Research Article

Quanqing Li* and Wenming Zou

# The existence and multiplicity of the normalized solutions for fractional Schrödinger equations involving Sobolev critical exponent in the $L^2$-subcritical and $L^2$-supercritical cases

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**Abstract:** This paper is devoted to investigate the existence and multiplicity of the normalized solutions for the following fractional Schrödinger equation:

$$\begin{cases} (-\Delta)^s u + \lambda u = \mu |u|^{p-2}u + |u|^{2_s^*-2}u, & x \in \mathbb{R}^N, \\ u > 0, & \int\limits_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2, \end{cases} \tag{P}$$

where $0 < s < 1$, $a, \mu > 0$, $N \geq 2$, and $2 < p < 2_s^*$. We consider the $L^2$-subcritical and $L^2$-supercritical cases. More precisely, in $L^2$-subcritical case, we obtain the multiplicity of the normalized solutions for problem $(P)$ by using the truncation technique, concentration-compactness principle, and genus theory. In $L^2$-supercritical case, we obtain a couple of normalized solution for $(P)$ by using a fiber map and concentration-compactness principle. To some extent, these results can be viewed as an extension of the existing results from Sobolev subcritical growth to Sobolev critical growth.

**Keywords:** normalized solutions, $L^2$-subcritical, $L^2$-supercritical, Sobolev critical exponent

**MSC 2020:** 35J20, 35J50, 35J70

## 1 Introduction

Over the past two decades, there has been a lot of interest in the following fractional Schrödinger equation:

$$(-\Delta)^s u + V(x)u = f(u), \quad x \in \Omega,$$

where $0 < s < 1$, $(-\Delta)^s$ denotes the fractional Laplacian of order $s$, $V : \mathbb{R}^N \to \mathbb{R}$ is an external potential function, $f(u)$ is the nonlinearity, and $\Omega \subset \mathbb{R}^N$ is a bounded or unbounded domain. It was introduced by Laskin [20,21] and comes from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths, where the Feynman path integral leads to the classical Schrödinger equation

*** Corresponding author: Quanqing Li,** Department of Mathematics, Honghe University, Mengzi, Yunnan 661100, China, e-mail: liquanqing1987@sina.com
**Wenming Zou:** Department of Mathematics, Tsinghua University, Beijing, Beijing 100084, China, e-mail: wzou@math.tsinghua.edu.cn

and the path integral over Lévy trajectories leads to the fractional Schrödinger equation. Such kind of equation is of particular interest in fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes [3]. It also appeared in several areas such as optimization, finance, phase transitions, stratified materials, crystal dislocation, flame propagation, conservation laws, materials science and water waves. This is one of the reasons why, recently, fractional problems are widely studied by more and more scholars.

Especially recently, the following time-dependent fractional Schrödinger equation

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} = (-\Delta)^s\psi - \mu|\psi|^{p-2}\psi - |\psi|^{q-2}\psi, & (t,x) \in [0,T^*) \times \mathbb{R}^N, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{1.1}$$

attracts much attention, where $2 < p < q < 2_s^*$, $0 < T^* \leq +\infty$, $\psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is a wave function that represents the quantum mechanical probability amplitude for a given unit mass particles to have position $x$ at time $t$ (the corresponding probability density is $|\psi(t,x)|^2$), $\mu > 0$ stands focusing situation and $\mu < 0$ stands defocusing situation, $\lambda \in \mathbb{R}$ is a frequency. Furthermore, (1.1) can describe the dynamics of a Bose-Einstein condensate in $\mathbb{R}^N$, in which all the quantum and particles are in the same $\psi(t,x)$. If we consider initial data in $H^s(\mathbb{R}^N)$ (see below for its definition), then (1.1) enjoys mass and energy conservation law. That is, if we set

$$M(\psi(t,x)) = \int_{\mathbb{R}^N} |\psi(t,x)|^2 \mathrm{d}x$$

and

$$E(\psi(t,x)) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\psi(t,x)|^2 \mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^N} |\psi(t,x)|^p \mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^N} |\psi(t,x)|^q \mathrm{d}x,$$

then

$$M(\psi(t,x)) = M(\psi(0,x)) = M(\psi_0)$$

and

$$E(\psi(t,x)) = E(\psi(0,x)) = E(\psi_0).$$

Mathematically, it is of great interest to consider standing waves for (1.1), whose solutions are of the form $e^{i\lambda t}u(x)$, where the real-valued function $u$ solves

$$(-\Delta)^s u + \lambda u = \mu|u|^{p-2}u + |u|^{q-2}u, \quad x \in \mathbb{R}^N. \tag{1.2}$$

Now, there exist two substantially different view points in terms of the frequency $\lambda$ in (1.2). One is to regard the frequency $\lambda$ as a given constant. In this situation, solutions of equation (1.2) are critical points of the corresponding action functional $J_\lambda(u)$ on $H^s(\mathbb{R}^N)$, where

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x + \frac{1}{2}\lambda\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p \mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^N} |u|^q \mathrm{d}x.$$

In this case, many scholars are more concerned about ground state solutions, which are important for both physical and mathematical points of view since they share further properties, like stability, positivity, and symmetry. They can be defined as minimizers of the aforementioned functional $J_\lambda$ among its nontrivial critical points, i.e., the minimizers of

$$m_\lambda := \{J_\lambda(u) : u \in H^s(\mathbb{R}^N)\backslash\{0\}, J_\lambda'(u) = 0\}.$$

Equivalently, they can be defined as minimizers of $J_\lambda$ on the associated Nehari manifold

$$\mathcal{N}_\lambda := \{u \in H^s(\mathbb{R}^N)\backslash\{0\} : \langle J_\lambda'(u), u\rangle = 0\}$$

(see [32]).

The other one is to regard the frequency $\lambda$ as an unknown quantity to problem (1.2). In this situation, it is natural to prescribe the value of the mass so that $\lambda$ can be interpreted as a Lagrange multiplier. As mentioned earlier, $|\psi(t,x)|^2$ represents the probability density of a single particle appearing in space $x$ at time $t$. Hence, it seems appropriate to investigate the solutions that satisfy the normalized condition $\int_{\mathbb{R}^N}|\psi(t,x)|^2\mathrm{d}x = 1$. For the $n$ body system, the wave function for the whole condensate becomes $\widetilde{\psi}(t,x) = \sqrt{n}\psi(t,x)$, and so the wave function is normalized according to the total number of the particles, i.e., $\int_{\mathbb{R}^N}|\widetilde{\psi}(t,x)|^2\mathrm{d}x = n$ (see [34]). But for convenience and extension, the normalized condition in mathematics is always assumed to hold for any positive constant $c > 0$, i.e., $\int_{\mathbb{R}^N}|\psi(t,x)|^2\mathrm{d}x = c$. Accordingly, $\int_{\mathbb{R}^N}|u|^2\mathrm{d}x = c$. At this time, to study the solution of equation (1.2) satisfying the normalized condition $\int_{\mathbb{R}^N}|u|^2\mathrm{d}x = c$, it remains to consider the critical point of the functional

$$E(u) = \frac{1}{2}\int\limits_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x - \frac{\mu}{p}\int\limits_{\mathbb{R}^N}|u|^p\mathrm{d}x - \frac{1}{q}\int\limits_{\mathbb{R}^N}|u|^q\mathrm{d}x.$$

On the constraint manifold

$$S(c) := \left\{u \in H^s(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N}|u|^2\mathrm{d}x = c\right\}.$$

Nowadays, some physicists are very interested in the solutions satisfying $\int_{\mathbb{R}^N}|u|^2\mathrm{d}x = c > 0$ for *a priori* given $c$. This is not only because the wave function $\psi(t,x)$ enjoys mass conservation law but also because the mass admits a clear physical meaning. For example, from a physical point of view, the mass $\|u\|^2_{L^2(\mathbb{R}^N)}$ may represent the number of particles of each component in Bose-Einstein condensates or the power supply in the nonlinear optics framework. In addition, such solutions can give a better insight into the dynamical properties, like orbital stability or instability, and can describe attractive Bose-Einstein condensates, $\mu > 0$ can represent the strength of the attractive interaction among the cold atoms. This type of solution is usually called prescribed $L^2$-norm solutions or normalized solutions in mathematics.

Comparing with research on fixed-frequency solutions, relatively fewer results about normalized solutions have been obtained, but it starts gaining much more attention in recent years. To the best of our knowledge, Jeanjean [18] first considered such type of problem, where he considered a semilinear elliptic equation:

$$-\Delta u = \lambda u + g(u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $N \geq 1$, $\lambda \in \mathbb{R}$, and $g$ satisfies some suitable conditions. In the light of a minimax procedure, he showed that for each $c > 0$, equation (1.3) admits at least a couple $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times \mathbb{R}^-$ of weak solution satisfying $\int_{\mathbb{R}^N}|u_c|^2\mathrm{d}x = c$. But, afterward, there was a little progress about the study of normalized solutions for a long time. One of the main reasons is that it is hard to prove the boundedness of constrained Palais-Smale sequence when the functional is unbounded from below on the constraint manifold. Recently, Bellazzini et al. [6] obtained the existence and instability of standing waves for (1.3). Furthermore, Bartsch and de Valeriola [5] obtained infinitely many normalized solutions for (1.3). For more references dealing with applications, we can refer to [1,11,23,27,29,30] and their references therein.

