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Stability of a Quadratic Functional Equation

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Abstract

This paper deals with the Ulam-Hyers stability of a quadratic functional equation

$$q\left(x-\frac{y+z}{2}\right)=\frac{1}{2}\left(q(x-z)+q(x-y)\right)-\frac{1}{4}q(z-y)$$

using direct and fixed point methods in fuzzy normed space.

Keywords and phrase: : Fuzzy normed space, Quadratic functional equation, Generalized Hyers-Ulam-Rassias stability, Fixed point method

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1. INTRODUCTION

One of the most interesting questions in the theory of functional analysis concerning the Ulam stability problem of functional equations is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation?

In 1940, Ulam [29] posed the famous Ulam stability problem. In 1941, Hyers [12] solved the well-known Ulam stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. He gave rise to the stability theory for functional equations. In 1950, Aoki [2] generalized Hyers' theorem for approximately additive functions. In 1978, Rassias [25] provided a generalized version of Hyers for approximately linear mappings. In addition, Rassias [24, 27] generalized the Hyers stability result by introducing two weaker conditions controlled by a product of different powers of norms and a mixed product-sum of powers of norms, respectively.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is said to be **quadratic functional equation** because the quadratic function $f(x) = ax^2$ is a solution of the functional equation (1.1).

This paper established the Ulam-Hyers stability of a quadratic functional equation

$$q\left(x - \frac{y+z}{2}\right) = \frac{1}{2}\left(q(x-z) + q(x-y)\right) - \frac{1}{4}q(z-y) \tag{1.2}$$

using the direct and fixed point methods in fuzzy normed space.

2. PRELIMINARIES

A.K. Katsaras [17] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [11, 19, 35]. In particular, T. Bag and S.K. Samanta [6], following S.C. Cheng and J.N. Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [18]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [7].

We use the definition of fuzzy normed spaces given in [6] and [22, 23, 24, 25].

Definition 2.1. Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

$(F1)$ $N(x,c) = 0$ for $c \leq 0$;

$(F2)$ $x = 0$ if and only if $N(x,c) = 1$ for all $c > 0$;

$(F3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;

$(F4)$ $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;

$(F5)$ $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{t\to\infty} N(x, t) = 1$;
$(F6)$ for $x \neq 0, N(x, \cdot)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(X, t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t'.

Example 2.2. Let $(X, || \cdot ||)$ be a normed linear space. Then

$$N(x,t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 2.3. Let (X, N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim\limits_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim\limits_{n\to\infty} x_n = x$.

Definition 2.4. A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

Definition 2.5. Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Definition 2.6. A mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ covering to x_0 in X, the sequence $f\{x_n\}$ converges to $f(x_0)$. If f is continuous at each point of $x_0 \in X$ then f is said to be continuous on X.

The stability of various functional equations in fuzzy normed spaces were investigated in [3, 4, 15, 21, 22, 23, 24, 25, 29, 32].

Hereafter throughout this paper, assume that $X, (Z, N')$ and (Y, N') are linear space, fuzzy normed space and fuzzy Banach space, respectively. We use the following abbreviation for a given function $f : X \to Y$ by

$$D_q(x, y, z) = q\left(x - \frac{y + z}{2}\right) - \frac{1}{2}\left(q(x - z) + q(x - y)\right) + \frac{1}{4}q(z - y)$$

for all $x, y, z \in X$.

3. FUZZY STABILITY RESULTS: DIRECT METHOD

Now, we investigate the generalized Ulam-Hyers stability of the functional equation (1.2) in fuzzy normed space using direct method.

Theorem 3.1. *Let* $\varsigma \in \{-1, 1\}$ *be fixed and let* $\vartheta : X^3 \to Z$ *be a mapping with* $0 < \left(\frac{d}{4}\right)^{\varsigma} < 1$

$$N\left(\vartheta\left(2^{\varsigma}x, 2^{\varsigma}y, 2^{\varsigma}z\right), r\right) \geq N\left(d^{\varsigma}\vartheta\left(x, y, z\right), r\right) \tag{3.1}$$

for all $x, y, z \in X$ *and all* $d > 0$ *and*

$$\lim_{n\to\infty} N'\left(\vartheta\left(2^{\varsigma n}x, 2^{\varsigma n}y, 2^{\varsigma n}z\right), 4^{\varsigma n}r\right) = 1 \tag{3.2}$$

