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Dhyan ARANHA & Piotr PSTRĄGOWSKI

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THE INTRINSIC NORMAL CONE FOR ARTIN STACKS

by Dhyan ARANHA & Piotr PSTRĄGOWSKI

ABSTRACT. — We extend the construction of the normal cone of a closed embedding of schemes to any locally morphism of finite type of higher Artin stacks and show that in the Deligne-Mumford case our construction recovers the relative intrinsic normal cone of Behrend and Fantechi. We characterize our extension as the unique one satisfying a short list of axioms, and use it to construct the deformation to the normal cone. As an application of our methods, we associate to any morphism of Artin stacks equipped with a choice of a global perfect obstruction theory a relative virtual fundamental class in the Chow group of Kresch.

RÉSUMÉ. — Nous étendons la construction du cône normal d'une immersion fermée de schémas à tout morphisme localement de type fini de champs d'Artin supérieurs et montrons que dans le cas de Deligne-Mumford notre construction retrouve le cône normal intrinsèque relatif de Behrend et Fantechi. Nous caractérisons notre extension comme l'unique satisfaisant une courte liste d'axiomes, et l'utilisons pour construire la déformation du cône normal. Comme application de nos méthodes, nous associons à tout morphisme de champs d'Artin muni d'un choix d'une théorie d'obstruction parfaite globale une classe fondamentale virtuelle relative dans le groupe de Chow de Kresch.

1. Introduction

Moduli spaces often have an expected or so called “virtual” dimension at each point, which is a lower bound for the actual dimension. An important example due to Kontsevich [14] is given by the moduli stack $\overline{\mathcal{M}}_{g,n}(V, \beta)$ of stable maps of degree $\beta \in H_2(V)$ from n -marked prestable curves of genus g into a smooth projective variety V . This is a proper Deligne–Mumford stack whose actual dimension at a given point (C, f) will in general be larger than its virtual dimension, given by

$$3g - 3 + n + \chi(C, f^*T_V) = (1 - g)(\dim V - 3) - \beta(\omega_V) + n.$$

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The moduli of stable curves is used to define Gromov–Witten invariants of V , and one of the key ingredients is to be able to construct a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(V, \beta)]^{\text{vir}} \in A_{(1-g)(\dim V - 3) - \beta(\omega_V) + n}(\overline{\mathcal{M}}_{g,n}(V, \beta)).$$

of the expected dimension. There is a general procedure for constructing such classes whenever the moduli space in question is a Deligne–Mumford stack equipped with a choice of a perfect obstruction theory due to Behrend and Fantechi [5].

The construction of the above virtual fundamental classes allowed Behrend to construct Gromov–Witten invariants for arbitrary smooth projective varieties and arbitrary genus satisfying the axioms of Kontsevich and Manin [4, 15]. Virtual fundamental classes are also foundational objects of several other enumerative theories, such as those of Donaldson–Thomas, Pandharipande–Thomas, and Vafa–Witten invariants [7, 22, 26].

In this paper, we extend the methods of Behrend and Fantechi to the setting of higher Artin stacks. Reducing to the classical case, where we have access to the Chow groups of Kresch, we are then able to construct a virtual fundamental class in a wide context.

THEOREM 1.1. — *Let $\mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of finite type Artin stacks. Suppose that \mathcal{Y} is of pure dimension r and that we have a perfect obstruction theory $\mathcal{E} \rightarrow L_{\mathcal{X}/\mathcal{Y}}$ which admits a global resolution. Then, there is a well-defined virtual fundamental class $[\mathcal{X} \rightarrow \mathcal{Y}, \mathcal{E}]^{\text{vir}} \in \text{CH}_{r+\chi(\mathcal{E})}(\mathcal{X})$ in the Chow group of \mathcal{X} .*

To mention a couple of examples to which Theorem 1.1 applies, we have

- (1) the moduli of twisted stable maps whose target is an Artin stack, see Example 8.12 and
- (2) the 0-truncation of any quasi-smooth morphism of derived Artin stacks, in particular the moduli spaces arising in Donaldson–Thomas theory, see Example 8.14.

The theory of twisted stable maps is of particular importance, as it allows one to construct generalizations of Gromov–Witten invariants; this will be explored in forthcoming work.

To obtain the needed virtual fundamental class, Behrend and Fantechi associate to any morphism $\mathcal{X} \rightarrow \mathcal{Y}$ of Deligne–Mumford type the *intrinsic normal cone* $\mathcal{C}_{\mathcal{X}/\mathcal{Y}}$, which is a closed substack of the normal sheaf. Informally, the virtual fundamental class is then obtained by intersecting the class of the normal cone with the zero section of abelian cone of the chosen perfect obstruction theory, mirroring a classical construction of Fulton [8].

In general, the intrinsic normal cone of a Deligne–Mumford stack is only Artin rather than Deligne–Mumford, and likewise it turns out that the natural definition of the intrinsic normal cone $\mathcal{C}_{\mathcal{X}}$ where \mathcal{X} is Artin forces the cone to be a *higher* Artin stack; that is, an étale sheaf on the site of schemes valued in the ∞ -category of spaces rather than in groupoids. Thus, to obtain the correct generalization we are forced to work in the setting of higher algebraic stacks.

Since we work with ∞ -categories, it is often easier to uniquely characterize a given construction rather than to write it down directly. This is exactly what we do, and so our work offers some conceptual clarification even in the classical context.

Let us say that a *relative higher Artin stack* $\mathcal{X} \rightarrow \mathcal{Y}$ is a locally of finite type morphism of higher Artin stacks. We denote the ∞ -category of relative higher Artin stacks with morphisms given by commutative squares by RelArt . We will say a morphism of relative higher Artin stacks

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{Y}' \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

is *smooth* if both vertical arrows are smooth and *surjective* if both vertical arrows are surjective.

If $\mathcal{X} \rightarrow \mathcal{Y}$ is a relative Artin stack, then its *normal sheaf* $\mathcal{N}_{\mathcal{X}\mathcal{Y}} := \mathbb{V}_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{Y}}[-1])$ is defined as the abelian cone associated to the shift of the cotangent complex. Our first result provides a unique characterization of this construction.

THEOREM 1.2 (Theorem 5.11). — *The normal sheaf functor $\mathcal{N} : \text{RelArt} \rightarrow \text{Art}$ is characterized uniquely by the following properties:*

- (1) *If $U \hookrightarrow V$ is a closed embedding of schemes, then $\mathcal{N}_U V$ coincides with the normal sheaf in the classical sense, that is, $\mathcal{N}_U V \simeq \mathbb{V}_U(I/I^2)$, where I is the ideal sheaf*
- (2) *\mathcal{N} preserves coproducts.*
- (3) *\mathcal{N} preserves smooth and smoothly surjective maps.*
- (4) *\mathcal{N} commutes with pullbacks along smooth morphisms.*

It is not difficult to see that any functor satisfying the above properties is a cosheaf on RelArt with respect to the topology determined by smoothly surjective maps, and so Theorem 1.2 is strongly related to the flat descent for the cotangent complex [6].