If we consider the time-dependent nonlinear fractional Schrödinger equation

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} = (-\Delta)^s\psi - |\psi|^{r-2}\psi, & (t,x) \in [0,T^*) \times \mathbb{R}^N, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{1.4}$$

it is well known to us that there is a natural scaling invariance associated with (1.4). Precisely, the scaling

$$\psi_\omega(t,x) = \omega^{\frac{2s}{r-2}}\psi(\omega^{2s}t, \omega x)$$

leaves (1.4) invariant for all $\omega > 0$. A simple calculation tells us that

$$\int_{\mathbb{R}^N} |\psi_\omega(t,x)|^2 \mathrm{d}x = \omega^{\frac{4s}{\bar{r}-2}-N} \int_{\mathbb{R}^N} |\psi(\omega^{2s}t,x)|^2 \mathrm{d}x.$$

According to the mass conservation law, $\bar{r} = 2 + \frac{4s}{N}$ can leave the mass invariant. That is why $2 + \frac{4s}{N}$ is called $L^2$-critical exponent or mass critical exponent, which is the threshold exponent for many dynamical properties such as global existence, blow-up, the stability, or instability of ground states. And it strongly affects the geometrical structure of the corresponding functional. Hence, the study of normalized solutions is attracting much attention of more and more researchers. Recently, Du et al. [13] studied the existence, nonexistence, and mass concentration of normalized solutions for nonlinear fractional Schrödinger equations:

$$(-\Delta)^s u + V(x)u = \mu u + af(u), \quad x \in \mathbb{R}^N,$$

where $f$ is a Sobolev subcritical nonlinearity. Chen and Liu [8] studied the asymptotic behavior of ground states for the fractional Schrödinger equation with combined $L^2$-critical and $L^2$-subcritical nonlinearities

$$(-\Delta)^s u + \lambda u = \mu|u|^{q-2}u + |u|^{p-2}u, \quad x \in \mathbb{R}^N$$

with prescribed mass $\|u\|_2 = a > 0$, where $\mu \in \mathbb{R}$, $2 < q < p < 2 + \frac{4s}{N}$, $N \geq 2$. Some scholars call this type of ground states as normalized ground states. Feng et al. [16] studied the existence and the instability of normalized standing waves for the fractional Schrödinger equation:

$$i\frac{\partial \psi}{\partial t} = (-\Delta)^s \psi - |\psi|^{p-2}\psi, \quad x \in \mathbb{R}^N,$$

where $2 + \frac{4s}{N} < p < 2_s^*$. Dinh [10] studied the existence and nonexistence of normalized solutions for the fractional Schrödinger equation:

$$(-\Delta)^s u + V(x)u = |u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $2 < p < 2_s^*$ if $N > 2s$ and $2 < p < +\infty$ if $N \leq 2s$. In addition, by studying the normalized solutions of the fractional Schrödinger equation:

$$(-\Delta)^s u + \lambda u = |x|^{-b}|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

Liu [22] considered the stability and instability of standing waves for the following inhomogeneous fractional Schrödinger equation:

$$i\frac{\partial \psi}{\partial t} = (-\Delta)^s \psi - |x|^{-b}|\psi|^{p-2}\psi, \quad x \in \mathbb{R}^N,$$

whose $L^2$- critical exponent is $2 + \frac{4s-2b}{N}$. More precisely, in $L^2$- subcritical case, i.e., $2 < p < 2 + \frac{4s-2b}{N}$, they proved that the standing waves are orbitally stable by applying the profile decomposition of bounded sequences in $H^s(\mathbb{R}^N)$ and variational methods; in $L^2$-critical case, i.e., $p = 2 + \frac{4s-2b}{N}$, they showed that the standing waves are strongly unstable by the blow-up method. Yang [33] studied the existence and asymptotic properties of normalized solutions for the fractional Choquard equation:

$$(-\Delta)^s u + \lambda u = |u|^{q-2}u + \mu[I_\alpha * |u|^p]|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $\mu > 0$, $N \geq 2$, $s \in (0,1)$, $\alpha \in (0,N)$, $q \in \left(2 + \frac{4s}{N}, \frac{2N}{N-2s}\right]$ and $p \in \left[1 + \frac{2s+\alpha}{N}, \frac{N+\alpha}{N-2s}\right)$, and $I_\alpha(x) = |x|^{\alpha-N}$. By using a refined version of the min-max principle, they showed that the aforementioned problem admitted a mountain pass type solution $u_\mu$ for some $\lambda < 0$ and gave some asymptotic properties of the solutions. Li and Luo [24] considered the existence and multiplicity of normalized solutions for a class of nonlinear fractional Choquard equation:

$$(-\Delta)^s u + \lambda u = [I_\alpha * |u|^p]|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $N \geq 3$, $s \in (0,1)$, $\alpha \in (0,N)$, and $p \in \left(\max\{1, \frac{2s+\alpha}{N}\}, \frac{N+\alpha}{N-2s}\right)$. Feng et al. [14] studied blow-up criteria and instability of normalized standing waves for the fractional Choquard equation:

$$i\frac{\partial\psi}{\partial t} = (-\Delta)^s\psi - [I_\alpha * |\psi|^p]|\psi|^{p-2}\psi, \quad x \in \mathbb{R}^N,$$

by using localized virial estimates and the profile decomposition theory in $H^s(\mathbb{R}^N)$.

From the aforementined commentaries, the existing work is mainly focused on the existence of normalized solutions for the fractional Schrödinger equations with the Sobolev subcritical growth. A natural question is whether we can obtain some results for critical fractional Schrödinger equations. As for the multiplicity, even existence of normalized solutions for the fractional Schrödinger equation with critical Sobolev exponent $2_s^*$, as far as we know, there are no results in this direction. In this article, we shall give some answers about this topic. Motivated by the works aforementioned and [1,2,26], where they all considered classical local semilinear equations, we address the study of normalized solutions for nonlocal and Sobolev critical probelm $(P)$.

For convenience, we define the homogeneous fractional Sobolev space:

$$\mathcal{D}^{s,2}(\mathbb{R}^N) := \left\{u \in L^{2_s^*}(\mathbb{R}^N) : |(-\Delta)^{\frac{s}{2}}u| \in L^2(\mathbb{R}^N)\right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} := \int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x.$$

Propositions 3.4 and 3.6 in [12] imply that

$$2C_{N,s}^{-1}\int\limits_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}(\xi)|^2 \mathrm{d}\xi = 2C_{N,s}^{-1}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^2_{L^2(\mathbb{R}^N)} = [u]^2_{H^s(\mathbb{R}^N)},$$

where $\hat{u}(\xi)$ is the Fourier transform of $u$ and

$$[u]_{H^s(\mathbb{R}^N)} := \left(\int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}.$$

Define

$$H^s(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x < +\infty\right\}$$

with the norm

$$\|u\| = \left(\int\limits_{\mathbb{R}^N} [|(-\Delta)^{\frac{s}{2}}u|^2 + |u|^2]\mathrm{d}x\right)^{\frac{1}{2}}.$$

Moreover, the best fractional critical Sobolev constant is given by

$$S := \inf_{u \in \mathcal{D}^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}}{\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)}}.$$

We are now in a position to state the main results of this paper.

