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FROM TENSOR TO VECTOR OF GRAVITATION

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ABSTRACT

Different gravitational force models are used for determining the satellites' orbits. The satellite gravity gradiometry (SGG) data contain this gravitational information and the satellite accelerations can be determined from them. In this study, we present that amongst the elements of the gravitational tensor in the local north-oriented frame, all of the elements are suitable for this purpose except T_{xy}. Three integral formulae with the same kernel function are presented for recovering the accelerations from the SGG data. The kernel of these integrals is well-behaving which means that the contribution of the far-zone data is not very significant to their integration results; but this contribution is also dependent on the type of the data being integrated. Our numerical studies show that the standard deviations of the differences between the accelerations recovered from T_{zz}, T_{xz} and T_{yz} and those computed by an existing Earth´s gravity model reduce by increasing the cap size of integration. However, their root mean squared errors increase for recovering T_y from T_{yz}. Larger cap sizes than 5 is recommended for recovering T_x and T_z but smaller ones for T_y.

Keywords: Orbit integration, satellite gravity gradiometry, satellite acceleration

1. INTRODUCTION

In satellite orbit integration a series of force models are integrated twice with respect to time for delivering the satellite's velocity and position vectors. Generally, the equations of motion of a satellite are expressed by a second-order vector differential equation containing these force models. The accuracies of these models play an important role in a successful orbit integration process. Today, satellites are launched for different purposes, including the Earth's gravity field determination and in order to obtain a high resolution gravity model different techniques are used. Satellite gravity gradiometry (SGG) is the technique which was used in the recent European Space Agency (ESA) satellite mission, the gravity field and steady-state ocean circulation explorer (GOCE) (ESA 1999). Since the SGG data are more sensitive to the local futures of the Earth's gravity field, we expect to derive high-resolution gravity models from their analyses.

Satellite orbit analysis is a well-known technique for the gravitational field recovery cf. e.g., Kaula (1966), Visser (1992) and Sneeuw (1992). It is important to consider that, although the SGG technique (Rummel et al. 1993, Keller and Sharifi 2005, Sharifi 2006) is used, the satellite's orbit should be determined as precise as possible so that the extracted perturbations can be analysed without worrying about the possible biases in the gravity field solution. Precise orbit determination of satellites (Parrot 1989, Santos 1994, Su 2000 and Wolf 2000) can be done in different ways, such as kinematic, dynamic and reduced dynamic, etc.; see e.g. Rim and Schutz (2001). Numerical integration of an orbit has some benefits with respect to the analytical one (Kaula 1966), as it is not restricted to the mathematical models of the perturbing forces. For details of the algorithms; see, e.g. Su (2000), Wolf (2000), Eshagh (2003, 2005, 2009a), Eshagh and Najafi-Alamdari (2006 and 2007) and Somodi and Földvary (2011). Eshagh et al. (2009) simplified the mathematical formulae of equations of motion of satellites and applied them for a study on the orbit integration of GOCE, the gravity field recovery and climate experiment (GRACE) (Tapley et al. 2005) and the Challenging Minisatellite Payload (CHAMP) (Reigber et al. 1999 and 2004) missions. In all methods for orbit determination, the acceleration of the satellite should be computed before the integration process using the force models. In fact, the satellite acceleration is the results of interactions of all gravitational and non-gravitational accelerations, but since some of the forces of the second category are practically removed from the satellite observations, therefore, we can assume that the satellite undergoes the gravitational accelerations; e.g. in the case of GOCE satellite the non-gravitational accelerations were compensated by a drag free and attitude control system (ESA 1999). The SGG data can be converted to these gravitational accelerations and in this case, we do not have to use corresponding force models. However, the important issue is to check the quality of these data prior to any application.

So far, different efforts have been done to validate the SGG data. The simplest method is the direct comparison of the observed SGG data with the corresponding generated ones using an existing Earth's gravity model (EGM); (see Eshagh and Abdollahzadeh 2010 and 2011). Also, Haagmans et al. (2002) and Kern and Haagmans (2004), Mueller et al. (2004) and Wolf (2007) used the extended Stokes and Hotine formulae for using the terrestrial gravimetric data for this purpose. Bouman et al. (2003) has set up a calibration model based on the instrument (gradiometer) characteristics to validate the SGG data. Least-squares collocation can be used for the same purpose; see e.g. Tscherning et al. (2006). Zielinski and Petrovskaya (2003) proposed a balloon-borne gradiometer to fly at 20-40 km altitude simultaneously with satellite mission and proposed downward continuation of satellite data and comparing them with balloon-borne data. Bouman and Koop (2003) presented an along-track interpolation method to detect the outliers. Their idea is to compare the along-track interpolated gradients with the measured ones. Pail (2003) proposed a combined adjustment method supporting high quality gravity field information within the well-surveyed test area for the continuation of the local gravity field upward and validating the SGG data. Bouman et al. (2004) stated that a high-degree EGM should be taken into account and to remove the greater part of the systematic errors. Kern and Haagmans (2004) and Kern et al. (2005) presented an algorithm for detecting the outliers in the SGG data in the time domain. Visser (2009) tried to estimate the biases and scale factors of the accelerometers used in the GOCE spacecraft by using its precise orbit. The stochastic modification was used by Eshagh (2010a) to modify the second-order radial derivative of the extended Stokes formula. Eshagh (2011a) also modified the second-order radial derivative of the Abel-Poisson formula in a least-squares sense to generate the second-order radial gradient at satellite level using an EGM and geoid model. The least-squares modification of the vertical-horizontal and horizontal-horizontal derivatives of the extended Stokes formula was done by Eshagh and Romeshkani (2011) and Romeshkani (2011) based on the theoretical study by Eshagh (2010b). Bouman et al. (2011) presented the gravity gradients of GOCE along its orbit in the local north-oriented frame (LNOF) which are suitable for different applications.

