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ON THE NORM CONTINUITY OF THE HK-FOURIER TRANSFORM

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Abstract. In this work we study the Cosine Transform operator and the Sine Transform operator in the setting of Henstock-Kurzweil integration theory. We show that these related transformation operators have a very different behavior in the context of Henstock-Kurzweil functions. In fact, while one of them is a bounded operator, the other one is not. This is a generalization of a result of E. Liflyand in the setting of Lebesgue integration.

1. Introduction

If f belongs to the space of real valued Lebesgue integrable functions, $L^1(\mathbb{R})$, the Fourier transform is defined for every real number s as

$$\mathcal{F}_1(f)(s) := \int_{\mathbb{R}} e^{-isx} f(x)\, dx, \tag{1.1}$$

where the integral is taken in the Lebesgue sense. When f is in $L^2(\mathbb{R})$, the Fourier transform of f can be defined as

$$\mathcal{F}_2(f)(s) := \lim_{n\to\infty} \int_{\mathbb{R}} e^{-isx} f_n(x)\, dx, \tag{1.2}$$

where the limit is taken in the norm topology of $L^2(\mathbb{R})$ and $(f_n)_{n\geq 1}$ is a sequence in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $\|f_n - f\|_2 \to 0$, as $n \to \infty$.

While the operator $\mathcal{F}_1$ has an integral representation on its domain, the operator $\mathcal{F}_2$ shares this property only on a dense subspace of its domain. This happens also for the Fourier transform operator $\mathcal{F}_p$ defined on $L^p(\mathbb{R})$ for $1 < p < 2$. Recently, it was shown in [13] that having an integral representation implies additional properties for the Fourier transform operator. Pointwise continuity and the Riemann-Lebesgue lemma were shown to be valid on a larger subspace of the domain of each of the operators $\mathcal{F}_p$ for $1 < p \leq 2$. The proof relies on the Henstock-Kurzweil integral which has the remarkable property that every Lebesgue integrable function is also integrable in the setting of the Henstock-Kurzweil theory with values of both integrals coinciding.

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If $f \in L^2(\mathbb{R})$, then $e^{-isx}f(x)$ is not necessarily Lebesgue integrable. However, the $\mathcal{HK}$-Fourier transform operator defined by the same formula (1.1) is well defined as a Henstock-Kurzweil integral for each $s \neq 0$ and any function of bounded variation vanishing at infinity [11, 10]. See definition below. We say "$\mathcal{HK}$-Fourier transform" in order to emphasize the use of Henstock-Kurzweil integral [17]. Furthermore, it was shown in [13] that $\mathcal{F}_p$ and the $\mathcal{HK}$-Fourier transform operator coincide in the intersections of their domains.

In this paper we look at norm continuity of the Henstock-Kurzweil Fourier transform operator. There is also a pending question concerning pointwise continuity of a $\mathcal{HK}$-Fourier transform function $\mathcal{F}_{HK}f(s)$ at the origin. We have not answered this question but we show below that there is a type of smoothness even at $s = 0$ in the case of the "real part" of the $\mathcal{HK}$-Fourier transform operator, namely the Cosine transform operator.

2. Henstock-Kurzweil Fourier transform

Definition 2.1. For each real valued function f defined over $\mathbb{R}$, it is said that f is of bounded variation over a closed interval $I \subseteq \mathbb{R}$ if and only if

$$\mathrm{Var}(f, I) := \sup_P \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| < \infty,$$

where the supreme is taken over all (finite) partitions P of I. If $I = \mathbb{R}$, then f is of bounded variation if and only if

$$\mathrm{Var}(f, \mathbb{R}) := \lim_{\substack{t\to+\infty \\ s\to-\infty}} \mathrm{Var}(f, [s, t])$$

exists in $\mathbb{R}$. We will denote the set of bounded variation functions over an interval $I \subseteq \mathbb{R}$ as $BV(I)$. If $I \subseteq \mathbb{R}$ is an unbounded interval, we define $BV_o(I)$ as the subspace of $BV(I)$ consisting of the functions which have limit zero at $\pm\infty$:

$$\lim_{\substack{I\ni t\to\infty \\ I\ni s\to-\infty}} |f(s)| + |f(t)| = 0.$$

In [17] the Henstock-Kurzweil integral was employed to study the Fourier transform. Later, in [11, 10], it was proved that (1.1) makes sense as a Henstock-Kurzweil integral over the space $BV_o(\mathbb{R})$. The norm in $BV(I)$ is taken as

$$\|f\|_{BV(I)} := \lim_{\substack{t\to+\infty \\ s\to-\infty}} \mathrm{Var}(f, [s, t] \cap I) + |f(u)|$$

where $u \in I$ is chosen arbitrarily, but fixed. Since $f(\pm\infty) = 0$, one norm in $BV_o(\mathbb{R})$ can be defined by

$$\|f\|'_{BV(\mathbb{R})} := \mathrm{Var}(f, \mathbb{R}).$$

Note that over $BV_o(\mathbb{R})$ the norms $\|\cdot\|'_{BV(\mathbb{R})}$ and $\|\cdot\|_{BV(\mathbb{R})}$ are equivalent:

$$\|f\|'_{BV(\mathbb{R})} \leq \|f\|_{BV(\mathbb{R})} \leq 2\|f\|'_{BV(\mathbb{R})} \quad \big(\forall\ f \in BV_o(\mathbb{R})\big).$$