for all $x, y, z \in X$ *and all* $r > 0$. *Suppose that a mapping* $q : X \to Y$ *satisfies the inequality*

$$N\left(D_q\left(x, y, z\right), r\right) \geq N'\left(\vartheta\left(x, y, z\right), r\right) \tag{3.3}$$

for all $x, y, z \in X$ *and all* $r > 0$. *Then the limit*

$$Q(z) = N - \lim_{n\to\infty} \frac{q\left(2^{n\varsigma}z\right)}{4^{n\varsigma}} \tag{3.4}$$

exists for all $z \in X$ *and all* $r > 0$ *and the mapping* $Q : X \to Y$ *is a unique quadratic mapping satisfying (1.2) and*

$$N\left(q(z) - Q(z), r\right) \geq N'\left(\vartheta(0, -z, z), r|4 - d|\right) \tag{3.5}$$

for all $z \in X$ *and all* $r > 0$.

Proof. First assume $\varsigma = 1$. Replacing (x, y, z) by $(0, -z, z)$ in (3.3), we get

$$N\left(q(2z) - 4q(z), r\right) \geq N'\left(\vartheta(0, -z, z), r\right) \tag{3.6}$$

for all $z \in X$ and all $r > 0$. Replacing z by $2^n z$ in (3.6), we obtain

$$N\left(\frac{q(2^{n+1}z)}{2^2} - q(2^n z), \frac{r}{2^2}\right) \geq N'\left(\vartheta(0, -2^n z, 2^n z, r\right) \tag{3.7}$$

for all $z \in X$ and all $r > 0$. Using (3.1), $(F3)$ in (3.7), we arrive

$$N\left(\frac{q(2^{n+1}z)}{2^2} - q(2^n z), \frac{r}{4}\right) \geq N'\left(\vartheta(0, -z, z, \frac{r}{d^n}\right) \tag{3.8}$$

for all $z \in X$ and all $r > 0$. It is easy to verify from (3.8), that

$$N\left(\frac{q(2^{n+1}z)}{2^{2(n+1)}} - \frac{q(2^n z)}{2^{2n}}, \frac{r}{2^2 \cdot 2^{2n}}\right) \geq N'\left(\vartheta(0,-z,z), \frac{r}{d^n}\right) \tag{3.9}$$

holds for all $z \in X$ and all $r > 0$. Replacing r by $d^n r$ in (3.9), we get

$$N\left(\frac{q(2^{n+1}z)}{2^{2(n+1)}} - \frac{q(2^n z)}{2^{2n}}, \frac{d^n\, r}{2^{2(n+1)}}\right) \geq N'\left(\vartheta(0,-z,z), r\right) \tag{3.10}$$

for all $z \in X$ and all $r > 0$. It is easy to see that

$$\frac{q(2^n z)}{2^{2n}} - q(z) = \sum_{i=0}^{n-1}\left[\frac{q(2^{i+1}z)}{2^{2(i+1)}} - \frac{q(2^i x)}{2^{2i}}\right] \tag{3.11}$$

for all $z \in X$. From equations (3.10) and (3.11), we have

$$\begin{aligned} N\left(\frac{q(2^n z)}{2^{2n}} - q(z), \sum_{i=0}^{n-1}\frac{d^i\, r}{2^{2(i+1)}}\right) & \\ &\geq min \bigcup_{i=0}^{n-1}\left\{\frac{q(2^{i+1}z)}{2^{2(i+1)}} - \frac{q(2^i z)}{2^{2i}}, \frac{d^i\, r}{2^{2(i+1)}}\right\} \\ &\geq min \bigcup_{i=0}^{n-1}\left\{N'\left(\vartheta(0,-z,z), r\right)\right\} \\ &\geq N'\left(\vartheta(0,-z,z), r\right) \end{aligned} \tag{3.12}$$

for all $z \in X$ and all $r > 0$. Replacing z by $2^m z$ in (3.12) and using (3.1), $(F3)$, we obtain

$$N\left(\frac{q(2^{n+m}z)}{2^{2(n+m)}} - \frac{q(2^m z)}{2^{2m}}, \sum_{i=0}^{n-1}\frac{d^i\, r}{2^{2(i+m)}}\right) \geq N'\left(\vartheta(0,-z,z), \frac{r}{d^m}\right) \tag{3.13}$$