All of these methods are based on the fact that the EGM and terrestrial gravity data are reliable enough to validate the SGG data at satellite level especially those of GOCE. The successful performance GOCE showed that its SGG data have good quality for the gravity field recovery goals. Therefore, they can be considered as reliable sources of gravimetric information and used for other purposes. Eshagh and Romeshkani (2013) tried to use these data for quality description of those terrestrial data lacking any information about their quality. Here, the idea is to use the SGG data instead of the force models for generation of the satellite accelerations as they have inherently included most of the gravitational information required for orbit integration. The non-gravitational effects are, today, technically removed from the observations. Therefore, we expect to obtain better and realistic gravitational information than those the models contain. The main issue is how to use the SGG data for this goal. The idea of this paper comes from a study which was done by Bobojc and Drozyner (2003) for using the SGG data for orbit determination in a stochastic way. However, our main goal is to find deterministic and integral relations to recover the satellite accelerations from these reliable SGG data.

2. EQUATIONS OF MOTION OF A SATELLITE

The Earth's gravitational potential is expressed by the following spherical harmonic series (cf. Heiskanen and Moritz 1967, p. 107):

$$V = \frac{GM}{R}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1}\sum_{m=-n}^{n} v_{nm}Y_{nm}\left(\theta,\lambda\right) \quad \text{where} \quad R < r \tag{1a}$$

and GM stands for the geocentric gravitational constant, R the semi-major axis of the reference ellipsoid, r the geocentric distance of any point outside the Earth's surface, $Y_{nm}\left(\theta,\lambda\right)$ the fully-normalised spherical harmonics of degree n and order m, with arguments of co-latitude θ and longitude λ and v_{nm} is the spherical harmonic coefficients of the gravity field. The ratio R / r is always smaller than 1 and by getting a power of $n + 1$ it becomes smaller and smaller for higher degrees.

The equations of motion of a satellite form a second-order vector differential equation and by integrating it twice, the velocity and position of the satellite are derived. This differential equation is solved by integrating some available force models acting on the satellite. The most significant force is the gravitational attraction of the Earth and if the satellite is assumed as a point mass, its accelerations in the LNOF, which is defined as a frame whose z-axis is pointing radially upward, the x-axis points towards the north and the frame is right handed, can be computed by (Hwang and Lin 1998, Eshagh 2008):

$$V_r = -\frac{GM}{R^2}\sum_{n=0}^{\infty}(n+1)\left(\frac{R}{r}\right)^{n+2}\sum_{m=-n}^{n} v_{nm}Y_{nm}(\theta,\lambda) \tag{1b}$$

$$\frac{V_\theta}{r} = \frac{GM}{R^2}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+2}\sum_{m=-n}^{n} v_{nm}\frac{\partial Y_{nm}(\theta,\lambda)}{\partial\theta} \tag{1c}$$

$$\frac{V_\lambda}{r\sin\theta} = \frac{GM}{R^2}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+2}\sum_{m=-n}^{n} v_{nm}\frac{\partial Y_{nm}(\theta,\lambda)}{\sin\theta\,\partial\lambda}. \tag{1d}$$

These three accelerations are the elements of the vector of gravitation **v**:

$$\mathbf{v} = \begin{pmatrix} V_x & V_y & V_z \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} \frac{V_\theta}{r} & \frac{V_\lambda}{r\sin\theta} & V_r \end{pmatrix}^{\mathrm{T}}, \tag{1e}$$

where $(.)^{\mathrm{T}}$ stands for the transpose operator.