Definition 2.2. Let $0 < p < \infty$ and $X \subset \mathbb{R}$. For any Lebesgue measurable function $f : X \to \mathbb{R}$ we define

$$\|f\|_p := \left\{ \int_X |f|^p \, dx \right\}^{1/p}.$$

The real vector space of functions f such that $\|f\|_p < \infty$ is denoted by $\mathcal{L}^p(X)$ and $\mathcal{W}_p$ denotes the subspace of function on which $\|\cdot\|_p$ vanishes.

For real numbers $p \geq 1$, $\|\cdot\|_p$ is a seminorm on $\mathcal{L}^p(X)$ and induces a norm in the quotient space $\mathcal{L}^p(X)/W_p$. We will denote the completion of this space with respect to its norm by $L^p(X)$. Similarly, for $p \geq 1$ we define $\mathcal{L}^p(X,\mathbb{C})$ and $L^p(X,\mathbb{C})$ by considering functions $f : X \to \mathbb{C}$. For $p = \infty$ and $f : X \to \mathbb{R}$, we define $\|f\|_\infty$ to be the essential supremum of $|f|$, and $\mathcal{L}^\infty(\mathbb{R})$ denotes the vector space of all Lebesgue measurable functions f for which $\|f\|_\infty < \infty$.

If $A \subsetneq X$ is a Lebesgue measurable set and m denotes the Lebesgue measure, then given a Lebesgue measurable function f defined on A such that $m(X \setminus A) = 0$, we will denote by the same symbol f the trivial extension of f to a (measurable) function on X. That is, we extend the function as zero on $X \setminus A$. Furthermore, for a function $f \in \mathcal{L}^p(X)$, or $f \in \mathcal{L}^p(X,\mathbb{C})$, we will denote by the same symbol f the (unique) element that defines the function in $L^p(X)$ or in $L^p(X,\mathbb{C})$, respectively.

In order to introduce the definition of the Henstock-Kurzweil integral, we consider the system of extended real numbers $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, and for each interval $[a,b] \subseteq \overline{\mathbb{R}}$ a gauge function, i.e., a map $\gamma : [a,b] \to (0,\infty)$.

Definition 2.3. Given a gauge function $\gamma : [a,b] \to (0,\infty)$, a tagged partition

$$P = \left\{([x_{i-1},x_i],t_i) : t_i \in [x_{i-1},x_i]\right\}_{i=1}^{n}$$

of $[a,b]$ is called γ-fine according to the following cases:
For $a \in \mathbb{R}$ and $b = \infty$,

(1) $a = x_0, b = x_n = t_n = \infty$.
(2) $[x_{i-1},x_i] \subset [t_i - \gamma(t_i), t_i + \gamma(t_i)]$, for all $i = 1,2,\ldots,n-1$.
(3) $[x_{n-1},\infty] \subset [1/\gamma(t_n),\infty]$.

For $a = -\infty$ and $b \in \mathbb{R}$,

(1) $a = x_0 = t_1 = -\infty, b = x_n$.
(2) $[x_{i-1},x_i] \subset [t_i - \gamma(t_i), t_i + \gamma(t_i)]$, for all $i = 2,\ldots,n$.
(3) $[-\infty,x_1] \subset [-\infty,-1/\gamma(t_1)]$.

For $a = -\infty$ and $b = \infty$,

(1) $b = x_n = t_n = \infty, a = x_0 = t_1 = -\infty$.
(2) $[x_{i-1},x_i] \subset [t_i - \gamma(t_i), t_i + \gamma(t_i)]$, for all $i = 2,3,\ldots,n-1$.
(3) $[x_{n-1},\infty] \subset [1/\gamma(t_n),\infty]$ and $[-\infty,x_1] \subset [-\infty,-1/\gamma(t_1)]$.

For $a, b \in \mathbb{R}$,

(1) $[x_{i-1},x_i] \subset [t_i - \gamma(t_i), t_i + \gamma(t_i)]$, for all $i = 1,2,\ldots,n$.