for all $z \in X$ and all $r > 0$ and all $m, n \geq 0$. Replacing r by $d^m r$ in (3.13), we get

$$N\left(\frac{q(2^{n+m}z)}{2^{2(n+m)}} - \frac{q(2^m z)}{2^{2m}}, \sum_{i=m}^{m+n-1}\frac{d^i\, r}{2^{2i}}\right) \geq N'\left(\vartheta(0,-z,z), r\right) \tag{3.14}$$

for all $z \in X$ and all $r > 0$ and all $m, n \geq 0$. Using $(F3)$ in (3.14), we obtain

$$N\left(\frac{q(2^{n+m}z)}{2^{2(n+m)}} - \frac{q(2^m z)}{2^{2m}}, r\right) \geq N'\left(\vartheta(0,-z,z), \frac{r}{\sum_{i=m}^{m+n-1}\frac{d^i}{2^{2(i+1)}}}\right) \tag{3.15}$$

for all $z \in X$ and all $r > 0$ and all $m, n \geq 0$. Since $0 < d < 2^2$ and $\sum_{i=0}^{n}\left(\frac{d}{2^2}\right)^i < \infty$, the cauchy criterion for convergence and $(F5)$ implies that $\left\{\frac{q(2^n z)}{2^{2n}}\right\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $Q(z) \in Y$. So one can define the mapping $Q : X \to Y$ by

$$Q(z) = N - \lim_{n\to\infty} \frac{q(2^n z)}{2^{2n}}$$

for all $z \in X$. Letting $m = 0$ in (3.15), we get

$$N\left(\frac{q(2^n z)}{2^{2n}} - q(z), r\right) \geq N'\left(\vartheta(0, -z, z), \frac{r}{\sum_{i=0}^{n-1} \frac{d^i}{2^{2i}}}\right) \tag{3.16}$$

for all $z \in X$ and all $r > 0$. Letting $n \to \infty$ in (3.16) and using $(F6)$, we arrive

$$N\left(q(z) - Q(z), r\right) \geq N'\left(\vartheta(0, -z, z), r(2^2 - d)\right)$$

for all $z \in X$ and all $r > 0$. To prove Q satisfies the functional equation (1.2), replacing (x, y, z) by $(2^n x, 2^n y, 2^n z)$ in (3.3), respectively , we obtain

$$N\left(\frac{1}{2^n} D_q(2^n x, 2^n y, 2^n z), r\right) \geq N'\left(\vartheta(2^n x, 2^n y, 2^n z), 2^{2n} r\right) \tag{3.17}$$

for all $r > 0$ and all $x, y, z \in X$. Now,

$$\begin{aligned}
&N\left(Q\left(x - \frac{y+z}{2}\right) - \frac{1}{2}\left(Q(x-z) + Q(x-y)\right) + \frac{1}{4}Q(z-y), r\right) \\
&\quad \geq min\left\{N\left(Q\left(x - \frac{y+z}{2}\right) - \frac{1}{2^{2n}} q\left(2^n\left(x - \frac{y+z}{2}\right)\right), \frac{r}{5}\right),\right. \\
&\qquad N\left(-\frac{1}{2}Q(x-z) + \frac{1}{2^{2n}2} q\left(2^n\left(x-z\right)\right), \frac{r}{5}\right), \\
&\qquad N\left(-\frac{1}{2}Q(x-y) + \frac{1}{2^{2n}2} q\left(2^n\left(x-y\right)\right), \frac{r}{5}\right), \\
&\qquad N\left(\frac{1}{4}Q(z-y) - \frac{1}{2^{2n}4} q\left(2^n\left(z-y\right)\right), \frac{r}{5}\right), \\
&\qquad N\left(\frac{1}{2^{2n}} q\left(2^n\left(x - \frac{y+z}{2}\right)\right) - \frac{1}{2^{2n}2} q\left(2^n\left(x-z\right)\right)\right. \\
&\qquad\qquad \left.\left. - \frac{1}{2^{2n}2} q\left(2^n\left(x-y\right)\right) + \frac{1}{2^{2n}4} q\left(2^n\left(z-y\right)\right), \frac{r}{5}\right)\right\}
\end{aligned} \tag{3.18}$$

for all $x, y, z \in X$ and all $r > 0$. Using (3.17) and $(F5)$ in (3.18), we arrive