Since we work only with discrete rings, the abelian cone associated to a quasi-coherent sheaf depends only on its coconnective part, which one can in fact recover from the abelian cone. Thus, Theorem 1.2 can be interpreted as saying that the “naive” cotangent complex $(L_{\mathcal{X}/\mathcal{Y}})_{\leq 1}$ is already determined by its behaviour on closed embeddings of schemes.

THEOREM 1.3 (Theorems 6.2, 6.3). — *There exists a unique functor $\mathcal{C} : \text{RelArt} \rightarrow \text{Art}$, called the normal cone, which satisfies the following properties:*

- (1) *If $U \hookrightarrow V$ is a closed embedding of schemes, then $\mathcal{C}_U V$ coincides with the classical normal cone, that is, $\mathcal{C}_U V \simeq \text{Spec}_U(\bigoplus I^k/I^{k+1})$, where I is the ideal sheaf.*
- (2) *\mathcal{C} preserves coproducts.*
- (3) *\mathcal{C} preserves smooth and smoothly surjective maps.*
- (4) *\mathcal{C} commutes with pullbacks along smooth morphisms of relative Artin stacks.*

Moreover, there is a natural map $\mathcal{C}_{\mathcal{X}}\mathcal{Y} \hookrightarrow \mathcal{N}_{\mathcal{X}}\mathcal{Y}$ which is a closed embedding for an arbitrary relative higher Artin stack $\mathcal{X} \rightarrow \mathcal{Y}$.

Note that Theorem 1.3 is qualitatively different from our axiomatization of the normal sheaf, where we have the construction using the cotangent complex, as part of the statement is that the needed functor exists. Rather, Theorem 1.2 should be thought of as suggesting that the above set of axioms on the normal cone is the right one. This is further evidenced by the following comparison with the construction of Behrend and Fantechi.

THEOREM 1.4 (Theorem 6.9). — *Let $\mathcal{X} \rightarrow \mathcal{Y}$ be relatively Deligne–Mumford morphism of Artin stacks of finite type. Then, the normal cone $\mathcal{C}_{\mathcal{X}}\mathcal{Y}$ coincides with the relative intrinsic normal cone of Behrend and Fantechi.*

The proofs of Theorem 1.2 and Theorem 1.3 use what we call *adapted cosheaves*. Roughly, a functor $F : \mathcal{C} \rightarrow \mathcal{H}$ from an ∞ -site into an ∞ -topos is an adapted cosheaf if it satisfies descent and preserves pullbacks along a distinguished class of geometric morphisms which contains all coverings. Our main result shows that an adapted cosheaves are stable under left Kan extension from a generating subcategory.

In our case, \mathcal{C} is the ∞ -category of relative higher Artin stacks, the distinguished class of maps is given by smooth morphisms, and the generating subcategory is the category of closed embeddings of schemes. This method is very general, and allows one to construct other functors related to the normal cone, for example the deformation space.

THEOREM 1.5 (Theorem 7.2). — *For any relative higher Artin stack $\mathcal{X} \rightarrow \mathcal{Y}$ there exists a higher Artin stack $M_{\mathcal{X}}^{\circ} \mathcal{Y}$ which fits into a commutative diagram*

$$\begin{array}{ccc} \mathcal{X} \times \mathbb{P}^1 & \hookrightarrow & M_{\mathcal{X}}^{\circ} \mathcal{Y} \\ & \searrow & \swarrow \\ & \mathbb{P}^1 & \end{array}$$

where both vertical arrows are flat and such that

- (1) over $\mathbb{A}^1 \simeq \mathbb{P}^1 - \{\infty\}$, the horizontal arrow is equivalent to $\mathcal{X} \times \mathbb{A}^1 \hookrightarrow \mathcal{Y} \times \mathbb{A}^1$ and
- (2) over $\{\infty\}$, the horizontal arrow is equivalent to $\mathcal{X} \hookrightarrow C_{\mathcal{X}} \mathcal{Y}$.

We point out that the assumptions of being Artin in the classical sense and of the existence of a global resolution appearing in Theorem 1.1 stem only from the fact that we are not aware of a theory of *integral* Chow groups for higher Artin stacks which has the needed properties.

In the rational case, the needed Chow groups were constructed in great generality by Khan, see [12], so that our methods yield the needed virtual fundamental class for any choice of perfect obstruction theory. Similarly, these virtual fundamental classes always exist in K -theory, see Remark 8.10.

Lastly, the restriction to morphisms locally of finite type comes from the fact that any such morphism of higher Artin stacks admits a smooth surjection from a closed embedding of schemes. We believe it is likely that the normal cone satisfies the analogues of the axioms of Theorem 1.3 with the class of smooth maps replaced by that of flat maps, as that is the case for the normal sheaf. If that was the case, the locally of finite type assumption could be removed throughout.

1.1. Notation and conventions

In the sequel we will use the term *Artin stack* to refer to what we called a higher Artin stack in the introduction, see Definition 2.5. Under this convention, classical Artin stacks correspond to what we call 1-*Artin* stacks.

To tackle coherence difficulties inherent in working with functors valued in spaces, we will use the framework of ∞ -*categories*, as developed by Joyal and Lurie. The standard reference is [19].

Throughout this paper, we will be working over a fixed field k . Note that even though we work with ∞ -categories, we will be indexing our stacks

using the category of *discrete* commutative k -algebras, which we denote by CAlg_k . Derived analogues of commutative rings will appear only indirectly.

1.2. Relation to other works

As mentioned in the introduction, Adeel Khan has constructed Chow groups of derived higher Artin stacks rationally, using motivic homotopy theory [12]. In the same work, Khan associates a canonical virtual fundamental class to any quasi-smooth morphism of derived Artin stacks using the axiomatics of bivariant homology theories. It follows from his work that in this context, the image of Khan's fundamental class in the Chow group of the classical truncation agrees with the one constructed in this paper, where the perfect obstruction theory is the one induced by the chosen quasi-smooth derived enhancement.

1.3. Acknowledgements

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Lastly, after putting this work on the Arxiv, Adeel Khan alerted us about [12] which had been put up some days before ours, see above. We would like to thank Adeel for very graciously inviting us to Regensburg to speak about our work as well as for the time he took to explain his work to us.

2. Preliminaries

The goal of this section is to introduce the reader to the language we will use throughout this paper. Our aim is to simply collect all the background material that will be needed to understand the main results. To this end, we have taken a minimalist approach, providing references to more thorough treatments in the literature.

2.1. Higher Artin stacks

In algebraic geometry, especially in moduli theory, one often cares about functors which are not naturally valued in sets, but rather in groupoids; such functors are then given geometric interpretation through the theory of algebraic stacks.