Definition 2.4. A function $f : [a,b] \to \mathbb{R}$ is said to be Henstock-Kurzweil integrable, if and only if there exists $A \in \mathbb{R}$ obeying that for each $\varepsilon > 0$, there is a gauge function $\gamma_\varepsilon : [a,b] \to (0,\infty)$ such that if $P = \left\{([x_{i-1},x_i],t_i) : t_i \in [x_{i-1},x_i]\right\}_{i=1}^{n}$ is a γ_ε-fine partition of $[a,b]$, then

$$\left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - A\right| < \varepsilon. \tag{2.1}$$

The number A is the integral of f over [a,b] and it is denoted by

$$\int_a^b f(x)\,dx = A.$$

Using the convention $0 \cdot (\pm\infty) = 0$, an extra condition for f is $f(\pm\infty) = 0$ [1].

The space of Henstock-Kurzweil integrable functions defined on an interval $I \subseteq \overline{\mathbb{R}}$ will be denoted by $\mathcal{HK}(I)$. Two fundamental theorems over the Henstock-Kurzweil integral are the following, the cases on $[-\infty, \infty]$ and $[-\infty, b]$ are analogous, see [1].

Theorem 2.5 (Multiplier Theorem). *If $f \in \mathcal{HK}([a,\infty])$ and $\varphi \in BV([a,\infty])$, then $f\varphi \in \mathcal{HK}([a,\infty])$ and*

$$\int_a^\infty f\varphi = \lim_{b\to\infty}\left[\varphi(b)\int_a^b f dt - \int_a^b F d\varphi\right]. \tag{2.2}$$

The second integral on the right side of the equation is a Riemann-Stieljes integral and

$$F(x) = \int_a^x f$$

is an integral in the sense of Henstock-Kurzweil.

Theorem 2.6 (Hake's Theorem). *$f \in \mathcal{HK}([a,\infty])$ if and only if, for all c, ε such that $c > a$, $c - a > \varepsilon > 0$, it holds that $f \in \mathcal{HK}([a+\varepsilon, c])$ and*

$$\lim_{\substack{\varepsilon\to 0\\ c\to\infty}} \int_{a+\varepsilon}^c f(t)\,dt$$

exists. This limit is the Henstock-Kurzweil integral of f on $[a,\infty]$.

The space $\mathcal{HK}(I)$ is a seminormed space with the Alexiewicz seminorm, which is defined as

$$\|f\|_{HK(I)} = \sup\left\{\left|\int_c^d f(x)\,dx\right| : [c,d] \subset I\right\}.$$

The quotient space $\mathcal{HK}(I)/\mathcal{W}(I)$ will be denoted by $HK(I)$, where $\mathcal{W}(I)$ is the subspace of $\mathcal{HK}(I)$ on which the Alexiewicz seminorm vanishes [3]. By $HK(\mathbb{R},\mathbb{C})$ will be denoted the space

$$HK(\mathbb{R},\mathbb{C}) := \{f_1 + if_2 \ : \ f_1, f_2 \in HK(\mathbb{R})\}$$

with norm

$$\|f_1 + if_2\|_{HK(\mathbb{R},\mathbb{C})} := \|f_1\|_{HK(\mathbb{R})} + \|f_2\|_{HK(\mathbb{R})}.$$

The completion of the spaces $HK(\mathbb{R})$ and $HK(\mathbb{R},\mathbb{C})$ with respective given norms will be denoted by $\widehat{HK}(\mathbb{R})$ and $\widehat{HK}(\mathbb{R},\mathbb{C})$.

Let us consider $\mathcal{S}(\mathbb{R})$, the Schwartz space of real valued functions defined on $\mathbb{R}$. We know that the Fourier transform operators $\mathcal{F}_1$ and $\mathcal{F}_2$ are well defined on $\mathcal{S}(\mathbb{R})$ and $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and have an integral representation given by (1.1) valid for every $s \in \mathbb{R}$. Because of their density in $L^2(\mathbb{R})$, both spaces are used to extend the Fourier transform over $L^2(\mathbb{R})$, see [4] and [16]. We also know that $HK(\mathbb{R}) \cap BV(\mathbb{R})$ is a dense subspace of $L^2(\mathbb{R})$. An important point is

$$HK(\mathbb{R}) \cap BV(\mathbb{R}) \subsetneq L^1(\mathbb{R}),$$

and for $f \in BV_o(\mathbb{R})$ the integral in (1.1) is well defined as a Henstock-Kurzweil integral for each $s \neq 0$. This means that on a dense subspace of $L^2(\mathbb{R})$, not contained in $L^1(\mathbb{R})$, the Fourier transform operator $\mathcal{F}_2$ is represented by an integral. Furthermore, a similar asseveration holds true for the Fourier transform operator with domain $L^2(\mathbb{R},\mathbb{C})$. See [11] and [13].

For any unbounded subset $X \subset \mathbb{R}$, we denote by $C_\infty(X)$ the space of complex valued continuous functions on X vanishing at infinity [14].