$$\begin{aligned}
&N\left(Q\left(x - \frac{y+z}{2}\right) - \frac{1}{2}\left(Q(x-z) + Q(x-y)\right) + \frac{1}{4}Q(z-y), r\right) \\
&\quad \geq min\left\{1, 1, 1, 1, N'\left(\vartheta(2^n x, 2^n y, 2^n z), 2^{2n} r\right)\right\} \\
&\quad \geq N'\left(\vartheta(2^n x, 2^n y, 2^n z), 2^{2n} r\right)
\end{aligned} \tag{3.19}$$

for all $x, y, z \in X$ and all $r > 0$. Letting $n \to \infty$ in (3.19) and using (3.2), we see that

$$N\left(Q\left(x - \frac{y+z}{2}\right) - \frac{1}{2}\left(Q(x-z) + Q(x-y)\right) + \frac{1}{4}Q(z-y), r\right) = 1 \tag{3.20}$$

for all $x, y, z \in X$ and all $r > 0$. Using $(F2)$ in the above inequality gives

$$Q\left(x - \frac{y+z}{2}\right) = \frac{1}{2}\left(Q(x-z) + Q(x-y)\right) - \frac{1}{4}Q(z-y)$$

for all $x, y, z \in X$. Hence, Q satisfies the quadratic functional equation (1.2). In order to prove $Q(z)$ is unique, let $Q'(z)$ be another quadratic functional equation satisfying (1.2) and (3.5). Hence,

$$\begin{aligned} N(Q(z) - Q'(z), r) &= N\left(\frac{Q(2^n z)}{2^{2n}} - \frac{Q'(2^n z)}{2^{2n}}, r\right) \\ &\geq min\Big\{N\left(\frac{Q(2^n z)}{2^{2n}} - \frac{q(2^n z)}{2^{2n}}, \frac{r}{2}\right), N\left(\frac{q(2^n z)}{2^{2n}} - \frac{Q'(2^n z)}{2^{2n}}, \frac{r}{2}\right)\Big\} \\ &\geq N'\left(\vartheta(0, -2^n z, 2^n z), \frac{r\, 2^{2n}(2^2 - d)}{2}\right) \\ &\geq N'\left(\vartheta(0, -z, z), \frac{r\, 2^{2n}(2^2 - d)}{2d^n}\right) \end{aligned}$$

for all $z \in X$ and all $r > 0$. Since

$$\lim_{n\to\infty} \frac{r\, 2^{2n}(2^2 - d)}{2d^n} = \infty,$$

we obtain

$$\lim_{n\to\infty} N'\left(\vartheta(0, -z, z), \frac{r\, 2^{2n}(2^2 - d)}{2d^n}\right) = 1.$$

Thus

$$N(Q(z) - Q'(z), r) = 1$$

for all $z \in X$ and all $r > 0$, hence $Q(z) = Q'(z)$. Therefore $Q(z)$ is unique.

For $\varsigma = -1$, we can prove the result by a similar method. This completes the proof of the theorem. $\square$

From Theorem 3.1, we obtain the following corollaries concerning the Ulam-Hyers stability for the functional equation (1.2).

Corollary 3.2. *Suppose that a mapping $q : X \to Y$ satisfies the inequality*

$$N\left(D_q(x, y, z), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left\{||x||^s + ||y||^s + ||z||^s\right\}, r\right), & s \neq 2; \\ N'\left(\epsilon\left\{||x||^s||y||^s||z||^s + (||x||^{3s} + ||y||^{3s} + ||z||^{3s})\right\}, r\right), & s \neq \frac{2}{3}; \end{cases} \tag{3.21}$$

for all $x, y, z \in X$ and all $r > 0$, where ϵ, s are constants. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N\left(q(z) - Q(z), r\right) \geq \begin{cases} N'\left(\epsilon, 3r\right), \\ N'\left(2\epsilon||z||^{s}, r|2^{2} - 2^{s}|\right), \\ N'\left(2\epsilon||z||^{3s}, r|2^{2} - 2^{3s}|\right) \end{cases} \tag{3.22}$$

for all $z \in X$ and all $r > 0$.

4. FUZZY STABILITY RESULTS: FIXED POINT METHOD

In this section, the authors present the generalized Ulam-Hyers stability of the functional equation (1.2) in fuzzy normed space using fixed point method.

Now we will recall the fundamental results in fixed point theory.