The orbit integration should be performed in an inertial frame, which its z-axis is towards the Earth's pole, the x-axis towards the Vernal equinox on the equator and the frame is right handed. However, the satellite accelerations are presented in the LNOF, therefore, they should be transferred to the inertial frame by (Hwang and Lin 1998, Eshagh et al. 2009):

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\lambda^* & \cos\theta\cos\lambda^* & -\sin\lambda^* \\ \sin\theta\sin\lambda^* & \sin\theta\sin\lambda^* & \cos\lambda^* \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} V_r \\ \frac{V_\theta}{r} \\ \frac{V_\lambda}{r\sin\theta} \end{pmatrix} \tag{1f}$$

where $\ddot{x}$, $\ddot{y}$ and $\ddot{z}$ are the satellite accelerations in the inertial frame and $\lambda^* = \lambda + \mathrm{GAST}$. GAST stands for Greenwich Apparent Sidereal Time which can be computed using astronomical data of Doodson or Delauney arguments (IERS conventions 2010).

3. TENSOR AND VECTOR OF GRAVITATION

By taking the derivatives of the vector of gravitation towards x, y and z axes in the LNOF, the tensor of gravitation becomes:

$$\mathbf{T} = \begin{pmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{xy} & V_{yy} & V_{yz} \\ V_{xz} & V_{yz} & V_{zz} \end{pmatrix}. \tag{2a}$$

The elements of this tensor have the following relations to the Earth's gravitational potential (Reed 1973, Petrovskaya and Vershkov 2006):

$$V_{zz} = V_{rr} \tag{2b}$$

$$V_{xz} = \frac{V_\theta}{r^2} - \frac{V_{r\theta}}{r} \tag{2c}$$

$$V_{yz} = \frac{V_\lambda}{r^2 \sin\theta} - \frac{V_{r\lambda}}{r \sin\theta} \tag{2d}$$

$$V_{xx} = \frac{V_r}{r} + \frac{V_{\theta\theta}}{r^2} \tag{2e}$$

$$V_{yy} = \frac{V_r}{r} + \frac{V_\theta}{r^2 \tan\theta} + \frac{V_{\lambda\lambda}}{r^2 \sin^2\theta} \tag{2f}$$

$$V_{xy} = \frac{V_{\theta\lambda}}{r^2 \sin\theta} + \frac{\cos\theta V_\lambda}{r^2 \sin^2\theta} \; . \tag{2g}$$

By looking at these relations we can simply find out that there are similarities between V_{zz} and V_{xz} and V_{yz} and the elements of the vector of gravitation V_z, V_x and V_y, respectively. Therefore, here and after, we continue our discussion with V_{zz} , V_{xz} and V_{yz} which have the following spherical harmonic expressions (Petrovskaya and Vershkov 2006):

$$V_{zz} = \frac{GM}{R^3} \sum_{n=0}^{\infty} (n+1)(n+2) \left(\frac{R}{r} \right)^{n+3} \sum_{m=-n}^{n} v_{nm} Y_{nm}(\theta, \lambda) \tag{2h}$$

$$V_{xz} = \frac{GM}{R^3} \sum_{n=0}^{\infty} (n+2) \left(\frac{R}{r} \right)^{n+3} \sum_{m=-n}^{n} v_{nm} \frac{\partial Y_{nm}(\theta, \lambda)}{\partial \theta} \tag{2i}$$

$$V_{yz} = \frac{GM}{R^3} \sum_{n=0}^{\infty} (n+2) \left(\frac{R}{r} \right)^{n+3} \sum_{m=-n}^{n} v_{nm} \frac{\partial Y_{nm}(\theta, \lambda)}{\sin\theta \, \partial \lambda} \, . \tag{2j}$$

Eshagh (2008, 2009b) presented alternative formulae for them which are simpler to use than the original ones.

3.1. DETERMINATION OF GRAVITATION VECTOR FROM GRAVITY GRADIENT TENSOR

As already explained, derivation of the tensor of gravitation is done by taking the derivatives of the vector of gravitation, therefore, this vector should be derived by integrating the tensor of gravitation. However, suitable relations for the integration should be found, which we try to present them in the following subsections.

3.2.HARMONIC RELATIONS

In order to find the harmonic relations between V_{zz} , V_{xz} and V_{yz} and V_z, V_x and V_y, let us, first, write Eqs. (1b) and (2h) in the following forms:

$$V_{z,n} = -\frac{GM}{R^2}(n+1)\left(\frac{R}{r}\right)^{n+2} V_n \tag{3a}$$

$$V_{zz,n} = \frac{GM}{R^3}(n+1)(n+2)\left(\frac{R}{r}\right)^{n+3} V_n \tag{3b}$$

The common harmonic in these relations is V_n. If we derive V_n from Eq. (3b) and insert it back into Eq. (3a) and simplify the result, we have:

$$V_{z,n} = -\frac{r}{n+2} V_{zz,n} . \tag{3c}$$

Similarly, from Eq. (1c) and (2i) we can write:

$$V_{xz,n} = \frac{GM}{R^3}(n+2)\left(\frac{R}{r}\right)^{n+3} V_{\theta,n} \tag{3d}$$

$$\frac{V_{\theta,n}}{r} = \frac{GM}{R^2}\left(\frac{R}{r}\right)^{n+2} V_{\theta,n} \ . \tag{3e}$$

