}, \ \zeta \in U.$$

Let $\zeta = \|z\|$. Then,

$$\cos\beta\|z\|^2 \min\{g(\|z\|), \ g(-\|z\|)\} \le Re\left\{e^{-i\beta}\left\langle J_f^{-1}(z)f(z), z\right\rangle\right\}$$
$$\le \cos\beta\|z\|^2 \max\{g(\|z\|), g(-\|z\|)\}. \qquad \square$$

3. Main results

THEOREM 3.1 *Let $\alpha \in [0, 1)$ and let $f: B^n \to \mathbb{C}^n$ be a g-almost star-like of order α. Then,*

$$\|z\| \exp\left(\int_0^{\|z\|}\left[\frac{1}{\alpha + (1-\alpha)\max\{g(x), g(-x)\}} - 1\right]\frac{1}{x}dx\right)$$
$$\le \|f(z)\|$$
$$\le \|z\| \exp\left(\int_0^{\|z\|}\left[\frac{1}{\alpha + (1-\alpha)\min\{g(x), g(-x)\}} - 1\right]\frac{1}{x}dx\right).$$

Proof Let $h(z) = J_f^{-1}(z)f(z)$, $z \in B^n$. Then, $h \in \mathcal{M}$. And let $v(z, t)$ be the solution of the initial value problem (2.2) corresponding to the above function h. For any $0 \le t < t'$, let $v(t) = v(z, t)$. Then,

$$|\|v(t)\| - \|v(t')\|| \le \|v(t) - v(t')\| \le \|\int_t^{t'}\frac{dv(\tau)}{d\tau}d\tau\| \le \int_t^{t'}\|\frac{dv(\tau)}{d\tau}\|d\tau$$
$$= \int_t^{t'}\| - h(v(\tau))\|d\tau.$$

Since $\|v(t)\|$ is continuous, the above inequality implies that $\|v(t)\|$ is absolutely continuous for $t \in [0, +\infty)$, and thus $\|v(t)\|$ is differentiable almost everywhere on $[0, +\infty)$. Since

$$\frac{d\|v(t)\|}{dt} = \frac{1}{\|v(t)\|}Re\left\langle\frac{\partial v(t)}{\partial t}, \ v(t)\right\rangle$$
$$= -\frac{1}{\|v(t)\|}Re\langle h(v(t)), \ v(t)\rangle,$$

and by Lemma 2.8,

$$
\begin{aligned}
&-\alpha\|v(t)\| - (1-\alpha)\|v(t)\| \max\{g(\|v(t)\|), g(-\|v(t)\|)\} \\
&\quad \le \frac{d\|v(t)\|}{dt} \\
&\quad \le -\alpha\|v(t)\| - (1-\alpha)\|v(t)\| \min\{g(\|v(t)\|), g(-\|v(t)\|)\}.
\end{aligned}
$$

Thus $\|v(t)\|$ is a decreasing function on $[0, +\infty)$. From the right-hand side of the above inequality,

$$
\begin{aligned}
&-\int_{\|z\|}^{\|v(t)\|} \frac{1}{\alpha + (1-\alpha)\min\{g(x), g(-x)\}} \frac{1}{x} dx \\
&\quad = -\int_0^t \frac{1}{\alpha + (1-\alpha)\min\{g(\|v(\tau)\|), g(-\|v(\tau)\|)\}} \frac{1}{\|v(\tau)\|} \frac{d\|v(\tau)\|}{d\tau} d\tau \\
&\quad \ge \int_0^t d\tau.
\end{aligned}
$$

It yields that

$$
-\int_{\|z\|}^{\|v(t)\|} \left[\frac{1}{\alpha + (1-\alpha)\min\{g(x), g(-x)\}} - 1\right] \frac{1}{x} dx - \ln\|v(t)\| + \ln\|z\| \ge t,
$$

i.e.

$$
\begin{aligned}
&\|z\| \exp\left(-\int_{\|z\|}^{\|v(t)\|} \left[\frac{1}{\alpha + (1-\alpha)\min\{g(x), g(-x)\}} - 1\right] \frac{1}{x} dx\right) \\
&\quad \ge e^t \|v(t)\|.
\end{aligned}
$$

