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Weighted sums of generalized polygonal numbers with coefficients 1 or 2

by

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1. Introduction. For any positive integer $m \geq 3$, the *m-gonal numbers* are the integers of the form

$$P_m(x) = (m-2) \cdot \left(\frac{x^2 - x}{2}\right) + x \quad \text{for } x \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

In 1638, Fermat claimed that every non-negative integer is written as the sum of m m-gonal numbers, that is, there exists an $\boldsymbol{x} = (x_1, \ldots, x_m) \in \mathbb{N}_0^m$ such that

$$\sum_{i=1}^{m} P_m(x_i) = N$$

for any $N \in \mathbb{N}_0$. Later, in 1770, Lagrange proved the four square theorem, which is exactly the case $m = 4$ of Fermat's assertion. In 1796, Gauss proved the so called Eureka Theorem, which is the case $m = 3$, and finally, Cauchy proved the general case $m \geq 5$ in 1815. Nathanson (see [12] and [13, pp. 3–33]) simplified Cauchy's theorem and provided the proof of a slightly stronger version. Fermat's polygonal number theorem was generalized in many directions.

In 1830, Legendre refined Fermat's polygonal number theorem and proved that any integer $N \geq 28(m-2)^3$ with $m \geq 5$ is written as

$$P_m(x_1) + P_m(x_2) + P_m(x_3) + P_m(x_4) + \delta_m(N),$$

where $x_1, x_2, x_3, x_4 \in \mathbb{N}_0$, $\delta_m(N) = 0$ if m is odd, and $\delta_m(N) \in \{0, 1\}$ if m is even. Nathanson [13, p. 33] simplified the proofs of Legendre's theorem. Recently, Meng and Sun [11] strengthened Legendre's theorem by showing that if $m \equiv 2 \pmod 4$ with $m \geq 5$, then any integer $N \geq 28(m-2)^2$

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can be written as the above with $\delta_m(N) = 0$, while if $m \equiv 0 \pmod 4$ with $m \geq 5$, there are infinitely many positive integers not of the form $P_m(x_1) + P_m(x_2) + P_m(x_3) + P_m(x_4)$ with $x_1, x_2, x_3, x_4 \in \mathbb{N}_0$.

On the other hand, Guy [6] considered Fermat's polygonal number theorem for more general numbers $P_m(x)$ with $x \in \mathbb{Z}$, which are called *generalized m-gonal numbers*. For a positive integer $m \geq 3$, $\boldsymbol{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$, and $\boldsymbol{x} = (x_1, \ldots, x_k) \in \mathbb{Z}^k$, we define the sum

$$P_{m,\boldsymbol{a}}(\boldsymbol{x}) := \sum_{i=1}^{k} a_i P_m(x_i). \tag{1.1}$$

We say the sum $P_{m,\boldsymbol{a}}$ *represents* an integer N if $P_{m,\boldsymbol{a}}(\boldsymbol{x}) = N$ has an integer solution $\boldsymbol{x} \in \mathbb{Z}^k$, and we write $N \to P_{m,\boldsymbol{a}}$. The sum $P_{m,\boldsymbol{a}}$ is called *universal* if it represents every non-negative integer. Guy [6] asked for which $k \in \mathbb{N}$ the equation

$$\sum_{i=1}^{k} P_m(x_i) = N$$

has an integer solution $x_1, \ldots, x_k \in \mathbb{Z}$ for any $N \in \mathbb{N}_0$, that is, what is the minimal number k_m such that the sum $P_{m,(1,\ldots,1)}$ (1 is repeated k_m times) is universal. He explained that $k_m = 3$ for $m \in \{3, 5, 6\}$ and $k_4 = 4$, and showed that $k_m \geq m - 4$ for $m \geq 8$, using the simple observation that the smallest generalized m-gonal number other than 0 and 1 is $m - 3$.

Later, Sun [16] proved that $P_{8,(1,1,1,1)}$ is universal, which implies $k_8 = 4$, and also explained in the introduction that $k_7 = 4$. Indeed, note that $P_{7,(1,1,1)}$ cannot represent 10, and one may show that $P_{7,(1,1,1,1)}$ is universal; thanks to Legendre's theorem, one needs only check that any integers less than $3500 = 28(7-2)^3$ are represented by $P_{7,(1,1,1,1)}$. In the same manner, one may verify that $k_9 = 5$. Recently in [1], it was shown that $k_m = m - 4$ for $m \geq 10$ (see the proof of Theorem 3.2 for another proof). Therefore, the value k_m is determined for any integer $m \geq 3$.

On the other hand, Kane and his collaborators [1] considered the specific case when

$$\boldsymbol{a} = \boldsymbol{a}_{r,r-1,k} = (1, \ldots, 1, r, \ldots, r),$$

where 1 is repeated $r - 1$ times and r is repeated $k - r + 1$ times, and determined the minimal number k, denoted $k_{m,r,r-1}$, such that $P_{m,\boldsymbol{a}_{r,r-1,k}}$ is universal. In particular, they proved that $k_{m,2,1} = \lfloor m/2 \rfloor$ for any $m \geq 14$.

Motivated by this, in this article, we study the representations of the sum (1.1) with coefficients 1 or 2. For simplicity, for any non-negative integers α and β, we denote

$$(1^\alpha, 2^\beta) = (\overbrace{1, \ldots, 1}^{\alpha \text{ times}}, \overbrace{2, \ldots, 2}^{\beta \text{ times}}),$$

where 1 is repeated α times, and 2 is repeated β times. The following theorem is the main result of this paper.

THEOREM 1.1. *For any positive integer $m \geq 10$, the sum $P_{m,(1^\alpha,2^\beta)}$ is universal if and only if it represents 1, $m-4$, and $m-2$. Moreover, the sum $P_{m,(1^\alpha,2^\beta)}$ is universal if and only if it represents*

$$1, 3, 5, 10, 19, \text{ and } 23 \text{ if } m = 7, \quad \text{and} \quad 1, 5, 7, \text{ and } 34 \text{ if } m = 9.$$

Note that Theorem 1.1 is complete in the sense that for each $m = 3, 4, 5, 6, 8$, there is a criterion for determining the universality of an arbitrary sum $P_{m,\boldsymbol{a}}$ (see Remark 1.3(1)). On the other hand, Theorem 1.1 will be proved by using Lemma 3.1 and Theorem 3.2. When we prove Theorem 3.2, Lemma 2.2 will be systematically applied for the case when $m \geq 19$, however, the same strategy does not work for $m \leq 18$. Moreover, neither Lemma 3.1 nor Theorem 3.2 covers the case of $m = 7$. Therefore, in order to deal with the cases for those small positive integers, we need the following theorem, which is analogous to that of Legendre.