By solving Eq. (3d) for $V_{\theta,n}$ and substitute it into Eq. (3e) and after simplifications we obtain:

$$\frac{V_{\theta,n}}{r} = \frac{r}{n+2} V_{xz,n} \ . \tag{3f}$$

In a very similar manner, we can derive the following relation from Eq. (1d) and Eq. (2j):

$$\frac{V_{\lambda,n}}{r\sin\theta} = \frac{r}{n+2} V_{yz,n} . \tag{3g}$$

As observed, the harmonics of $V_{xz,n}$, $V_{yz,n}$ have the same coefficients for their conversions to $\frac{V_{\theta,n}}{r}$ and $\frac{V_{\lambda,n}}{r\sin\theta}$, respectively. This coefficient is the same for conversion of $V_{zz,n}$ and $V_{z,n}$ but with a negative sign.

On the other hand, it should be stated that since the Earth gravitational potential is harmonic outside the Earth, therefore, one can write:

$$V_{zz} = -\left(V_{xx} + V_{yy}\right) \ . \tag{3h}$$

This means that V_{xx} and V_{yy} can be also used for recovering V_z using Eq. (2h). Therefore, all elements of the gravitational tensor are useful for our purpose except V_{xy} .

3.3. INTEGRAL RELATIONS

From the harmonic relations derived in the previous section, similar integral formulae can be obtained for recovering V_z, V_x and V_y from V_{zz}, V_{xz} and V_{yz}, respectively. Here, the following formula for any function U on the unit sphere σ is introduced. The Laplace harmonic U_n of the function U over this sphere is (Heiskanen and Moritz 1967, p. 34):

$$U_n = \frac{2n+1}{4\pi} \iint_{\sigma} U' P_n(\cos\psi)\, d\sigma \quad , \tag{4a}$$

where $P_n(\cos\psi)$ is the Legendre polynomial of degree n, ψ stands for the geocentric spherical angle between the computation and integration points, $d\sigma$ the surface integration element and the prime over U stands for integration points.

According to Eqs. (3c), (3f) and (3g) the Laplace harmonic of the vector of gravitation is:

$$\mathbf{v}_n = \frac{r}{n+2} \begin{pmatrix} V_{xz,n} & V_{yz,n} & -V_{zz,n} \end{pmatrix}^{\mathrm{T}} , \tag{4b}$$

By taking the summation over n from 0 to ∞ from both sides of (4b) we obtain:

$$\mathbf{v} = \frac{r}{4\pi} \iint_{\sigma} K(\psi) \begin{pmatrix} V'_{xz} & V'_{yz} & -V'_{zz} \end{pmatrix}^{\mathrm{T}} d\sigma \ \text{ where } \quad K(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{n+2} P_n(\cos\psi) . \tag{4c}$$

Eq. (4c) is an integral formula for deriving the vector of gravitation from V_{zz} , V_{xz} and V_{yz} . The kernel of this integral formula is in spectral form which requires the time-consuming generation of the Legendre polynomials, but finding a closed-form formula for this function is possible. To do so, let us write the kernel in the following form:

$$K(\psi) = 2\sum_{n=0}^{\infty} P_n(\cos\psi) - 3\sum_{n=0}^{\infty} \frac{P_n(\cos\psi)}{n+2} \tag{4d}$$

As we already know (Martinec 2003):

$$\sum_{n=0}^{\infty} P_n(\cos\psi) = \frac{1}{2\sin\frac{\psi}{2}} \tag{4e}$$

$$\sum_{n=0}^{\infty}\frac{P_n(\cos\psi)}{n+2}=2\sin\frac{\psi}{2}+\cos\psi\ln\left(4\sin\frac{\psi}{2}\left(\sin\frac{\psi}{2}+1\right)\right) \tag{4f}$$

Finally, the closed-form formula of the kernel function becomes:

$$K(\psi)=\frac{1}{\sin\frac{\psi}{2}}-6\sin\frac{\psi}{2}-3\cos\psi\ln\left(4\sin\frac{\psi}{2}\left(\sin\frac{\psi}{2}+1\right)\right). \tag{4g}$$