For some purposes (in our case, of giving a well-behaved definition of an intrinsic normal cone of an algebraic stack) even the category of functors valued in groupoids is not sufficient [18, 28]. One is then naturally led to consider functors valued in *spaces*; the geometric interpretation of such functors is given by the theory of *higher* algebraic stacks, first studied by Simpson [25].

Remark 2.1. — For the purpose of this paper, the word *space* will be synonymous with the word ∞ -*groupoid*; that is, an ∞ -category where all of the morphisms are invertible [19, 1.2.5.1].

The need for a suitable theory of higher groupoids needed to define higher algebraic stacks was recognized by Simpson [25]. More precisely, to define n -stacks one needs n -groupoids, which we can identify with n -truncated ∞ -groupoids, or what we will call in this paper *n -truncated spaces*.

DEFINITION 2.2. — *The category $\mathcal{A}ff$ of affine k -schemes is the opposite of the category $\mathcal{C}Alg_k$ of discrete k -algebras. We will consider $\mathcal{A}ff$ as a Grothendieck site with respect to the étale topology.*

A scheme can be identified with a particular sheaf of sets over $\mathcal{A}ff$; similarly, an algebraic stack over k can be identified with an appropriate sheaf of groupoids. As explained above, when working with higher stacks, we instead allow sheaves valued in spaces.

DEFINITION 2.3. — *A prestack X is a presheaf over $\mathcal{A}ff$ valued in the ∞ -category \mathcal{S} of spaces; that is, it is a functor $X : \mathcal{A}ff^{op} \rightarrow \mathcal{S}$. We say a prestack X is a stack if it is a sheaf with respect to the étale topology. We denote the ∞ -categories of (pre)stacks by $\mathcal{P}rStk$ and Stk .*

Note that any set can be considered as a discrete space, so that any presheaf of sets gives rise to a prestack as above. Moreover, in this case the ∞ -categorical sheaf condition reduces to the usual one.

Remark 2.4. — Recall that if X is a presheaf of sets on the site of affine schemes, then X is a sheaf if and only if it preserves products and for any étale surjection $U \rightarrow V$ of affine k -schemes, the diagram $X(V) \rightarrow X(U) \rightrightarrows X(U \times_V U)$ is a limit.

In the case of a presheaf of spaces, to only consider the two-fold intersections is not enough, and one instead requires that the whole diagram

$$X(V) \longrightarrow X(U) \rightrightarrows X(U \times_V U) \rightrightarrows X(U \times_V U \times_V U) \dots$$

induced by the Čech nerve of $U \rightarrow V$ is a limit diagram of spaces.

Note that in our definition of a stack, we allow sheaves of spaces, but we still index them by the classical category of discrete k -algebras, rather than a derived variant. Thus, we are working within the framework of classical, rather than derived, algebraic geometry.

Nevertheless, a lot of the definitions we will work with are analogous to the ones which became standard in derived algebraic geometry. In particular, our notion of an Artin stack is analogous to the one appearing in Lurie's thesis [18], see also [25, 29] for earlier treatments.

DEFINITION 2.5. — *We define n -Artin stacks and smooth n -Artin stacks inductively as follows:*

- (1) *We say a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of stacks is a relative 0-Artin stack if for any map $g : \mathrm{Spec}(A) \rightarrow \mathcal{Y}$, the fiber product $\mathrm{Spec}(A) \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space.*
- (2) *We say that relative 0-Artin stack $f : \mathcal{X} \rightarrow \mathcal{Y}$ is smooth if each of the associated maps $\mathrm{Spec}(A) \times_{\mathcal{Y}} \mathcal{X} \rightarrow \mathrm{Spec}(A)$ is smooth as a morphism of algebraic spaces.*
- (3) *For $n > 0$, we say a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of stacks is a relative n -Artin stack if for any map $\mathrm{Spec}(A) \rightarrow \mathcal{Y}$ there exists a smooth surjection $U \rightarrow \mathrm{Spec}(A) \times_{\mathcal{Y}} \mathcal{X}$ which is a relative $(n - 1)$ -Artin stack, where U is an algebraic space.*
- (4) *We say that a relative n -Artin stack $f : \mathcal{X} \rightarrow \mathcal{Y}$ is smooth if for every $\mathrm{Spec}(A) \rightarrow \mathcal{X}$ there exists a smooth surjection $U \rightarrow \mathrm{Spec}(A) \times_{\mathcal{Y}} \mathcal{X}$ as in the previous item, such that $U \rightarrow \mathrm{Spec}(A)$ is a smooth morphism of schemes.*
- (5) *We say that a stack \mathcal{X} is an n -Artin stack if it is a relative n -Artin stack over $\mathrm{Spec}(k)$ and will refer to an Artin stack as a stack \mathcal{X} which is n -Artin for some n .*

We denote the ∞ -category Artin stacks by Art .

The next proposition is a collection of basic properties pertaining to higher Artin stacks.

PROPOSITION 2.6. — *We have that*

- (1) *Any relative n -Artin stack is also a relative m -Artin stack for any $m \geq n$.*

- (2) A pullback of a (smooth) relative n -Artin stack is (smooth) relative n -Artin
- (3) Let $\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$ be a pair of composable morphisms. If both f and g are (smooth) relative n -Artin stacks, then so is $g \circ f$.
- (4) Suppose that $n > 0$ and that we are given morphisms $\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z}$, where $\mathcal{X} \rightarrow \mathcal{Y}$ is an $(n - 1)$ -submersion and $\mathcal{X} \rightarrow \mathcal{Z}$ is a relative n -Artin stack. Then $\mathcal{Y} \rightarrow \mathcal{Z}$ is a relative n -Artin stack.
- (5) Let $\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$ be a composable pair of morphisms and $n \geq 1$. If $g \circ f$ is a relative $(n - 1)$ -Artin stack and g is a relative n -Artin stack, then f is a relative $(n - 1)$ -Artin stack.

Proof. — This is [18, 5.1.4]. □

Observe that one of the pleasant consequences of Proposition 2.6 is that any morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of n -Artin stacks is automatically a relative n -Artin stack.

Example 2.7. — Let \mathbb{G}_m denote the multiplicative group scheme over k . Classically, we can form the quotient 1-Artin stack $\mathcal{B}\mathbb{G}_m := [\mathrm{Spec}(k) // \mathbb{G}_m]$. From a homotopy-theoretic perspective, the stack $\mathcal{B}\mathbb{G}_m$ is the étale sheafification of the presheaf

$$\begin{aligned} \mathcal{B}\mathbb{G}_m : \mathrm{Aff}^{\mathrm{op}} &\longrightarrow \mathcal{S} \\ R &\longmapsto K(R^\times, 1) \end{aligned}$$

where $K(R^\times, 1)$ is the first Eilenberg–MacLane space of the abelian group of units of R . Since \mathbb{G}_m is abelian, it is very natural to consider prestacks

$$R \longmapsto K(R^\times, n)$$

for any $n \geq 1$. We define $\mathcal{B}^n\mathbb{G}_m$ to be étale sheafification of the presheaf defined by the above formula, one can show that it is an n -Artin stack.