**Theorem 1.1.** *If $2 < p < 2 + \frac{4s}{N}$, for given $k \in \mathbb{N}$, there exists $\alpha > 0$ independent of $k$ and $\mu_k := \mu(k)$ such that problem $(P)$ possesses at least $k$ couples $(u_j, \lambda_j) \in H^s(\mathbb{R}^N) \times \mathbb{R}$ of weak solutions for $\mu \geq \mu_k$ and $a \in \left(0, \left(\frac{\alpha}{\mu}\right)^{\frac{2s}{2N-p(N-2s)}}\right]$ with $\int_{\mathbb{R}^N} |u_j|^2 \mathrm{d}x = a^2$, $\lambda_j > 0$ for all $j \in [1,k]$.*

**Theorem 1.2.** *If $2 + \frac{4s}{N} < p < 2_s^*$, there exists $\mu^* = \mu^*(a) > 0$ such that as $\mu \geq \mu^*$, problem $(P)$ possesses a couple $(u_a, \lambda_a) \in H^s(\mathbb{R}^N) \times \mathbb{R}$ of weak solutions with $\int_{\mathbb{R}^N} |u_a|^2 \mathrm{d}x = a^2$, $\lambda_a > 0$.*

**Remark 1.1.** It is well known that problem $(P)$ on whole space $\mathbb{R}^N$ is invariant under translations. Obviously, for any $u \in H^s(\mathbb{R}^N)$ and $y \in \mathbb{R}^N$, the sequence $\{u_n\} := \{u(x + ny)\} \subset H^s(\mathbb{R}^N)$ is a bounded minimizing sequence that cannot be precompact in any $L^t(\mathbb{R}^N)$ for $2 < t < 2_s^*$. So, roughly speaking, the problem $(P)$ possesses bounded minimizing sequences that do not converge. This is caused by the invariance of $\mathbb{R}^N$ with respect to translations. How to overcome this difficulty? A natural way is to guess that translational invariance is the only reason that leads to the lack of compactness, so we try to work in a space of functions where translations are not allowed. This is possible in this case, because the problem $(P)$ is also invariant under rotations, so we can take the space of radial functions $H^s_{\mathrm{rad}}(\mathbb{R}^N)$ as the working space, where

$$H^s_{\mathrm{rad}}(\mathbb{R}^N) := \{u \in H^s(\mathbb{R}^N) : u \text{ is radially decreasing}\}.$$

To be precise, we will consider the functional $I : H^s_{\mathrm{rad}}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \frac{\mu}{p} \int_{\mathbb{R}^N} |u|^p \mathrm{d}x - \frac{1}{2_s^*} \int_{\mathbb{R}^N} |u|^{2_s^*} \mathrm{d}x,$$

restricted to the following sphere in $L^2(\mathbb{R}^N)$

$$S(a) := \{u \in H^s_{\mathrm{rad}}(\mathbb{R}^N) : \|u\|_2 = a\}.$$

Here, we point out that there is another way to overcome this difficulty, i.e., by adding a perturbation term $V(x)u$, where $V$ is usually assumed to be coercive, see [9,10,13,17].

Furthermore, Sobolev critical exponent $2_s^*$ also leads to the lack of compactness. Even the embedding of the radially symmetric space of $H^s_{\mathrm{rad}}(\mathbb{R}^N)$ into $L^{2_s^*}(\mathbb{R}^N)$ is not compact. This is one of the most dramatic cases of loss of compactness and has been studied intensively in the last decades, starting with the pioneering paper [7]. In [25], Luo and Zhang studied the normalized solutions of the fractional Schrödinger equation:

$$(-\Delta)^s u + \lambda u = \mu |u|^{q-2} u + |u|^{p-2} u, \quad x \in \mathbb{R}^N,$$

where $2 < q < p < 2_s^*$. Under different assumptions on $q < p$, $\mu \in \mathbb{R}$, they obtained some existence and nonexistence results about the normalized solutions. Compared with [25], we consider the case $p = 2_s^*$ and multiplicity of normalized solutions. We point out that the Sobolev critical case (i.e., $p = 2_s^*$) is much more challenging and less straightforward since $H^s_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is not compact. On the other hand, since $H^s_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is also not compact, we see that the weak limit of Palais-Smale sequences could leave the constraint manifold $S(a)$. Hence, in both cases, we have to show that the Lagrange multipliers are negative, which is vital in obtaining the compactness. With the aid of the compactness-compactness principle in fractional Sobolev spaces, we overcome the difficulty.

Next, no matter $2 < p < 2 + \frac{4s}{N}$ or $2 + \frac{4s}{N} < p < 2_s^*$, the energy functional $I(u)$ on the constraint manifold $S(a)$ is all unbounded from below. Hence, it is unlikely to obtain a solution to problem $(P)$ by minimizing problem. Naturally, we would hope to overcome this difficulty by finding other ways. Recently, much attention is paid to the existence of normalized solutions when the corresponding energy functional is unbounded from below on the constraint manifold. In the $L^2$-subcritical case, motivated by [1,2,26], we adopt a truncation technique that ensures the truncation functional is bounded from below and coercive. In the $L^2$-supercritical case, although $I$ admits a mountain-pass geometry on $S(a)$ that leads to the existence of Palais-Smale sequence, we cannot obtain the boundedness of the Palais-Smale sequence. Motivated by [18], by introducing a fiber map $\tau * u = e^{\frac{N}{2}\tau} u(e^\tau x)$, which ensures that $\tilde{I}(\tau, u) := I(\tau * u)$ on $S(a) \times \mathbb{R}$ possesses the same type of geometric structure as $I$ on $S(a)$, together with the additional property of the Palais-Smale sequence for $I$, we obtain the boundedness of the Palais-Smale sequence.

Finally, since problem $(P)$ is nonlocal, which brings new mathematical difficulties that make the study of such type of equations particularly interesting, some fine estimates are necessary.

## 2 Proof of Theorem 1.1

We first recall the definition of genus. Let $X$ ba a Banach space and $A$ be a subset of $X$. The set $A$ is said to be symmetric if $u \in A$ implies that $-u \in A$. Denote by $\Sigma$ the family of closed symmetric subsets $A$ of $X$ such that $0 \notin A$, i.e.,

$$\Sigma = \{a \subset X\backslash\{0\} : A \text{ is closed and symmetric with respect to the origin}\}.$$

For $A \in \Sigma$, define

$$\gamma(A) = \begin{cases} 0, & \text{if } A = \varnothing, \\ \inf\{k \in \mathbb{N} : \exists \text{ an odd } \varphi \in C(A, \mathbb{R}^k\backslash\{0\})\}, \\ +\infty, & \text{if no such odd map}, \end{cases}$$

and $\Sigma_k = \{A \in \Sigma : \gamma(A) \geq k\}$. In the following, we give some lemmas that are necessary for us to prove Theorem 1.1.

**Lemma 2.1.** ([8], Lemma 2.3, or [10], Lemma 2.7) *Let $N \geq 2$. The embedding $H^s_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N)$ is compact for any $2 < t < 2^*_s$.*

**Lemma 2.2.** (Fractional Gagliardo-Nirenberg inequality) [15] *Let $u \in H^s(\mathbb{R}^N)$ and $2 < t < 2^*_s$, then*

$$\|u\|_t^t \leq C_{s,N,t} \left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^{\frac{N(t-2)}{2s}} \|u\|_2^{t-\frac{N(t-2)}{2s}}.$$

**Remark 2.1.** The sharp constant $C_{s,N,t} > 0$ can be obtained by minimizing the corresponding "Weinstein functional" [31] given by

$$J(u) = \frac{\left(\int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right)^{\frac{N(t-2)}{4s}} \cdot \left(\int\limits_{\mathbb{R}^N} |u|^2 \mathrm{d}x\right)^{\frac{(2s-N)(t-2)}{4s}+1}}{\int\limits_{\mathbb{R}^N} |u|^t \mathrm{d}x}$$

defined for $u \in H^s(\mathbb{R}^N)$ with $u \not\equiv 0$. To be exact, $\frac{1}{C_{s,N,t}} = \inf_{u \in H^s(\mathbb{R}^N)\backslash\{0\}} J(u)$.

**Lemma 2.3.** ([28], Theorem 5) *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $\{u_n\}$ be a sequence in $\mathcal{D}^{s,2}(\Omega)$ weakly converging to $u$ as $n \to \infty$ and such that*

$$|(-\Delta)^{\frac{s}{2}} u_n|^2 \rightharpoonup \mu \text{ and } |u_n|^{2^*_s} \rightharpoonup \nu \text{ in } \mathcal{M}(\mathbb{R}^N)$$

*as $n \to \infty$. Then, either $u_n \to u$ in $L^{2^*_s}_{\mathrm{loc}}(\mathbb{R}^N)$ or there exists a (at most countable) set of distinct points $\{x_j\}_{j\in J} \subset \overline{\Omega}$ and positive numbers $\{\nu_j\}_{j\in J}$ such that*

$$\nu = |u|^{2^*_s} \mathrm{d}x + \Sigma_{j\in J} \nu_j \delta_{x_j}.$$

*If, in addition, $\Omega$ is bounded, then there exists a positive measure $\tilde{\mu} \in \mathcal{M}(\mathbb{R}^N)$ with spt$\tilde{\mu} \subset \overline{\Omega}$ and positive numbers $\{\mu_j\}_{j\in J}$ such that*

$$\mu = |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x + \tilde{\mu} + \Sigma_{j\in J} \mu_j \delta_{x_j} \quad \text{and} \quad \nu_j \leq S(\mu_j)^{\frac{2^*_s}{2}},$$

*where $\delta_{x_j}$ denotes the Dirac delta function at $x_j$.*

**Lemma 2.4.** ([35], Lemma 3.3) *Let $\{u_n\} \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u$ in $D^{s,2}(\mathbb{R}^N)$,*

$$|(-\Delta)^{\frac{s}{2}} u_n|^2 \rightharpoonup \mu \quad \text{and} \quad |u_n|^{2^*_s} \rightharpoonup \nu \text{ in } \mathcal{M}(\mathbb{R}^N)$$

*as $n \to \infty$, and define*

$$\mu_\infty := \lim_{R\to+\infty} \limsup_{n\to\infty} \int_{|x|>R} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x$$

