

APARTMENT CLASSES OF INTEGRAL SYMPLECTIC GROUPS

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ABSTRACT. In this note we present an alternative proof of a theorem of Gunnells, which states that the Steinberg module of $\mathrm{Sp}_{2n}(\mathbb{Q})$ is a cyclic $\mathrm{Sp}_{2n}(\mathbb{Z})$ -module, generated by integral apartment classes.

1. INTRODUCTION

Consider the vector space \mathbb{Q}^{2n} equipped with the standard symplectic form ω , i.e. the skew-symmetric, non-degenerate bilinear form which on the standard basis $\{\vec{e}_1, \vec{f}_1, \dots, \vec{e}_n, \vec{f}_n\}$ evaluates as

$$\begin{aligned}\omega(\vec{e}_i, \vec{e}_j) &= \omega(\vec{f}_i, \vec{f}_j) = 0 \text{ for } i, j \in \{1, \dots, n\}, \\ \omega(\vec{e}_i, \vec{f}_j) &= 0 \text{ for } i \neq j \in \{1, \dots, n\}, \\ \omega(\vec{e}_i, \vec{f}_i) &= -\omega(\vec{f}_i, \vec{e}_i) = 1 \text{ for } i \in \{1, \dots, n\}.\end{aligned}$$

The symplectic group $\mathrm{Sp}_{2n}(\mathbb{Q})$ is the group of \mathbb{Q} -linear automorphisms of \mathbb{Q}^{2n} that preserve ω . Restricting to \mathbb{Z}^{2n} , we obtain the symplectic module $(\mathbb{Z}^{2n}, \omega)$ and the integral symplectic group $\mathrm{Sp}_{2n}(\mathbb{Z})$. We may use the standard basis to identify $\mathrm{Sp}_{2n}(\mathbb{Q})$ and $\mathrm{Sp}_{2n}(\mathbb{Z})$ with $2n \times 2n$ -matrix groups.

This work concerns *the symplectic Steinberg module* St_n^ω , an important $\mathrm{Sp}_{2n}(\mathbb{Z})$ -representation (the rational dualizing module of $\mathrm{Sp}_{2n}(\mathbb{Z})$ [BS73]) that can be constructed as follows: A subspace $V \subseteq \mathbb{Q}^{2n}$ is called isotropic if $\omega|_V$ is zero. The Tits building of type C_n over \mathbb{Q} is the poset T_n^ω of all nontrivial proper isotropic subspaces V of $(\mathbb{Q}^{2n}, \omega)$ ordered by the inclusion of subspaces. This poset admits a natural $\mathrm{Sp}_{2n}(\mathbb{Z})$ -action because symplectic matrices map isotropic subspaces to isotropic subspaces. A theorem of Solomon–Tits [Sol69] implies that $T_n^\omega \simeq \bigvee S^{n-1}$ has the homotopy type of a bouquet of $(n-1)$ -spheres. The Steinberg module of $\mathrm{Sp}_{2n}(\mathbb{Q})$ is the $\mathrm{Sp}_{2n}(\mathbb{Z})$ -module that arises as the reduced top-degree homology of the symplectic Tits building,

$$\mathrm{St}_n^\omega := \tilde{H}_{n-1}(T_n^\omega; \mathbb{Z}).$$

The goal of this work is to give an alternative proof of a theorem of Gunnells [Gun00, 4.11. Theorem], which shows that St_n^ω is a cyclic $\mathrm{Sp}_{2n}(\mathbb{Z})$ -module and describes an explicit set of generators for St_n^ω .

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Theorem 1.1 (Gunnells). *There exists an $\mathrm{Sp}_{2n}(\mathbb{Z})$ -equivariant surjection*

$$[-] : \mathbb{Z}[\mathrm{Sp}_{2n}(\mathbb{Z})] \rightarrow \mathrm{St}_n^\omega$$

for all $n \geq 1$ such that the set of generators of St_n^ω given by $\{[M] : M \in \mathrm{Sp}_{2n}(\mathbb{Z})\}$ is equal to the set of integral apartment classes (see [Definition 4.2](#)).

[Theorem 1.1](#) is the special case $\mathcal{O} = \mathbb{Z}$ of Gunnells' result [[Gun00](#), 4.11 Theorem], which allows any Euclidean ring of integers \mathcal{O} of a number field K/\mathbb{Q} .

Our primary interest in St_n^ω stems from applications to the study of the rational cohomology of $\mathrm{Sp}_{2n}(\mathbb{Z})$: Using ideas contained in [[BS73](#); [CFP19](#)], Gunnells' generating set for St_n^ω can be used to prove that the rational cohomology of $\mathrm{Sp}_{2n}(\mathbb{Z})$ vanishes in its virtual cohomological dimension $\mathrm{vcd}_n = n^2$,

$$H^{n^2}(\mathrm{Sp}_{2n}(\mathbb{Z}); \mathbb{Q}) = 0$$

if $n \geq 1$ (see Brück–Patz–Sroka [[Sro21](#), Chapter 5] or, for a more general version, Brück–Santos Rego–Sroka [[BSS22](#)]). In a sequel that is joint work with Peter Patzt [[BPS23](#)], we build on the techniques developed in this note to determine the relations between all integral apartment classes, i.e. between the generators of St_n^ω appearing in [Theorem 1.1](#). This yields a presentation of the symplectic Steinberg module St_n^ω for $n \geq 1$. We use this presentation to prove that the rational cohomology of $\mathrm{Sp}_{2n}(\mathbb{Z})$ vanishes one degree below its virtual cohomological dimension,

$$H^{n^2-1}(\mathrm{Sp}_{2n}(\mathbb{Z}); \mathbb{Q}) = 0$$

if $n \geq 2$ (see Brück–Patz–Sroka [[BPS23](#)] for $n \geq 3$; and Igusa [[Igu62](#)], Hain [[Hai02](#)] and Hulek–Tommasi [[HT12](#)] for $n \leq 4$).

Our motivation for writing the present note is threefold: Gunnells' algorithmic strategy of proof for [Theorem 1.1](#) in [[Gun00](#)] was inspired by work of Ash–Rudolph for special linear groups [[AR79](#)]. Our first aim here is to implement an idea of Putman [[Put21](#)] and present a new, more geometric argument in the style of recent work on special linear groups [[CFP19](#); [CP17](#)]. In fact, our strategy to study integral symplectic groups relies on and uses results obtained in [[CFP19](#); [CP17](#)]. This idea is also prominent in the sequel [[BPS23](#)], which relies on results contained in [[CFP19](#); [CP17](#); [Brü+22](#)]. Our second aim is to develop and showcase some of the techniques used in [[BPS23](#)] at a simpler example. Our approach to [Theorem 1.1](#) requires to show that a certain simplicial complex \mathcal{IA}_n is highly connected. The difficult parts of this connectivity calculation have been carried out by Putman in [[Put09](#)]. However, Putman informed us that [[Put09](#)] contains small gaps. Our third aim in this article is to explain how these can be filled.

Outline. [Section 2](#) introduces a new poset, the restricted Tits building $T_n^\omega(W)$, and studies its connectivity properties. This poset is a variant of the symplectic Tits building T_n^ω defined above. In [Section 3](#), we define the complex $\mathcal{I}_n^{\sigma,\delta}$, which Putman introduced in [[Put09](#)], and the complex \mathcal{IA}_n , which we use in our proof of Gunnells' theorem. We show that Putman's connectivity results for $\mathcal{I}_n^{\sigma,\delta}$ imply that \mathcal{IA}_n is highly connected as well. Furthermore, we explain how one can combine connectivity results obtained by Church–Putman [[CP17](#)] with our results about the restricted Tits building to give an alternative proof of the first steps of Putman's connectivity calculation for $\mathcal{I}_n^{\sigma,\delta}$. This fixes the gaps in Putman's argument (see [Remark 3.6](#) and [Remark 3.7](#)). In [Section 4](#), we define the integral apartment class map appearing in Gunnells' theorem. [Section 5](#) contains the new proof of [Theorem 1.1](#).

Acknowledgments. This article is based on Chapter 5 of Sroka's PhD Thesis [[Sro21](#)] written at the University of Copenhagen. Essentially, all results presented here are contained in [[Sro21](#)]. It is a pleasure to thank Andrew Putman for posing the question that led to this work [[Put20](#)] and for sharing his idea for a new proof

of Gunnells' theorem [Put21]. We thank Peter Patzt for helpful discussions and comments, and the department of the University of Copenhagen for the excellent working conditions. RJS would like to thank his PhD advisor Nathalie Wahl for many fruitful and clarifying conversations about [Sro21, Chapter 5]. We thank the anonymous referee for their careful reading and helpful suggestions.

2. THE RESTRICTED TITS BUILDING

This section introduces and studies a new poset, the restricted Tits building $T_n^\omega(W)$. In the next section, we use this poset to fix gaps in an argument contained in [Put09]. The results for $T_n^\omega(W)$ presented here are also used in the sequel to this work [BPS23]. We assume that $n \geq 1$ throughout this section.