To see this in the basic case of $n = 2$, note that we have an equivalence of stacks

$$\mathcal{B}^2\mathbb{G}_m \simeq \varinjlim (\dots \mathcal{B}\mathbb{G}_m \times \mathcal{B}\mathbb{G}_m \rightrightarrows \mathcal{B}\mathbb{G}_m \rightrightarrows \mathrm{Spec}(k)).$$

We claim that the induced map $\mathrm{Spec}(k) \rightarrow \mathcal{B}^2\mathbb{G}_m$ is an 2-submersion, it is clearly surjective. Furthermore, since the diagram

$$\begin{array}{ccc} \mathcal{B}\mathbb{G}_m & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathcal{B}^2\mathbb{G}_m. \end{array}$$

is a pullback diagram and the maps $\mathcal{B}\mathbb{G}_m \rightarrow *$ are smooth relative 1-Artin stacks on the account of their fibers being \mathbb{G}_m , we conclude that $\mathrm{Spec}(k) \rightarrow \mathcal{B}^2\mathbb{G}_m$ is a 2-submersion. More generally, in the discussion above we could replace \mathbb{G}_m with any smooth abelian group scheme.

DEFINITION 2.8. — *Suppose that P is a property of morphisms of schemes over k which is local both in the source and target in the smooth topology. Then, we say a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of Artin stacks has property P if for any $\mathrm{Spec}(A) \rightarrow \mathcal{Y}$ there exists a smooth surjection $S \rightarrow \mathrm{Spec}(A) \times_{\mathcal{Y}} \mathcal{X}$ from a scheme such that $S \rightarrow \mathrm{Spec}(A)$ has property P .*

Example 2.9. — Properties local both in the source and target in the smooth topology to which we might want to apply Definition 2.8 include being *locally of finite type*, *flat* and *smooth*. Note that in the smooth case the resulting notion will coincide with that of a smooth relative Artin stack of Definition 2.5, as expected.

A lot of the constructions in this note will be done relative to a fixed stack, that is, will take place in the overcategory Stk/\mathcal{X} . This ∞ -category can be itself described as an ∞ -category of sheaves in a standard way, as we now describe.

Remark 2.10. — If \mathcal{C} is a small ∞ -category and $\mathcal{X} \in \mathrm{PSh}(\mathcal{C})$ is a presheaf, there is a canonical equivalence

$$\mathrm{PSh}(\mathcal{C})/\mathcal{X} \simeq \mathrm{PSh}(\mathcal{C}/\mathcal{X}),$$

between the overcategory of the presheaves and presheaves on the overcategory [19, 5.1.6.12]. Under this equivalence, an object $\mathcal{F} \in \mathrm{PSh}(\mathcal{C})/\mathcal{X}$ corresponds to an object which assigns to a morphism $f : U \rightarrow \mathcal{X}(U)$, which we can identify with a point $f \in X(U)$, the fiber product $\{f\} \times_{\mathcal{X}(U)} \tilde{\mathcal{F}}(U)$. Moreover, if \mathcal{C} is an ∞ -site, then there is an induced topology on the overcategory \mathcal{C}/\mathcal{X} and the above equivalence restricts to one of the form $\mathrm{Sh}(\mathcal{C})/\mathcal{X} \simeq \mathrm{Sh}(\mathcal{C}/\mathcal{X})$.

In our situation, we will take \mathcal{C} to be the site $\mathcal{A}ff$, and so we deduce that for an arbitrary stack $\mathcal{X} \in \mathrm{Stk}$ there is a canonical equivalence $\mathrm{Stk}/\mathcal{X} \simeq \mathrm{Sh}(\mathcal{A}ff/\mathcal{X})$. In this note we will use this equivalence implicitly, blurring the distinction between the two ∞ -categories.

2.2. Quasi-coherent sheaves

If R is a ring, we can associate to it the derived ∞ -category $\mathcal{D}(R)$, which is an ∞ -categorical enhancement of the classical unbounded derived category. This ∞ -category is stable and admits a canonical t -structure whose

heart $\mathcal{D}(R)^\heartsuit := \mathcal{D}(R)_{\geq 0} \cap \mathcal{D}(R)_{\leq 0}$ is given by the abelian category of R -modules.

One advantage of working with stable ∞ -categories, rather than triangulated categories, is that the former can be glued together in a controlled manner. This allows one to give a transparent definition of a quasi-coherent sheaf on an Artin stack, which we now review.

DEFINITION 2.11. — *Let $\mathcal{X} \in \text{Stk}$ be a stack. We define the stable ∞ -category $\text{QCoh}(\mathcal{X})$ of quasi-coherent sheaves on \mathcal{X} as the limit*

$$\text{QCoh}(\mathcal{X}) := \varprojlim_{\text{Spec}(A) \rightarrow \mathcal{X}} \mathcal{D}(A)$$

taken over the category of affine schemes equipped with a map into \mathcal{X} , with the maps between module ∞ -categories given by extension of scalars.

Example 2.12. — If $\mathcal{X} \simeq \text{Spec}(A)$ is affine, then the ∞ -category of affines over \mathcal{X} has a terminal object given by the identity and we obtain

$$\text{QCoh}(\text{Spec}(A)) \simeq \mathcal{D}(A).$$

In particular, notice that according to this convention a quasi-coherent sheaf on $\text{Spec}(A)$ is an object of the derived ∞ -category rather than a discrete A -module.

According to Definition 2.11, a quasi-coherent sheaf \mathcal{F} on \mathcal{X} consists of an assignment of an object $\mathcal{F}(\text{Spec}(A)) \in \mathcal{D}(A)$ for each map $\eta : \text{Spec}(A) \rightarrow \mathcal{X}$, equivalently, for each point $\eta \in X(A)$. This data is required to be compatible in the sense that we have distinguished equivalences $B \otimes_A \mathcal{F}(\text{Spec}(A)) \simeq \mathcal{F}(\text{Spec}(B))$ for each composite $\text{Spec}(B) \rightarrow \text{Spec}(A) \rightarrow \mathcal{X}$, as well as higher coherence data.

More formally, we define $\text{QCoh}(\mathcal{X})$ as follows. One can construct an ∞ -category \mathcal{D} whose objects are pairs (A, M) , where $A \in \text{CAlg}_k$ is a discrete k -algebra and $M \in \mathcal{D}(A)$, and such that the obvious functor $\mathcal{D} \rightarrow \text{CAlg}_k$ is a coCartesian fibration. Then, $\text{QCoh}(\mathcal{X})$ is given by the ∞ -category of CoCartesian sections of the pullback fibration $\mathcal{D} \times_{\text{CAlg}_k} (\text{CAlg}_k)_{/\mathcal{X}} \rightarrow (\text{CAlg}_k)_{/\mathcal{X}}$.