*and*

$$\nu_\infty := \lim_{R\to+\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^{2_s^*} \mathrm{d}x.$$

*The quantities $\mu_\infty$ and $\nu_\infty$ are well defined and satisfy*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x = \int_{\mathbb{R}^N} \mathrm{d}\mu + \mu_\infty$$

*and*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2_s^*} \mathrm{d}x = \int_{\mathbb{R}^N} \mathrm{d}\nu + \nu_\infty.$$

**Lemma 2.5.** ([35], Lemma 3.4) *Let $\{u_n\} \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u$ in $D^{s,2}(\mathbb{R}^N)$,*

$$|(-\Delta)^{\frac{s}{2}} u_n|^2 \rightharpoonup \mu \quad \text{and} \quad |u_n|^{2_s^*} \rightharpoonup \nu \text{ in } \mathcal{M}(\mathbb{R}^N)$$

*as $n \to \infty$. Then $\nu_j \le (S^{-1}\mu(\{x_j\}))^{\frac{2_s^*}{2}}$ for any $j \in J$ and $\nu_\infty \le (S^{-1}\mu_\infty)^{\frac{2_s^*}{2}}$.*

For $u \in S(a)$, by Lemma 2.2 and Sobolev embedding theorem, it easy to see that

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \frac{\mu}{p} \int_{\mathbb{R}^N} |u|^p \mathrm{d}x - \frac{1}{2_s^*} \int_{\mathbb{R}^N} |u|^{2_s^*} \mathrm{d}x \\ &\ge \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \frac{\mu}{p} a^{p-\frac{N(p-2)}{2s}} C_{s,N,p} \left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^{\frac{N(p-2)}{2s}} - \frac{1}{2_s^* \cdot S^{\frac{2_s^*}{2}}} \left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^{2_s^*} \\ &:= M\left(\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2\right), \end{aligned}$$

where

$$M(t) = \frac{1}{2} t^2 - \frac{\mu}{p} a^{p-\frac{N(p-2)}{2s}} C_{s,N,p} t^{\frac{N(p-2)}{2s}} - \frac{1}{2_s^* \cdot S^{\frac{2_s^*}{2}}} t^{2_s^*}.$$

By $2 < p < 2 + \frac{4s}{N}$, we obtain $\frac{N(p-2)}{2s} < 2 < 2_s^*$, and there exists $\alpha > 0$ such that as $\mu a^{p-\frac{N(p-2)}{2s}} \le \alpha$, the function $M(\cdot)$ attains its positive local maximum. More precisely, there exists two constants $0 < R_1 < R_2 < +\infty$ such that $M(\cdot) < 0$ in the interval $(0, R_1)$ or $(R_2, +\infty)$, and $M(\cdot) > 0$ in the interval $(R_1, R_2)$. Let $\tau(\cdot) \in C^\infty(\mathbb{R}^+, [0, 1])$ be a nonincreasing function such that $\tau(t) = 1$ for $t \le R_1$ and $\tau(t) = 0$ for $t \ge R_2$.

Define the truncated functional as follows:

$$I_\tau(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \frac{\mu}{p} \int_{\mathbb{R}^N} |u|^p \mathrm{d}x - \frac{\tau\left(\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2\right)}{2_s^*} \int_{\mathbb{R}^N} |u|^{2_s^*} \mathrm{d}x.$$

For $u \in S(a)$, again by Lemma 2.2 and Sobolev embedding theorem, one has

$$\begin{aligned} I_\tau(u) &\ge \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \frac{\mu}{p} a^{p-\frac{N(p-2)}{2s}} C_{s,N,p} \left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^{\frac{N(p-2)}{2s}} - \frac{\tau\left(\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2\right)}{2_s^* \cdot S^{\frac{2_s^*}{2}}} \left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^{2_s^*} \\ &:= \widetilde{M}\left(\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2\right), \end{aligned}$$

where

$$\widetilde{M}(t) = \frac{1}{2}t^2 - \frac{\mu}{p}a^{p-\frac{N(p-2)}{2s}}C_{s,N,p}t^{\frac{N(p-2)}{2s}} - \frac{\tau(t)}{2_s^* \cdot S^{\frac{2_s^*}{2}}}t^{2_s^*}.$$

Then by the definition of $\tau(\cdot)$, we know that when $a \in \left(0, \left(\frac{\alpha}{\mu}\right)^{\frac{2s}{2N-p(N-2s)}}\right]$, $\widetilde{M}(\cdot) < 0$ in the interval $(0, R_1)$ and $\widetilde{M}(\cdot) > 0$ in the interval $(R_1, +\infty)$. In what follows, we always assume $a \in \left(0, \left(\frac{\alpha}{\mu}\right)^{\frac{2s}{2N-p(N-2s)}}\right]$. Without loss of generality, we may assume that

$$\frac{1}{2}r^2 - \frac{1}{2_s^* \cdot S^{\frac{2_s^*}{2}}}r^{2_s^*} \geq 0 \quad \text{for } r \in [0, R_1] \quad \text{and} \quad R_1^2 < S^{\frac{N}{2s}}. \tag{2.1}$$

**Lemma 2.6.**
*(i)* $I_\tau \in C^1(H^s_{\mathrm{rad}}(\mathbb{R}^N), \mathbb{R})$.
*(ii)* *$I_\tau$ is coercive and bounded from below on $S(a)$. Moreover, if $I_\tau \leq 0$, then $\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2 \leq R_1$ and $I_\tau(u) = I(u)$.*
*(iii)* *$I_\tau|_{S(a)}$ satisfies the $(\mathrm{PS})_c$ condition for all $c < 0$.*

**Proof.** *(i)* and *(ii)* can be proved by using of a standard argument. For *(iii)*, let $\{u_n\}$ be a $(\mathrm{PS})_c$ sequence of $I_\tau$ restricted to $S(a)$ with $c < 0$, i.e., $I_\tau(u_n) \to c < 0$ and $\|I_\tau'|_{S(a)}(u_n)\| \to 0$ as $n \to \infty$. By *(ii)*, $\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2 \leq R_1$ for large $n$, and $\{u_n\}$ is also a $(\mathrm{PS})_c$ sequence of $I|_{S(a)}$ with $c < 0$. Then, $\{u_n\}$ is bounded in $H^s_{\mathrm{rad}}(\mathbb{R}^N)$. Hence, extracting subsequences if necessary, there exists $u \in H^s_{\mathrm{rad}}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^s_{\mathrm{rad}}(\mathbb{R}^N)$ and $u_n \to u$ in $L^t(\mathbb{R}^N)$ for $2 < t < 2_s^*$ and $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^N$. Since $2 < p < 2 + \frac{4s}{N} < 2_s^*$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p \mathrm{d}x = \int_{\mathbb{R}^N} |u|^p \mathrm{d}x.$$

Furthermore, $u \not\equiv 0$. Otherwise, $\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x = 0$, and whence by (2.1), we see that

$$\begin{aligned} 0 > c &= \lim_{n\to\infty} I(u_n) \\ &= \lim_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^N}|u_n|^{2_s^*}\mathrm{d}x\right] \\ &\geq \lim_{n\to\infty}\left[\frac{1}{2}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2 - \frac{\mu}{p}\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x - \frac{1}{2_s^* S^{\frac{2_s^*}{2}}}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^{2_s^*}\right] \\ &\geq -\frac{\mu}{p}\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x = 0, \end{aligned}$$

a contradiction. On the other hand, let $\Phi(v) := \frac{1}{2}\int_{\mathbb{R}^N}|v|^2\mathrm{d}x$, $\forall v \in H^s(\mathbb{R}^N)$, then $S(a) = \Phi^{-1}\left(\left\{\frac{a^2}{2}\right\}\right)$. By Proposition 5.12 in [32], there exists $\lambda_n \in \mathbb{R}$ such that

$$\|I'(u_n) - \lambda_n\Phi'(u_n)\| \to 0$$

as $n \to \infty$, which means that

$$(-\Delta)^s u_n - \mu|u_n|^{p-2}u_n - |u_n|^{2_s^*-2}u_n = \lambda_n u_n + o(1) \text{ in } (H^s_{\mathrm{rad}}(\mathbb{R}^N))^*, \tag{2.2}$$

where $(H^s_{\mathrm{rad}}(\mathbb{R}^N))^*$ is the dual space of $H^s_{\mathrm{rad}}(\mathbb{R}^N)$. Therefore, for $\varphi \in H^s_{\mathrm{rad}}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}\varphi\mathrm{d}x - \mu\int_{\mathbb{R}^N}|u_n|^{p-2}u_n\varphi\mathrm{d}x - \int_{\mathbb{R}^N}|u_n|^{2_s^*-2}u_n\varphi\mathrm{d}x = \lambda_n\int_{\mathbb{R}^N}u_n\varphi\mathrm{d}x + o(1)\|\varphi\|. \tag{2.3}$$