Recall that the symplectic Tits building T_n^ω is the poset of nontrivial isotropic subspaces of \mathbb{Q}^{2n} ordered by inclusion of subspaces. The order complex of this poset, which we also denote by $T_n^\omega(\mathbb{Q})$, is an ordered simplicial complex with k -simplices given by the following set of flags

$$\{V_0 \subsetneq \cdots \subsetneq V_k : 0 \neq V_i \subsetneq \mathbb{Q}^{2n} \text{ isotropic subspace}\}.$$

This complex has dimension $n - 1$ and the i -th face of a k -simplex is obtained by omitting the i -th isotropic subspace V_i of the flag.

Definition 2.1. We define

$$W = \left\langle \vec{e}_1, \vec{f}_1, \dots, \vec{e}_{n-1}, \vec{f}_{n-1}, \vec{e}_n \right\rangle_{\mathbb{Q}} \subseteq \mathbb{Q}^{2n}$$

to be the subspace of \mathbb{Q}^{2n} spanned by all standard basis vectors apart from f_n .

We denote by $T_n^\omega(W)$ the subposet of the symplectic Tits building T_n^ω consisting of isotropic subspaces $V \in T_n^\omega$ that are contained in W .

Recall from [Qui78, p.116–117] that a poset P is Cohen–Macaulay of dimension d if the associated order complex P is d -spherical, i.e. d -dimensional and homotopy equivalent to a wedge of d -spheres, and the link of each k -simplex in P is $(d - k - 1)$ -spherical. The main result of this section is the following theorem.

Theorem 2.2. $T_n^\omega(W)$ is a contractible Cohen–Macaulay poset of dimension $n - 1$.

For any subspace $H \subseteq \mathbb{Q}^{2n}$, let

$$H^\perp = \{\vec{v} \in \mathbb{Q}^{2n} : \omega(\vec{v}, \vec{h}) = 0 \text{ for all } \vec{h} \in H\}$$

denote the symplectic complement of H in \mathbb{Q}^{2n} . The following two observations are the main ingredients of the proof of Theorem 2.2.

Lemma 2.3. If $V \in T_n^\omega(W)$, then $\langle \vec{e}_n \rangle_{\mathbb{Q}} + V \in T_n^\omega(W)$.

Proof. Observe that $W \subseteq \langle \vec{e}_n \rangle_{\mathbb{Q}}^\perp$, hence $V \subseteq \langle \vec{e}_n \rangle_{\mathbb{Q}}^\perp$. It follows that $\langle \vec{e}_n \rangle_{\mathbb{Q}} + V \subseteq W$ is isotropic. Hence, $\langle \vec{e}_n \rangle_{\mathbb{Q}} + V \in T_n^\omega(W)$. \square

Given a poset P , the upper link $P_{>x}$ and the lower link $P_{<x}$ of an element $x \in P$ are the subposets of P containing all elements $y \in P$ that satisfy $y > x$ and $y < x$ respectively. The interval (x, z) between two elements $x \leq z \in P$ is the subposet of P consisting of all $y \in P$ that satisfy $x < y < z$. The next observation concerns certain upper links in $T_n^\omega(W)$ and is similar to Lemma 4.2 of Sprehn–Wahl [SW20].

Lemma 2.4. If $\langle \vec{e}_n \rangle_{\mathbb{Q}} \subseteq Q \in T_n^\omega(W)$, then

$$T_n^\omega(W)_{>Q} \cong T^\omega(\langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp)_{>Q \cap \langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp}.$$

For the case $Q = \langle \vec{e}_n \rangle_{\mathbb{Q}}$, we set $T^\omega(\langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp)_{>Q \cap \langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp} := T^\omega(\langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp)$.

Proof. Note that any $V \in T_n^\omega(W)_{>Q}$ admits a direct sum decomposition $V = \langle \vec{e}_n \rangle_{\mathbb{Q}} \oplus (V \cap \langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp)$. The poset maps

$$T_n^\omega(W)_{>Q} \rightarrow T^\omega(\langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp)_{>Q \cap \langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp} : V \mapsto V \cap \langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp$$

and

$$T^\omega(\langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp)_{>Q \cap \langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp} \rightarrow T_n^\omega(W)_{>Q} : V \mapsto \langle \vec{e}_n \rangle_{\mathbb{Q}} \oplus V$$

are therefore inverses of each other. \square

Lemma 2.5. $T_n^\omega(W)$ is contractible.

Proof. The poset map

$$f : T_n^\omega(W) \rightarrow T_n^\omega(W) : V \mapsto \langle \vec{e}_n \rangle_{\mathbb{Q}} + V$$

is well-defined by Lemma 2.3 and satisfies $V \subseteq f(V)$. It follows from [Qui78, §1.5] that $T_n^\omega(W)$ is homotopy equivalent to $\text{im}(f)$ and that $\text{im}(f)$ is contractible using the cone point $\langle \vec{e}_n \rangle_{\mathbb{Q}}$. \square

Lemma 2.6. $T_n^\omega(W)$ is a Cohen–Macaulay poset of dimension $n - 1$.

Proof. This proof uses the characterization of Cohen–Macaulay posets given in [Qui78, Proposition 8.6.]: Let $Q' \subseteq Q \in T_n^\omega(W)$. We need to see that the lower link $T_n^\omega(W)_{<Q}$ is $(\dim Q - 2)$ -spherical, the interval (Q', Q) is $(\dim Q - \dim Q' - 2)$ -spherical and the upper link $T_n^\omega(W)_{>Q}$ is $(n - \dim Q - 1)$ -spherical.

Connectivity of the lower link and the interval: Note that $T_n^\omega(W)_{<Q}$ is the poset of nontrivial proper subspaces of Q . This is exactly a Tits building $T(Q)$ of type $A_{\dim(Q)-1}$, which is known to be a Cohen–Macaulay poset of dimension $(\dim Q - 2)$ (see [Sol69] and [Bro89, IV.5 Remark 2]). Therefore, $T_n^\omega(W)_{<Q}$ is $(\dim Q - 2)$ -spherical and (Q', Q) is $((\dim Q - 2) - (\dim Q' - 1) - 1) = (\dim Q - \dim Q' - 2)$ -spherical.

Connectivity of the upper link: We consider two cases.

1. Assume that $\langle \vec{e}_n \rangle_{\mathbb{Q}} \not\subseteq Q$. Then $\langle \vec{e}_n \rangle_{\mathbb{Q}} + Q \in T_n^\omega(W)_{>Q}$ is a cone point of the image of the monotone poset map $f : V \mapsto \langle \vec{e}_n \rangle_{\mathbb{Q}} + V$ on $T_n^\omega(W)_{>Q}$. It follows from [Qui78, §1.5] that $T_n^\omega(W)_{>Q}$ is contractible and in particular $(n - \dim Q - 1)$ -spherical.
2. Assume that $\langle \vec{e}_n \rangle_{\mathbb{Q}} \subseteq Q$. Then Lemma 2.4 yields the identification

$$T_n^\omega(W)_{>Q} \cong T^\omega(\langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp)_{>Q \cap \langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp}.$$

But $T^\omega(\langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp)$ is Cohen–Macaulay of dimension $(n - 2)$ (see [Sol69] and [Bro89, IV.5 Remark 2]). Therefore, $T_n^\omega(W)_{>Q \cap \langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp}$ is spherical of dimension

$$(n - 2) - (\dim(Q \cap \langle \vec{e}_n, \vec{f}_n \rangle_{\mathbb{Q}}^\perp) - 1) - 1 = n - \dim Q - 1. \quad \square$$

3. PUTMAN'S CONNECTIVITY RESULTS REVISITED AND THE COMPLEX \mathcal{IA}_n^m

This section introduces the simplicial complex \mathcal{IA}_n that plays a key role in our proof of Gunnells' Theorem 1.1. The simplicial complex \mathcal{IA}_n contains a subcomplex $\mathcal{I}_n^{\sigma, \delta} \hookrightarrow \mathcal{IA}_n$, which has been studied by Putman in [Put09]. The goal of this section is twofold. After defining simplicial complexes related to $\mathcal{I}_n^{\sigma, \delta}$ and \mathcal{IA}_n , we outline the strategy that Putman used in [Put09] to prove that $\mathcal{I}_n^{\sigma, \delta}$ is highly connected. Our first goal is to give alternative proofs for the first steps of Putman's argument, filling gaps in [Put09] (see Remark 3.6 and Remark 3.7). For this, we combine the results for the restricted Tits building $T_n^\omega(W)$ obtained in the last section with connectivity calculations of Church–Putman [CP17]. Our second goal is to show

that the complex \mathcal{IA}_n can be constructed from $\mathcal{I}_n^{\sigma,\delta}$ by attaching simplices along highly connected links. As a consequence, Putman's connectivity result for $\mathcal{I}_n^{\sigma,\delta}$ implies that \mathcal{IA}_n is highly connected as well. The high-connectivity of \mathcal{IA}_n and the link structure of \mathcal{IA}_n are exactly the properties that make the induction argument in the proof of Gunnells' theorem in [Section 5](#) work.

Notation. Throughout this subsection, we assume that $m, n \in \mathbb{N}$ denote natural numbers satisfying $m + n \geq 1$. We consider $\mathbb{Z}^{2(m+n)} \subset \mathbb{Q}^{2(m+n)}$ equipped with the standard symplectic form ω and denote its standard symplectic basis by $\{\vec{e}_1, \vec{f}_1, \dots, \vec{e}_{m+n}, \vec{f}_{m+n}\}$. Given a primitive vector $\vec{v} \in \mathbb{Z}^{2(m+n)}$, we write $v = \langle \vec{v} \rangle_{\mathbb{Z}}$ for the rank-1 summand it spans in $\mathbb{Z}^{2(m+n)}$. Similarly, given a rank-1 summand v of $\mathbb{Z}^{2(m+n)}$, we write \vec{v} for some choice of primitive vector in v . Note that there are exactly two such choices, the other one being $-\vec{v}$.