Theorem 4.1. *(Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx, Ty) \leq Ld(x, y)$ *for some (Lipschitz constant) $L < 1$. Then,*
(i) The mapping T has one and only fixed point $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* is globally attractive, that is

(A2) $lim_{n\to\infty} T^n x = x^*$, *for any starting point $x \in X$;*
(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L}\ d(T^n x, T^{n+1}x), \forall\ n \geq 0, \forall\ x \in X;$

(A4) $d(x, x^*) \leq \frac{1}{1-L}\ d(x, x^*), \forall\ x \in X.$

Theorem 4.2. *[20](The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*
$(B1)$ $\quad d(T^n x, T^{n+1}x) = \infty \quad \forall\ n \geq 0,$
or
$(B2)$ *there exists a natural number n_0 such that:*
(i) *$d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;*
(ii)*The sequence $(T^n x)$ is convergent to a fixed point y^* of T*
(iii) *y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;*
(iv) *$d(y^*, y) \leq \frac{1}{1-L}\ d(y, Ty)$ for all $y \in Y$.*

In order to prove the stability results we define the following:
δ_i is a constant such that

$$\delta_i = \begin{cases} 2 & if \quad i = 1, \\ \frac{1}{2} & if \quad i = 0 \end{cases}$$

and Ω is the set such that

$$\Omega = \{g \mid g : X \to Y, g(0) = 0\}.$$

Theorem 4.3. *Let $q : X \to Y$ be a mapping for which there exist a mapping $\vartheta : X^3 \to Z$ with the condition*

$$\lim_{n\to\infty} N'\left(\vartheta\left(\mu_i^n x, \mu_i^n y, \mu_i^n z\right), \mu_i^{2n} r\right) = 1 \tag{4.1}$$

forall $x, y, z \in X, r > 0$ and satisfying the functional inequality

$$N\left(D_q(x, y, z), r\right) \geq N'\left(\vartheta(x, y, z), r\right) \tag{4.2}$$

forall $x, y, z \in X, r > 0$. If there exists $L = L(i) > 0$ such that the function

$$z \to \gamma(z) = \vartheta\left(0, -\frac{z}{2}, \frac{z}{2}\right),$$

has the property

$$N'\left(\frac{L\gamma(\mu_i z)}{\mu_i^2}, r\right) = N'\left(\gamma(z), r\right), \ \forall\, z \in X, r > 0. \tag{4.3}$$

Then there exists unique quadratic mapping $Q : X \to Y$ satisfying the functional equation (1.2) and

$$N\left(q(z) - Q(z), r\right) \geq N'\left(\frac{L^{1-i}}{1-L}\gamma(z), r\right) \ \forall\, z \in X, r > 0. \tag{4.4}$$

Proof. Let d be a general metric on Ω, such that

$$d(g, h) = inf\left\{K \in (0, \infty) | N\left(g(z) - h(z), r\right) \geq N'\left(\varsigma(z), Kr\right), z \in X, r > 0\right\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tg(z) = \frac{1}{\delta_i^2} g(\delta_i z)$, for all $z \in X$. For $g, h \in \Omega$, we have $d(g, h) \leq K$

$$\begin{aligned}
\Rightarrow \quad & N\left(g(z) - h(z), r\right) \geq N'\left(K\gamma(z), r\right), \ \ \forall z \in X, r > 0 \\
\Rightarrow \quad & N\left(\frac{g(\delta_i z)}{\delta_i^2} - \frac{h(\delta_i z)}{\delta_i^2}, r\right) \geq N'\left(K\gamma(\delta_i z), \delta_i^2 r\right), \ \ \forall z \in X, r > 0 \\
\Rightarrow \quad & N\left(Tg(z) - Th(z), r\right) \geq N'\left(KL\gamma(z), r\right), \ \ \forall z \in X, r > 0 \\
\Rightarrow \quad & d\left(Tg(z), Th(z)\right) \leq KL, \ \ \forall z \in X \\
\Rightarrow \quad & d\left(Tg, Th\right) \leq Ld(g, h)
\end{aligned} \tag{4.5}$$

for all $g, h \in \Omega$. There fore T is strictly contractive mapping on Ω with Lipschitz constant L. Replacing (x, y, z) by $(0, -z, z)$ in (4.2), we get

$$N\left(q(2z) - 4q(z), r\right) \geq N'\left(\vartheta(0, -z, z), r\right) \tag{4.6}$$