THEOREM 1.2. *Let $m \geq 5$ and N be integers. Let $\boldsymbol{a}$ be one of the vectors in*

$$\{(1,1,1,1), (1,1,1,2), (1,1,2,2), (1,2,2,2)\},$$

and put $C_{\boldsymbol{a}} = \frac{1}{8}, \frac{1}{10}, \frac{1}{3}$, and $\frac{7}{8}$ accordingly. Then we have the following:

(1) *Every integer $N \geq C_{\boldsymbol{a}}(m-2)^3$ is represented by $P_{m,\boldsymbol{a}}$, unless*

$$\boldsymbol{a} \in \{(1,1,1,1), (1,1,2,2)\} \quad \text{and} \quad m \equiv 0 \pmod 4 \quad \text{with } m > 8.$$

(2) *In each exceptional case, there are infinitely many positive integers which are not represented by $P_{m,\boldsymbol{a}}$.*

REMARK 1.3. (1) In [10], Kane and Liu showed that there exists a unique minimal positive integer γ_m such that for any $\boldsymbol{a} \in \mathbb{N}^k$, $P_{m,\boldsymbol{a}}$ is universal if and only if it represents every $N \leq \gamma_m$.

For the case when $3 \leq m \leq 9$ with $m \notin \{7, 9\}$, the value γ_m is known: $\gamma_3 = \gamma_6 = 8$ (Bosma and Kane [3]), $\gamma_4 = 15$ (the Conway–Schneeberger fifteen theorem, see [2, 4]), $\gamma_5 = 109$ (Ju [8]), and $\gamma_8 = 60$ (Ju and Oh [9]), so those theorems give us criteria for $P_{m,(1^\alpha,2^\beta)}$ to be universal. It seems to be difficult to obtain the values γ_m for $m = 7, 9$.

(2) Generalizing the number $k_{m,r,r-1}$, for any $r \in \mathbb{N}$, let us define the number

$$k_{m,r} := \min\{k \mid P_{m,\boldsymbol{a}} \text{ is universal for some } \boldsymbol{a} \in \mathbb{N}^k_{\leq r}\},$$

where $\mathbb{N}_{\leq r} = \{a \in \mathbb{N} \mid a \leq r\}$. Then $k_{m,r} \leq k_{m,r,r-1}$ follows from the definition. In particular, by Theorem 3.2, $k_{m,2,1} = k_{m,2} = \lfloor m/2 \rfloor$ for any odd integer m with $m \geq 11$, while $k_{m,2} = \lfloor m/2 \rfloor - 1 < \lfloor m/2 \rfloor = k_{m,2,1}$ for any even integer m with $m \geq 10$, and $k_{9,2} = 4 < 5 = k_{9,2,1}$.

(3) Theorem 1.2(1) will be proved with the aid of Lemmas 4.1–4.3. In those lemmas, the following system of diophantine equations is considered:

$$\begin{cases} a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = a, \\ a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b, \end{cases}$$

where $a, a_1, a_2, a_3, a_4 \in \mathbb{N}$ and $b \in \mathbb{Z}$. We study the solvability of the above equation over $\mathbb{Z}$ by connecting it with the existence of a representation of a binary $\mathbb{Z}$-lattice by a diagonal quaternary $\mathbb{Z}$-lattice with a certain constraint. When $(a_1, a_2, a_3, a_4) = (1, 1, 1, 1)$, the above equation was considered by Goldmakher and Pollack [5], and our approach was taken in this case by Hoffmann [7]. Hence, our strategy could be considered as a generalization of the method used in [7].

(4) In addition to what we introduced previously, Meng and Sun [11] also showed that if $m \not\equiv 0 \pmod 4$, then any $N \geq 1628(m-2)^3$ can be written as

$$P_m(x_1) + P_m(x_2) + 2P_m(x_3) + 2P_m(x_4) \quad \text{with } x_1, x_2, x_3, x_4 \in \mathbb{N}_0,$$

while if $m \equiv 0 \pmod 4$, then there are infinitely many positive integers not of the above form. Therefore, the statement "any sufficiently large positive integer is represented by $P_{m,\boldsymbol{a}}$ over $\mathbb{Z}$" has nothing to prove if we weaken the condition $x_i \in \mathbb{N}_0$ to $x_i \in \mathbb{Z}$, but Theorem 1.2(1) gives improvements on the constants $C_{\boldsymbol{a}}$. On the other hand, Theorem 1.2(2) tells us something more.

The rest of the paper is organized as follows. In Section 2, we introduce a geometric language and the theory of $\mathbb{Z}$-lattices which are used to prove our theorems. In Section 3, we classify all the universal sums $P_{m,(1^\alpha,2^\beta)}$ and prove Theorem 1.1. Finally, in Section 4, we prove Theorem 1.2, giving information on the integers represented by each of the sums $P_{m,(1,1,1,1)}$, $P_{m,(1,1,1,2)}$, $P_{m,(1,1,2,2)}$, and $P_{m,(1,2,2,2)}$.