Definition 3.1. Let \mathcal{V}_{m+n} be the set

$$\mathcal{V}_{m+n} := \{v \subseteq \mathbb{Z}^{2(m+n)} : v \text{ is a rank-1 summand of } \mathbb{Z}^{2(m+n)}\}.$$

A subset $\Delta = \{v_0, \dots, v_k\} \subset \mathcal{V}_{m+n}$ of $k + 1$ lines in $\mathbb{Z}^{2(m+n)}$ is called

- a *standard* simplex if $\langle \vec{v}_i : 0 \leq i \leq k \rangle_{\mathbb{Z}}$ is an isotropic rank- $(k + 1)$ summand of $\mathbb{Z}^{2(m+n)}$;
- a *2-additive* simplex if $\vec{v}_0 = \pm \vec{v}_1 \pm \vec{v}_2$ and $\Delta \setminus \{v_0\}$ is a standard $(k - 1)$ -simplex;
- a σ simplex if $\omega(\vec{v}_k, \vec{v}_{k-1}) = \pm 1$, $\omega(\vec{v}_k, \vec{v}_i) = 0$ for $0 \leq i \leq k - 2$ and $\Delta \setminus \{v_k\}$ is a standard $(k - 1)$ -simplex;
- a *mixed* simplex if $\Delta \setminus \{v_0\}$ is a σ simplex, $\Delta \setminus \{v_k\}$ is a 2-additive simplex and $\omega(v_0, v_k) = 0$.

Definition 3.2. The simplicial complexes $\mathcal{I}_{m+n}, \mathcal{I}_{m+n}^{\delta}, \mathcal{I}_{m+n}^{\sigma,\delta}$ and \mathcal{IA}_{m+n} have \mathcal{V}_{m+n} as their vertex set and

- the simplices of \mathcal{I}_{m+n} are all standard;
- the simplices of $\mathcal{I}_{m+n}^{\delta}$ are all either standard or 2-additive;
- the simplices of $\mathcal{I}_{m+n}^{\sigma,\delta}$ are all either standard, 2-additive or σ ;
- the simplices of \mathcal{IA}_{m+n} are all either standard, 2-additive, σ or mixed.

Definition 3.3. Let X_{m+n} denote the complex $\mathcal{I}_{m+n}, \mathcal{I}_{m+n}^{\delta}, \mathcal{I}_{m+n}^{\sigma,\delta}$ or \mathcal{IA}_{m+n} . We define X_n^m to be the full subcomplex of $\text{Link}_{X_{m+n}}(\{e_1, \dots, e_m\})$ on the vertex set of lines $v \in \text{Link}_{X_{m+n}}(\{e_1, \dots, e_m\})$ satisfying the following:

1. $\vec{v} \notin \langle \vec{e}_1, \dots, \vec{e}_m \rangle_{\mathbb{Z}}$.
2. For $1 \leq i \leq m$, we have $\omega(\vec{e}_i, \vec{v}) = 0$, i.e. there are no σ edges between v and the vertices of $\{e_1, \dots, e_m\}$.

Definition 3.4. Let $W = \langle \vec{e}_1, \vec{f}_1, \dots, \vec{e}_{m+n-1}, \vec{f}_{m+n-1}, \vec{e}_{m+n} \rangle_{\mathbb{Q}} \subseteq \mathbb{Q}^{2(m+n)}$ be as in [Section 2](#). We write $\mathcal{I}_n^m(W)$ and $\mathcal{I}_n^{\delta,m}(W)$ for the full subcomplexes of \mathcal{I}_n^m and $\mathcal{I}_n^{\delta,m}$, respectively, on the set of vertices contained in W .

The complexes $\mathcal{I}_n^m(W), \mathcal{I}_n^m, \mathcal{I}_n^{\delta,m}, \mathcal{I}_n^{\delta,m}(W)$ and $\mathcal{I}_n^{\sigma,\delta,m}$ have been defined and studied by Putman [[Put09](#), Section 6]. The next theorem lists the five steps of Putman's proof that $\mathcal{I}_n^{\sigma,\delta,m}$ is spherical.

Theorem 3.5 (Putman, [[Put09](#), Proposition 6.13 and 6.11]). *Let $m \geq 0$ and $n \geq 1$, then:*

1. $\mathcal{I}_n^m(W)$ is $(n - 2)$ -connected.
2. \mathcal{I}_n^m is $(n - 2)$ -connected.
3. $\mathcal{I}_n^{\delta,m}(W)$ is $(n - 1)$ -connected.
4. $\mathcal{I}_n^{\delta,m} \hookrightarrow \mathcal{I}_n^{\sigma,\delta,m}$ is the zero map on π_k for $0 \leq k \leq n - 1$.

5. $\mathcal{I}_n^{\sigma, \delta, m}$ is n -dimensional and $(n-1)$ -connected.¹

Putman made us aware that the proof of [Put09, Proposition 6.13] contains some small gaps. They occur in the proof of [Item 1](#) and [Item 3](#) of [Theorem 3.5](#), and are explained in the next remark.

Remark 3.6. [Put09, Proofs of Proposition 6.13.1 and 6.13.3] assert – without proof – that certain isomorphisms of simplicial complexes exist, but it seems unclear why this would be the case. Using the notation of [Put09], the claims are as follows.

1. [Put09, Page 632. Proof of Proposition 6.13, first conclusion, third paragraph.] asserts that $\text{link}_{\mathcal{L}^{\Delta^k, W}(g)}(\phi(\Delta')) \cong \mathcal{L}^{k+m', W}(g)$.
2. [Put09, Page 634. Proof of Proposition 6.13, third conclusion. Step 1.] asserts that $\text{link}_{\mathcal{L}_\delta^{\Delta^k, W}(g)}(\phi(t)) \cong \mathcal{L}^{k+(m'-1), W}(g)$. [Put09, Page 634. Proof of Proposition 6.13, third conclusion. Step 2, Case 2.] asserts an isomorphism $\mathcal{L}^{\Delta^k \cup \{v\}, W}(g) \cong \mathcal{L}^{k+1, W}(g)$. [Put09, Page 634. Proof of Proposition 6.13, third conclusion. Step 3.] asserts that $\mathcal{L}_\delta^{\Delta^k \cup \{\phi(x)\}, W}(g) \cong \mathcal{L}_\delta^{k+1, W}(g)$.

We use the first claim to illustrate why these assertions are difficult to verify. Translated to the notation of the present note, it is as follows:

1. Let $0 \leq k \leq n-2$ and Δ a k -simplex in $\mathcal{I}_n^m(W)$. Then

$$\text{Link}_{\mathcal{I}_n^m(W)}(\Delta) \cong \mathcal{I}_{n-k-1}^{m+k+1}(W).$$

Let $m = 0$, $n \geq 2$ and consider the 0-simplex $\Delta = e_n \in \mathcal{I}_n(W)$. Then the claim asserts that

$$\text{Link}_{\mathcal{I}_n(W)}(e_n) \cong \text{Link}_{\mathcal{I}_n(W)}(e_1).$$

For \mathcal{I}_n , i.e. if the vertex set has *not* been restricted using W , it is indeed easy to see that there is an isomorphism $\text{Link}_{\mathcal{I}_n}(e_n) \cong \text{Link}_{\mathcal{I}_n}(e_1)$. However, while there is an equality $\text{Link}_{\mathcal{I}_n(W)}(e_n) = \text{Link}_{\mathcal{I}_n}(e_n)$, the inclusion $\text{Link}_{\mathcal{I}_n(W)}(e_1) \subsetneq \text{Link}_{\mathcal{I}_n}(e_1)$ is strict because e.g. the vertex f_n is not contained in $\text{Link}_{\mathcal{I}_n(W)}(e_1)$. Therefore the first assertion about links in $\mathcal{I}_n(W)$ has the following consequence:

$$\text{Link}_{\mathcal{I}_n}(e_1) \cong \text{Link}_{\mathcal{I}_n}(e_n) = \text{Link}_{\mathcal{I}_n(W)}(e_n) \cong \text{Link}_{\mathcal{I}_n(W)}(e_1) \subsetneq \text{Link}_{\mathcal{I}_n}(e_1),$$

i.e. it identifies $\text{Link}_{\mathcal{I}_n}(e_1)$ with a proper subcomplex of itself. It is not clear why such an identification would exist. Similar issues arise if one tries to verify the other claims.