Example 2.13. — The *structure sheaf* $\mathcal{O}_{\mathcal{X}}$ of a stack \mathcal{X} is the quasi-coherent sheaf given by

$$\mathcal{O}_{\mathcal{X}}(\text{Spec}(A) \rightarrow \mathcal{X}) := A,$$

where we consider the right hand side as an element of the heart of $\mathcal{D}(A)$.

Remark 2.14. — Note that Definition 2.11 makes sense already when \mathcal{X} is a prestack, but one can show that the ∞ -category of quasi-coherent sheaves is the same on a prestack and its stackification. In other words, formation of QCoh satisfies descent with respect to the étale topology, in fact, even with respect to the flat topology [21, 6.2.3.1].

Remark 2.15. — If \mathcal{X} is Artin, then one can replace the indexing ∞ -category in Definition 2.11 by the category of those affines $\mathrm{Spec}(A) \rightarrow \mathcal{X}$ which are smooth over \mathcal{X} , see [9, 1.4.2].

As a limit of stable, presentable ∞ -categories, $\mathrm{QCoh}(\mathcal{X})$ is stable and presentable for any stack \mathcal{X} . Moreover, it is functorial; for any morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of stacks we have an induced adjunction

$$f^* \dashv f_* : \mathrm{QCoh}(\mathcal{Y}) \rightleftarrows \mathrm{QCoh}(\mathcal{X}).$$

Using the informal description given above, f^* is defined by $(f^*\mathcal{F})(\mathrm{Spec}(A)) := \mathcal{F}(\mathrm{Spec}(A))$, and its right adjoint exists for abstract reasons. Notice in particular that if $f : \mathrm{Spec}(A) \rightarrow \mathcal{X}$ is a map from an affine scheme, then as an object of $\mathrm{QCoh}(\mathrm{Spec}(A)) \simeq \mathcal{D}(A)$, the pullback $f^*\mathcal{F}$ corresponds to $\mathcal{F}(\mathrm{Spec}(A))$.

The ∞ -category $\mathrm{QCoh}(\mathcal{X})$ admits a canonical t -structure in which \mathcal{F} is connective if and only if $\mathcal{F}(\mathrm{Spec}(A))$ is connective for any morphism $\mathrm{Spec}(A) \rightarrow \mathcal{X}$. In general, this t -structure is not well-behaved, but the situation is much better in the Artin case.

LEMMA 2.16. — *Let \mathcal{X} be Artin. Then, $\mathrm{QCoh}(\mathcal{X})$ admits a t -structure in which a quasi-coherent sheaf is (co)connective if and only if for any smooth atlas $p : \mathrm{Spec}(U) \rightarrow \mathcal{X}$, the quasi-coherent sheaf $p^*\mathcal{F}$ is (co)connective.*

Proof. — This is [18, 5.2.4]. □

Note that by definition, the pullback functor $f^* : \mathrm{QCoh}(\mathcal{Y}) \rightarrow \mathrm{QCoh}(\mathcal{X})$ preserves connective objects, which implies formally that its right adjoint f_* preserves coconnective objects. If f is a smooth morphism of Artin stacks, then by Lemma 2.16 above f^* also preserves coconnectivity.

DEFINITION 2.17. — *Let P be a property of objects of the derived ∞ -category of a ring which is stable under arbitrary base-change. Then, if \mathcal{X} is a stack and $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})$, we say \mathcal{E} has property P if $f^*\mathcal{E} \in \mathcal{D}(A)$ has property P for any $f : \mathrm{Spec}(A) \rightarrow \mathcal{X}$.*

The properties to which Definition 2.17 applies which will be of interest to us are the properties of being perfect, perfect of given amplitude and

perfect up to order n . These properties can be defined in a homotopy-invariant way, see [20, 21], but for the convenience of the reader we will rephrase them in terms of chain complexes. We remind the reader that we are using *homological* grading conventions.

DEFINITION 2.18. — *Let A be a discrete k -algebra and M an object of $\mathcal{D}(A)$. We say that M is*

- (1) *perfect if it can be represented by a bounded chain complex of finitely generated projectives.*
- (2) *perfect of amplitude $[a, b]$ if it can be represented by a bounded chain complex of finitely generated projectives which can be chosen to vanish outside of degrees $d \in [a, b]$.*
- (3) *perfect to order n if it can be represented by a chain complex which is bounded from below and consists of finitely generated projectives in degrees $d \leq n$.*

2.3. Cotangent complex

Throughout the paper, we will need some basic properties of the cotangent complex. The latter is most naturally defined and constructed in the setting of derived algebraic geometry, and since several thorough references exist in the latter context, we will keep our exposition to the minimum.

Remark 2.19 (Derived algebraic geometry in this paper). — While we introduce the cotangent complex in the language of derived algebraic geometry, as is most natural, the rest of the paper will not use any derived notions. A reader already familiar with the cotangent complex can safely skip this subsection.

The construction of the cotangent complex for commutative rings first appeared in the work of André and Quillen [1, 23]. Roughly speaking given a commutative k -algebra A the cotangent complex $L_{A/k}$ controls the deformation theory of A . More precisely there is a bijection between $\mathrm{Ext}_A^1(L_{A/k}, A)$ and infinitesimal deformations of A and the obstruction to pass to a higher order deformations lives in $\mathrm{Ext}_A^2(L_{A/k}, A)$. This was later globalized to the case of schemes by Grothendieck and Illusie [10, 11].

In order to construct the cotangent complex of (higher) Artin stacks rigorously, it is useful to be able to find a universal property that characterizes it uniquely. To this end we take our inspiration from the Kähler differentials:

The Kähler differentials enjoy a universal property which stems from the fact that given a k -algebra A and an A -module M the A -module $\Omega_{A/k}$ corepresents k -linear derivations from A to M . Already in this case it follows from the work of André and Quillen that

$$\mathrm{Hom}_{\mathrm{CAlg}_A}(A, A \oplus M) \simeq \mathrm{Hom}_A(\Omega_{A/k}, M) \simeq \mathrm{Ext}_A^0(L_{A/k}, M).$$

Moreover, they show that

$$\mathrm{Hom}_{\mathcal{H}(A)}(A, A \oplus M[i]) \simeq \mathrm{Ext}_A^i(L_A, M)$$

Where $\mathcal{H}(A)$ is the homotopy category of simplicial A -algebras and $A \oplus M[i]$ is the trivial square zero extension of A by the Eilenberg–MacLane space $K(M, i)$. In particular $A \oplus M[i]$ is generally not discrete. This suggests that to formulate a universal property for the cotangent complex simplicial commutative rings and ∞ -categories must come into the picture even if we only care about discrete commutative k -algebras.