Especially,

$$\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2 - \mu\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x - \int_{\mathbb{R}^N}|u_n|^{2_s^*}\mathrm{d}x = \lambda_n a^2 + o(1).$$

The boundedness of $\{\|u_n\|\}$ yields that $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Then, up to a subsequence, there exists $\lambda_a \in \mathbb{R}$ such that $\lambda_n \to \lambda_a$ as $n \to \infty$. By (2.2), using a standard argument, we can conclude that

$$(-\Delta)^s u - \mu|u|^{p-2}u - |u|^{2_s^*-2}u = \lambda_a u. \tag{2.4}$$

Indeed, for any $\varphi \in H^s_{\mathrm{rad}}(\mathbb{R}^N)$, it follows by the definition of weak convergence that

$$\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}\varphi\mathrm{d}x \to \int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi\mathrm{d}x$$

as $n \to \infty$. Noting that $\lambda_n \to \lambda_a$ as $n \to \infty$, we easily deduce

$$\lambda_n\int_{\mathbb{R}^N}u_n\varphi\mathrm{d}x \to \lambda_a\int_{\mathbb{R}^N}u\varphi\mathrm{d}x$$

as $n \to \infty$. Furthermore, since $\{|u_n|^{2_s^*-2}u_n\}$ is bounded in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^N)$ and

$$|u_n(x)|^{2_s^*-2}u_n(x) \to |u(x)|^{2_s^*-2}u(x) \quad \text{a.e. on } \mathbb{R}^N,$$

then

$$|u_n|^{2_s^*-2}u_n \rightharpoonup |u|^{2_s^*-2}u \quad \text{in } L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^N),$$

which yields that

$$\int_{\mathbb{R}^N}|u_n|^{2_s^*-2}u_n\varphi\mathrm{d}x \to \int_{\mathbb{R}^N}|u|^{2_s^*-2}u\varphi\mathrm{d}x$$

as $n \to \infty$.

In the following, we show $\lambda_a < 0$. In fact, since $u$ is a weak solution of (2.4), we have the following Pohozaev identity:

$$0 = \frac{N+2-2s}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 - \frac{N+2}{2}\lambda_a\int_{\mathbb{R}^N}|u|^2\mathrm{d}x - \frac{N+p}{p}\mu\int_{\mathbb{R}^N}|u|^p\mathrm{d}x - \frac{N+2-2s}{2}\int_{\mathbb{R}^N}|u|^{2_s^*}\mathrm{d}x.$$

Moreover,

$$\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 - \lambda_a\int_{\mathbb{R}^N}|u|^2\mathrm{d}x - \mu\int_{\mathbb{R}^N}|u|^p\mathrm{d}x - \int_{\mathbb{R}^N}|u|^{2_s^*}\mathrm{d}x = 0.$$

Consequently, by the aforementioned two identities, we deduce that

$$s\cdot\lambda_a\int_{\mathbb{R}^N}|u|^2\mathrm{d}x = \frac{p(N-2s)-2N}{2p}\mu\int_{\mathbb{R}^N}|u|^p\mathrm{d}x < 0$$

since $2 < p < 2 + \frac{4s}{N} < 2_s^*$ and $u \not\equiv 0$, which indicates that $\lambda_a < 0$.

In the sequel, we shall prove $u_n \to u$ in $L^{2_s^*}(\mathbb{R}^N)$ by using the concentration-compactness principle in fractional Sobolev spaces. In fact, since $\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2 \le R_1$ for large $n$, by Prohorov's theorem (see Theorem 8.6.2 in [4]), there exist two positive measures $\mu, \nu \in \mathcal{M}(\mathbb{R}^N)$ such that

$$|(-\Delta)^{\frac{s}{2}}u_n|^2 \rightharpoonup \mu \quad \text{and} \quad |u_n|^{2_s^*} \rightharpoonup \nu \quad \text{in } \mathcal{M}(\mathbb{R}^N) \tag{2.5}$$

as $n \to \infty$. Then, Lemmas 2.3–2.5 hold and by Lemma 2.3, either $u_n \to u$ in $L^{2_s^*}_{\mathrm{loc}}(\mathbb{R}^N)$ or there exists a (at most countable) set of distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^N$ and positive numbers $\{\nu_j\}_{j\in J}$ such that

$$\nu = |u|^{2_s^*}\mathrm{d}x + \Sigma_{j\in J}\nu_j\delta_{x_j}.$$

If the latter occurs, we can also deduce that $u_n \to u$ in $L^{2_s^*}_{\mathrm{loc}}(\mathbb{R}^N)$. We divide the proof into three steps.

Step 1: We prove that $\mu(\{x_j\}) = \nu_j$, where $\mu(\{x_j\})$ comes from Lemma 2.5.

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function with $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B_{\frac{1}{2}}(0)$, $\varphi \equiv 0$ in $\mathbb{R}^N\backslash B_1(0)$. For any $\rho > 0$, set $\varphi_\rho(x) := \varphi\left(\frac{x-x_j}{\rho}\right)$. Then

$$\varphi_\rho(x) = \begin{cases} 1, & |x - x_j| \le \frac{1}{2}\rho, \\ 0, & |x - x_j| \ge \rho. \end{cases}$$

By the boundedness of $\{u_n\}$ in $H^s_{\mathrm{rad}}(\mathbb{R}^N)$, we know that $\{\varphi_\rho u_n\}$ is also bounded in $H^s_{\mathrm{rad}}(\mathbb{R}^N)$. So

$$o(1) = \langle I'(u_n), \varphi_\rho u_n\rangle = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(\varphi_\rho u_n)\mathrm{d}x - \mu\int_{\mathbb{R}^N}\varphi_\rho|u_n|^p\mathrm{d}x - \int_{\mathbb{R}^N}\varphi_\rho|u_n|^{2_s^*}\mathrm{d}x. \tag{2.6}$$

It is easy to see that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(\varphi_\rho u_n)\mathrm{d}x \\
&= \frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{[u_n(x) - u_n(y)][\varphi_\rho(x)u_n(x) - \varphi_\rho(y)u_n(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&= \frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2\varphi_\rho(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{[u_n(x) - u_n(y)][\varphi_\rho(x) - \varphi_\rho(y)]u_n(x)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&:= I_1 + I_2.
\end{aligned}$$

For $I_1$, by (2.5),

$$\lim_{\rho\to 0}\lim_{n\to\infty} I_1 = \lim_{\rho\to 0}\lim_{n\to\infty}\frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2\varphi_\rho(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y = \lim_{\rho\to 0}\int_{\mathbb{R}^N}\varphi_\rho\mathrm{d}\mu = \mu(\{x_j\}).$$

Moreover,

$$\begin{aligned}
I_2 &= \frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{[u_n(x) - u_n(y)][\varphi_\rho(x) - \varphi_\rho(y)]u_n(x)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&\le \frac{C_{N,s}}{2}\left(\int_{\mathbb{R}^{2N}} \frac{|u_n(x)|^2|\varphi_\rho(x) - \varphi_\rho(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} \\
&\le C\frac{C_{N,s}}{2}\left(\int_{\mathbb{R}^{2N}} \frac{|u_n(x)|^2|\varphi_\rho(x) - \varphi_\rho(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}.
\end{aligned}$$

Similar computations of Lemma 3.4 in [35] show that

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^{2N}} \frac{|u_n(x)|^2|\varphi_\rho(x) - \varphi_\rho(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y = 0.$$

Consequently,

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(\varphi_\rho u_n)\mathrm{d}x = \mu(\{x_j\}).$$

By the definition of $\varphi_\rho$ and the absolute continuity of the Lebesgue integral, one obtains

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\varphi_\rho|u_n|^p\mathrm{d}x=\lim_{\rho\to 0}\int_{\mathbb{R}^N}\varphi_\rho|u|^p\mathrm{d}x=\lim_{\rho\to 0}\int_{|x-x_j|\le\rho}\varphi_\rho|u|^p\mathrm{d}x=0.$$

Again by (2.5), one obtains

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\varphi_\rho|u_n|^{2_s^*}\mathrm{d}x=\lim_{\rho\to 0}\int_{\mathbb{R}^N}\varphi_\rho\mathrm{d}\nu=\nu(\{x_j\})=\nu_j.$$

Summing up, from (2.6), taking the limit over $n$, and then the limit as $\rho\to 0$, we obtain

$$\mu(\{x_j\})=\nu_j.$$

Step 2: We prove that $\mu_\infty=\nu_\infty$, where $\mu_\infty$ and $\nu_\infty$ come from Lemma 2.4.