Remark 3.7. The proofs of [Put09, Proposition 6.13.2 and 6.13.4], i.e. [Item 2](#) and [Item 4](#) of [Theorem 3.5](#), use identifications which are analogous to the assertions described in [Remark 3.6](#). The important difference is that in [Put09, Proposition 6.13.2 and 6.13.4] these are identifications of complexes whose vertex set has *not* been restricted using $W_{m+n} = \langle \vec{e}_1, \vec{f}_1, \dots, \vec{e}_{m+n-1}, \vec{f}_{m+n-1}, \vec{e}_{m+n} \rangle_{\mathbb{Q}} \subseteq \mathbb{Q}^{2(m+n)}$. Hence, these identifications can easily be verified. The proofs of [Put09, Proposition 6.13.2 and 6.13.4] furthermore rely on [Put09, Proposition 6.13.1 and 6.13.3]. Since we provide alternative arguments for these two statements, the proofs of [Put09, Proposition 6.13.2 and 6.13.4] are unaffected by the discussion in [Remark 3.6](#). Similarly, the proof of [Put09, Proposition 6.11], i.e. [Item 5](#) in [Theorem 3.5](#), is not affected.

We now start working towards an alternative proof of [Put09, 6.13.1 and 6.13.3], i.e. [Item 1](#) and [Item 3](#) of [Theorem 3.5](#), using the restricted Tits building introduced in the previous subsection and connectivity calculations obtained by Church–Putman [CP17]. This fixes the gaps in [Put09] outlined in [Remark 3.6](#).

¹[Put09, Theorem 6.11] states this result only for $m = 0$. Its proof uses [Item 2](#) and [Item 4](#) for $m = 0$. For $m > 0$, the same argument works if one uses [Item 2](#) and [Item 4](#) for $m > 0$.

Definition 3.8. Let $W_{m+n} = \langle \vec{e}_1, \vec{f}_1, \dots, \vec{e}_{m+n-1}, \vec{f}_{m+n-1}, \vec{e}_{m+n} \rangle_{\mathbb{Q}} \subseteq \mathbb{Q}^{2(m+n)}$ be as in Section 2. We write

$$T_n^{\omega, m} := (T_{m+n}^{\omega})_{>\langle \vec{e}_1, \dots, \vec{e}_m \rangle_{\mathbb{Q}}} \text{ and } T_n^{\omega, m}(W) := T(W_{m+n})_{>\langle \vec{e}_1, \dots, \vec{e}_m \rangle_{\mathbb{Q}}}$$

for the upper links of the isotropic subspace $\langle \vec{e}_1, \dots, \vec{e}_m \rangle_{\mathbb{Q}}$ in T_{m+n}^{ω} and $T_{m+n}^{\omega}(W)$, respectively.

Lemma 3.9. *Let $m \geq 0$ and $n \geq 1$. $T_n^{\omega, m}$ and $T_n^{\omega, m}(W)$ are Cohen–Macaulay posets of dimension $(n-1)$. Furthermore, $T_n^{\omega, m}(W)$ is contractible.*

Proof. For the upper links in the symplectic Tits building $T_n^{\omega, m}$, this follows from the Solomon–Tits Theorem (see [Sol69] and [Bro89, IV.5 Remark 2]) and [Qui78, Proposition 8.6]. For the upper links in the restricted Tits building $T_n^{\omega, m}(W)$, it follows from Theorem 2.2 and [Qui78, Proposition 8.6]. The contractibility of $T_n^{\omega, m}(W) = T_{m+n}^{\omega}(W)_{>\langle \vec{e}_1, \dots, \vec{e}_m \rangle_{\mathbb{Q}}}$ follows from Item 1 in the proof of Lemma 2.6 because $\langle \vec{e}_{m+n} \rangle_{\mathbb{Q}} \not\subseteq \langle \vec{e}_1, \dots, \vec{e}_m \rangle_{\mathbb{Q}}$. \square

Definition 3.10. Given an isotropic subspace $V \in T_n^{\omega, m}$, we obtain an isotropic summand $V \cap \mathbb{Z}^{2(m+n)}$ of $\mathbb{Z}^{2(m+n)}$ (see e.g. [CP17, Lemma 2.4]) properly containing $\langle \vec{e}_1, \dots, \vec{e}_m \rangle_{\mathbb{Z}}$.

1. Let $\mathcal{B}^m(V \cap \mathbb{Z}^{2(m+n)})$ be the full subcomplex of \mathcal{I}_n^m on the vertices satisfying $v \subseteq V \cap \mathbb{Z}^{2(m+n)}$.
2. Let $\mathcal{BA}^m(V \cap \mathbb{Z}^{2(m+n)})$ be the full subcomplex of $\mathcal{I}_n^{\delta, m}$ on the vertices satisfying $v \subseteq V \cap \mathbb{Z}^{2(m+n)}$.

Let $\mathbb{Z}^{m+n} = \langle e_1, \dots, e_{m+n} \rangle_{\mathbb{Q}} \cap \mathbb{Z}^{2(m+n)}$. By results of Maazen [Maa79] and Church–Putman [CP17], the complex $\mathcal{B}^m(\mathbb{Z}^{m+n})$ is Cohen–Macaulay of dimension $(n-1)$. The connectivity properties of $\mathcal{BA}^m(\mathbb{Z}^{m+n})$ have also been studied by Church–Putman [CP17]. We summarize the results contained in [CP17] in the following theorem.

Theorem 3.11 ([CP17, Theorem 4.2 and Theorem C]). *Let $m \geq 0$ and $n \geq 1$.*

1. $\mathcal{B}^m(\mathbb{Z}^{m+n})$ is $(n-2)$ -connected and Cohen–Macaulay of dimension $(n-1)$.
2. $\mathcal{BA}^m(\mathbb{Z}^{m+n})$ is $(n-1)$ -connected.

The following two results allow us to relate the complexes introduced above. The first is a result of Quillen [Qui78]. For this, we recall that the height $h(y)$ of an element y in a poset P is the length l of the longest chain of the form $y_0 < \dots < y_l = y$ in P . If no such l exists, we put $h(y) = \infty$.

Theorem 3.12 ([Qui78, Corollary 9.7]). *Let $f : X \rightarrow Y$ be a poset map which is strictly increasing (if $x < x'$, then $f(x) < f(x')$). Assume that Y is Cohen–Macaulay of dimension n and that the poset fibers $f_{\leq y} = \{x \in X : f(x) \leq y\}$ are Cohen–Macaulay of dimension $h(y)$ for all $y \in Y$. Then X is Cohen–Macaulay of dimension n .*

The second result is a generalization of Quillen’s [Qui78, Theorem 9.1] due to van der Kallen–Looijenga.

Theorem 3.13 ([KL11, Corollary 2.2]). *Let $f : X \rightarrow Y$ be a poset map, $\theta \in \mathbb{Z}$, and $t : Y \rightarrow \mathbb{Z}$ an increasing (if $y' < y$, then $t(y') < t(y)$) but bounded function. Suppose that for every $y \in Y$, the poset fiber $f_{\leq y} = \{x \in X : f(x) \leq y\}$ is $(t(y)-2)$ -connected and that the upper link $Y_{>y}$ is $(\theta - t(y) - 1)$ -connected. Then the map f is θ -connected.*

We are now ready to formulate our alternative arguments for the first items of Theorem 3.5. These arguments are completely formal and analogous to the proof of

e.g. [KL11, Proposition 1.2.]; the major difference being the input from [Theorem 2.2](#) and [Theorem 3.11](#).

Lemma 3.14. *Let $m \geq 0$ and $n \geq 1$. The complex $\mathcal{I}_n^m(W)$ is Cohen–Macaulay of dimension $n - 1$. In particular, [Item 1 of Theorem 3.5](#) holds.*

Lemma 3.15. *Let $m \geq 0$ and $n \geq 1$. The complex \mathcal{I}_n^m is Cohen–Macaulay of dimension $n - 1$. In particular, [Item 2 of Theorem 3.5](#) holds.*

[Lemma 3.15](#) can easily be deduced from [Put09, Proposition 6.13.2], i.e. [Item 2 of Theorem 3.5](#), or [KL11, Proposition 1.2]. Since it is used in the proof that \mathcal{IA}_n^m is highly connected at the end of this section and in the sequel [BPS23], we included a short argument.

Proof of [Lemma 3.14](#) and [Lemma 3.15](#). Let $P(\mathcal{I}_n^m(W))$ and $P(\mathcal{I}_n^m)$ denote the simplex posets of $\mathcal{I}_n^m(W)$ and \mathcal{I}_n^m , respectively. The two lemmas follow by considering the poset maps

$$f : P(\mathcal{I}_n^m(W)) \rightarrow T_n^{\omega,m}(W) : \Delta \mapsto \langle \vec{e}_1, \dots, \vec{e}_m \rangle_{\mathbb{Q}} \oplus \langle \Delta \rangle_{\mathbb{Q}}$$

and

$$f : P(\mathcal{I}_n^m) \rightarrow T_n^{\omega,m} : \Delta \mapsto \langle \vec{e}_1, \dots, \vec{e}_m \rangle_{\mathbb{Q}} \oplus \langle \Delta \rangle_{\mathbb{Q}},$$

respectively, and invoking [Theorem 3.12](#). Let $V \in T_n^{\omega,m}(W)$ or $V \in T_n^{\omega,m}$. The application of Quillen’s result relies on the facts that $h(V) = \dim(V) - m - 1$, that $T_n^{\omega,m}(W)$ and $T_n^{\omega,m}$ are Cohen–Macaulay posets of dimension $(n - 1)$ (see [Lemma 3.9](#)) and the observation that $f_{\leq V} = \mathcal{B}^m(V \cap \mathbb{Z}^{2(m+n)})$, which is Cohen–Macaulay of dimension $(\dim(V) - m - 1)$ by [Theorem 3.11](#). \square

Finally, we formulate an alternative argument for [Item 3 of Theorem 3.5](#).