A *derived stack* is an étale sheaf on the opposite of the ∞ -category CAlg_k^{an} of *animated k -algebras*, where the latter is the ∞ -category underlying the model category of simplicial commutative k -algebras [21, 29]. Any commutative k -algebra determines a discrete animated ring, and through left Kan extension one obtains a functor $\iota : \mathrm{Stk} \hookrightarrow d\mathrm{Stk}$ which can be shown to be fully faithful. Moreover, for any stack \mathcal{X} in our sense we have $\mathrm{QCoh}(\mathcal{X}) \simeq \mathrm{QCoh}(\iota\mathcal{X})$.

DEFINITION 2.20 ([21, 17.2.4.2]). — *One says that a morphism $\mathcal{X} \rightarrow \mathcal{Y}$ of derived stacks admits an algebraic cotangent complex if there exists an almost connective quasi-coherent sheaf $L_{\mathcal{X}/\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{X})$ such that for any animated k -algebra A , any point $\eta \in \mathcal{X}(A)$ and any $M \in \mathcal{D}(A)_{\geq 0}$, there is a natural equivalence*

$$\mathrm{map}_{\mathcal{D}(A)_{\geq 0}}(\eta^* L_{\mathcal{X}/\mathcal{Y}}, M) \simeq \mathrm{fib}_\eta(\mathcal{X}(A \oplus M) \longrightarrow \mathcal{X}(A) \times_{\mathcal{Y}(A)} \mathcal{Y}(A \oplus M)).$$

In other words, the algebraic cotangent complex $L_{\mathcal{X}/\mathcal{Y}}$ corepresents derivations in animated k -algebras.

DEFINITION 2.21. — *If $\mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of stacks, then we say it admits a cotangent complex if the associated morphism $\iota\mathcal{X} \rightarrow \iota\mathcal{Y}$ of derived stacks admits an algebraic cotangent complex. In this case, the cotangent complex $L_{\mathcal{X}/\mathcal{Y}}$ is the image of $L_{\iota\mathcal{X}/\iota\mathcal{Y}}$ under the equivalence $\mathrm{QCoh}(\mathcal{X}) \simeq \mathrm{QCoh}(\iota\mathcal{X})$.*

It follows from our definition that if $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $g : \mathcal{Y} \rightarrow \mathcal{Z}$ are a composable pair of morphisms of stacks which admit cotangent complexes,

then we have a canonical cofibre sequence

$$f^*L_{\mathcal{Y}/\mathcal{Z}} \longrightarrow L_{\mathcal{X}/\mathcal{Z}} \longrightarrow L_{\mathcal{X}/\mathcal{Y}}$$

of quasi-coherent sheaves on \mathcal{X} [21, 17.2.5.2]. However, some care must be taken with base-change properties.

WARNING 2.22. — *The inclusion $\mathrm{CAlg}_k \hookrightarrow \mathrm{CAlg}_k^{\mathrm{an}}$ of k -algebras into animated k -algebras does not preserve pushouts, which are given by the tensor product in the source and the derived tensor product in the target. It follows that the embedding $\iota : \mathrm{Stk} \hookrightarrow d\mathrm{Stk}$ of stacks into derived stacks does not preserve pullbacks, and the cotangent complex of Definition 2.21 does not satisfy arbitrary base-change in the same way its derived analogue does.*

Remark 2.23. — One can show that the embedding $i : \mathrm{Stk} \hookrightarrow d\mathrm{Stk}$ commutes with pullbacks along all flat morphisms. It follows that the cotangent complex of Definition 2.21 satisfies flat base-change.

We will now give existence and finiteness statements for the cotangent complex.

PROPOSITION 2.24. — *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a relative n -Artin stack. Then, f admits a cotangent complex $L_{\mathcal{X}/\mathcal{Y}}$ which is $(-n)$ -connective and perfect to order -1 . If f is smooth, then $L_{\mathcal{X}/\mathcal{Y}}$ is perfect of non-positive amplitude.*

Proof. — It is not difficult to see from our inductive definition that if f is relative n -Artin, then the associated morphism $\iota(\mathcal{X} \rightarrow \mathcal{Y})$ of derived stacks is n -representable in the sense of Lurie. The two statements are then given by [21, I.1.2.5.3, I.1.3.3.7]. \square

PROPOSITION 2.25. — *Suppose that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of Artin stacks which is locally of finite type. Then, $L_{\mathcal{X}/\mathcal{Y}}$ is perfect to order 0.*

Proof. — Since the cotangent complex satisfies smooth base-change, we may assume that $\mathcal{Y} = Y$ is affine. Choose a smooth surjection $p : X \rightarrow \mathcal{X}$ from a scheme, it is then enough to show that $p^*L_{\mathcal{X}/Y}$ is perfect to order 0. We have a cofibre sequence

$$p^*L_{\mathcal{X}/Y} \longrightarrow L_{X/Y} \longrightarrow L_{X/X},$$

and since the last term is perfect of non-positive amplitude, we see it is enough to show that $L_{X/Y}$ is perfect to order 0.

Since our notion of perfect of order 0 is local, we may further assume that $X \rightarrow Y$ is a morphism of affine schemes of finite type. The statement is then clear, since $L_{X/Y}$ is connective and $h_0(L_{X/Y}) \simeq \Omega_{X/Y}^0$ is finitely generated. \square

We close with a basic example of a calculation of the cotangent complex.

Example 2.26. — We will identify the cotangent complex of the stack $\mathcal{B}^2\mathbb{G}_m$ of Example 2.7. Let $i : \mathrm{Spec}(k) \rightarrow \mathcal{B}^2\mathbb{G}_m$ be the base-point and consider the diagram

$$\begin{array}{ccccc}
 & & & & \\
 & & & & \\
 \mathrm{Spec}(k) & \xrightarrow{\delta} & \mathcal{B}\mathbb{G}_m & \longrightarrow & \mathrm{Spec}(k) \\
 & \searrow & \downarrow & & \downarrow i \\
 & & \mathrm{Spec}(k) & \xrightarrow{i} & \mathcal{B}^2\mathbb{G}_m.
 \end{array}$$

where the square is a pullback. It follows that i is smooth, and so Remark 2.23 implies that $\delta^*L_{\mathcal{B}\mathbb{G}_m/\mathrm{Spec}(k)} \simeq L_{\mathrm{Spec}(k)/\mathcal{B}^2\mathbb{G}_m}$. We then deduce from the cofibre sequence of cotangent complexes that

$$i^*L_{\mathcal{B}^2\mathbb{G}_m} \simeq \delta^*L_{\mathcal{B}\mathbb{G}_m}[-1] \simeq k[-2].$$

In fact, one can show more generally that for any smooth abelian group scheme G we have $i^*L_{\mathcal{B}^n G} \simeq \mathfrak{g}^\vee[-n]$, where \mathfrak{g} is the Lie algebra.