The kernel function $K(\psi)$ is singular at $\psi=0$. Lots of studies have been done about geodetic integrals which are singular at the computation point. Schwartz et al. (1990) used the planar approximation of the kernel function as he believed that for a relatively dense data this approximation is good enough for integration. The classical spherical method was presented by Martinec (1998) which removes the data at the computation point, so that the result of integration at that point becomes zero and after that restore it by assuming that the data is constant in a small integration domain around the computation point. Novak et al. (2001) used this idea for precise geoid computation. Hirt et al. (2011) presented the idea of using mean kernel instead of the point kernel at the computation point. In fact, they have developed the idea of de Min (1994) for numerical integration of Stokes's function and generalised it to all geodetic integrals; for more references about the singularity problems the reader is referred to these papers and the references therein. Here, we use a similar idea and divide the integration domain into two parts. The first part is the whole unit sphere except a small cap around the computation point. The size of the cap depends on the resolution of the available data, the denser data the smaller cap. This means that the integration performs for all data points except the computation point. It is assumed that the data, here, the elements of the tensor of gravitation, is constant over σ_0 and we can take them outside the integrals:

$$\mathbf{v}=\frac{r}{4\pi}\iint_{\sigma-\sigma_0}K(\psi)\begin{pmatrix}V'_{xz}\\V'_{yz}\\-V'_{zz}\end{pmatrix}d\sigma+\frac{r}{4\pi}\begin{pmatrix}V'_{xz}\\V'_{yz}\\-V'_{zz}\end{pmatrix}\iint_{\sigma_0}K(\psi)d\sigma \tag{4h}$$

The second integral in the r.h.s of Eq. (4h) is simplified to (cf. Heiskanen and Moritz 1967, P. 262):

$$\frac{r}{2}\begin{pmatrix}V'_{xz}\\V'_{yz}\\-V'_{zz}\end{pmatrix}\int_0^{\psi_0}K(\psi)\sin\psi d\psi=\frac{r}{2}\begin{pmatrix}V'_{xz}\\V'_{yz}\\-V'_{zz}\end{pmatrix}\left(\zeta^2\left(6\ln\left(4\zeta\left(\zeta+1\right)\right)-3\right)\left(1-\zeta^2\right)+2\zeta\left(3\zeta-2\right)-2\right) \tag{4i}$$

where $\zeta=\cos\psi_0$.

The integral formula (4h) means that the singular point is removed from the first integral term in the r.h.s of (4h). One can simply consider a value of 0 for this term at the singular point. The effect of the removed singular point is restored to the result afterwards by the second term in the r.h.s. of (4h).

4. GRAVITY GRADIENTS IN ORBITAL FRAME

The real SGG data are measured in the gradiometric frame and not the LNOF. Here, we reformulate the integral formulae in terms of the gravity gradients in the orbital frame, which is one step closer to the gradiometer frame. This frame is defined as: its *w*-axis is towards the Earth centre and perpendicular to the orbit, *u*-axis is towards the motion of the satellite and *v*-axis is considered in such a way that the frame becomes left handed. If we assume that the orbit of the satellite is circular the satellite track azimuth can be computed by (Vermeer 1990, Petrovskaya and Vershkov 2006):

$$\cos\alpha = \frac{\sin I}{\sin\theta} \tag{6a}$$

where I stands for the orbital inclination. The transformation of V_{xz} and V_{yz} in the LNOF to their equivalents in the orbital frame V_{uw} and V_{vw} is (Petrovskaya and Vershkov 2006):

$$V_{uw} = \cos\alpha V_{xz} + \sin\alpha V_{yz} \tag{6b}$$

$$V_{vw} = -\sin\alpha V_{xz} + \cos\alpha V_{yz} \quad . \tag{6c}$$

By assuming that V_{xz} and V_{yz} are unknown and V_{uw} and V_{vw} are given, Eqs. (6b) and (6c) are considered as a system of equations with the following solution:

$$V_{xz} = \sin\alpha V_{vw} - \cos\alpha V_{uw} \tag{6d}$$

$$V_{yz} = \cos\alpha V_{vw} + \sin\alpha V_{uw} \quad . \tag{6e}$$

Eshagh (2011b) showed that the following relation holds for V_{zz}:

$$V_{zz} = -V_{uu} - V_{vv} \tag{6f}$$

By substituting Eqs. (6d), (6e) and (6f) into Eq. (4c) we find the following integral relation between the gravity gradients in the orbital frame and the vector of gravitation:

$$\mathbf{v} = \frac{r}{4\pi}\iint_{\sigma} K(\psi) \begin{pmatrix} \sin\alpha V_{vw} - \cos\alpha V_{uw} \\ \cos\alpha V_{vw} + \sin\alpha V_{uw} \\ V'_{uu} + V'_{vv} \end{pmatrix} d\sigma \quad . \tag{6g}$$

In order to derive integral formulae of the equations of motion of a satellite in terms of the gravity gradients in the orbital frame it is enough to insert Eq. (6g) into Eq. (1f). Concerning the singularity problem one can simply use the technique presented for Eq. (4h). We do not present this procedure here to shorten the paper.

5. NUMERICAL STUDIES

Our numerical study is divided into two parts. In the first part the behaviour of the kernel function (4g) is presented and discussed and in the second part, the gravity gradients are generated over Fennoscandia and used in the integral formula (4h) for testing the feasibility of the idea in practice.