Definition 2.7. The $\mathcal{HK}$–Fourier transform exists for every $s \neq 0$ and is defined by

$$\mathcal{F}_{HK} : BV_o(\mathbb{R}) \to C_\infty(\mathbb{R}\backslash\{0\}),$$
$$\mathcal{F}_{HK}(f)(s) := \int_{-\infty}^{\infty} e^{-isx} f(x)\, dx,$$

where the integral is a Henstock-Kurzweil integral.

$(\mathcal{F}_{HK} f)(s)$ is well defined for $s \neq 0$ and continuous except possibly at zero. See [11] and [17, Example 3(d)].

We define the norm

$$\|f\|_{HK(\mathbb{R})\cap BV(\mathbb{R})} := \|f\|_{HK(\mathbb{R})} + \|f\|_{BV(\mathbb{R})}. \tag{2.3}$$

The next proposition is a corollary of [12, Theorem 1].

Proposition 1. *The $\mathcal{HK}$-Fourier Transform operator with domain $HK(\mathbb{R}) \cap BV(\mathbb{R})$ and codomain $L^2(\mathbb{R}, \mathbb{C})$ is a bounded operator.*

Proof. From the Plancherel Theorem and [12, Theorem 1] we get

$$\|\mathcal{F}_{HK} f\|_2 \; = \; \sqrt{2\pi} \cdot \|f\|_2 \; \leq \; \sqrt{\pi} \cdot \|f\|_{HK(\mathbb{R})\cap BV(\mathbb{R})}.$$ □

The Henstock-Kurzweil Fourier Sine Transform is given by

$$\mathcal{F}^s_{HK}(f)(s) := \int_{-\infty}^{\infty} \sin(sx) f(x)\, dx \quad (s \geq 0). \tag{2.4}$$

Example 1. *Let*

$$f(x) = \begin{cases} -1 & \text{if } x \in [-1, 0), \\ \;\;1 & \text{if } x \in [0, 1], \\ \;\;0 & \text{otherwise.} \end{cases}$$

We have that $f \in HK(\mathbb{R}) \cap BV(\mathbb{R})$ and for $s > 0$, it follows that

$$\begin{aligned} \mathcal{F}^s_{HK}(f)(s) &= \int_0^{\infty} 2\sin(sx) f(x)\, dx \\ &= \int_0^1 2\sin(sx)\, dx \\ &= 2\,\frac{1-\cos(s)}{s}. \end{aligned}$$

Note that this function is not an element of $HK(\mathbb{R})$. Therefore, the image of the space $HK(\mathbb{R}) \cap BV(\mathbb{R})$ under the action of $\mathcal{F}^s_{HK}$ is not contained in $HK(\mathbb{R})$.

The previous example shows that the $\mathcal{HK}$-Fourier Sine transform cannot be defined as a bounded operator from $BV_o(\mathbb{R})$ into $HK(\mathbb{R})$. However, for the $\mathcal{HK}$-Fourier Cosine transform

$$\mathcal{F}^c_{HK}(f)(s) := \int_{-\infty}^{\infty} \cos(sx) f(x)\, dx \tag{2.5}$$

a different situation occurs.

The integrability of the Fourier Cosine and Sine transforms of functions in $BV_o(\mathbb{R})$ is a problem that has been attacked in different ways. The aim is to obtain a wide variety of subspaces of $BV_o(\mathbb{R})$ where the transforms are integrable. In [9], Liflyand studied the integrability of these transforms in the sense of Lebesgue. Among others, he showed that when $f \in BV_o(\mathbb{R})$ is locally absolutely continuous with its derivative in a space $\mathcal{W}$, then the Fourier transform of f belongs to $L^1(\mathbb{R})$. Here $\mathcal{W}$ being the subspace of functions $g \in L^1(\mathbb{R})$ such that

$$\frac{(\mathcal{F}_1 g)(s)}{s} \in L^1(\mathbb{R}).$$

In analogy with the above we take the space

$$\Lambda = \{g \in HK(\mathbb{R}) \cap BV(\mathbb{R}) \mid \frac{(\mathcal{F}_{HK} g)(s)}{s} \in HK(\mathbb{R}) \},$$

which is not empty because $\mathcal{S}(\mathbb{R}) \subset \Lambda$. Let $AC_{loc}(\mathbb{R})$ be the space of locally absolutely continuous functions on $\mathbb{R}$. In this setting, we provide the next proposition.