2. Preliminaries. In this section, we introduce several definitions, notations and well-known results on quadratic forms in the more convenient geometric language of quadratic spaces and lattices. A *$\mathbb{Z}$-lattice* $L = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_k$ of rank k is a free $\mathbb{Z}$-module equipped with a non-degenerate symmetric bilinear form B such that $B(v_i, v_j) \in \mathbb{Q}$ for any $1 \leq i, j \leq k$. The corresponding quadratic map is defined by $Q(v) = B(v, v)$ for any $v \in L$. We say a $\mathbb{Z}$-lattice L is *positive definite* if $Q(v) > 0$ for any non-zero vector $v \in L$, and *integral* if $B(v, w) \in \mathbb{Z}$ for any $v, w \in L$. Throughout this article, we always assume that a $\mathbb{Z}$-lattice is positive definite and integral. If $B(v_i, v_j) = 0$ for any $i \neq j$, then we simply write

$$L = \langle Q(v_1), \ldots, Q(v_k) \rangle.$$

The corresponding quadratic form in k variables is defined by

$$f_L(x_1,\dots,x_k)=\sum_{1\le i,j\le k} B(v_i,v_j)x_ix_j.$$

For two $\mathbb{Z}$-lattices ℓ and L, we say ℓ is *represented* by L, written $\ell\to L$, if there is a linear map $\sigma:\ell\to L$ such that

$$B(\sigma(x),\sigma(y))=B(x,y) \quad \text{for any } x,y\in\ell.$$

Such a linear map σ is called a *representation* from ℓ to L. When $\ell\to L$ and $L\to\ell$, we say ℓ and L are *isometric* to each other, and we write $\ell\cong L$. For any prime p, we define the localization of L at p by $L_p=L\otimes_{\mathbb{Z}}\mathbb{Z}_p$. We say ℓ is *locally represented* by L if there is a local representation $\sigma_p:\ell_p\to L_p$ which preserves the bilinear forms for any prime p. For a $\mathbb{Z}$-lattice L, we define the *genus* $\mathrm{gen}(L)$ of L as

$$\mathrm{gen}(L)=\{K \text{ on } \mathbb{Q}L \mid K_p\cong L_p \text{ for any prime } p\},$$

where $\mathbb{Q}L=\{\alpha v\mid \alpha\in\mathbb{Q},\ v\in L\}$ is the quadratic space on which L lies. The isometric relation induces an equivalence relation on $\mathrm{gen}(L)$, and we call the number of different equivalence classes in $\mathrm{gen}(L)$ the *class number* of L.

Any unexplained notation and terminology can be found in [15].

The following is the well-known local-global principle for $\mathbb{Z}$-lattices.

THEOREM 2.1. *Let ℓ and L be $\mathbb{Z}$-lattices. If ℓ is locally represented by L, then $\ell\to L'$ for some $L'\in\mathrm{gen}(L)$. Moreover, if the class number of L is* 1, *then $\ell\to L$ if and only if ℓ is locally represented by L.*

Proof. See [15, Example 102:5]. ∎

Note that in case when ℓ is a unary $\mathbb{Z}$-lattice $\langle n\rangle$, $\ell\to L$ if and only if $n=f_L(\boldsymbol{x})$ is solvable over $\mathbb{Z}$, and ℓ is locally represented by L if and only if $n=f_L(\boldsymbol{x})$ is solvable over $\mathbb{Z}_p$ for any prime p. The following lemma plays an important role in the proof of Theorem 3.2, hence also in the proof of Theorem 1.1.

LEMMA 2.2. *The sum $P_{m,(1,2,2,2)}$ represents every integer in the set*

$$\{(m-2)N\in\mathbb{N}_0 : N\neq 2^{2s}(8t+1) \text{ for any } s,t\in\mathbb{N}_0\}.$$

Proof. Consider $(x_1,\dots,x_4)\in\mathbb{Z}^4$ in the hyperplane $x_1+2x_2+2x_3+2x_4=0$. Then

$$\begin{aligned}P_{m,(1,2,2,2)}(x_1,x_2,x_3,x_4)&=\tfrac{m-2}{2}\big((-2x_2-2x_3-2x_4)^2+2x_2^2+2x_3^2+2x_4^2\big)\\&=(m-2)(3x_2^2+3x_3^2+3x_4^2+4(x_2x_3+x_3x_4+x_4x_2)).\end{aligned}$$

Note that the $\mathbb{Z}$-lattice L of rank 3 to which the ternary quadratic form $3x_2^2+\cdots$ in the last equation corresponds has class number 1. Moreover, one may check that L locally represents every integer not of the form $2^{2s}(8t+1)$. Therefore, the lemma follows from Theorem 2.1. ∎

3. Main theorem

LEMMA 3.1. *Let $m \geq 9$ be a positive integer and let α, β be non-negative integers. Assume that $P_{m,(1^\alpha,2^\beta)}$ is universal. Then*

(1) *$P_{m,(1^{\alpha'},2^{\beta'})}$ is universal for any integers $\alpha' \geq \alpha$ and $\beta' \geq \beta$,*
(2) *$P_{m,(1^{\alpha+2\beta'},2^{\beta-\beta'})}$ is universal for any integer $0 \leq \beta' \leq \beta$,*
(3) *$\alpha \geq \max(m-2\beta-4, 1)$,*
(4) *if $\beta = \lfloor m/2 \rfloor - 2$, then $\alpha \geq 2$.*

Proof. Statements (1) and (2) are obvious. On the other hand, since $P_{m,(1^\alpha,2^\beta)}$ represents 1, we have $\alpha \geq 1$. Note that the smallest generalized m-gonal number other than 0 and 1 is $m-3$. So, in order for the equation

$$m-4 = \sum_{i=1}^{\alpha} P_m(x_i) + \sum_{i=\alpha+1}^{\alpha+\beta} 2P_m(x_i)$$

to have a solution $\boldsymbol{x} \in \mathbb{Z}^{\alpha+\beta}$, we should have $\alpha+2\beta \geq m-4$. This proves (3). Now assume that $\beta = \lfloor m/2 \rfloor - 2$. Then $\alpha \geq 1$ by (3). If the equation

$$m-2 = P_m(x_1) + \sum_{i=2}^{1+\beta} 2P_m(x_i)$$

had a solution, then we should have $P_m(x_1) \in \{0, 1, m-3\}$ and $2P_m(x_i) \in \{0, 2\}$ for each i with $2 \leq i \leq 1+\beta$. However, this is impossible. Therefore, we should have $\alpha \geq 2$. ∎

THEOREM 3.2. *Let $m \geq 9$ be a positive integer and let α and β be non-negative integers. Then the sum $P_{m,(1^\alpha,2^\beta)}$ is universal if and only if*

$$\alpha \geq \begin{cases} 1 & \text{if } \beta \geq \lfloor m/2 \rfloor - 1, \\ 2 & \text{if } \beta = \lfloor m/2 \rfloor - 2, \\ m-2\beta-4 & \text{if } 0 \leq \beta \leq \lfloor m/2 \rfloor - 3, \end{cases}$$

unless $m = 9$ and $\beta = 3$, in which case $P_{9,(1^\alpha,2^3)}$ is universal if and only if $\alpha \geq 2$.