Proof of [Item 3 of Theorem 3.5](#). We will show that $P(\mathcal{I}_n^{\delta,m}(W))$, the simplex poset of $\mathcal{I}_n^{\delta,m}(W)$, is $(n - 1)$ -connected. There is a poset map

$$f : P(\mathcal{I}_n^{\delta,m}(W)) \rightarrow T_n^{\omega,m}(W) : \Delta \mapsto \langle \vec{e}_1, \dots, \vec{e}_m \rangle_{\mathbb{Q}} + \langle \Delta \rangle_{\mathbb{Q}}.$$

Let $\theta = n$ and define $t : T_n^{\omega,m}(W) \rightarrow \mathbb{Z} : V \mapsto \dim(V) - m + 1$. By [Lemma 3.9](#), we know that $T_n^{\omega,m}(W)_{>V}$ is $((n - 1) - (\dim(V) - m - 1) - 2) = (\theta - t(V) - 1)$ -connected. Furthermore, $f_{\leq V} = P(\mathcal{BA}^m(V \cap \mathbb{Z}^{2(m+n)}))$ is $(\dim(V) - m - 1) = (t(V) - 2)$ -connected by [Theorem 3.11](#) for $\dim(V) \geq 1 + m$. Therefore [Theorem 3.13](#) implies that f is n -connected. By [Lemma 3.9](#), the target is contractible, hence $\mathcal{I}_n^{\delta,m}(W)$ is $(n - 1)$ -connected. \square

We end this section by explaining how Putman’s connectivity result for $\mathcal{I}_n^{\sigma,\delta,m}$ (see [Theorem 3.5](#)) implies high-connectivity of \mathcal{IA}_n^m .

Corollary 3.16. *If $m \geq 0$ and $n \geq 1$, then \mathcal{IA}_n^m is $(n - 1)$ -connected.*

Proof. By [Item 5 of Theorem 3.5](#), the subcomplex $X_0 = \mathcal{I}_n^{\sigma,\delta,m}$ of $X_1 = \mathcal{IA}_n^m$ is $(n - 1)$ -connected. We will apply the standard link argument explained in [HV17, §2.1] and [HV17, Corollary 2.2] to conclude that X_1 is $(n - 1)$ -connected as well. Let B be the set of *minimal* mixed simplices contained in X_1 , i.e. B is the set of simplices $\Delta = \Delta' * \Theta$ in \mathcal{IA}_n^m consisting of a σ edge $\Theta = \{v, w\}$ and a 2-additive simplex of the form $\Delta' = \{\langle \pm \vec{v}_1 \pm \vec{v}_2 \rangle, v_1, v_2\}$ or $\Delta' = \{\langle \pm \vec{e}_i \pm \vec{v}_1 \rangle, v_1\}$ where $e_i \in \{e_1, \dots, e_m\}$. Here, we call $\Delta' = \{\langle \pm \vec{e}_i \pm \vec{v}_1 \rangle, v_1\}$ a 2-additive simplex in \mathcal{IA}_n^m if $\{\langle \pm \vec{e}_i \pm \vec{v}_1 \rangle, v_1, e_i\}$ is a 2-additive simplex in \mathcal{IA}_{m+n} . We note that, by [Definition 3.3](#), a simplex Δ in \mathcal{IA}_n^m is mixed (i.e. $\Delta \cup \{e_1, \dots, e_n\}$ is mixed in \mathcal{IA}_{m+n}) if and only if Δ has a unique face contained in B . This property implies that B is a set of bad simplices in the sense of [HV17, §2.1]: [HV17, §2.1, Condition (1)] holds, since any simplex in $X_1 = \mathcal{IA}_n^m$ with no face in B has to be

in $X_0 = \mathcal{I}_n^{\sigma, \delta, m}$. [HV17, §2.1, Condition (2)] holds, because if two faces of a simplex in $X_1 = \mathcal{I}\mathcal{A}_n^m$ are in B these faces need to be equal. For $\Delta \in B$, the complex of simplices that are good for Δ is therefore given by $\text{Link}_{X_1}^{\text{good}}(\Delta) = \text{Link}_{X_1}(\Delta)$. Now, $\text{Link}_{X_1}(\Delta) \cong \mathcal{I}_{n-\dim(\Delta)+1}^{m+\dim(\Delta)-2}$ and, by Lemma 3.15, this is (even more than) $(n - \dim(\Delta) - 2)$ -connected for every minimal mixed simplex Δ . Since $X_0 = \mathcal{I}_n^{\sigma, \delta, m}$ is $(n - 1)$ -connected (see Theorem 3.5), [HV17, Corollary 2.2] therefore implies that $X_1 = \mathcal{I}\mathcal{A}_n^m$ is $(n - 1)$ -connected as well. \square

4. SYMPLECTIC INTEGRAL APARTMENT CLASSES

Following [Gun00, Section 3], we explain the construction of the symplectic integral apartment class map appearing in Theorem 1.1,

$$[-] : \mathbb{Z}[\text{Sp}_{2n}(\mathbb{Z})] \rightarrow \text{St}_n^\omega = \tilde{H}_{n-1}(T_n^\omega; \mathbb{Z}).$$

The image

$$[M] \in \text{St}_n^\omega = \tilde{H}_{n-1}(T_n^\omega; \mathbb{Z})$$

of an integral symplectic matrix $M \in \text{Sp}_{2n}(\mathbb{Z})$ under this map is called its *integral apartment class*. To define these homology classes, we use the following notation and observations.

Definition 4.1. Let $\llbracket n \rrbracket := \{1, \bar{1}, \dots, n, \bar{n}\}$. A nonempty subset $I \subseteq \llbracket n \rrbracket$ is called a standard subset if for all $1 \leq a \leq n$ it holds that $\{a, \bar{a}\} \not\subseteq I$. We denote by $\partial\beta_n$ the simplicial complex whose vertex set is $\llbracket n \rrbracket$ and whose k -simplices are the standard subsets $I \subset \llbracket n \rrbracket$ of size $k + 1$.

Observe that $\partial\beta_1 = \{1, \bar{1}\} \cong S^0$ and that the inclusion of vertex sets $\llbracket n \rrbracket \subseteq \llbracket n+1 \rrbracket$ induces an inclusion of simplicial complexes $\partial\beta_n \hookrightarrow \partial\beta_{n+1}$ for any $n \in \mathbb{N}$. It is readily verified that $\partial\beta_{n+1}$ is exactly the simplicial join $\partial\beta_n * \{n+1, \bar{n+1}\}$. It follows that $\partial\beta_n \cong *_1^n S^0$ is a simplicial sphere of dimension $n - 1$ and that $\partial\beta_{n+1} = \partial\beta_n * \{n+1, \bar{n+1}\}$ is obtained from $\partial\beta_n$ by suspension. We fix a fundamental class $\xi = \xi_0 \in \tilde{H}_0(\partial\beta_1; \mathbb{Z})$ once and for all. Using the suspension isomorphism, this class gives rise to fundamental classes $\xi = \xi_{n-1} \in \tilde{H}_{n-1}(\partial\beta_n; \mathbb{Z})$ for all $n \in \mathbb{N}$.

Given an integral symplectic matrix $M \in \text{Sp}_{2n}(\mathbb{Z})$, its column vectors form a symplectic basis of \mathbb{Q}^{2n} . We may index the $2n$ column vectors from left to right by $\llbracket n \rrbracket = \{1, \bar{1}, \dots, n, \bar{n}\}$. By the definition of the symplectic form ω , this indexing $M = (\tilde{M}_a)_{a \in \llbracket n \rrbracket}$ has the property that if $I \in \partial\beta_n$ is a simplex, then

$$M_I = \langle \{\tilde{M}_a : a \in I\} \rangle_{\mathbb{Q}}$$

is an isotropic subspace of \mathbb{Q}^{2n} . This implies that for every $M \in \text{Sp}_{2n}(\mathbb{Z})$, we can define a poset map

$$\partial M : P(\partial\beta_n) \rightarrow T_n^\omega : I \mapsto M_I,$$

where $P(\partial\beta_n)$ denotes the poset of simplices of $\partial\beta_n$. Note that, passing to the associated order complexes of these posets, ∂M defines a simplicial embedding. Since the order complex of the poset $P(\partial\beta_n)$ is the barycentric subdivision of $\partial\beta_n$, it follows that the image of the map ∂M is a subcomplex that is homeomorphic to an $(n - 1)$ -sphere. Such a subcomplex is called an *integral apartment* of the Tits building T_n^ω . Taking homology, we obtain a map

$$\partial M_\star : \tilde{H}_{n-1}(P(\partial\beta_n); \mathbb{Z}) \rightarrow \text{St}_n^\omega.$$

Barycentric subdivision of simplicial complexes comes with a natural homology isomorphism on chain level $b : C_\star \rightarrow C_\star \circ P$, where C_\star assigns an ordered simplicial complex its simplicial chain complex with trivial \mathbb{Z} -coefficients. Using the induced isomorphism

$$b : \tilde{H}_{n-1}(\partial\beta_n; \mathbb{Z}) \rightarrow \tilde{H}_{n-1}(P(\partial\beta_n); \mathbb{Z}),$$

we obtain a unique class $b(\xi) \in \tilde{H}_{n-1}(P(\partial\beta_n); \mathbb{Z})$ for every $n \in \mathbb{N}$.