Figure 1 shows the behaviour of the kernel function (4g) to a geocentric angle of 5°. The kernel is rather well-behaving (Eshagh 2011b) as it has its maximum value around the computation point and decreases fast to zero. As the plot illustrates, the contribution of the far-zone data should not be significant, but some practical tests are required as the contribution of the far-zone data depends on the type of data being integrated as well.

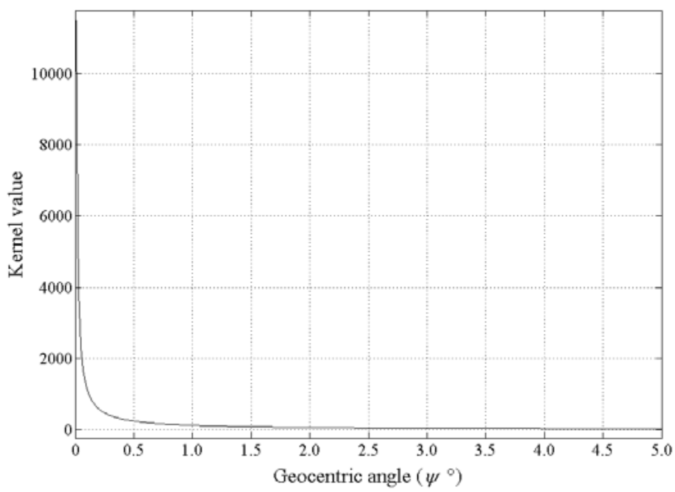

Figure 1. Behaviour of $K(\psi)$

Here, Fennoscandia, limited between latitudes of 50° N to 75° N and longitudes of 0° N to 35° N, is considered as the test area. We use the nonsingular expressions of the gravity gradients presented by Eshagh (2009b) as well as the vectorised models presented for their programming by Eshagh and Abdollahzadeh (2010, 2011). We considered a larger area by 9° than Fennoscandia as required for integration. EGM08 (Pavlis et al. 2008) to degree and order 360 is used as the reference gravity model. It should be stated that in this numerical study, the second-order derivatives of the disturbing potential (T) are considered as the gravity gradients and the first-order ones as the accelerations. This means that the normal gravity field, in this study GRS80 (Moritz 1980, 2000), has been already subtracted from EGM08. This reduction is useful for better visualisation of the gradients and accelerations but in practice the gravity gradients of the true gravity field are measured. Nevertheless, the removed normal gravity field can always be restored to the derived accelerations after the integration process.

Figure 2 shows the maps of the gravity gradients and the accelerations at a constant elevation of 250 km over Fennoscandia, generated directly in the nodes of a $0.5^\circ \times 0.5^\circ$ grid by their spherical harmonic expansions and EGM08 to degree and order 360. Figure 2a is the map of T_{xz} and Figure 2b is that of T_x, and similarly Figures 2c and 2d are those of T_{yz} and T_y and Figures 2e and 2f of T_{zz} and $-T_z$. The maps of the accelerations are very similar to those of the gradients, but they are smoother as expected. The statistics of the gradients and accelerations are presented in Table 1. The maximum and minimum values of T_{zz} are 0.6 E and -0.4 E, respectively, while they are 0.5 E and -0.3 for T_{xz} and 0.3 E and -0.5 E for T_{yz}. T_{zz} has the largest STD of 0.2 E while STD is 0.1 for the rest of them. The maximum and minimum values are 16.0 mGal and -6.6 mGal, respectively, for T_x while they are 0.6 and -24.3 and 1.2 and -33.3 for T_y and T_z. The mean values of T_y and T_z are -13.7 mGal and -15.2 mGal which are both negative and large. This means that the number of positive and negative accelerations over the area is not in balance, or in other words, the number of negative values are larger than that of positive one. This causes that the mean values become negative. However, this is not the case for T_x as its mean value is 2.7 mGal. The largest STD is related to T_z which is equal to 8.8 mGal while they are 5.4 and 4.7 in unit of mGal for T_y and T_x.

Table 1. Statistics of gradients [E] and accelerations [mGal]

	Max	Mean	Min	STD		Max	Mean	Min	STD
T_{xz}	0.5	0.0	-0.3	0.1	T_x	16.0	2.7	-6.6	4.7
T_{yz}	0.3	-0.1	-0.5	0.1	T_y	0.6	-13.7	-24.3	5.4
T_{zz}	0.6	0.1	-0.4	0.2	T_z	1.2	-15.2	-33.3	8.8

Table 2. Statistics of differences between accelerations generated from EGM08 and those from gravity gradients using integral formulae (4h), Columns 3-7 contain the results in the case of the removing of the zero- and first-degrees from the kernel function of integral formulae (4h). Unit: 1 mGal