Proposition 2. *Suppose that $g \in HK(\mathbb{R}) \cap BV(\mathbb{R}) \cap AC_{loc}(\mathbb{R})$. If g is also in Λ, then $\mathcal{F}_{HK} g \in HK(\mathbb{R})$.*

Proof. The proof for the $\mathcal{HK}$-Fourier Sine transform is obtained from the Multiplier Theorem and the equality

$$(\mathcal{F}^s_{HK} g)(s) = \frac{1}{s}\int_{-\infty}^{\infty} \cos(sx) dg(x) = \frac{1}{s}\int_{-\infty}^{\infty} \cos(sx) g'(x) dx.$$

A similar formula for $(\mathcal{F}^c_{HK} g)(s)$ is valid, which proves the proposition. □

This shows that taking into account the Henstock-Kurzweil integration theory, the subspace of $BV_o(\mathbb{R})$ where the $\mathcal{HK}$-Fourier transform of each of its elements is integrable, it is larger than the one considered by Liflyand.

For $f \in BV_o(\mathbb{R})$, the integral

$$\int_0^{\infty} f(x)\, dx,$$

might not exist, so that $(\mathcal{F}^c_{HK} f)(s)$ is not well defined at the point $s = 0$. Without resorting to the condition over the space Λ, we prove in theorem 1 below that $\mathcal{F}^c_{HK}$ can be extended to a bounded linear transformation from $BV_o(\mathbb{R})$ into $HK(\mathbb{R})$. We will need some lemmas.

We set $\mathbb{R}^+ = [0, \infty)$ and

$$\|f\|'_{HK(\mathbb{R}^+)} := \sup_{0 \le b < \infty} \left| \int_0^b f(x)\, dx \right|. \tag{2.6}$$

Lemma 2.8. *Suppose that $f \in HK(\mathbb{R})$ is an even function. Then*

$$\|f\|_{HK(\mathbb{R})} = 2\|f\|'_{HK(\mathbb{R}^+)}.$$

Proof. This follows from elementary properties of the integral $\int_a^b f(x)\, dx$ and consideration of the cases $a \cdot b \ge 0$ or $a \cdot b < 0$. □

Remark 1. The Sine Integral function, see [6] and [15], is given by

$$\mathrm{Si}(v) = \frac{2}{\pi}\int_0^v \frac{\sin y}{y} dy.$$

It has the following properties:

(a) $\mathrm{Si}(0) = 0$,
(b) $\lim_{v\to\infty} \mathrm{Si}(v) = 1$,
(c) $\mathrm{Si}(v) \leq \mathrm{Si}(\pi)$ for all $v \in [0, \infty]$.

Let us consider the set of functions $\Omega := \{ \ h_t : \mathbb{R} \to \mathbb{R} \mid t \in \mathbb{R} \ \}$ with

$$h_t(x) := \begin{cases} x^{-1}\sin(tx) & \text{if } x \neq 0, \\ t & \text{if } x = 0. \end{cases}$$

Note that $h_{-t}(x) = -h_t(x)$. For given $0 \leq u \leq v$ and $0 < t$, we make the change of variable $y = tx$. It follows that

$$\left| \int_u^v \frac{\sin(tx)}{x}\, dx \right| = \left| \int_{tu}^{tv} \frac{\sin(y)}{y}\, dy \right| .$$

Therefore, because h_t is an even function, we have that Ω is a bounded set in $HK(\mathbb{R})$ and

$$\pi\,\mathrm{Si}(\pi) \ := \sup\{ \ \| \ h_t \ \|_{HK(\mathbb{R})} \ : \ h_t \in \Omega \ \}. \tag{2.7}$$

Lemma 2.9. *Let $0 < a < b$ and $f \in BV(\mathbb{R})$. Then*

$$\left| \int_u^v \left[\frac{\sin(bx) - \sin(ax)}{x} \right] f(x)\, dx \right| \leq 4\pi\,\mathrm{Si}(\pi) \ \|f\|_{BV(\mathbb{R})},$$

for each compact interval $[u, v] \subset \mathbb{R}$.

Proof. We have as a consequence of the Multiplier Theorem:

$$\left| \int_u^v \left[\frac{\sin(bx) - \sin(ax)}{x} \right] f(x)\, dx \right| \leq 2\|x^{-1}\left[\sin(bx) - \sin(ax)\right] \|_{HK(\mathbb{R})}\|f\|_{BV(\mathbb{R})}.$$

Therefore, applying (2.7) we obtain the proof. ☐

As we already mentioned, $(\mathcal{F}^c_{HK} f)(s)$ might not be well defined at the point $s = 0$. However, it does have certain regularity even there. This regularity implies the Bounded Linear Transformation theorem for the $\mathcal{HK}$-Fourier Cosine Transform operator.