Definition 4.2. The *symplectic integral apartment class* $[M] \in \text{St}_n^\omega$ of $M \in \text{Sp}_{2n}(\mathbb{Z})$ is defined to be the value of $\partial M_\star : \tilde{H}_{n-1}(P(\partial\beta_n); \mathbb{Z}) \rightarrow \text{St}_n^\omega$ at $b(\xi)$,

$$[M] := \partial M_\star(b(\xi)).$$

This defines a map

$$[-] : \mathbb{Z}[\text{Sp}_{2n}(\mathbb{Z})] \rightarrow \text{St}_n^\omega : M \mapsto [M]$$

which we called the *symplectic integral apartment class map*.

Remark 4.3. The construction of symplectic apartment classes described above also works if one starts with an element in the rational symplectic group $M \in \text{Sp}_{2n}(\mathbb{Q})$. This leads to the definition of a symplectic *rational* apartment class map

$$[-] : \mathbb{Z}[\text{Sp}_{2n}(\mathbb{Q})] \rightarrow \text{St}_n^\omega.$$

It follows from the proof of the Solomon–Tits Theorem (see [Sol69] or [Bro89, Section IV.5, Theorem 2]) that this map is a surjection. Gunnells’ theorem states that its restriction to the “much smaller” group ring $\mathbb{Z}[\text{Sp}_{2n}(\mathbb{Z})]$ is still a surjection. The “smallness” of $\mathbb{Z}[\text{Sp}_{2n}(\mathbb{Z})]$ can be illustrated from a building theoretic perspective: Every apartment in the complete system of apartments of the building T_n^ω (in the sense of [Bro89, Section IV.4]) can be obtained by a $\text{Sp}_{2n}(\mathbb{Q})$ -translation, but not by a $\text{Sp}_{2n}(\mathbb{Z})$ -translation, of the standard apartment.

Remark 4.4. Gunnells’ proof for [Theorem 1.1](#) was inspired by work of Ash–Rudolph [AR79] and based on the content of [Remark 4.3](#). The general strategy is to devise an algorithm that takes as input a *rational* apartment classes $[M] \in \text{St}_n^\omega$ for $M \in \text{Sp}_{2n}(\mathbb{Q})$ and outputs a linear combination of *integral* apartment classes that is equal to $[M]$.

Remark 4.5. The integral apartment classes introduced in [Definition 4.2](#) are called “unimodular symbols” in [Gun00], and the rational apartment classes in [Remark 4.3](#) are called “symplectic modular symbols” in [Gun00]. More generally, the terminology that we use in this note is close to that of [CFP19; CP17; Brü+22; BSS22], while the terminology in Gunnells’ paper [Gun00] is close to that of [AR79].

5. A NEW PROOF OF GUNNELLS’ THEOREM

In this final section, we present a new proof of Gunnells’ [Theorem 1.1](#), i.e. we prove that the integral apartment class map

$$[-] : \mathbb{Z}[\text{Sp}_{2n}(\mathbb{Z})] \rightarrow \text{St}_n^\omega$$

is surjective. Our strategy is to factor it into a composition of four maps and then verify that each of these is a surjection. This is analogous to the strategy employed by Church–Farb–Putman in [CFP19]. Gunnells’ theorem then follows from the following two propositions, whose proof we will explain in the remainder of this work.

Proposition 5.1. *If $n \geq 1$, there exists a commutative diagram of the following shape*

$$\begin{array}{ccc}
 \mathbb{Z}[\text{Sp}_{2n}(\mathbb{Z})] & & \\
 \downarrow \alpha & & \searrow [-] \\
 H_n(\mathcal{I}\mathcal{A}_n, \mathcal{I}_n^\delta) & \xrightarrow{\delta} & \tilde{H}_{n-1}(\mathcal{I}_n^\delta) \\
 & & \downarrow b \\
 & & \tilde{H}_{n-1}(P(\mathcal{I}_n^\delta)) \xrightarrow{s_\star} \tilde{H}_{n-1}(T_n^\omega) = \text{St}_n^\omega
 \end{array}$$

where $[-] : \mathbb{Z}[\mathrm{Sp}_{2n}(\mathbb{Z})] \rightarrow \mathrm{St}_n^\omega$ is the integral apartment class map, δ is the connecting morphism of the long exact sequence of the pair $(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ and b is the homology isomorphism coming from barycentric subdivision.

The morphisms α and s_* in the statement of [Proposition 5.1](#) are defined below.

Proposition 5.2. *If $n \geq 1$, then the maps occurring in [Proposition 5.1](#) satisfy:*

1. $\alpha : \mathbb{Z}[\mathrm{Sp}_{2n}(\mathbb{Z})] \rightarrow H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ is a surjection.
2. $\delta : H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta) \rightarrow \tilde{H}_{n-1}(\mathcal{I}_n^\delta)$ is a surjection.
3. $b : \tilde{H}_{n-1}(\mathcal{I}_n^\delta) \rightarrow \tilde{H}_{n-1}(P(\mathcal{I}_n^\delta))$ is an isomorphism.
4. $s_* : \tilde{H}_{n-1}(P(\mathcal{I}_n^\delta)) \rightarrow \mathrm{St}_n^\omega$ is an isomorphism.

To define the morphisms α and s_* in the statement of [Proposition 5.1](#), we start by introducing a simplicial complex, which is closely related to the complex $\partial\beta_n$ occurring in the definition of the apartment class map.

Definition 5.3. We call a nonempty subset $I \subset \llbracket n \rrbracket$ a σ subset, if $\{n, \bar{n}\} \subset I$ and for all $1 \leq a \leq n-1 : \{a, \bar{a}\} \not\subset I$. Let β_n be the simplicial complex with vertex set $\llbracket n \rrbracket$ and k -simplices subsets $I \subset \llbracket n \rrbracket$ of size $k+1$, which are either standard (see [Definition 4.1](#)) or σ subsets.

Note that $\beta_1 \cong D^1$. Furthermore, $\beta_n \cong (*_1^{n-1}S^0) * D^1$ is homeomorphic to a disc of dimension n whose boundary sphere is triangulated by the subcomplex $\partial\beta_n \subset \beta_n$, i.e.

$$(|\beta_n|, |\partial\beta_n|) \cong (D^n, S^{n-1})$$

The definition of the map α involves the following construction: Let $M = (\vec{M}_a)_{a \in \llbracket n \rrbracket} \in \mathrm{Sp}_{2n}(\mathbb{Z})$. Given a k -simplex I of β_n , we find an associated simplex $M_I^\alpha = \{\langle \vec{M}_a \rangle_{\mathbb{Z}} : a \in I\}$ of \mathcal{IA}_n : If I is a standard subset, then M_I^α is a standard simplex. If I is σ subset, then M_I^α is a σ simplex. The resulting map

$$M^\alpha : \beta_n \rightarrow \mathcal{IA}_n$$

is a simplicial embedding and the boundary

$$\partial M^\alpha : \partial\beta_n \rightarrow \mathcal{IA}_n$$

of this simplicial disc is contained in \mathcal{I}_n^δ , i.e.

$$\partial M^\alpha : \partial\beta_n \rightarrow \mathcal{I}_n^\delta \rightarrow \mathcal{IA}_n.$$

Definition 5.4. The value $\alpha(M)$ of the map $\alpha : \mathbb{Z}[\mathrm{Sp}_{2n}(\mathbb{Z})] \rightarrow H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ at a matrix $M = (\vec{M}_a)_{a \in \llbracket n \rrbracket} \in \mathrm{Sp}_{2n}(\mathbb{Z})$ is defined to be the image of the fundamental class $\xi \in \tilde{H}_{n-1}(\partial\beta_n)$ under the composition

$$\tilde{H}_{n-1}(\partial\beta_n) \xleftarrow{\cong} H_n(\beta_n, \partial\beta_n) \xrightarrow{(M^\alpha, \partial M^\alpha)_*} H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta),$$

where the first isomorphism is the connecting morphism associated to the pair $(\beta_n, \partial\beta_n)$.

Finally, we define the map s_* .

Definition 5.5. $s_* : \tilde{H}_{n-1}(P(\mathcal{I}_n^\delta)) \rightarrow \mathrm{St}_n^\omega$ is the map induced in homology by the spanning map

$$s : P(\mathcal{I}_n^\delta) \rightarrow T_n^\omega : \Delta \mapsto \langle \Delta \rangle_{\mathbb{Q}},$$

where $P(\mathcal{I}_n^\delta)$ denotes the poset of simplices of \mathcal{I}_n^δ .