Let $\psi\in C^\infty(\mathbb{R}^N)$ be a cut-off function with $0\le\psi\le 1$, $\psi\equiv 0$ in $B_{\frac{1}{2}}(0)$, $\psi\equiv 1$ in $\mathbb{R}^N\backslash B_1(0)$. For any $R>0$, set

$$\psi_R(x):=\psi\left(\frac{x}{R}\right)=\begin{cases}0, & |x|\le\frac{1}{2}R,\\ 1, & |x|\ge R.\end{cases}$$

Again by the boundedness of $\{u_n\}$ in $H^s_{\mathrm{rad}}(\mathbb{R}^N)$, we see that $\{\psi_Ru_n\}$ is also bounded in $H^s_{\mathrm{rad}}(\mathbb{R}^N)$. As a consequence,

$$o(1)=\langle I'(u_n),\psi_Ru_n\rangle=\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(\psi_Ru_n)\mathrm{d}x-\mu\int_{\mathbb{R}^N}\psi_R|u_n|^p\mathrm{d}x-\int_{\mathbb{R}^N}\psi_R|u_n|^{2_s^*}\mathrm{d}x. \tag{2.7}$$

It is easy to calculate that

$$\begin{aligned}&\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(\psi_Ru_n)\mathrm{d}x\\
&=\frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}}\frac{[u_n(x)-u_n(y)][\psi_R(x)u_n(x)-\psi_R(y)u_n(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&=\frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2\psi_R(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y+\frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}}\frac{[u_n(x)-u_n(y)][\psi_R(x)-\psi_R(y)]u_n(x)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&:=I_3+I_4.\end{aligned}$$

For $I_3$, again by (2.5),

$$\lim_{R\to\infty}\lim_{n\to\infty}I_3=\lim_{R\to\infty}\lim_{n\to\infty}\frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2\psi_R(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y=\mu_\infty.$$

Furthermore,

$$\begin{aligned}I_4&=\frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}}\frac{[u_n(x)-u_n(y)][\psi_R(x)-\psi_R(y)]u_n(x)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&\le\frac{C_{N,s}}{2}\left(\int_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}\cdot\left(\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}\\
&\le C\frac{C_{N,s}}{2}\left(\int_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}.\end{aligned}$$

From the aforementioned proof, it is easy to see that

$$\lim_{R\to\infty}\lim_{n\to\infty}\int\limits_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y=\lim_{R\to\infty}\lim_{n\to\infty}\int\limits_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|[1-\psi_R(x)]-[1-\psi_R(y)]|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y=0.$$

Consequently,

$$\lim_{R\to 0}\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(\psi_R u_n)\mathrm{d}x=\mu_\infty.$$

Arguing as in the proof of Lemma 3.3 of [35], one obtains

$$\lim_{R\to\infty}\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\psi_R|u_n|^p\mathrm{d}x=\lim_{R\to\infty}\int\limits_{\mathbb{R}^N}\psi_R|u|^p\mathrm{d}x=\lim_{R\to\infty}\int\limits_{|x|>\frac{1}{2}R}\psi_R|u|^p\mathrm{d}x=0$$

and

$$\lim_{R\to\infty}\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\psi_R|u_n|^{2_s^*}\mathrm{d}x=\nu_\infty.$$

And so, (2.7) yields that $\mu_\infty=\nu_\infty$.

Step 3: We prove that $\nu_j=0$ for any $j\in J$ and $\nu_\infty=0$. We argue by contradiction. Suppose that there exists $j_0\in J$ such that $\nu_{j_0}>0$ or $\nu_\infty>0$. Steps 1 and 2 of Lemma 2.5 imply that

$$\nu_{j_0}\le(S^{-1}\mu\{x_{j_0}\})^{\frac{2_s^*}{2}}=(S^{-1}\nu_{j_0})^{\frac{2_s^*}{2}}$$

or

$$\nu_\infty\le(S^{-1}\mu_\infty)^{\frac{2_s^*}{2}}=(S^{-1}\nu_\infty)^{\frac{2_s^*}{2}}.$$

It yields that $\nu_{j_0}\ge S^{\frac{N}{2s}}$ or $\nu_\infty\ge S^{\frac{N}{2s}}$. If the former case is valid, then

$$\begin{aligned}R_1^2&\ge\lim_{\rho\to 0}\lim_{n\to\infty}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2\ge S\lim_{\rho\to 0}\lim_{n\to\infty}\|u_n\|_{2_s^*}^2\\&\ge\lim_{\rho\to 0}\lim_{n\to\infty}\left(\int\limits_{\mathbb{R}^N}\varphi_\rho|u_n|^{2_s^*}\mathrm{d}x\right)^{\frac{2}{2_s^*}}=S\lim_{\rho\to 0}\left(\int\limits_{\mathbb{R}^N}\varphi_\rho\mathrm{d}\nu\right)^{\frac{2}{2_s^*}}\\&=S\cdot\nu_{j_0}^{\frac{2}{2_s^*}}\ge S^{\frac{N}{2s}},\end{aligned}$$

which contradicts with (2.1). If the last case is true, then

$$R_1^2\ge\lim_{\rho\to 0}\lim_{n\to\infty}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2\ge\mu_\infty=\nu_\infty\ge S^{\frac{N}{2s}},$$

which also contradicts with (2.1).

Consequently, by Lemma 2.3, we know that $u_n\to u$ in $L^{2_s^*}_{\mathrm{loc}}(\mathbb{R}^N)$, which together with Lemma 2.4 yields that $u_n\to u$ in $L^{2_s^*}(\mathbb{R}^N)$,

Taking into account of (2.3)–(2.4), we obtain

$$\lim_{n\to\infty}\left[\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2-\lambda_a\|u_n\|_2^2\right]=\lim_{n\to\infty}\left[\mu\|u_n\|_p^p+\|u_n\|_{2_s^*}^{2_s^*}+o(1)\right]=\mu\|u\|_p^p+\|u\|_{2_s^*}^{2_s^*}=\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2-\lambda_a\|u\|_2^2. \tag{2.8}$$

Since $\lambda_a<0$,

$$
\begin{aligned}
-\lambda_a\|u\|_2^2 &\le \liminf_{n\to\infty} -\lambda_a\|u_n\|_2^2 \le \limsup_{n\to\infty} -\lambda_a\|u_n\|_2^2 \\
&\le \limsup_{n\to\infty} -\lambda_a\|u_n\|_2^2 + \liminf_{n\to\infty} \left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2 - \left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 \\
&\le \limsup_{n\to\infty}\left[\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2 - \lambda_a\|u_n\|_2^2\right] - \left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2 \\
&= -\lambda_a\|u\|_2^2,
\end{aligned}
$$

and so

$$\lim_{n\to\infty} -\lambda_a\|u_n\|_2^2 = -\lambda_a\|u\|_2^2.$$

Then,

$$\lim_{n\to\infty} \|u_n\|_2^2 = \|u\|_2^2,$$

and by (2.8), one obtain

$$\lim_{n\to\infty} \left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2 = \left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2.$$

Hence, $u_n \to u$ in $H^s_{\mathrm{rad}}(\mathbb{R}^N)$ and $\|u\|_2 = a$. This completes the proof. □

For $\varepsilon > 0$, set

$$I_\tau^{-\varepsilon} = \{u \in H^s_{\mathrm{rad}}(\mathbb{R}^N) \cap S(a) : I_\tau(u) \le -\varepsilon\} \subset H^s_{\mathrm{rad}}(\mathbb{R}^N).$$

By the fact that $I_\tau$ is even and continuous on $H^s_{\mathrm{rad}}(\mathbb{R}^N)$, $I_\tau^{-\varepsilon}$ is closed and symmetric. Then, we have the following lemma, whose proof is similar to Lemma 3.2 in [2].

**Lemma 2.7.** *Given $n \in \mathbb{N}$, there exist $\varepsilon_n := \varepsilon(n) > 0$ and $\mu_n := \mu(n) > 0$ such that as $0 < \varepsilon \le \varepsilon_n$ and $\mu \ge \mu_n$, $\gamma(I_\tau^{-\varepsilon}) \ge n$.*

Set

$$\Sigma_k := \{D \subset H^s_{\mathrm{rad}}(\mathbb{R}^N) \cap S(a) : D \quad \text{is closed and symmetric},\ \gamma(D) \ge k\},$$

and

$$c_k := \inf_{D\in\Sigma_k}\sup_{u\in D} I_\tau(u) > -\infty$$

for all $k \in \mathbb{N}$ by Lemma 2.6(*ii*). To prove Theorem 1.1, let us define

$$K_c := \{u \in H^s_{\mathrm{rad}}(\mathbb{R}^N) \cap S(a) : I'_\tau(u) = 0, I_\tau(u) = c\}.$$

Then, the following lemmas hold.