Proof. The "only if" part follows immediately from Lemma 3.1(3)–(4), and the fact that $P_{9,(1^1,2^3)}$ cannot represent 34. Now we prove the "if" part. Note that if we prove that $P_{m,(1^{m-2\beta-4},2^\beta)}$ is universal when $\beta = \lfloor m/2 \rfloor - 3$, then Lemma 3.1(2) implies that it is also universal for any $0 \leq \beta \leq \lfloor m/2 \rfloor - 3$. Moreover, for m even, if $P_{m,(1^2,2^{(m-6)/2})}$ is universal, then so is $P_{m,(1^2,2^{(m-4)/2})}$ by Lemma 3.1(1). Hence, in view of Lemma 3.1(1), it is enough to prove that

(i) $P_{m,(1^1,2^{\lfloor m/2 \rfloor - 1})}$ for any $m \geq 10$, $P_{9,(1^2,2^3)}$, and $P_{9,(1^1,2^4)}$ are universal,
(ii) $P_{m,(1^2,2^{(m-5)/2})}$ and $P_{m,(1^3,2^{(m-7)/2})}$ are universal for any odd integer m,
(iii) $P_{m,(1^2,2^{(m-6)/2})}$ is universal for any even integer m.

First, we prove (i). The statement for any $m \geq 14$ is proved in [1, Theorem 1.1(3)] (see [1, Section 4] for the proof). For any $9 \leq m \leq 13$, note that $\lfloor m/2 \rfloor - 1 \geq 3$. By Theorem 1.2(1), we know that $P_{m,(1^1,2^3)}$ represents every integer $N \geq \frac{7}{8}(m-2)^3$. Therefore, by checking (by a computer program) whether or not the integers less than $\frac{7}{8}(m-2)^3$ are represented by $P_{m,(1^1,2^3)}$, one may determine the set $E(P_{m,(1^1,2^3)})$ of all integers that are not represented by $P_{m,(1^1,2^3)}$. From this set, one may conclude what we want; for example, we have $E(P_{9,(1^1,2^3)}) = \{34\}$, so 34 is represented by both $P_{9,(1^2,2^3)}$ and $P_{9,(1^1,2^4)}$. Hence they are universal.

Next, we prove (ii) and (iii). For any $9 \leq m \leq 18$, one may similarly prove that the sums are universal by determining the set $E(P_{m,(1^1,2^3)})$, $E(P_{m,(1^2,2^2)})$, or $E(P_{m,(1^3,2^1)})$ with the aid of Theorem 1.2(1). Now, we assume $m \geq 19$. We first prove the universality of $P_{m,(1^2,2^{(m-5)/2})} = P_{m,(1^1,2^3)} + P_{m,(1^1,2^{(m-11)/2})}$ for any odd integer m with $m \geq 19$. Let N be a non-negative integer and let

$$\begin{aligned} R_1 = \{0, 1, \ldots, m-10, 2m-11, 3m-12, 4m-13, 4m-12,& \\ 3m-9, 2m-6, m-3\},& \\ R_2 = \{r + 2(m-2) \mid r \in R_1\} \quad \text{and} \quad R = R_1 \cup R_2.& \end{aligned}$$

Note that R_i is a complete set of residues modulo $m-2$ for each $i = 1, 2$, and one may check that any integer $r \in R$ is represented by $P_{m,(1^1,2^{(m-11)/2})}$. Also, one may check that every integer $N < 6m - 17$ is represented by $P_{m,(1^2,2^{(m-5)/2})}$. Assume that $N \geq 6m - 17$. For each $i = 1, 2$, there is a unique $r_i \in R_i$ such that

$$N \equiv r_i \pmod{m-2} \quad \text{and} \quad N - r_i \geq 0.$$

Write $N - r_i = c_i(m-2)$. Since $r_2 - r_1 = 2(m-2)$, we have $c_1 - c_2 = 2$, hence for some $i_0 \in \{1, 2\}$, c_{i_0} is not of the form $2^{2s}(8t+1)$ for any $s, t \in \mathbb{N}_0$. Therefore, by Lemma 2.2, $N - r_{i_0}$ is represented by $P_{m,(1^1,2^3)}$, hence $N = (N - r_{i_0}) + r_{i_0}$ is represented by $P_{m,(1^2,2^{(m-5)/2})}$.

To prove the universality of $P_{m,(1^3,2^{(m-7)/2})} = P_{m,(1^1,2^3)} + P_{m,(1^2,2^{(m-13)/2})}$ for any odd integer m with $m \geq 19$, and $P_{m,(1^2,2^{(m-6)/2})} = P_{m,(1^1,2^3)} + P_{m,(1^1,2^{(m-12)/2})}$ for any even integer m with $m \geq 19$, we take

$$\begin{aligned} R_1 = \{0, 1, \ldots, m-11, 2m-12, 3m-13, 4m-14, 5m-15,& \\ 4m-12, 3m-9, 2m-6, m-3\}.& \end{aligned}$$

Then one may show the universality by repeating the same argument. ∎

Proof of Theorem 1.1. The proof is nothing but combining Lemma 3.1 and Theorem 3.2 appropriately. When $m \geq 10$, assume that $P = P_{m,(1^\alpha,2^\beta)}$ represents $1, m-4$, and $m-2$. Since $1 \to P$, we have $\alpha \geq 1$. Moreover, since $m - 4 \to P$, we have $\alpha + 2\beta \geq m - 4$ (see the proof of Lemma 3.1).