3. The abelian cone associated to a quasi-coherent sheaf

In this section we study the *abelian cone functor*, a contravariant analogue of the h^1/h^0 functor of Behrend and Fantechi. The main result of this section is Theorem 3.10, which establishes that the abelian cone is Artin under certain conditions.

DEFINITION 3.1. — *Let \mathcal{X} be a stack and $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})$ a quasi-coherent sheaf. Then, the abelian cone associated to \mathcal{E} is the prestack over \mathcal{X} defined by the formula*

$$\mathbb{V}_{\mathcal{X}}(\mathrm{Spec}(A) \xrightarrow{f} \mathcal{X}) := \mathrm{map}_{\mathcal{D}(A)}(f^*\mathcal{E}, A),$$

where the latter is the mapping space in $\mathrm{QCoh}(\mathrm{Spec}(A)) \simeq \mathcal{D}(A)$. This construction yields the abelian cone functor $\mathbb{V}_{\mathcal{X}} : \mathrm{QCoh}(\mathcal{X})^{\mathrm{op}} \rightarrow \mathrm{PrStk}/_{\mathcal{X}}$.

We hope the next example eases the reader with the knowledge this is a straightforward extension of the construction of relative spectrum over a scheme.

Example 3.2. — Let S be a scheme and $\mathcal{E} \in \mathrm{QCoh}(S)^\heartsuit$ be a quasi-coherent sheaf in the classical sense. We claim that $\mathbb{V}_S(\mathcal{E})$ is the relative spectrum of the symmetric algebra on \mathcal{E} .

We have $\mathbb{V}_S(\mathcal{E})(\mathrm{Spec}(A) \rightarrow S) = \mathrm{map}_{\mathcal{D}(A)}(f^*\mathcal{E}, A)$, where $f^* : \mathrm{QCoh}(S) \rightarrow \mathcal{D}(A)$ is the pullback functor between stable ∞ -categories, which classically corresponds to the derived pullback in the sense that $h_i(f^*\mathcal{E}) \simeq R_i f^*\mathcal{E}$. Since \mathcal{E} is connective, so is $f^*\mathcal{E}$, and since A is a discrete commutative ring, it is also coconnective when considered as a module over itself. Since $(f^*\mathcal{E})_{\leq 0} \simeq R_0 f^*\mathcal{E}$, we deduce using the t -structure axioms that

$$\begin{aligned} \mathrm{map}_{\mathcal{D}(A)}(f^*\mathcal{E}, A) &\simeq \mathrm{Hom}_{\mathrm{Mod}_A}(R_0 f^*\mathcal{E}, A) \\ &\simeq \mathrm{map}_{\mathrm{Sch}/S}(\mathrm{Spec}(A), \mathrm{Spec}_S(\mathrm{Sym}(R_0 f^*\mathcal{E}))), \end{aligned}$$

which is what we wanted to show.

PROPOSITION 3.3. — *Let \mathcal{X} be a stack. Then, $\mathbb{V}_{\mathcal{X}}(\mathcal{E})$ is a stack for any $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})$.*

Proof. — We have to show that for any commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(A) & \xrightarrow{p} & \mathrm{Spec}(B) \\ & \searrow f & \swarrow g \\ & \mathcal{X} & \end{array}$$

where p is an étale covering of rings, the diagram

$$\begin{aligned} \mathbb{V}_{\mathcal{X}}(\mathrm{Spec}(A) \rightarrow \mathcal{X}) &\longrightarrow \mathbb{V}_{\mathcal{X}}(\mathrm{Spec}(B) \rightarrow \mathcal{X}) \\ &\rightrightarrows \mathbb{V}_{\mathcal{X}}(\mathrm{Spec}(B \otimes_A B) \rightarrow \mathcal{X}) \rightrightarrows \dots \end{aligned}$$

induced by the Čech nerve of p is a limit diagram of spaces. Unwinding the definitions, we see that we have to prove that

$$\begin{aligned} \mathrm{map}_{\mathcal{D}(B)}(g^*\mathcal{E}, B) &\longrightarrow \mathrm{map}_{\mathcal{D}(A)}(f^*\mathcal{E}, A) \\ &\rightrightarrows \mathrm{map}_{\mathcal{D}(A \otimes_B A)}((f \otimes f)^*\mathcal{E}, A \otimes_B A) \rightrightarrows \dots \end{aligned}$$

is a limit. Using the adjunction between pullback and pushforward, it is enough to verify that

$$g_*B \longrightarrow f_*A \rightrightarrows (f \otimes f)_*(A \otimes_B A) \rightrightarrows \dots$$

is a limit diagram in $\mathrm{QCoh}(\mathcal{X})$. Since g_* is a right adjoint and $f_* \simeq g_* \circ p_*$, this follows from the classical fact that if p is faithfully flat, in particular étale, then

$$B \longrightarrow p_*A \rightrightarrows (p \otimes p)_*(A \otimes_B A) \rightrightarrows \dots$$

is a limit of B -modules. □

Notice that since the mapping space between any two A -modules admits a canonical lift to a connective spectrum, in fact a connective \mathbb{Z} -module, the cone $C_{\mathcal{X}}(\mathcal{E})$ is canonically an abelian stack over \mathcal{X} , that is, an abelian group object in $\mathrm{Stk}/_{\mathcal{X}}$.

LEMMA 3.4. — *The cone functor $\mathbb{V}_{\mathcal{X}} : \mathrm{QCoh}(\mathcal{X})^{op} \rightarrow \mathrm{Ab}(\mathrm{Stk}/_{\mathcal{X}})$ takes colimits to limits.*

Proof. — It is enough to observe that for any $f : \mathrm{Spec}(A) \rightarrow \mathcal{X}$ we have

$$\begin{aligned} \mathbb{V}_{\mathcal{X}}(\varinjlim \mathcal{E}_i)(f) &\simeq \mathrm{map}_A(f^* \varinjlim \mathcal{E}_i, A) \simeq \mathrm{map}_A(\varinjlim f^* \mathcal{E}_i, A) \\ &\simeq \varprojlim \mathrm{map}_A(\mathcal{E}_i, A) \simeq \varprojlim \mathbb{V}_{\mathcal{X}}(\mathcal{E}_i)(f), \end{aligned}$$

where we've used that $f^* : \mathrm{QCoh}(\mathcal{X}) \rightarrow \mathcal{D}(A)$ is a left adjoint. \square

LEMMA 3.5. — *The cone construction satisfies base change in the sense that for any morphism $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ of stacks and $\mathcal{E} \in \mathrm{QCoh}(\mathcal{Y})$ there's a canonical equivalence $\mathcal{X} \times_{\mathcal{Y}} \mathbb{V}_{\mathcal{Y}}(\mathcal{E}) \simeq \mathbb{V}_{\mathcal{X}}(\varphi^* \mathcal{E})$.*