Cap size		Max	Mean	Min	STD	RMS	Max	Mean	Min	STD	RMS
1°	T_x	3.6	-1.1	-12.6	3.4	3.6	3.3	-1.4	-12.8	3.4	3.7
	T_y	8.0	-2.0	-5.0	2.3	3.1	9.3	-0.7	-3.7	2.4	2.5
	T_z	16.9	9.1	1.2	3.4	9.7	17.2	9.5	1.5	3.4	10.1
3°	T_x	5.1	-0.4	-11.4	3.7	3.8	4.8	-0.7	-11.7	3.7	3.8
	T_y	4.5	-4.4	-7.4	2.3	5.0	6.1	-2.8	-5.7	2.3	3.6
	T_z	13.7	6.1	-0.4	2.9	6.7	14.5	6.8	0.3	2.9	7.4
5°	T_x	5.2	-0.5	-10.9	3.7	3.7	4.9	-0.8	-11.2	3.7	3.8
	T_y	1.9	-6.3	-9.4	2.3	6.7	3.8	-4.4	-7.5	2.2	5.0
	T_z	9.5	3.2	-2.1	2.4	3.4	10.7	4.4	-0.9	2.4	5.0
7°	T_x	6.5	0.0	-10.2	3.9	3.9	6.0	-0.4	-10.6	3.9	3.9
	T_y	0.1	-7.7	-10.9	2.3	8.0	2.2	-5.6	-8.8	2.3	6.0
	T_z	6.4	0.9	-3.6	2.0	2.2	8.0	2.5	-2.0	2.0	3.2

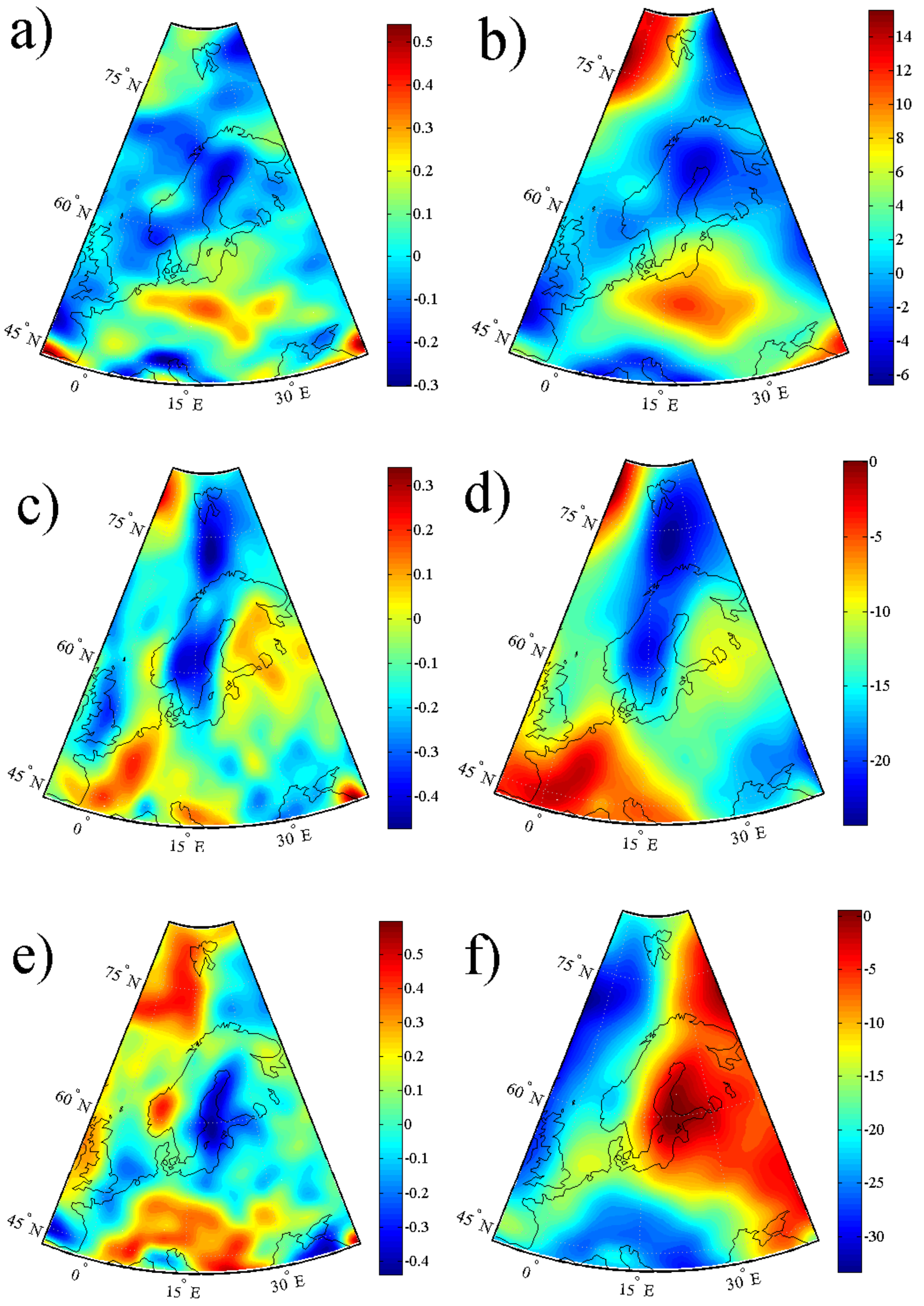

Fig. 2. Maps of gravity gradients and accelerations a) T_{xz} , c) T_{yz} and e) T_{zz} [E], b) T_x , d) T_y and f) -T_z [mGal]