Thus, by Theorem 3.2, P is universal unless $\beta = \lfloor m/2 \rfloor - 2$. In the case when $\beta = \lfloor m/2 \rfloor - 2$, we should have $\alpha \geq 2$ in order for $P_{m,(1^\alpha,2^{\lfloor m/2 \rfloor - 2})}$ to represent $m-2$ (see the proof of Lemma 3.1), and therefore $P_{m,(1^\alpha,2^{\lfloor m/2 \rfloor - 2})}$ is universal by Theorem 3.2.

When $m = 9$, one may similarly show that if $P = P_{9,(1^\alpha,2^\beta)}$ represents $1, 5$, and 7, then it is universal, except for $P_{9,(1,2,2,2)}$. Using Theorem 1.2, we may verify that $E(P_{9,(1,2,2,2)}) = \{34\}$, and so both $P_{9,(1^2,2^3)}$ and $P_{9,(1^1,2^4)}$ are universal. Therefore, we conclude that if P represents $1, 5, 7$, and 34, then it is universal.

When $m = 7$, one may show that if $P = P_{7,(1^\alpha,2^\beta)}$ represents $1, 3$, and 5 then P should contain $P_{7,(1,1,1)}, P_{7,(1,1,2)}$, or $P_{7,(1,2,2)}$, and they do not represent $10, 23$, or 19, respectively. On the other hand, using Theorem 1.2, we may verify that each of the sums $P_{7,(1,1,1,1)}, P_{7,(1,1,1,2)}, P_{7,(1,1,2,2)}$, and $P_{7,(1,2,2,2)}$ is universal. Therefore, we conclude that if P represents $1, 3, 5, 10, 19$, and 23, then it is universal. ■

4. Representations of quaternary sums $P_{m,(1^\alpha,2^\beta)}$. In this section, we prove Theorem 1.2. Throughout this section, let us set several notations. For each $\boldsymbol{a} = (a_1, a_2, a_3, a_4) \in \mathbb{N}^4$, we put $A = A_{\boldsymbol{a}} = \sum_{i=1}^4 a_i$, and we define the quaternary diagonal $\mathbb{Z}$-lattice $L_{\boldsymbol{a}}$ with basis $\{w_1, w_2, w_3, w_4\}$ by

$$L_{\boldsymbol{a}} = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3 + \mathbb{Z}w_4 = \langle a_1, a_2, a_3, a_4 \rangle.$$

Let

$$S := \{(1,1,1,1), (1,1,1,2), (1,1,2,2), (1,2,2,2)\},$$

and for each $\boldsymbol{a} \in S$, we define the set of integers

$$E_{\boldsymbol{a}} = \begin{cases} \{2^{2s}(8t+7) \mid s \in \mathbb{N}_0, t \in \mathbb{Z}\} & \text{if } \boldsymbol{a} = (1,1,1,1) \text{ or } (1,1,2,2), \\ \{5^{2s+2}(5t \pm 2) \mid s \in \mathbb{N}_0, t \in \mathbb{Z}\} & \text{if } \boldsymbol{a} = (1,1,1,2), \\ \{2^{2s}(16t+14) \mid s \in \mathbb{N}_0, t \in \mathbb{Z}\} & \text{if } \boldsymbol{a} = (1,2,2,2). \end{cases}$$

For a binary $\mathbb{Z}$-lattice $\ell = \mathbb{Z}v_1 + \mathbb{Z}v_2$, we write $\ell = [Q(v_1), B(v_1, v_2), Q(v_2)]$.

The following lemmas will play crucial roles in proving Theorem 1.2(1).

Lemma 4.1. *Let $\boldsymbol{a} = (a_1, a_2, a_3, a_4) \in \mathbb{N}^4$, $a \in \mathbb{N}$, and $b \in \mathbb{Z}$. Assume that the system of diophantine equations*

$$\begin{cases} a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = a, \\ a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b \end{cases} \tag{4.1}$$

has an integer solution $x_1, x_2, x_3, x_4 \in \mathbb{Z}$. Then

(1) *$a \equiv b \pmod 2$ and $Aa - b^2 \geq 0$,*
(2) *the integer $N := \frac{m-2}{2}(a-b) + b$ is represented by $P_{m,(a_1,a_2,a_3,a_4)}$.*

Proof. Since $x_i^2 \equiv x_i \pmod 2$, we necessarily have $a \equiv b \pmod 2$, and the inequality $Aa - b^2 \geq 0$ is nothing but the Cauchy–Schwarz inequality. Moreover, note that

$$\begin{aligned}\frac{m-2}{2}(a-b) + b &= \sum_{i=1}^{4} a_i\left(\frac{m-2}{2}(x_i^2 - x_i) + x_i\right)\\ &= P_{m,(a_1,a_2,a_3,a_4)}(x_1, x_2, x_3, x_4).\end{aligned}$$

This proves the lemma. ∎

LEMMA 4.2. *Let $\boldsymbol{a} \in S$, and let a and b be integers such that*

$$a \equiv b \pmod 2 \quad \textit{and} \quad Aa - b^2 > 0.$$

Then the following are equivalent:

(1) *The system* (4.1) *has an integer solution $x_1, x_2, x_3, x_4 \in \mathbb{Z}$.*
(2) *There exists a representation $\sigma : [A, b, a] \to L_{\boldsymbol{a}}$ such that*

$$\sigma(v_1) = w_1 + w_2 + w_3 + w_4.$$

(3) *The binary $\mathbb{Z}$-lattice $[A, b, a]$ is represented by the quaternary $\mathbb{Z}$-lattice $L_{\boldsymbol{a}}$.*
(4) *The positive integer $Aa - b^2$ is not contained in $E_{\boldsymbol{a}}$.*

Proof. We first prove (3)⇔(4). Note that the class number of $L_{\boldsymbol{a}}$ is 1 for any $\boldsymbol{a} \in S$. Therefore, by Theorem 2.1, $[A, b, a]$ is represented by $L_{\boldsymbol{a}}$ if and only if $[A, b, a]$ is locally represented by $L_{\boldsymbol{a}}$. By [14, Theorems 1 and 3], one may check, under the given assumptions on a and b, that $[A, b, a]$ is locally represented by $L_{\boldsymbol{a}}$ if and only if $Aa - b^2 \notin E_{\boldsymbol{a}}$.