5.1. **Proof of Proposition 5.1.** Let $M \in \mathrm{Sp}_{2n}(\mathbb{Z})$. We need to verify that

$$[M] = (s_* \circ b \circ \delta \circ \alpha)(M).$$

Consider the following diagram:

$$\begin{array}{ccccc} H_n(\beta_n, \partial\beta_n) & \xrightarrow{\cong} & \tilde{H}_{n-1}(\partial\beta_n) & \xrightarrow{b} & \tilde{H}_{n-1}(P(\partial\beta_n)) \\ \downarrow (M^\alpha, \partial M^\alpha)_* & & \downarrow \partial M^\alpha_* & & \downarrow P(\partial M^\alpha)_* \\ H_n(\mathcal{I}\mathcal{A}_n, \mathcal{I}_n^\delta) & \xrightarrow{\delta} & \tilde{H}_{n-1}(\mathcal{I}_n^\delta) & \xrightarrow{b} & \tilde{H}_{n-1}(P(\mathcal{I}_n^\delta)) \end{array}$$

The left square commutes because the connecting morphism of the long exact sequence of a pair is a natural transformation. The right square commutes because $b : C_* \rightarrow C_* \circ P$ is a natural homology isomorphism. It follows that

$$(b \circ \delta \circ \alpha)(M) = (P(\partial M^\alpha)_* \circ b)(\xi) \in \tilde{H}_{n-1}(P(\mathcal{I}_n^\delta)).$$

To complete the proof, we need to see that

$$(s \circ P(\partial M^\alpha) \circ b)_*(\xi) = [M],$$

where $[M] = (\partial M \circ b)_*(\xi)$ is as in Definition 4.2. This holds because the composition $(s \circ P(\partial M^\alpha))$ defined in this section is equal to the map ∂M defined in the paragraph before Definition 4.2.

5.2. **Proof of Proposition 5.2.** The arguments for Item 2, Item 3 and Item 4 of Proposition 5.2 are similar to the arguments used by Church–Farb–Putman in the setting of $\mathrm{SL}_n(\mathbb{Z})$ [CFP19]. However, while the analogue of the surjectivity of the map $\alpha_n : \mathbb{Z}[\mathrm{Sp}_{2n}(\mathbb{Z})] \rightarrow H_n(\mathcal{I}\mathcal{A}_n, \mathcal{I}_n^\delta)$ is rather immediate for special linear groups, this step (i.e. Item 1 of Proposition 5.2) is more involved for symplectic groups. The reason is that apartments in the Tits building of type A_{n-1} , which is used in the argument for $\mathrm{SL}_n(\mathbb{Z})$, have the same combinatorial structure as the boundary of an $(n-1)$ -simplex $\partial\Delta^{n-1}$ and can therefore be “filled” by gluing in a single simplex of dimension $n-1$. Apartments of the Tits building of type C_n , which we use here for $\mathrm{Sp}_{2n}(\mathbb{Z})$, have a different simplicial structure. They are modelled by the complex $\partial\beta_n$ and require multiple simplices to be “filled”. In the complex $\mathcal{I}\mathcal{A}_n$, this is achieved by σ simplices. Observe that σ simplices already occur in dimension one. Therefore, and in contrast to the analogous situation for special linear groups (see [CFP19, Step 1, 2.3 Proof of Theorem B]), the relative chain complex $C_*(\mathcal{I}\mathcal{A}_n, \mathcal{I}_n^\delta)$ is nontrivial in degree $\star = n-1$.

Proof of Item 2 of Proposition 5.2. Corollary 3.16 implies that $\tilde{H}_{n-1}(\mathcal{I}\mathcal{A}_n) = 0$. Hence, the long exact sequence of the pair $(\mathcal{I}\mathcal{A}_n, \mathcal{I}_n^\delta)$ implies that $\delta : H_n(\mathcal{I}\mathcal{A}_n, \mathcal{I}_n^\delta) \rightarrow \tilde{H}_{n-1}(\mathcal{I}_n^\delta)$ is surjective. \square

Proof of Item 3 of Proposition 5.2. The map $b : \tilde{H}_{n-1}(\mathcal{I}_n^\delta) \rightarrow \tilde{H}_{n-1}(P(\mathcal{I}_n^\delta))$ is an isomorphism by definition. It is induced by a natural homology isomorphism of chain complexes. \square

Proof of Item 4 of Proposition 5.2. To verify that the map $s_* : \tilde{H}_{n-1}(P(\mathcal{I}_n^\delta)) \rightarrow \tilde{H}_{n-1}(T_n^\omega) = \mathrm{St}_n^\omega$ is an isomorphism, we can apply Theorem 3.13 once more. Let $\theta = n$. Let $V \in T_n^\omega$ and set $t(V) = \dim(V) + 1$. Lemma 3.9 for $m = 0$ implies that the upper link $(T_n^\omega)_{>V}$ is $((n-1) - (\dim(V) - 1) - 2) = (\theta - t(V) - 1)$ -connected. Theorem 3.11 for $m = 0$ implies that the lower fiber $f_{\leq V} = \mathcal{B}\mathcal{A}(V \cap \mathbb{Z}^n)$ is $(\dim(V) - 1) = (t(V) - 2)$ -connected. Therefore, it follows from Theorem 3.13 that $s : P(\mathcal{I}_n^\delta) \rightarrow T_n^\omega$ is n -connected and, hence, that the map $s_* : \tilde{H}_{n-1}(P(\mathcal{I}_n^\delta)) \rightarrow \tilde{H}_{n-1}(T_n^\omega) = \mathrm{St}_n^\omega$ is an isomorphism. \square

Let $n \in \mathbb{N}$. In the following, E_n^ω denotes the set of all σ edges in \mathcal{IA}_n . The proof of [Item 1](#) of [Proposition 5.2](#) is by induction on $n \geq 1$. The base case $n = 1$ is the content of the next lemma.

Lemma 5.6. *If $n = 1$, then $\alpha_n : \mathbb{Z}[\mathrm{Sp}_{2n}(\mathbb{Z})] \rightarrow H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ is surjective.*

Proof. For $2n = 2$, it follows that \mathcal{IA}_1 is a one-dimensional connected simplicial complex, that all edges are σ edges and that $\mathcal{I}_1^\delta = \mathcal{I}_1$ is exactly the 0-skeleton of \mathcal{IA}_1 .² In particular,

$$H_1(\mathcal{IA}_1, \mathcal{I}_1^\delta) \cong \bigoplus_{\Delta \in E_1^\omega} \mathbb{Z}.$$

Given some $M \in \mathrm{Sp}_2(\mathbb{Z})$ with $M = (\vec{v}, \vec{w})$, we see that $M^\alpha(\beta_1) \subset \mathcal{IA}_1$ is exactly the σ edge $\Delta = \{v, w\}$ and $M^\alpha(\partial\beta_1) \subset \mathcal{I}_1^\delta$ is exactly the boundary of this edge. Hence, under the identification above, α_1 maps the symplectic matrix M to a generator of the \mathbb{Z} -summand indexed by $\Delta = \{v, w\}$. Given any σ edge $\Delta = \{v, w\}$, we have that $\omega(\vec{v}, \vec{w}) = \pm 1$. Thus, for some choice of signs $(\pm\vec{v}, \pm\vec{w}) \in \mathrm{Sp}_2(\mathbb{Z})$. It follows that α_n is surjective for $n = 1$. \square

Proof of [Item 1](#) of [Proposition 5.2](#). To see that the map

$$\alpha_n : \mathbb{Z}[\mathrm{Sp}_{2n}(\mathbb{Z})] \rightarrow H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$$

is surjective, we perform an induction on $n \geq 1$. The induction beginning $n = 1$ is [Lemma 5.6](#). Let $n > 1$ and assume that [Item 1](#) of [Proposition 5.2](#) holds for $1 \leq k \leq n - 1$. We deduce the surjectivity of the map α_n in two steps.

The first step is to show that the target $H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ is a direct sum of “smaller” Steinberg modules. For this, we observe that \mathcal{IA}_n is obtained from \mathcal{I}_n^δ via the following pushout diagram:

$$\begin{array}{ccc} \bigsqcup_{\Delta \in E_n^\omega} \Sigma^1 \mathrm{Link}_{\mathcal{IA}_n}(\Delta) & \xrightarrow{\cong} & \bigsqcup_{\Delta \in E_n^\omega} \mathrm{Star}_{\mathcal{IA}_n}(\Delta) \cap \mathcal{I}_n^\delta & \longrightarrow & \mathcal{I}_n^\delta \\ & & \downarrow & & \downarrow \\ & & \bigsqcup_{\Delta \in E_n^\omega} \mathrm{Star}_{\mathcal{IA}_n}(\Delta) & \longrightarrow & \mathcal{IA}_n \end{array}$$

In particular, excision implies that

$$(1) \quad H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta) \cong \bigoplus_{\Delta \in E_n^\omega} H_n(\mathrm{Star}_{\mathcal{IA}_n}(\Delta), \Sigma^1 \mathrm{Link}_{\mathcal{IA}_n}(\Delta)).$$

The contractibility of $\mathrm{Star}_{\mathcal{IA}_n}(\Delta)$ implies that the connecting morphism of the pair

$$(\mathrm{Star}_{\mathcal{IA}_n}(\Delta), \Sigma^1 \mathrm{Link}_{\mathcal{IA}_n}(\Delta))$$

is an isomorphism

$$(2) \quad H_n(\mathrm{Star}_{\mathcal{IA}_n}(\Delta), \Sigma^1 \mathrm{Link}_{\mathcal{IA}_n}(\Delta)) \xrightarrow{\delta_n} \tilde{H}_{n-1}(\Sigma^1 \mathrm{Link}_{\mathcal{IA}_n}(\Delta)).$$