Theorem 1. *The $\mathcal{HK}$-Fourier Cosine Transform can be extended to a bounded linear transformation from $BV_o(\mathbb{R})$ into $HK(\mathbb{R})$.*

Proof. We know that

$$\int_{-\infty}^{\infty} \cos(sx) f(x)\, dx = \lim_{M\to\infty} \int_{-M}^{M} \cos(sx) f(x)\, dx.$$

Also, for $s > 0$ the map $x \mapsto \cos(sx)$ is in $HK[-M, M]$ for $0 < M < \infty$. Then, by the Multiplier Theorem,

$$\left| \int_{-M}^{M} \cos(sx) f(x)\, dx \right| \leq 2\| \cos(s\cdot)\|_{HK[-M,M]}\| \ f \ \|_{BV[-M,M]}.$$

Where, for any fixed $s > 0$,

$$\begin{aligned}\|\cos(s\cdot)\|_{HK[-M,M]} &= 2\sup_{0\le u\le v\le M}\left|\int_u^v \cos(sx)\,dx\right| \\ &= 2\sup_{0\le u\le v\le M}\left|\int_{su}^{sv}\frac{\cos(y)}{s}\,dy\right| \\ &= 2\sup_{0\le u\le v\le M}\left|\frac{\sin(sv)-\sin(su)}{s}\right| \\ &\le \frac{4}{|s|}.\end{aligned}$$

Then for each compact interval $[a,b]\subset\mathbb{R}\setminus\{0\}$, we have by Lebesgue Dominated Convergence Theorem:

$$\begin{aligned}\int_a^b (\mathcal{F}^c_{HK}f)(s)ds &= \int_a^b \lim_{M\to\infty}\int_{-M}^{M}\cos(sx)f(x)\,dxds \\ &= \lim_{M\to\infty}\int_a^b\int_{-M}^{M}\cos(sx)f(x)\,dxds.\end{aligned}$$

By Fubini's Theorem,

$$\begin{aligned}\left|\lim_{M\to\infty}\int_a^b\int_{-M}^{M}\cos(sx)f(x)\,dxds\right| &= \left|\lim_{M\to\infty}\int_{-M}^{M}\int_a^b\cos(sx)f(x)\,dsdx\right| \\ &= \left|\lim_{M\to\infty}\int_{-M}^{M}\frac{\sin(bx)-\sin(ax)}{x}f(x)\,dx\right|.\end{aligned} \tag{2.8}$$

Therefore, by Lemma 2.9, we get that

$$\left|\int_a^b \mathcal{F}^c_{HK}(f)(s)ds\right| \le 2\left\|x^{-1}[\sin(bx)-\sin(ax)]\right\|_{HK(\mathbb{R})}\ \|f\|_{BV(\mathbb{R})}. \tag{2.9}$$

Moreover, since $\mathcal{F}^c_{HK}(f)$ is an even function,

$$\|\mathcal{F}^c_{HK}(f)\|_{HK(\mathbb{R})} = 2\sup_{0\le b}\left|\int_0^b \mathcal{F}^c_{HK}(f)(s)\,ds\right|. \tag{2.10}$$

(2.9) with $a=0$ and (2.10) yield:

$$\|\mathcal{F}^c_{HK}(f)\|_{HK(\mathbb{R})} \le 4\sup_{0\le b}\ \|\ h_b\ \|_{HK(\mathbb{R})}\ \ \|f\|_{BV(\mathbb{R})} = 4\pi\,\mathrm{Si}(\pi)\ \|f\|_{BV(\mathbb{R})}. \tag{2.11}$$

Therefore, the norm of $\mathcal{F}^c_{HK}(f)$ is finite. To show that $\mathcal{F}^c_{HK}(f)$ belongs to $HK[0,1]$, we prove the existence of the limit

$$\lim_{b\to 0^+}\int_b^1 \mathcal{F}^c_{HK}(f)(s)\,ds. \tag{2.12}$$

Given $\varepsilon > 0$ take $R > 0$ great enough such that

$$\left|\int_{R<|x|}\frac{\sin(bx)-\sin(b'x)}{x}f(x)\,dx\right| \le 4\pi\,\mathrm{Si}(\pi)\ \|f\|_{BV(R<|x|)} \le \frac{\varepsilon}{2}. \tag{2.13}$$

For this R, we have

$$\begin{aligned}\left|\int_{|x|<R} \frac{\sin(bx)-\sin(b'x)}{x} f(x)dx\right| \\ \leq 2\|\frac{\sin(bx)-\sin(b'x)}{x}\|_{HK([-R,R])}\ \|f\|_{BV([-R,R])} \\ \leq C\ (b+b') < \frac{\varepsilon}{2}\end{aligned} \tag{2.14}$$

if $b,\ b' < \delta$ for some positive δ. Here C is a constant depending only on f and R. By using (2.8), and the two previous estimations one proves the existence of (2.12).