Next, we prove (1)⇔(2). Assume there exist $x_1, x_2, x_3, x_4 \in \mathbb{Z}$ satisfying (4.1). Define a linear map $\sigma : [A, b, a] \to L_{\boldsymbol{a}}$ by

$$\sigma(v_1) = w_1 + w_2 + w_3 + w_4 \quad \text{and} \quad \sigma(v_2) = \sum_{i=1}^{4} x_i w_i.$$

Then $\sigma : [A, b, a] \to L$ is a representation since we have

$$\begin{cases} Q(\sigma(v_1)) = A = Q(v_1), \\ Q(\sigma(v_2)) = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = a = Q(v_2), \\ B(\sigma(v_1), \sigma(v_2)) = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b = B(v_1, v_2), \end{cases}$$

from (4.1). This proves (1)⇒(2), and (2)⇒(1) can also be easily proved.

Finally, we prove (2)⇔(3). We need only prove (3)⇒(2). Assume that there is a representation $\tau : [A, b, a] \to L_{\boldsymbol{a}}$. By changing the sign of w_i for $1 \leq i \leq 4$ or by interchanging w_i and w_j for $1 \leq i, j \leq 4$ with $a_i = a_j$ if necessary, we may assume that either $\tau(v_1) = w_1 + w_2 + w_3 + w_4$ or

$$\tau(v_1) = \begin{cases} 2w_1 & \text{if } \boldsymbol{a} = (1, 1, 1, 1), \\ 2w_1 + w_2 & \text{if } \boldsymbol{a} = (1, 1, 1, 2), \\ 2w_1 + w_3 & \text{if } \boldsymbol{a} = (1, 1, 2, 2). \end{cases}$$

In the former case, we are done by taking $\sigma = \tau$. To deal with the latter case, let $\tau(v_2) = \sum_{i=1}^{4} y_i w_i$ $(y_i \in \mathbb{Z})$.

First, we consider the case when $\boldsymbol{a} = (1,1,1,2)$ and $\tau(v_1) = 2w_1 + w_2$. Consider the $\mathbb{Q}$-linear map σ_T from $\mathbb{Q}L_{\boldsymbol{a}}$ to itself defined by

$$\sigma_T(w_j) = \sum_{i=1}^{4} t_{ij} w_i \text{ for each } 1 \le j \le 4, \text{ where } T = (t_{ij}) = \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & -2 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

Then $\sigma_T \in O(\mathbb{Q}L_{\boldsymbol{a}})$. If we let $\sigma = \sigma_T \circ \tau$, then

$$\sigma(v_1) = \sigma_T(2w_1 + w_2) = w_1 + w_2 + w_3 + w_4.$$

On the other hand, since $\tau : [A, b, a] \to L_{\boldsymbol{a}}$ is a representation, we have

$$y_1^2 + y_2^2 + y_3^2 + 2y_4^2 = a \quad \text{ and } \quad 2y_1 + y_2 = b.$$

Note that since $y_2^2 \equiv y_2 \equiv b \equiv a \pmod 2$, we have $y_1 \equiv y_1^2 \equiv y_3^2 \equiv y_3 \pmod 2$. Therefore, $\sigma(v_2) = \sigma_T(\sum_{i=1}^{4} y_i w_i) =: \sum_{i=1}^{4} x_i w_i \in L_{\boldsymbol{a}}$, since

$$(x_1, x_2, x_3, x_4) = \left(y_2, \frac{y_1 + y_3}{2} + y_4, \frac{y_1 + y_3}{2} - y_4, \frac{y_1 - y_3}{2}\right) \in \mathbb{Z}^4,$$

which implies that $\sigma : [A, b, a] \to L_{\boldsymbol{a}}$ is a representation that we want to find.

For each of the remaining two cases, one may follow the argument similar to the above to show that $\sigma = \sigma_T \circ \tau$ is a representation that we desired, by taking

$$T = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad \text{or} \quad \frac{1}{2}\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & -2 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

according as $(\boldsymbol{a}, \tau(v_1)) = ((1,1,1,1), 2w_1)$ or $((1,1,2,2), 2w_1 + w_3)$. ∎

Lemma 4.3. *Let $\boldsymbol{a} \in S$ and put $B_{\boldsymbol{a}} = 2, 2, 4, 7$ according as*

$$\boldsymbol{a} = (1,1,1,1), (1,1,1,2), (1,1,2,2), (1,2,2,2).$$

Let $m \ge 5$ be an integer and let I be a closed interval whose length is greater than or equal to $B_{\boldsymbol{a}}(m-2)$. Then for any integer N, there exists an integer $b \in I$ such that

$$N \equiv b \pmod{m-2} \quad \text{and} \quad Aa - b^2 \notin E_{\boldsymbol{a}}, \tag{4.2}$$

where $a = 2\left(\frac{N-b}{m-2}\right) + b$, unless $m \equiv 0 \pmod 4$ and $\boldsymbol{a} \in \{(1,1,1,1), (1,1,2,2)\}$.

Proof. For any integer N, let b_0 be the smallest integer in I such that $N \equiv b_0 \pmod{m-2}$. For an integer k, we define

$$b_k = b_0 + k(m-2), \quad a_k = 2\left(\frac{N - b_k}{m-2}\right) + b_k, \quad D_k = Aa_k - b_k^2.$$

Note that $a_k = a_0 + k(m-4) \in \mathbb{Z}$ for any integer k. We will show that

$$D_k \notin E_{\boldsymbol{a}} \quad \text{for some } 0 \le k \le B_{\boldsymbol{a}} - 1.$$

Then the lemma follows since $b = b_k$ satisfies (4.2) and the interval I contains $B_{\boldsymbol{a}}(m-2)$ consecutive integers.

Case 1: $\boldsymbol{a} = (1,1,1,1)$ and $m \not\equiv 0 \pmod 4$ ($E_{\boldsymbol{a}} = \{2^{2s}(8t+7) \mid s \in \mathbb{N}_0, t \in \mathbb{Z}\}$).