Let $t \to +\infty$, by Lemma 2.6,

$$
\begin{aligned}
&\|z\| \exp\left(\int_0^{\|z\|} \left[\frac{1}{\alpha + (1-\alpha)\min\{g(x), g(-x)\}} - 1\right] \frac{1}{x} dx\right) \\
&\quad \ge \|f(z)\|.
\end{aligned}
$$

Using the same arguments,

$$
\begin{aligned}
&\|z\| \exp\left(\int_0^{\|z\|} \left[\frac{1}{\alpha + (1-\alpha)\max\{g(x), g(-x)\}} - 1\right] \frac{1}{x} dx\right) \\
&\le \|f(z)\|.
\end{aligned}
$$

□

In particular, we can get the growth theorem for almost star-like mappings of order α on the unit ball B^n when $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$.

Corollary 3.2 *Let $\alpha \in [0, 1)$ and let $f: B^n \to \mathbb{C}^n$ be a almost star-like of order α. Then,*

$$
\frac{\|z\|}{(1 + (1-2\alpha)\|z\|)^{\frac{2(1-\alpha)}{1-2\alpha}}} \le \|f(z)\| \le \frac{\|z\|}{(1 - (1-2\alpha)\|z\|)^{\frac{2(1-\alpha)}{1-2\alpha}}}, \ \alpha \in [0, 1)\backslash\{\frac{1}{2}\}.
$$

$$
\|z\| \exp(-\|z\|) \le \|f(z)\| \le \|z\| \exp(\|z\|), \ \alpha = \frac{1}{2}.
$$

We can also obtain the growth theorem for almost star-like mappings of order α and type γ on B^n if let $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $\zeta \in U$, $\gamma \in (0, 1)$ in Theorem 3.1.

COROLLARY 3.3 *Let $\alpha \in [0, 1)$, $\gamma \in (0, 1)$ and let $f:B^n \to \mathbb{C}^n$ be a almost star-like of order α and type γ. Then,*

$$\begin{aligned}&\frac{\|z\|}{\left[1+(1-2\alpha(1-\gamma))\|z\|\right]^{\frac{2(1-\gamma)(1-\alpha)}{1-2\alpha(1-\gamma)}}}\\ &\quad\leq \|f(z)\|\\ &\quad\leq \frac{\|z\|}{\left[1-(1-2\alpha(1-\gamma))\|z\|\right]^{\frac{2(1-\gamma)(1-\alpha)}{1-2\alpha(1-\gamma)}}},\ 1-2\alpha(1-\gamma)\neq 0;\\ &\|z\|\exp[(2\gamma-1)\|z\|]\leq\|f(z)\|\leq\|z\|\exp[(1-2\gamma)\|z\|],\ 1-2\alpha(1-\gamma)=0.\end{aligned}$$

COROLLARY 3.4 *Let $\gamma \in (0, 1)$ and let $f:B^n \to \mathbb{C}^n$ be a star-like mapping of order γ. Then,*

$$\frac{\|z\|}{(1+\|z\|)^{2(1-\gamma)}} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^{2(1-\gamma)}}.$$

Proof Let $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $\zeta \in U$, $\gamma \in (0, 1)$. Then, g satisfies the conditions of Definition 2.1. Let $\alpha = 0$ in Definition 2.2. We have

$$\left\langle J_f^{-1}(z)f(z), \frac{z}{\|z\|^2}\right\rangle \in g(U),\ z\in B^n\backslash\{0\},$$

thus, it is that

$$\left|\left\langle J_f^{-1}(z)f(z), \frac{z}{\|z\|^2}\right\rangle - \frac{1}{2\gamma}\right| \leq \frac{1}{2\gamma},\ z\in B^n\backslash\{0\}.$$

So f is a star-like mapping of order γ. And letting $\alpha = 0$ in Theorem 3.1, the result can be obtained. □

The following corollary is due to Zhang and Feng (2013).

COROLLARY 3.5 *Let $\rho \in [0, 1)$ and let $f:B^n \to \mathbb{C}^n$ be a parabolic star-like mapping of order ρ. Then,*

$$\begin{aligned}&\|z\|\exp\left(\int_0^{\|z\|}\left[\frac{1}{1+\frac{4(1-\rho)}{\pi^2}\left(\log\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)^2}-1\right]\frac{1}{x}dx\right)\\ &\quad\leq\|f(z)\|\\ &\quad\leq\|z\|\exp\left(\int_0^{\|z\|}\left[\frac{1}{1+\frac{4(1-\rho)}{\pi^2}\left(\log\frac{1+i\sqrt{x}}{1-i\sqrt{x}}\right)^2}-1\right]\frac{1}{x}dx\right),\end{aligned}$$

where we choose the branch of the square root such that $\sqrt{1} = 1$, and the branch of the logarithm function such that $\log 1 = 0$.