for all $z \in X, r > 0$. Using (F3) in (4.6), we arrive

$$N\left(\frac{q(2z)}{2^2} - q(z), r\right) \geq N'\left(\vartheta(0, -z, z), 2^2 r\right) \tag{4.7}$$

for all $z \in X, r > 0$ with the help of (4.3) when $i = 0$, it follows from (4.7), we get

$$\begin{aligned} \Rightarrow \quad & N\left(\frac{q(2z)}{2^2} - q(z), r\right) \geq N'\left(L\gamma(z), r\right) \\ \Rightarrow \quad & d(Tq, q) \leq L = L^1 = L^{1-i}. \end{aligned} \tag{4.8}$$

Replacing z by $\frac{z}{2}$ in (4.6), we obtain

$$N\left(q(z) - 2^2 q\left(\frac{z}{2}\right), r\right) \geq N'\left(\vartheta\left(0, -\frac{z}{2}, \frac{z}{2}\right), r\right) \tag{4.9}$$

for all $z \in X, r > 0$ with the help of (4.3) when $i = 1$, it follows from (4.9), we get

$$\begin{aligned} \Rightarrow \quad & N\left(q(z) - 2^2 q\left(\frac{z}{2}\right), r\right) \geq N'\left(\gamma(z), r\right) \\ \Rightarrow \quad & d(q, Tq) \leq 1 = L^0 = L^{1-i}. \end{aligned} \tag{4.10}$$

Then from (4.8) and (4.10), we can conclude

$$d(q, Tq) \leq L^{1-i} < \infty.$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point Q of T in Ω such that

$$Q(z) = N - \lim_{k\to\infty} \frac{q(2^k z)}{2^{2k}}, \qquad \forall z \in X, r > 0. \tag{4.11}$$

Replacing (x, y, z) by $(\delta_i x, \delta_i y, \delta_i z)$ in (4.2), we arrive

$$N\left(\frac{1}{\delta_i^{2n}} D_q(\delta_i x, \delta_i y, \delta_i z), r\right) \geq N'\left(\vartheta(\delta_i x, \delta_i y, \delta_i z), \delta_i^{2n} r\right) \tag{4.12}$$

for all $r > 0$ and all $x, y, z \in X$

By proceeding the same procedure as in the Theorem 3.1, we can prove the mapping, $Q : X \to Y$ satisfies the functional equation (1.2).

By fixed point alternative, since Q is unique fixed point of T in the set

$$\Delta = \{q \in \Omega | d(q, Q) < \infty\},$$

therefore Q is a uniqe function such that

$$N\left(q(z) - Q(z), r\right) \geq N'\left(K\gamma(z), r\right) \tag{4.13}$$

for all $z \in X, r > 0$ and $K > 0$. Again using the fixed point alternative, we obtain

$$\begin{aligned}
& d(q, Q) \leq \frac{1}{1-L} d(q, Tq) \\
\Rightarrow \quad & d(q, Q) \leq \frac{L^{1-i}}{1-L} \\
\Rightarrow \quad & N\left(q(z) - Q(z), r\right) \geq N'\left(\frac{L^{1-i}}{1-L}\gamma(z), r\right)
\end{aligned} \tag{4.14}$$

for all $z \in X$ and $r > 0$. This completes the proof of the theorem. □

From Theorem 4.3, we obtain the following corollary concerning the stability for the functional equation (1.2).

Corollary 4.4. *Suppose that a mapping $q : X \to Y$ satisfies the inequality*

$$\begin{aligned}
& N\left(D_q(x, y, z), r\right) \\
& \quad \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon \left\{||x||^s + ||y||^s + ||z||^s\right\}, r\right), & s \neq 2; \\ N'\left(\epsilon \left\{||x||^s ||y||^s ||z||^s + (||x||^{3s} + ||y||^{3s} + ||z||^{3s})\right\}, r\right), & s \neq \frac{2}{3}; \end{cases}
\end{aligned} \tag{4.15}$$

for all $x, y, z \in X$ and all $r > 0$, where ϵ, s are constants. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N\left(q(z) - Q(z), r\right) \geq \begin{cases} N'\left(\epsilon, 3r\right), \\ N'\left(2\epsilon ||z||^s, |2^2 - 2^s| r\right), \\ N'\left(2\epsilon ||z||^{3s}, |2^2 - 2^{3s}| r\right) \end{cases} \tag{4.16}$$

for all $z \in X$ and all $r > 0$.