Similarly, we prove that $\mathcal{F}^c_{HK} f \in HK(\mathbb{R})$ by showing existence of

$$\lim_{b\to\infty} \int_1^b \mathcal{F}^c_{HK}(f)(s)\, ds. \tag{2.15}$$

We get as before the same estimation in (2.13), for any $b, b' > 0$ and $R > 0$ great enough. Now to show that the integral in (2.14) is small, we estimate the integral

$$\begin{aligned}\int_0^R \frac{\sin(bx)-\sin(b'x)}{x} f(x)dx &= \int_0^{bR} \frac{\sin(y)}{y} f(y/b)dy - \int_0^{b'R} \frac{\sin(y)}{y} f(y/b')dy \\ &= \int_{b'\delta}^{bR} \frac{\sin(y)}{y} f(y/b)dy - \int_{b'\delta}^{b'R} \frac{\sin(y)}{y} f(y/b')dy \\ &\quad + \int_0^{b'\delta} \frac{\sin(y)}{y} \big(f(y/b)-f(y/b')\big)dy.\end{aligned} \tag{2.16}$$

The main argument is to show that each of these integrals can be viewed as convergent alternating series. First we take f continuous and nonincreasing in $[0, R]$ such that $f(R) > 0$. Note that for a given $y > 0$ and $\tilde{y} = y + \pi$, then

$$\frac{\tilde{y}}{y} > 1 \geq \frac{f(\tilde{y}/b)}{f(y/b)} \qquad \Rightarrow \qquad \frac{|\sin(y)|}{y} f(y/b) > \frac{|\sin(\tilde{y}|)}{\tilde{y}} f(\tilde{y}/b).$$

It follows that the first two integrals on the right side of (2.16) tends to zero for $b \geq b' \gg 1$ great enough. The last integral on the right side of (2.16) can be written as

$$\int_0^{b'\delta} \frac{\sin(y)}{y} \big(f(y/b)-f(0)\big)dy - \int_0^{b'\delta} \frac{\sin(y)}{y} \big(f(y/b')-f(0)\big)dy.$$

Note that $|f(y/b)-f(0)|$ and $|f(y/b')-f(0)|$ are arbitrarily small whenever $|y| \leq b'\delta$ and $\delta > 0$ is small enough under the hypothesis that f is continuous. Because f is nonincreasing these two integrals define alternating series with respective initial terms

$$\int_0^{\pi} \frac{\sin(y)}{y} \big(f(y/b)-f(0)\big)dy \quad \text{and} \quad \int_0^{\pi} \frac{\sin(y)}{y} \big(f(y/b')-f(0)\big)dy.$$

This proves that for f nonincreasing and continuous the integral in (2.14) is arbitrarily small for $b \geq b'$ great enough. In fact this proves that for $f \in BV([-R, R]) \cap C([-R, R])$ then

$$\left|\int_{|x|<R} \frac{\sin(bx)-\sin(b'x)}{x} f(x)dx\right| < \varepsilon/2 \quad \text{for } b,\ b' \gg 1. \tag{2.17}$$

This assertion follows because any bounded variation and continuous function is the difference of positive nonincreasing continuous functions. Now we will show that (2.17) remains valid for any function of bounded variation in $[-R, R]$. Towards this goal, we suppose that f is piecewise constant. Due to convergence of the integral

$$\int_0^\infty \frac{\sin(t)}{t} dt,$$

one can prove easily that inequality (2.17) also holds true in this case. Furthermore, if f is the limit in the norm of the space $BV([-R, R])$ of a sequence of functions (f_n) which are piecewise constant, then (2.17) still remains valid. Now we use the Jordan decomposition of a bounded variation function which states that

$$\begin{aligned}
&f = f_1 + f_2,\\
&f_1 \in BV([-R, R]) \cap C([-R, R]),\\
&\|f_2 - P_n\|_{BV} \overset{n\to\infty}{\longrightarrow} 0, \quad (P_n) \text{ a sequence of piecewise constant functions.}
\end{aligned}$$

See [5], [18]. Summing up, the previous arguments applied to f_1 and f_2 give (2.15), which proves the theorem. □

This theorem has its implications to interpolation theory for the classical Fourier Transform on the space $L^p(\mathbb{R})$. From the Haussdorf-Young inequality ([14, Theorem 1.2.1]) and the sharp Hausdorff-Young inequality [2], [8, Theorem 5.7] for $p \in [1, 2]$ we have

$$\|\mathcal{F}^c_{HK}(f)\|_q \leq \gamma_p \|f\|_p,$$

where $1/p + 1/q = 1$ and

$$\gamma_p = \begin{cases} 1 & \text{if } p = 1,\\ (2\pi)^{\frac{1}{q}} (\frac{p-1}{p})^{\frac{p-1}{2p}} p^{\frac{1}{2p}} & \text{if } 1 < p \leq 2. \end{cases}$$

Also, we consider the spaces $\mathcal{L}^p(\mathbb{R}) \cap BV_o(\mathbb{R})$ and $L^q(\mathbb{R}) \cap HK(\mathbb{R})$ with given norms

$$\|f\|_{\mathcal{L}^p \cap BV_o} := \|f\|_p + \|f\|_{BV_o(\mathbb{R})}.$$

Similarly for $\|f\|_{L^q \cap HK}$.