The suspension isomorphism gives an identification

$$(3) \quad \tilde{H}_{n-1}(\Sigma^1 \mathrm{Link}_{\mathcal{IA}_n}(\Delta)) \xrightarrow{\Sigma^{-1}} \tilde{H}_{n-2}(\mathrm{Link}_{\mathcal{IA}_n}(\Delta)).$$

Observe that $\mathrm{Link}_{\mathcal{IA}_n}(\Delta) = \mathcal{I}^\delta(\Delta^\perp)$, where $\Delta^\perp := \langle \Delta \rangle^\perp \subset \mathbb{Q}^{2n}$. We therefore proved that

$$(4) \quad H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta) \cong \bigoplus_{\Delta \in E_n^\omega} \tilde{H}_{n-2}(\mathcal{I}^\delta(\Delta^\perp)) \cong \bigoplus_{\Delta \in E_n^\omega} \mathrm{St}^\omega(\Delta^\perp)$$

²In fact, $\mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$ and \mathcal{IA}_1 is isomorphic to the 1-dimensional complex of partial frames $\mathcal{B}(\mathbb{Z}^2)$. The complex $\mathcal{B}(\mathbb{Z}^2)$ is discussed in detail in the introduction of [\[CP17\]](#) (see paragraph “Improving connectivity: the complex of partial augmented frames”).

where the last isomorphism is obtained by invoking [Item 3](#) and [Item 4](#) of [Proposition 5.2](#) and $\text{St}^\omega(\Delta^\perp)$ denotes the Steinberg module of the symplectic subspace $\Delta^\perp \subset \mathbb{Q}^{2n}$. This completes the first step.

The second step of the proof that $\alpha_n : \mathbb{Z}[\text{Sp}_{2n}(\mathbb{Z})] \rightarrow H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ is surjective is to decompose the domain $\mathbb{Z}[\text{Sp}_{2n}(\mathbb{Z})]$ in a compatible way and identify the resulting map on each summand. This is the content of the following claim.

Claim 5.7. Let $\Delta = \{v, w\} \in E_n^\omega$ and let $\omega(\vec{v}, \vec{w}) = 1$, $\tilde{\Delta} = (\vec{v}, \vec{w})$ an ordered pair. Let $\mathbb{Z}[\text{Sp}(\tilde{\Delta}^\perp)] \subset \mathbb{Z}[\text{Sp}_{2n}(\mathbb{Z})]$ be the \mathbb{Z} -summand spanned by symplectic matrices $M \in \text{Sp}_{2n}(\mathbb{Z})$ satisfying $\vec{M}_n = \vec{v}$ and $\vec{M}_{\bar{n}} = \vec{w}$. The sequence of identifications above yields a map

$$[-]_{\tilde{\Delta}} : \mathbb{Z}[\text{Sp}(\tilde{\Delta}^\perp)] \rightarrow H_{n-1}(\mathcal{IA}(\Delta^\perp), \mathcal{I}^\delta(\Delta^\perp)) \rightarrow \tilde{H}_{n-2}(\mathcal{I}^\delta(\Delta^\perp)) \rightarrow \text{St}^\omega(\Delta^\perp)$$

that is exactly the integral apartment class map of the group $\text{Sp}(\Delta^\perp)$ of symplectic automorphisms of the summand $\Delta^\perp \subset \mathbb{Z}^{2n}$.

Before proving this claim, we explain how this finishes the proof of the induction step and hence of [Proposition 5.2](#). [Claim 5.7](#) implies that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{Z}[\text{Sp}_{2n}(\mathbb{Z})] & \xlongequal{\quad} & \bigoplus_{\tilde{\Delta}=(\vec{M}_n, \vec{M}_{\bar{n}})} \mathbb{Z}[\text{Sp}(\tilde{\Delta}^\perp)] \\ \downarrow \alpha_n & & \downarrow \oplus [-]_{\tilde{\Delta}} \\ H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta) & \xrightarrow{\cong} & \bigoplus_{\Delta \in E_n^\omega} \text{St}^\omega(\Delta^\perp) \end{array}$$

The induction hypothesis, [Claim 5.7](#) and [Item 2](#), [Item 3](#) and [Item 4](#) of [Proposition 5.2](#) imply that the integral apartment class maps occurring on the right hand side of the above diagram,

$$[-]_{\tilde{\Delta}} : \mathbb{Z}[\text{Sp}(\tilde{\Delta}^\perp)] \rightarrow \text{St}^\omega(\Delta^\perp),$$

are surjective. For any $\Delta \in E_n^\omega$, there exists an ordered pair $\tilde{\Delta} = (\vec{M}_n, \vec{M}_{\bar{n}})$ such that $\Delta = \{\langle \vec{M}_n \rangle_{\mathbb{Z}}, \langle \vec{M}_{\bar{n}} \rangle_{\mathbb{Z}}\}$. It follows that the right vertical map in the diagram is surjective. Therefore, α_n is surjective as well. \square

Proof of Claim 5.7. It suffices to consider the case where $(\vec{v}, \vec{w}) = (\vec{e}_n, \vec{f}_n)$ consists of the last symplectic pair of the standard symplectic basis. All other cases can be reduced to this case by applying a symplectic matrix that sends (\vec{v}, \vec{w}) to (\vec{e}_n, \vec{f}_n) . Let $\Delta = \{\langle \vec{e}_n \rangle_{\mathbb{Z}}, \langle \vec{f}_n \rangle_{\mathbb{Z}}\}$, $\tilde{\Delta} = (\vec{e}_n, \vec{f}_n)$ and $M \in \text{Sp}_{2n}(\mathbb{Z})$ a symplectic matrix with $\vec{M}_n = \vec{e}_n$ and $\vec{M}_{\bar{n}} = \vec{f}_n$. The symplectic relations imply that the \vec{e}_n - and \vec{f}_n -coordinates of all other column vectors $\vec{M}_a, \vec{M}_{\bar{a}}$, where $a \in \{1, \dots, n-1\}$, of M are zero. In particular, M corresponds to a unique element \tilde{M} of the symplectic group $\text{Sp}(\Delta^\perp)$ of the summand $\Delta^\perp \subset \mathbb{Z}^{2n}$ and vice versa. Recall that the class $\alpha(M) \in H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ was defined using the map of pairs:

$$(M^\alpha, \partial M^\alpha) : (\beta_n, \partial \beta_n) \rightarrow (\mathcal{IA}_n, \mathcal{I}_n^\delta)$$

This map factors through the pair

$$(\text{Star}_{\mathcal{IA}_n}(\Delta), \text{Star}_{\mathcal{IA}_n}(\Delta) \cap \mathcal{I}_n^\delta) \cong (\text{Star}_{\mathcal{IA}_n}(\Delta), \Sigma^1 \text{Link}_{\mathcal{IA}_n}(\Delta)).$$

The naturality of connecting morphisms yields a commutative diagram:

$$\begin{array}{ccccc} H_n(\beta_n, \partial \beta_n) & \xrightarrow{\delta} & \tilde{H}_{n-1}(\partial \beta_n) & \xrightarrow{\Sigma^{-1}} & \tilde{H}_{n-2}(\partial \beta_{n-1}) \\ \downarrow (M^\alpha, \partial M^\alpha)_* & & \downarrow \partial M_*^\alpha & & \downarrow \partial \tilde{M}_*^\alpha \\ H_n(\text{Star}_{\mathcal{IA}_n}(\Delta), \Sigma^1 \text{Link}_{\mathcal{IA}_n}(\Delta)) & \xrightarrow{\delta} & \tilde{H}_{n-1}(\Sigma^1 \text{Link}_{\mathcal{IA}_n}(\Delta)) & \xrightarrow{\Sigma^{-1}} & \tilde{H}_{n-2}(\text{Link}_{\mathcal{IA}_n}(\Delta)) \end{array}$$

Hence, under the identifications in Equation (1), Equation (2) and Equation (3), the class

$$\alpha(M) \in H_n(\mathcal{I}\mathcal{A}_n, \mathcal{I}^\delta)$$

is mapped to

$$\partial\widetilde{M}_*^\alpha(\xi_{n-2}) \in \widetilde{H}_{n-2}(\text{Link}_{\mathcal{I}\mathcal{A}_n}(\Delta)) = \widetilde{H}_{n-2}(\mathcal{I}^\delta(\Delta^\perp)).$$

The following commuting square proves that $\partial\widetilde{M}_*^\alpha(\xi_{n-2})$ is exactly $(\delta \circ \alpha_{n-1})(\widetilde{M})$:

$$\begin{array}{ccc} H_n(\beta_{n-1}, \partial\beta_{n-1}) & \xrightarrow{\delta} & \widetilde{H}_{n-1}(\partial\beta_{n-1}) \\ \downarrow (\widetilde{M}^\alpha, \partial\widetilde{M}^\alpha)_* & & \downarrow \partial\widetilde{M}_*^\alpha \\ H_n(\mathcal{I}\mathcal{A}(\Delta^\perp), \mathcal{I}^\delta(\Delta^\perp)) & \xrightarrow{\delta} & \widetilde{H}_{n-1}(\mathcal{I}^\delta(\Delta^\perp)) \end{array}$$

Hence, the final identification used in Equation (4) and Proposition 5.1 yield that $\alpha_n(M)$ is mapped to

$$(s_* \circ b \circ \delta \circ \alpha_{n-1})(\widetilde{M}) = [\widetilde{M}] \in \text{St}^\omega(\Delta^\perp) \quad \square$$

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