Proof. Setting

$$\vartheta(x, y, z) = \begin{cases} \epsilon, \\ \epsilon\left(||x||^s + ||y||^s + ||z||^s\right), \\ \epsilon\left\{||x||^s ||y||^s ||z||^s + ||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\} \end{cases}$$

for all $x, y, z \in X$. Then,

$$N'\left(\vartheta(\delta_i^n x, \delta_i^n y, \delta_i^n z), \delta_i^{2n} r\right)$$
$$= \begin{cases} N'\left(\frac{\epsilon}{\delta_i^{2n}}, r\right), \\ N'\left(\frac{\epsilon}{\delta_i^{2n}}\left(||\delta_i^n x||^s + ||\delta_i^n y||^s + ||\delta_i^n z||^s\right), r\right), \\ N'\left(\frac{\epsilon}{\delta_i^{2n}}\left\{||\delta_i^n x||^s ||\delta_i^n y||^s + ||\delta_i^n z||^s + ||\mu_i^n x||^{3s} + ||\mu_i^n y||^{3s} + ||\mu_i^n z||^{3s}\right\}, r\right), \end{cases}$$
$$= \begin{cases} \to 1 \text{ as } n \to \infty, \\ \to 1 \text{ as } n \to \infty, \\ \to 1 \text{ as } n \to \infty. \end{cases}$$

Thus, (4.1) is holds. But we have $\gamma(z) = \vartheta\left(0, \frac{z}{2}, \frac{z}{2}\right)$ has the property

$$N'\left(L\frac{1}{\delta_i^2}\gamma(\delta_i z), r\right) \geq N'\left(\gamma(z), r\right) \ \forall\, z \in X, r > 0.$$

Hence

$$N'\left(\gamma(z), r\right) = N'\left(\vartheta\left(0, \frac{z}{2}, \frac{z}{2}\right), r\right) = \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon 2^{1-s}||z||^s, r\right), \\ N'\left(\epsilon 2^{1-3s}||z||^{3s}, r\right). \end{cases}$$

Now,

$$N'\left(\frac{1}{\delta_i^2}\gamma(\delta_i z), r\right) = \begin{cases} N'\left(\frac{\epsilon}{\delta_i^2}, r\right), \\ N'\left(\frac{\epsilon}{\delta_i^2}\left(\frac{2}{2^s}\right)||\delta_i z||^s, r\right), \\ N'\left(\frac{\epsilon}{\delta_i^2}\left(\frac{2}{2^{3s}}\right)||\delta_i z||^{3s}, r\right), \end{cases} = \begin{cases} N'\left(\delta_i^{-2}\gamma(x), r\right), \\ N'\left(\delta_i^{s-2}\gamma(z), r\right), \\ N'\left(\delta_i^{3s-2}\gamma(z), r\right) \end{cases}$$

for all $z \in X$ and all $r > 0$. Hence the inequality (4.3) holds either, $L = 2^{s-2}$ for $s < 2$ if $i = 0$ and $L = 2^{2-s}$ for $s > 0$ if $i = 1$.

Case 1: $L = 2^{s-2}$ for $s < 2$ if $i = 0$

$$N\left(q(z) - Q(z), r\right) \geq N'\left(\epsilon\left(\frac{2^{s-2}}{1 - 2^{s-2}}\right)\gamma(z), r\right) = N'\left(2\epsilon||z||^s, \frac{r}{2^2 - 2^s}\right).$$

Case 2: $L = 2^{2-s}$ for $s > 2$ if $i = 1$

$$N\left(q(z) - Q(z), r\right) \geq N'\left(\epsilon\left(\frac{1}{1 - 2^{2-s}}\right)\gamma(z), r\right) = N'\left(2\epsilon||z||^s, \frac{r}{2^s - 2^2}\right).$$

Similarly, the inequality (4.3) holds either, $L = 2^{-2}$ if $i = 0$ and $L = 2^2$ if $i = 1$ for condition (i) and also the inequality (4.3) holds either $L = 2^{3s-2}$ for $s < \frac{2}{3}$ if $i = 0$ and $L = 2^{2-3s}$ for $s > \frac{2}{3}$ if $i = 1$ for condition (iii). Hence the proof is complete. $\square$

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