Proof Let $g(\zeta) = 1 + \frac{4(1-\rho)}{\pi^2}\left(\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^2$, $\zeta \in U$, $\rho \in [0, 1)$. Then, g is a biholomorphic function from U onto the parabolic region Ω_ρ in the right half-plane (see Ali, 2005), where

$$\Omega_\rho = \left\{w\in\mathbb{C}:|w-1|\leq(1-2\rho)+Re\{w\}\right\}.$$

And g satisfies the conditions of Definition 2.1.

Since f is a parabolic star-like mapping of order ρ, we have

$$\left|\left\langle J_f^{-1}(z)f(z), \frac{z}{\|z\|^2}\right\rangle - 1\right| \le (1-2\rho) + Re\left\{\left\langle J_f^{-1}(z)f(z), \frac{z}{\|z\|^2}\right\rangle\right\}, \ z \in B^n \backslash \{0\}.$$

Thus, it is that

$$\left\langle J_f^{-1}(z)f(z), \frac{z}{\|z\|^2}\right\rangle \in g(U), \ z \in B^n \backslash \{0\}.$$

Let $\alpha = 0$ and let $g(\zeta) = 1 + \frac{4(1-\rho)}{\pi^2}\left(\log \frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^2$ in Theorem 3.1. Then, the result can be obtained. □

THEOREM 3.6 *Let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and let $f: B^n \to \mathbb{C}^n$ be a g-spiral-like of type β. Then,*

$$\begin{aligned} &\|z\| \exp\left(\int_0^{\|z\|}\left[\frac{1}{\max\{g(x), g(-x)\}} - 1\right]\frac{1}{x}dx\right) \\ &\quad \le \|f(z)\| \\ &\quad \le \|z\| \exp\left(\int_0^{\|z\|}\left[\frac{1}{\min\{g(x), g(-x)\}} - 1\right]\frac{1}{x}dx\right). \end{aligned}$$

Proof Let $h(z) = e^{-i\beta}J_f^{-1}(z)f(z)$, $z \in B^n$. By Definition 2.4, we know $h \in \mathcal{N}$. And let $v(z, t)$ be the solution of the initial value problem (2.2) corresponding to the above function h. For any $0 \le t < t'$, let $v(t) = v(z, t)$. Then,

$$\begin{aligned} &\big|\|v(t)\| - \|v(t')\|\big| \le \|v(t) - v(t')\| \le \left\|\int_t^{t'} \frac{dv(\tau)}{d\tau}d\tau\right\| \le \int_t^{t'} \left\|\frac{dv(\tau)}{d\tau}\right\| d\tau \\ &\quad = \int_t^{t'} \|-h(v(\tau))\| d\tau. \end{aligned}$$

Since $\|v(t)\|$ is continuous, the above inequality implies that $\|v(t)\|$ is absolutely continuous for $t \in [0, +\infty)$, and thus $\|v(t)\|$ is differentiable almost everywhere on $[0, +\infty)$. Since

$$\begin{aligned} \frac{d\|v(t)\|}{dt} &= \frac{1}{\|v(t)\|} Re\left\langle \frac{\partial v(t)}{\partial t}, v(t)\right\rangle \\ &= -\frac{1}{\|v(t)\|} Re\langle h(v(t)), v(t)\rangle, \end{aligned}$$

and by Lemma 2.9,

$$\begin{aligned} -\cos\beta\|v(t)\| \max\{g(\|v(t)\|), g(-\|v(t)\|)\} &\le \frac{d\|v(t)\|}{dt} \\ &\le -\cos\beta\|v(t)\| \min\{g(\|v(t)\|), g(-\|v(t)\|)\}. \end{aligned}$$