**Lemma 2.8.** *If $c = c_k = c_{k+1} = \cdots = c_{k+r}$, then $\gamma(K_c) \ge r + 1$. In particular, $I_\tau$ possesses at least $r + 1$ nontrivial critical points.*

**Proof.** For $\varepsilon > 0$, it is easy to see that $I_\tau^{-\varepsilon} \in \Sigma$. For any $k \in \mathbb{N}$, by the previous lemma, there exists $\varepsilon_k = \varepsilon(k) > 0$ and $\mu_k = \mu(k) > 0$ such that if $0 < \varepsilon \le \varepsilon_k$ and $\mu \ge \mu_k$, we obtain $\gamma(I_\tau^{-\varepsilon}) \ge k$. Then $I_\tau^{-\varepsilon_k} \in \Sigma_k$, and

$$c_k \le \sup_{u\in I_\tau^{-\varepsilon_k}} I_\tau(u) = -\varepsilon_k < 0.$$

Suppose that $0 > c = c_k = c_{k+1} = \cdots = c_{k+r}$, then Lemma 2.6(*iii*) implies that $I_\tau$ satisfies the $(\mathrm{PS})_c$ condition. Hence, $K_c$ is a compact set. By Theorem 2.1 in [2] or Theorem 2.1 in [19], $I_\tau|_{S(a)}$ possesses at least $r + 1$ critical points. □

**Proof of Theorem 1.1.** By Lemma 2.6($ii$), the critical points of $I_\tau$ founded in Lemma 2.8 are the critical points of $I$. So Theorem 1.1 is proved. □

## 3 Proof of Theorem 1.2

In this section, we study case $2 + \frac{4s}{N} < p < 2_s^*$. Since $\frac{N(p-2)}{2s} > 2$, it follows that the truncated functional $I_\tau$ is still unbounded from below on $S(a)$. Therefore, we cannot use the truncation technique in Section 2 to study problem $(P)$.

For convenience, we set $f(t) = \mu|t|^{p-2}t + |t|^{2_s^*-2}t$ for all $t \in \mathbb{R}$ and introduce the following auxiliary functional:

$$\tilde{I} : S(a) \times \mathbb{R} \to \mathbb{R}, \quad (u, \tau) \mapsto I(\tau * u),$$

where $(\tau * u)(x) := e^{\frac{N}{2}\tau}u(e^\tau x)$. Then simple calculations show that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\tau * u)|^2 \mathrm{d}x = e^{2\tau s} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x$$

and

$$\int_{\mathbb{R}^N} |\tau * u|^q \mathrm{d}x = e^{\frac{q-2}{2}N\tau} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x, \quad \forall q \in [2, 2_s^*].$$

Then

$$\begin{aligned}\tilde{I}(u, \tau) &= I(\tau * u) = I(e^{\frac{N}{2}\tau}u(e^\tau x)) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\tau * u)|^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(\tau * u)\mathrm{d}x \\ &= \frac{1}{2}e^{2\tau s}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x - e^{-N\tau}\int_{\mathbb{R}^N} F(e^{\frac{N\tau}{2}}u(x))\mathrm{d}x \\ &= \frac{1}{2}e^{2\tau s}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x - \frac{\mu}{p}\cdot e^{\frac{p-2}{2}N\tau}\int_{\mathbb{R}^N} |u|^p \mathrm{d}x - \frac{1}{2_s^*}\cdot e^{\frac{2_s^*-2}{2}N\tau}\int_{\mathbb{R}^N} |u|^{2_s^*}\mathrm{d}x.\end{aligned}$$

Clearly, the aforementioned estimates imply Lemma 3.1.

**Lemma 3.1.** ([25], Lemma 5.1) *Let $u \in S(a)$ be arbitrary but fixed. Then*
($i$) $\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\tau * u)|^2 \mathrm{d}x \to 0$ *and* $\tilde{I}(u, \tau) \to 0$ *as* $\tau \to -\infty$.
($ii$) $\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\tau * u)|^2 \mathrm{d}x \to +\infty$ *and* $\tilde{I}(u, \tau) \to -\infty$ *as* $\tau \to +\infty$.

With the aid of fractional Gagliardo-Nirenberg inequality (see Lemma 2.2), we can obtain the next lemma.

**Lemma 3.2.** ([25], Lemma 5.2) *There exists $K(a) > 0$ sufficiently small such that*

$$I(u) > 0 \quad \text{for } u \in A \quad \text{and} \quad 0 < \sup_{u \in A} I(u) < \inf_{u \in B} I(u),$$

*where*

$$A := \left\{u \in S(a) : \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x \le K(a)\right\}$$

*and*

$$B := \left\{u \in S(a) : \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x = 2K(a)\right\}.$$

As a consequence of Lemmas 3.1 and 3.2, we see that for fixed $u_0 \in S(a)$, there exists two constants $\tau_1, \tau_2$ satisfying $\tau_1 < 0 < \tau_2$, such that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_1|^2 \mathrm{d}x < \frac{K(a)}{2}, \quad \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_2|^2 \mathrm{d}x > 2K(a)$$

and

$$I(u_1) > 0, \quad I(u_2) < 0,$$

where $u_1 := \tau_1 * u_0 \in S(a)$ and $u_2 := \tau_2 * u_0 \in S(a)$. In the following, denote the mountain pass level $\gamma_\mu(a)$ by

$$\gamma_\mu(a) := \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)),$$

where

$$\Gamma := \{g \in C([0,1], S(a)) : g(0) = u_1, g(1) = u_2\}.$$

Then for any $g \in \Gamma$,

$$\max_{t \in [0,1]} I(g(t)) > \max\{I(u_1), I(u_2)\}.$$

It yields that $\gamma_\mu(a) > 0$. About $\gamma_\mu(a)$, the following lemma holds.

**Lemma 3.3.** $\lim\limits_{\mu \to +\infty} \gamma_\mu(a) = 0$.

**Proof.** Taking $g_0(t) := [(1-t)\tau_1 + t\tau_2] * u_0 \in \Gamma$, then

$$\begin{aligned}
0 < \gamma_\mu(a) &\le \max_{t \in [0,1]} I(g_0(t)) = \max_{t \in [0,1]} I([(1-t)\tau_1 + t\tau_2] * u_0) \\
&= \max_{t \in [0,1]} \left\{ \frac{1}{2} e^{2[(1-t)\tau_1 + t\tau_2]s} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_0|^2 \mathrm{d}x - \frac{\mu}{p} \cdot e^{\frac{p-2}{2} N[(1-t)\tau_1 + t\tau_2]} \int_{\mathbb{R}^N} |u_0|^p \mathrm{d}x - \frac{1}{2_s^*} \cdot e^{\frac{2_s^*-2}{2} N[(1-t)\tau_1 + t\tau_2]} \int_{\mathbb{R}^N} |u_0|^{2_s^*} \mathrm{d}x \right\} \\
&\le \max_{r \ge 0} \left\{ \frac{1}{2} r^{2s} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_0|^2 \mathrm{d}x - \frac{\mu}{p} \cdot r^{\frac{N(p-2)}{2}} \int_{\mathbb{R}^N} |u_0|^p \mathrm{d}x \right\} \\
&\le C\left(\frac{1}{\mu}\right)^{\frac{4s}{N(p-2)-4s}} \to 0 \quad (\mu \to +\infty).
\end{aligned}$$

This completes the proof. □

By Proposition 2.2 in [18] and Proposition 5.4 in [25], there exists a sequence $\{u_n\} \subset S(a)$ satisfying

$$I(u_n) \to \gamma_\mu(a) \quad \text{and} \quad \|I'|_{S(a)}(u_n)\| \to 0 \quad \text{and} \quad Q(u_n) \to 0$$

as $n \to \infty$, where

$$Q(u_n) = s \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x + N \int_{\mathbb{R}^N} F(u_n) \mathrm{d}x - \frac{N}{2} \int_{\mathbb{R}^N} f(u_n) u_n \mathrm{d}x.$$

In the sequel, like Section 2, set $\Phi(v) := \frac{1}{2}\int_{\mathbb{R}^N} |v|^2 \mathrm{d}x, \quad \forall v \in H^s(\mathbb{R}^N)$, then $S(a) = \Phi^{-1}\left(\left\{\frac{a^2}{2}\right\}\right)$. By Proposition 5.12 in [32], there exists $\lambda_n \in \mathbb{R}$ such that

$$\|I'(u_n) - \lambda_n \Phi'(u_n)\| \to 0$$

as $n \to \infty$, and so

$$(-\Delta)^s u_n - f(u_n) = \lambda_n u_n + o(1) \quad \text{in } (H^s_{\mathrm{rad}}(\mathbb{R}^N))^*. \tag{3.1}$$

Therefore, for $\varphi \in H^s_{\mathrm{rad}}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi \mathrm{d}x - \int_{\mathbb{R}^N} f(u_n)\varphi \mathrm{d}x = \lambda_n \int_{\mathbb{R}^N} u_n \varphi \mathrm{d}x + o(1)\|\varphi\|. \tag{3.2}$$

We have the following two lemmas.