In [7, Theorem 6.3.1] it is proved that the space of bounded variation functions defined on a compact interval $[a, b]$ is a Banach space. With a few changes over that proof it is possible to show that the space $BV_0(\mathbb{R})$ is a Banach space. Therefore, $\mathcal{L}^p(\mathbb{R}) \cap BV_0(\mathbb{R})$ is a Banach space of real valued functions defined on $\mathbb{R}$, whereas elements in the Banach space $L^q(\mathbb{R}) \cap HK(\mathbb{R})$ are classes of functions.

Theorem 1 yields the following result.

Corollary 1. *The map $\mathcal{F}^c_{HK} : \mathcal{L}^p(\mathbb{R}) \cap BV_o(\mathbb{R}) \to L^q(\mathbb{R}) \cap HK(\mathbb{R})$ is continuous for $p \in [1, 2]$ and $1/p + 1/q = 1$.*

Proof. By the definition of the norm on $L^q(\mathbb{R}) \cap HK(\mathbb{R})$ we know

$$\|\mathcal{F}^c_{HK}(f)\|_{L^q \cap HK} = \|\mathcal{F}^c_{HK}(f)\|_q + \|\mathcal{F}^c_{HK}(f)\|_{HK(\mathbb{R})}.$$

Next by the Haussdorf-Young inequality we have

$$\|\mathcal{F}^c_{HK}(f)\|_q \leq \gamma_p \|f\|_p$$

and by Lemma 2.9,

$$\|\mathcal{F}^c_{HK}(f)\|_{HK(\mathbb{R})} \leq 4\pi \operatorname{Si}(\pi) \|f\|_{BV_o(\mathbb{R})}.$$

Therefore,

$$\|\mathcal{F}_{HK}^{c}(f)\|_{L^q \cap HK} \leq \max\left(\gamma_p, 4\pi\,\mathrm{Si}(\pi)\right)\|f\|_{\mathcal{L}^p\cap BV_o} = 4\pi\,\mathrm{Si}(\pi)\|f\|_{\mathcal{L}^p\cap BV_o}. \qquad \square$$

Proposition 3. *The operator* $\digamma : D(\digamma) \to HK(\mathbb{R},\mathbb{C})$, *with domain*

$$D(\digamma) = \{\ f \in L^2(\mathbb{R},\mathbb{C}) \mid \mathcal{F}_2 f \in HK(\mathbb{R},\mathbb{C})\ \} \subset L^2(\mathbb{R},\mathbb{C})$$

defined by

$$(\digamma f)(s) := (\mathcal{F}_2 f)(s)$$

is a densely defined closed operator.

Proof. The density of $D(\digamma)$ in $L^2(\mathbb{R},\mathbb{C})$ follows since: $\mathcal{S}(\mathbb{R},\mathbb{C}) \subset D(\digamma)$, and it is a dense subspace of $L^2(\mathbb{R},\mathbb{C})$. Moreover, $\mathcal{F}_2$ restricted to $\mathcal{S}(\mathbb{R},\mathbb{C})$ is a bijection onto $\mathcal{S}(\mathbb{R},\mathbb{C}) \subset HK(\mathbb{R},\mathbb{C})$. In order to prove that $\digamma$ is a closed operator, we take a sequence (f_n) in $D(\digamma)$ such that

$$f_n \to f \text{ in } L^2\text{-norm} \quad \text{and} \quad \digamma f_n \to \Upsilon \text{ in } HK\text{-norm}.$$

Both together must imply

$$f \in D(\digamma) \quad \text{and} \quad \digamma f = \Upsilon.$$

Note that Υ might belong to the completion of $HK(\mathbb{R},\mathbb{C})$. Since $\mathcal{F}_2$ is an unitary operator on $L^2(\mathbb{R},\mathbb{C})$, one has $\mathcal{F}_2 f_n \in L^2([s,t],\mathbb{C}) \cap L^1([s,t],\mathbb{C})$ for every $s,\ t \in \mathbb{R}$. Therefore,

$$\int_s^t \Upsilon := \lim_{n\to\infty}\int_s^t \digamma f_n = \lim_{n\to\infty}\int_s^t \mathcal{F}_2 f_n = \int_s^t \mathcal{F}_2 f.$$

One can use the Cauchy-Bunyakovsky-Schwarz inequality to prove that the last equality holds true for every $[s,\ t] \subset \mathbb{R}$. This shows that $\mathcal{F}_2 f = \Upsilon \in HK(\mathbb{R},\mathbb{C})$, proving that $f \in D(\digamma)$. $\square$

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