Thus, $\|v(t)\|$ is a decreasing function on $[0, +\infty)$. From the right-hand side of the above inequality,

$$
\begin{aligned}
&-\int_{\|z\|}^{\|v(t)\|} \frac{1}{\min\{g(x), g(-x)\}} \frac{1}{x} dx \\
&= -\int_0^t \frac{1}{\min\{g(\|v(\tau)\|), g(-\|v(\tau)\|)\}} \frac{1}{\|v(\tau)\|} \frac{d\|v(\tau)\|}{d\tau} d\tau \\
&\geq \cos\beta \int_0^t d\tau.
\end{aligned}
$$

It yields that

$$
-\int_{\|z\|}^{\|v(t)\|} \left[\frac{1}{\min\{g(x), g(-x)\}} - 1\right] \frac{1}{x} dx - \ln\|v(t)\| + \ln\|z\| \geq t\cos\beta,
$$

i.e.

$$
\begin{aligned}
&\|z\| \exp\left(-\int_{\|z\|}^{\|v(t)\|} \left[\frac{1}{\min\{g(x), g(-x)\}} - 1\right] \frac{1}{x} dx\right) \\
&\quad \geq e^{t\cos\beta}\|v(t)\|.
\end{aligned}
$$

Let $t \to +\infty$, by Lemma 2.7,

$$
\begin{aligned}
&\|z\| \exp\left(\int_0^{\|z\|} \left[\frac{1}{\min\{g(x), g(-x)\}} - 1\right] \frac{1}{x} dx\right) \\
&\quad \geq \|f(z)\|.
\end{aligned}
$$

Using the same arguments,

$$
\begin{aligned}
&\|z\| \exp\left(\int_0^{\|z\|} \left[\frac{1}{\max\{g(x), g(-x)\}} - 1\right] \frac{1}{x} dx\right) \\
&\quad \leq \|f(z)\|.
\end{aligned}
$$

□

In particular, letting $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$ in Theorem 3.6, we can obtain the growth theorem for spiral-like mappings of type β on B^n. This result is due to Hamada and Kohr, and we can refer to the Theorem 7.3.6 in Kohr and Liczberski (1998).

COROLLARY 3.7 *Let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and let $f:B^n \to \mathbb{C}^n$ be a spiral-like mapping of type β. Then,*

$$
\frac{\|z\|}{(1+\|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2}.
$$

Letting $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $\zeta \in U$, $\gamma \in (0, 1)$ in Theorem 3.6, we can also obtain the growth theorem for spiral-like mappings of type β and order γ on B^n.

COROLLARY 3.8 *(Feng et al., 2007) Let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\gamma \in (0, 1)$, and let $f:B^n \to \mathbb{C}^n$ be a spiral-like mapping of type β order γ. Then,*

$$
\frac{\|z\|}{(1+\|z\|)^{2(1-\gamma)}} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^{2(1-\gamma)}}.
$$

If $g(\zeta) = \frac{1+(1-2\gamma)\zeta}{1-\zeta}$, $\zeta \in U$, $\gamma \in (0, 1)$ in Theorem 3.6, we can obtain the growth theorem for almost spiral-like mappings of type β and order γ on B^n.

COROLLARY 3.9 *(Feng et al., 2007) Let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\gamma \in (0, 1)$, and let $f: B^n \to \mathbb{C}^n$ be a almost spiral-like mappings of type β and order γ. Then,*

$$\frac{\|z\|}{(1+(1-2\gamma)\|z\|)^{\frac{2(1-\gamma)}{1-2\gamma}}} \le \|f(z)\| \le \frac{\|z\|}{(1-(1-2\gamma)\|z\|)^{\frac{2(1-\gamma)}{1-2\gamma}}},\ \gamma \in [0, 1)\backslash\left\{\frac{1}{2}\right\}.$$

$$\|z\| \exp(-\|z\|) \le \|f(z)\| \le \|z\| \exp(\|z\|),\ \gamma = \frac{1}{2}.$$

Funding
This work is supported by the Doctoral Foundation of Pingdingshan University [grant number PXY-BSQD-2015005], the Key Scientific Research Projects in Universities of Henan Province [grant number 16B110010], and the Foster Foundation of Pingdingshan University [grant number PXY-PYJJ2016007].

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Citation information
Cite this article as: The growth theorems for subclasses of biholomorphic mappings in several complex variables, Xiaofei Zhang, *Cogent Mathematics* (2017), 4: 1339369.

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