Table 2 presents the statistics of the differences between the accelerations generated from EGM08 and those from the gravity gradients derived by the integral formula (4h). It shows that the STD of the differences is reduced by increasing the cap size of integration for T_z which means that by integrating more data the undulations of T_z over the area is presented better, whilst the STD does not change for T_y. The mean value of differences reduces with increasing the integration cap size for T_z and T_x, whilst it increases for T_y. The possible reason could be due to the fact that the global mean value of T_z is zero, because of exclusion of zero-degree harmonic, and it is difficult to recover that mean value using the locally-distributed data. The reduction of the mean value with the increase of cap size means that by considering larger cap size more data are integrated and the local mean value goes to the global one. A similar issue is seen for T_x but the opposite for T_y which shows that the increase of the cap size just adds systematic effects on the results and shifts them from the true values. The mean values are -2.0 mGal and -7.7 mGal when the cap size is 1° and 7°, respectively, meaning that when cap size is smaller the integration is more successful than the case it is larger. In order to investigate the integration issue we remove the zero- and first-degrees from the kernel function so that it becomes blind to them and not to try to recover the corresponding frequencies from the gradients. In the case of using a cap size of 1° we observe insignificant changes in the STD of the differences comparing to the case that the kernel includes these degrees. However, a significant reduction in the mean value of the differences is seen for T_y. The table shows that the exclusion of these degrees does not significantly influence T_{xz} and the RMS of the differences remains, more or less, the same by increasing the cap size. Again, we observe that the increase of the mean values by increasing the cap size for T_y, but removing these degrees from the kernel has had significant influence in the magnitude of the mean value. The table shows a reduction of 2 mGal in the mean values in a cap size of 7°. However, the mean values for T_z is reduced by the increase of the cap size at a slower rate than the case the kernel includes the zero- and first-degree terms.

Here, we selected Fennoscandia for presenting our results, but we already tested the situation over other areas and we observed similar issues there as well. Generally, the recovery of T_z from T_{zz} is successful when the cap size of integration is large and when the kernel of the integral includes the zero- and first-degree terms. The opposite is the generation of T_y from T_{yz}, and it is successful when the cap size of integration is small and the kernel excludes these terms. Deriving T_x from T_{xz} is successful whether or not the kernel have these terms and large cap sizes lead to successful integrations.

In this study, we considered all of the gravity gradients at the same level whilst in a real satellite gradiometry mission they are measured on the orbit. One can continue them downward/upward (Toth et al. 2004, 2005) to the mean orbital sphere over the study area and perform the integration afterwards. When the accelerations derived from them, they can be continued upward/downward to the original position in the orbit. Today, by the advances in technology these continuation processes are not time consuming, especially using interpolators and available computational software. Another important issue regarding the use of integral formulae is the cap size of integration, which means that a larger area than the desired one is required. In such a case, the orbit improvement over the areas which are close to the satellite polar gaps is not successful due to the lack of gravity gradients over them.

6. CONCLUSIONS

This study showed that amongst the elements tensor of gravitation in the local north-oriented frame, only T_{xy} is not suitable for recovering accelerations of T_z, T_x and T_y. Three integral formulae with the same kernel function were derived for deriving these accelerations from them. The derived closed-form formula of the kernel shows that it is a well-behaving kernel and the contribution of the far-zone data should not be significant, but our numerical studies showed that the type of data being integrated is also important to judge about the influence of the far-zone data. Having a well-behaving kernel is necessary, but not sufficient condition for judging about the significance of the far-zone data. This means that the integration within a certain cap size can be optimal for one gradient but not for another. We found out that the increase of cap size of integration leads to a more successful result for recovering T_z from T_{zz} and T_x from T_{xz}. The standard deviation of the differences between T_z generated directly from a gravity model and the one recovered from T_{zz} decreases by increasing the cap size so as the mean value of them. A similar situation is observed for the computation of T_x from T_{xz}. However, the situation is different for recovering T_y from T_{yz} as the mean value of differences increases by increasing the cap size of integration. Removing the zero- and first-degree terms of the kernel and integrating the gradients leads to no significant change in T_x but destroys the quality of recovering T_z and considerably improves T_y. Consequently, we recommend using the integral formulae with a larger integration domain than 5° for T_x and T_z but small one, around 1°, for T_y.

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