If $N \not\equiv 0 \pmod 2$ or $m \not\equiv 0 \pmod 2$, then one may note that b_k is an odd integer for some $k \in \{0,1\}$. Then $D_k = 4a_k - b_k^2 \equiv 3 \pmod 8$, since $a_k \equiv b_k \equiv 1 \pmod 2$. Hence, $D_k \notin E_{\boldsymbol{a}}$.

Otherwise, $N \equiv 0 \pmod 2$ and $m \equiv 2 \pmod 4$. Thus, $b_i \equiv a_i \equiv 0 \pmod 2$ for any integer i. Moreover, since $m-4 \equiv 2 \pmod 4$, we have $a_k \equiv 2 \pmod 4$ for some $k \in \{0,1\}$. Since $D_k \equiv 4$ or $8 \pmod{16}$, we have $D_k \notin E_{\boldsymbol{a}}$.

Case 2: $\boldsymbol{a} = (1,1,1,2)$ ($E_{\boldsymbol{a}} = \{5^{2s+2}(5t \pm 2) \mid s \in \mathbb{N}_0, t \in \mathbb{Z}\}$).

If $(m-2) \not\equiv 0 \pmod 5$, then $b_k \not\equiv 0 \pmod 5$ for some $k \in \{0,1\}$. Note that $D_k = 5a_k - b_k^2 \equiv \pm 1 \pmod 5$. Hence, $D_k \notin E_{\boldsymbol{a}}$.

Now assume that $(m-2) \equiv 0 \pmod 5$. Note that $N \equiv b_0 \pmod 5$. If $N \not\equiv 0 \pmod 5$, then $D_0 \equiv \pm 1 \pmod 5$, hence $D_0 \notin E_{\boldsymbol{a}}$. If $5 \mid N$, then $a_k \not\equiv 0 \pmod 5$ for some $k \in \{0,1\}$. Since $b_k \equiv b_0 \equiv 0 \pmod 5$, we have $5 \mid D_k$ but $25 \nmid D_k$, hence $D_k \notin E_{\boldsymbol{a}}$.

Case 3: $\boldsymbol{a} = (1,1,2,2)$ and $m \not\equiv 0 \pmod 4$ ($E_{\boldsymbol{a}} = \{2^{2s}(8t+7) \mid s \in \mathbb{N}_0, t \in \mathbb{Z}\}$).

If $N \not\equiv 0 \pmod 2$ or $m \not\equiv 0 \pmod 2$, then one may note that b_k is an odd integer for some $k \in \{0,1\}$. Then $D_k = 6a_k - b_k^2 \equiv 1$ or $5 \pmod 8$, since $a_k \equiv b_k \equiv 1 \pmod 2$. Hence $D_k \notin E_{\boldsymbol{a}}$.

Otherwise, $N \equiv 0 \pmod 2$ and $m \equiv 2 \pmod 4$. So, $b_i \equiv a_i \equiv 0 \pmod 2$, hence $D_i \equiv 0 \pmod 4$ for any integer i. Note that $D_{i_1} \equiv D_{i_2} \pmod{16}$ if and only if

$$4(i_1 - i_2)\left(3\left(\frac{m-4}{2}\right) - b_0\left(\frac{m-2}{2}\right) + (i_1+i_2)\left(\frac{m-2}{2}\right)^2\right) \equiv 0 \pmod{16}.$$

Since $m-4 \equiv 2 \pmod 4$ and $m-2 \equiv 0 \pmod 4$, this is equivalent to $i_1 \equiv i_2 \pmod 4$. Hence,

$$\{D_i \bmod 16 \mid i = 0,1,2,3\} = \{0,4,8,12\}.$$

Therefore, $D_k \notin E_{\boldsymbol{a}}$ for some $k \in \{0,1,2,3\}$.

Case 4: $\boldsymbol{a} = (1,2,2,2)$ ($E_{\boldsymbol{a}} = \{2^{2s}(16t+14) \mid s \in \mathbb{N}_0, t \in \mathbb{Z}\}$).

We will show that $D_k \notin E_{\boldsymbol{a}}$ for some integer k with $0 \le k \le 6$. We may assume that $D_0 \in E_{\boldsymbol{a}}$, since otherwise we are done. For any integer i, define

$$\varDelta_i = D_i - D_0 = 7(m-4)i - (b_0 + i(m-2))^2 + b_0^2.$$

4.1: $m \equiv 1 \pmod 2$. Note that $\varDelta_i \equiv 0 \pmod 2$ for any i. Moreover, for integers i_1, i_2 with $i_1 \equiv i_2 \pmod 2$, we have $\varDelta_{i_1} \equiv \varDelta_{i_2} \pmod 8$ if and only if

$$\left(\frac{i_1 - i_2}{2}\right)\bigl(7(m-4) - (m-2)((i_1+i_2)(m-2) + 2b_0)\bigr) \equiv 0 \pmod 4.$$

Since both $m-4$ and $m-2$ are odd, this is equivalent to $i_1 \equiv i_2 \pmod 8$. Hence, $\{\Delta_i \pmod 8 \mid i = 0,2,4,6\} = \{0,2,4,6\}$, so $\Delta_k \equiv 4 \pmod 8$ for some $k \in \{0,2,4,6\}$. Therefore, one may show that $D_k = D_0 + \Delta_k \notin E_{\boldsymbol{a}}$.

4.2: $m \equiv 2 \pmod 4$. In this case, one may easily show that $\Delta_2 \equiv 4 \pmod 8$. Hence, $D_2 = D_0 + \Delta_2 \notin E_{\boldsymbol{a}}$.

4.3: $m \equiv 4 \pmod 8$. If $b_0 \equiv 0 \pmod 2$, then $\Delta_1 \equiv 4 \pmod 8$, so $D_1 = D_0 + \Delta_1 \notin E_{\boldsymbol{a}}$. Now assume $b_0 \equiv 1 \pmod 2$. Then $\Delta_i \equiv 0 \pmod 8$ for any i. Moreover, for integers i_1, i_2 with $i_1 \equiv i_2 \pmod 2$, one may show that

$$\Delta_{i_1} \equiv \Delta_{i_2} \pmod{32} \iff i_1 \equiv i_2 \pmod 8.$$

Hence, $\{\Delta_i \pmod{32} \mid i = 0,2,4,6\} = \{0,8,16,24\}$.