**Lemma 3.4.** *There exists a constant $C = C(N, s, p) > 0$ such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \le C\gamma_\mu(a)$$

*and*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} f(u_n)u_n \mathrm{d}x \le C\gamma_\mu(a)$$

*and*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x \le C\gamma_\mu(a).$$

**Proof.** Since $I(u_n) \to \gamma_\mu(a)$ and $Q(u_n) \to 0$ as $n \to \infty$,

$$\begin{aligned} N\gamma_\mu(a) + o(1) &= NI(u_n) + Q(u_n) \\ &= \frac{N+2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x - \frac{N}{2} \int_{\mathbb{R}^N} f(u_n)u_n \mathrm{d}x \\ &= \frac{N+2s}{2} \left[ 2\gamma_\mu(a) + 2\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x + o(1) \right] - \frac{N}{2} \int_{\mathbb{R}^N} f(u_n)u_n \mathrm{d}x, \end{aligned}$$

hence,

$$\begin{aligned} 2s\gamma_\mu(a) + o(1) &= \frac{N}{2} \int_{\mathbb{R}^N} f(u_n)u_n \mathrm{d}x - (N+2s) \int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \\ &\ge \frac{Np}{2} \int_{\mathbb{R}^N} F(u_n)\mathrm{d}x - (N+2s) \int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \\ &= \frac{Np - 2(N+2s)}{2} \int_{\mathbb{R}^N} F(u_n)\mathrm{d}x, \end{aligned}$$

i.e.,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \le \frac{4s}{Np - 2(N+2s)} \gamma_\mu(a),$$

and then,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} f(u_n)u_n \mathrm{d}x \le C\gamma_\mu(a)$$

and

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x=\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left[2\int_{\mathbb{R}^N}F(u_n)\mathrm{d}x+2\gamma_\mu(a)+o(1)\right]\leq C\gamma_\mu(a).$$

This completes the proof. □

By Lemma 3.4, we can estimate $\lambda_n$ as follows.

**Lemma 3.5.** *$\{\lambda_n\}$ is bounded in $\mathbb{R}$ and $\limsup_{n\to\infty}|\lambda_n|\leq\frac{C}{a^2}\gamma_\mu(a)$ and*

$$\lambda_n=-\frac{1}{a^2}\cdot\frac{2N-(N-2s)p}{2sp}\cdot\mu\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x+o(1).$$

**Proof.** By (3.2) and the fact that $u_n\in S(a)$,

$$\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x-\int_{\mathbb{R}^N}f(u_n)u_n\mathrm{d}x=\lambda_n\int_{\mathbb{R}^N}|u_n|^2\mathrm{d}x+o(1)=\lambda_n a^2+o(1),$$

which indicates that

$$\lambda_n=\frac{1}{a^2}\left[\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x-\int_{\mathbb{R}^N}f(u_n)u_n\mathrm{d}x\right]+o(1).$$

By the boundedness of $\{u_n\}$ in $H^s_{\mathrm{rad}}(\mathbb{R}^N)$, we know that $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Lemma 3.4 means that $\limsup_{n\to\infty}|\lambda_n|\leq\frac{C}{a^2}\gamma_\mu(a)$. Moreover, combining with $Q(u_n)\to 0$ as $n\to\infty$, we see that

$$\begin{aligned}\lambda_n&=\frac{1}{a^2}\left[\frac{N}{2s}\int_{\mathbb{R}^N}f(u_n)u_n\mathrm{d}x-\frac{N}{s}\int_{\mathbb{R}^N}F(u_n)\mathrm{d}x-\int_{\mathbb{R}^N}f(u_n)u_n\mathrm{d}x\right]+o(1)\\&=\frac{1}{a^2}\left[\frac{N-2s}{2s}\int_{\mathbb{R}^N}f(u_n)u_n\mathrm{d}x-\frac{N}{s}\int_{\mathbb{R}^N}F(u_n)\mathrm{d}x\right]+o(1)\\&=\frac{1}{a^2}\left[\frac{N-2s}{2s}\int_{\mathbb{R}^N}\left(\mu|u_n|^p+|u_n|^{2_s^*}\right)\mathrm{d}x-\frac{N}{s}\int_{\mathbb{R}^N}\left(\frac{\mu}{p}|u_n|^p+\frac{1}{2_s^*}|u_n|^{2_s^*}\right)\mathrm{d}x\right]+o(1)\\&=\frac{1}{a^2}\left(\frac{N-2s}{2s}-\frac{N}{sp}\right)\int_{\mathbb{R}^N}\mu|u_n|^p\mathrm{d}x+o(1)\\&=-\frac{1}{a^2}\cdot\frac{2N-(N-2s)p}{2sp}\cdot\mu\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x+o(1).\end{aligned}$$

This completes the proof. □

From the boundedness of $\{u_n\}$ in $H^s_{\mathrm{rad}}(\mathbb{R}^N)$, up to a subsequence, there exists $u\in H^s_{\mathrm{rad}}(\mathbb{R}^N)$ such that $u_n\rightharpoonup u$ in $H^s_{\mathrm{rad}}(\mathbb{R}^N)$ and $u_n\to u$ in $L^t(\mathbb{R}^N)$ for $2<t<2_s^*$ and $u_n(x)\to u(x)$ a.e. on $\mathbb{R}^N$. Since $2+\frac{4s}{N}<p<2_s^*$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x=\int_{\mathbb{R}^N}|u|^p\mathrm{d}x.\tag{3.3}$$

**Lemma 3.6.** *There exists $\mu^*=\mu^*(a)>0$ such that $u\neq 0$ for all $\mu\geq\mu^*$.*

**Proof.** We argue by contradiction. Suppose that $u = 0$. Then taking into account of (3.3) and Lemma 3.5, one has $\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x = 0$ and $\lim_{n\to\infty}\lambda_n = 0$. Combining with (3.2), we obtain

$$\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x - \|u_n\|_{2_s^*}^{2_s^*} = o(1).$$

Up to a subsequence,

$$\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x \to l \geq 0$$

and

$$\|u_n\|_{2_s^*}^{2_s^*} \to l$$

as $n \to \infty$. If $l = 0$, we can deduce from the expression of $I(u_n)$ that $\gamma_\mu(a) = 0$. It is a contradiction. Hence, $l > 0$. By the definition of $S$, we have

$$S \leq \frac{\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x}{\|u_n\|_{2_s^*}^2} \to \frac{l}{l^{\frac{2}{2_s^*}}} = l^{\frac{2s}{N}}$$

as $n \to \infty$. It follows that $l \geq S^{\frac{N}{2s}}$. Consequently, by (3.3), we have

$$\begin{aligned}\gamma_\mu(a) &= \lim_{n\to\infty} I(u_n)\\ &= \lim_{n\to\infty}\left\{\frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^N}|u_n|^{2_s^*}\mathrm{d}x\right\}\\ &= \left(\frac{1}{2} - \frac{1}{2_s^*}\right)l = \frac{s}{N}l \geq \frac{s}{N}S^{\frac{N}{2s}},\end{aligned}$$

a contradiction to Lemma 3.3. □

Subsequently, by virtue of the concentration-compactness principle (see Section 2), we can obtain the following lemma. Since the proof is similar, we omit it here.

**Lemma 3.7.** *$u_n \to u$ in $L^{2_s^*}(\mathbb{R}^N)$ for $\mu \geq \mu^*$.*

**Proof of Theorem 1.2.** Fixed $\mu \geq \mu^*$. By Lemma 3.5, we may assume that $\lambda_n \to \lambda_a$ as $n \to \infty$. Combining with Lemma 3.6 and (3.3), it is easy to see that

$$\begin{aligned}\lim_{n\to\infty}\lambda_n &= -\frac{1}{a^2}\cdot\frac{2N-(N-2s)p}{2sp}\cdot\mu\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x\\ &= -\frac{1}{a^2}\cdot\frac{2N-(N-2s)p}{2sp}\cdot\mu\int_{\mathbb{R}^N}|u|^p\mathrm{d}x < 0,\end{aligned}$$

and so $\lambda_a < 0$. Arguing as the proof of that in Section 2, by (3.1), we have

$$(-\Delta)^s u - f(u) = \lambda_a u, \quad x \in \mathbb{R}^N.$$

Then, (3.3) yields that

$$
\begin{aligned}
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \lambda_a \|u\|_2^2 &= \int_{\mathbb{R}^N} f(u) u \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left[ \mu |u|^p + |u|^{2_s^*} \right] \mathrm{d}x \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \left[ \mu |u_n|^p + |u_n|^{2_s^*} \right] \mathrm{d}x \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \left[ |(-\Delta)^{\frac{s}{2}} u_n|^2 - \lambda_n u_n^2 \right] \mathrm{d}x \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \left[ |(-\Delta)^{\frac{s}{2}} u_n|^2 - \lambda_a u_n^2 \right] \mathrm{d}x.
\end{aligned}
$$

Since $\lambda_a < 0$, with a similar argument as the proof in Section 2, we can derive that

$$
\lim_{n\to\infty} \left\| (-\Delta)^{\frac{s}{2}} u_n \right\|_2^2 = \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^2 \quad \text{and} \quad \lim_{n\to\infty} \|u_n\|_2^2 = \|u\|_2^2.
$$

Hence, $u_n \to u$ in $H^s_{\mathrm{rad}}(\mathbb{R}^N)$ and $\|u\|_2 = a$. This completes the proof. □

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