If D_0 is of the form $4(16t+14)$, then $D_k = D_0 + \Delta_k \notin E_{\boldsymbol{a}}$ for some $k \in \{0,2,4,6\}$ with $\Delta_k \equiv 16 \pmod{32}$. Otherwise, for some $k \in \{0,2,4,6\}$ with $\Delta_k \equiv 8 \pmod{32}$, we have $D_k = D_0 + \Delta_k \notin E_{\boldsymbol{a}}$.

4.4: $m \equiv 0 \pmod 8$. The proof is quite similar to that of 4.3. ∎

Proof of Theorem 1.2(1). For each $\boldsymbol{a} \in S$, let $A = A_{\boldsymbol{a}}$ and $B = B_{\boldsymbol{a}}$, let

$$I = I_{\boldsymbol{a}} = \left[\frac{A}{2}\left(\frac{m-4}{m-2}\right) - \frac{B}{2}(m-2), \frac{A}{2}\left(\frac{m-4}{m-2}\right) + \frac{B}{2}(m-2)\right]$$

be a closed interval whose length is $B(m-2)$, and let $N \ge C_{\boldsymbol{a}}(m-2)^3$ be an integer. Then by Lemma 4.3, there exists an integer $b \in I$ such that

$$N \equiv b \pmod{m-2} \quad \text{and} \quad Aa - b^2 \notin E_{\boldsymbol{a}},$$

where $a = 2\left(\frac{N-b}{m-2}\right) + b$. Note that $a \equiv b \pmod 2$ and since $C_{\boldsymbol{a}} = \frac{B^2}{8A}$, we have

$$\max_{b \in I}\left[\left(\frac{m-2}{2A}\right)b^2 - \left(\frac{m-4}{2}\right)b\right] = \frac{B^2}{8A}(m-2)^3 - \frac{A(m-4)^2}{8(m-2)} < C_{\boldsymbol{a}}(m-2)^3$$

for any $m \ge 5$. Since $Aa - b^2 > 0$ if and only if $N > \left(\frac{m-2}{2A}\right)b^2 - \left(\frac{m-4}{2}\right)b$, we have $Aa - b^2 > 0$. Therefore, by Lemma 4.2, there are integers $x_1, \ldots, x_4$ such that

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = a \quad \text{and} \quad a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b.$$

Therefore, by Lemma 4.1, $N = \frac{m-2}{2}(a-b) + b$ is represented by $P_{m,\boldsymbol{a}}$. ∎

Proof of Theorem 1.2(2). Let $\boldsymbol{a}$ be either $(1,1,1,1)$ or $(1,1,2,2)$, $A = A_{\boldsymbol{a}}$ and let $m = 4l+4$ for some integer $l \ge 2$. Let N_0 be a positive integer such that

$$N_0 \nrightarrow P_{m,\boldsymbol{a}} \quad \text{and} \quad (2l+1)N_0 + Al^2 \equiv 0 \pmod 4.$$

Note that such an integer exists; for example, one may take $N_0 = 10$ when $\boldsymbol{a} = (1,1,2,2)$ and $l \ge 2$ is odd, and $N_0 = 8$ otherwise. Moreover, we put

$$n = \mathrm{ord}_{2l+1}(2) = \mathrm{ord}_{2l+1}(l+1),$$

where $\mathrm{ord}_b(a)$ is the smallest positive integer k such that $a^k \equiv 1 \pmod b$ for any positive integers a and b with $\gcd(a,b) = 1$.

We claim that for any $t \in \mathbb{N}_0$, the integer

$$N_t = N_{t,\boldsymbol{a}} := \frac{4^{nt}((2l+1)N_0 + Al^2) - Al^2}{2l+1}$$

is not represented by $P_{m,\boldsymbol{a}}$. Since $N_t \in \mathbb{N}$, the theorem follows directly from this claim. We will show that for $t \in \mathbb{N}$, $N_t \to P_{m,\boldsymbol{a}}$ implies $N_{t-1} \to P_{m,\boldsymbol{a}}$. Then since N_0 is not represented by $P_{m,\boldsymbol{a}}$, the claim follows. Note that for any integer N, we have

$$N = P_{m,\boldsymbol{a}}(x_1, x_2, x_3, x_4) \iff (2l+1)N + Al^2 = \sum_{i=1}^{4} a_i((2l+1)x_i - l)^2.$$

Assume that $N_t = P_{m,\boldsymbol{a}}(x_1, x_2, x_3, x_4)$ for some $x_1, x_2, x_3, x_4 \in \mathbb{Z}$. Then

$$4^n((2l+1)N_{t-1} + Al^2) = 4^{nt}((2l+1)N_0 + Al^2) = \sum_{i=1}^{4} a_i((2l+1)x_i - l)^2.$$

Since the left hand side is a multiple of 16, we have $(2l+1)x_i - l \equiv 0 \pmod 2$ for any $1 \le i \le 4$. Since $(2l+1)x_i - l)/2 \equiv -l(l+1) \pmod{2l+1}$, there exist $y_1, y_2, y_3, y_4 \in \mathbb{Z}$ such that

$$4^{n-1}((2l+1)N_{t-1} + Al^2) = \sum_{i=1}^{4} a_i((2l+1)y_i - l(l+1))^2.$$

Applying similar arguments recursively, we have

$$(2l+1)N_{t-1} + Al^2 = \sum_{i=1}^{4} a_i((2l+1)z_i - l(l+1)^n)^2$$

for some $z_1, z_2, z_3, z_4 \in \mathbb{Z}$. Since $(2l+1)z_i - l(l+1)^n \equiv -l \pmod{2l+1}$,

$$(2l+1)N_{t-1} + Al^2 = \sum_{i=1}^{4} a_i((2l+1)z_i' - l)^2$$

for some $z_1', z_2', z_3', z_4' \in \mathbb{Z}$, and so $N_{t-1} \to P_{m,\boldsymbol{a}}$. This proves the claim, and hence the theorem. ∎

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