Proof. — For any $f : \mathrm{Spec}(A) \rightarrow \mathcal{X}$ we have

$$\begin{aligned} (\mathcal{X} \times_{\mathcal{Y}} \mathbb{V}_{\mathcal{Y}}(\mathcal{E}))(f) &\simeq \mathbb{V}_{\mathcal{Y}}(\varphi \circ f) \simeq \mathrm{map}_A((\varphi \circ f)^* \mathcal{E}, A) \\ &\simeq \mathrm{map}_A(f^* \varphi^* \mathcal{E}, A) \simeq \mathbb{V}_{\mathcal{X}}(\varphi^* \mathcal{E})(f), \end{aligned}$$

which is what we wanted to show. \square

LEMMA 3.6. — *Let \mathcal{X} be a stack and $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})$. Then, $\mathbb{V}_{\mathcal{X}}(\mathcal{E}) \simeq \mathbb{V}_{\mathcal{X}}(\mathcal{E}_{\leq 0})$.*

Proof. — For $f : \mathrm{Spec}(A) \rightarrow \mathcal{X}$ we have

$$\mathbb{V}_{\mathcal{X}}(\mathcal{E})(f) \simeq \mathrm{map}_A(f^* \mathcal{E}, A) \simeq \mathrm{map}_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{E}, f_* A)$$

and since A is coconnective as a module over itself, the same is true for $f_* A$ and we write further

$$\begin{aligned} \mathrm{map}_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{E}, f_* A) &\simeq \mathrm{map}_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{E}_{\leq 0}, f_* A) \\ &\simeq \mathrm{map}_{\mathrm{QCoh}(\mathcal{X})}(f^* \mathcal{E}_{\leq 0}, A) \simeq \mathbb{V}_{\mathcal{X}}(\mathcal{E}_{\leq 0})(f). \quad \square \end{aligned}$$

LEMMA 3.7. — *Let \mathcal{X} be a stack and $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})$. If \mathcal{E} is connective, then $\mathbb{V}_{\mathcal{X}}(\mathcal{E}) \rightarrow \mathcal{X}$ is affine. If \mathcal{E} is bounded below, then the converse holds.*

Proof. — By Lemma 3.5, we can assume that $\mathcal{X} \simeq \mathrm{Spec}(A)$ is affine. To see the forward direction, observe that by Lemma 3.6, $\mathbb{V}_{\mathrm{Spec}(A)}(\mathcal{E}) \simeq \mathbb{V}_{\mathrm{Spec}(A)}(\mathcal{E}_{\leq 0})$. Since $\mathcal{E}_{\leq 0} \in \mathrm{QCoh}(\mathrm{Spec}(A))^{\heartsuit}$ by assumption, the statement follows from Example 3.2.

Now assume that \mathcal{E} is bounded below and that $\mathbb{V}_{\mathcal{X}}(\mathcal{E}) \rightarrow \mathcal{X}$ is affine, in particular it has discrete fibers. We first claim that for an arbitrary map $f : \text{Spec}(B) \rightarrow \mathcal{X}$ and a B -module M , the space $\text{map}_{\mathcal{D}(B)}(f^*\mathcal{E}, M)$ is discrete.

If we form the trivial square-zero extension $B \oplus M$ and consider the composite morphism $g : \text{Spec}(B \oplus M) \rightarrow \mathcal{X}$, it follows that the space

$$\mathbb{V}_{\mathcal{X}}(\mathcal{E})(\text{Spec}(B \oplus M)) = \text{map}_{\mathcal{D}(B \oplus M)}((B \oplus M) \otimes_B^L f^*\mathcal{E}, B \oplus M)$$

is discrete. Using the extension of scalars adjunction we can rewrite the right hand side as

$$\text{map}_{\mathcal{D}(B)}(f^*\mathcal{E}, B \oplus M) \simeq \text{map}_{\mathcal{D}(B)}(f^*\mathcal{E}, B) \times \text{map}_{\mathcal{D}(B)}(f^*\mathcal{E}, M)$$

and thus we deduce that $\text{map}_{\mathcal{D}(B)}(f^*\mathcal{E}, M)$ is discrete, as claimed.

Now suppose for a contradiction that $f^*\mathcal{E}$ is not connective and let $r < 0$ be the smallest integer such that $h_r(f^*\mathcal{E}) \neq 0$. Then

$$\pi_{-r} \text{map}_{\mathcal{D}(B)}(f^*\mathcal{E}, M) \simeq \pi_0 \text{map}_{\mathcal{D}(B)}((f^*\mathcal{E})_{\leq r}, M) \simeq \text{Ext}_B^0(h_r(f^*\mathcal{E}), M)$$

and since we've already shown that the left hand side vanishes for any M , we deduce that the same must be true for the right hand side. It follows that $h_r(f^*\mathcal{E}) = 0$, giving the desired contradiction and ending the argument. \square

We are nearing the main result of this section, which states that the cone functor produces higher Artin stacks when it is applied to quasi-coherent sheaves satisfying certain finiteness conditions. We begin with a simple lemma.

LEMMA 3.8. — *Let \mathcal{X} be a stack and let $\mathcal{E} \rightarrow \mathcal{E}' \rightarrow \mathcal{P}$ be a cofibre sequence of quasi-coherent sheaves on \mathcal{X} such that \mathcal{P} is perfect and of non-positive amplitude. Then, $\mathbb{V}_{\mathcal{X}}(\mathcal{E}') \rightarrow \mathbb{V}_{\mathcal{X}}(\mathcal{E})$ is surjective and $\mathbb{V}_{\mathcal{X}}(\mathcal{E}') \times_{C_{\mathcal{X}}(\mathcal{E})} \mathbb{V}_{\mathcal{X}}(\mathcal{E}') \simeq \mathbb{V}_{\mathcal{X}}(\mathcal{E}') \times_{\mathcal{X}} \mathbb{V}_{\mathcal{X}}(\mathcal{P})$.*

Proof. — By Lemma 3.5, we can assume that $\mathcal{X} \simeq \text{Spec}(A)$ is affine. If $f : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is a morphism of affine schemes, then the cofibre sequence $\Sigma^{-1}\mathcal{P} \rightarrow \mathcal{E} \rightarrow \mathcal{E}'$ induces a fibre sequence

$$\mathbb{V}(\mathcal{E}')(f) \longrightarrow \mathbb{V}(\mathcal{E})(f) \longrightarrow \mathbb{V}(\Sigma^{-1}\mathcal{P})(f)$$

of spaces. By definition, we have $\pi_0 \mathbb{V}(\Sigma^{-1}\mathcal{P})(f) \simeq \text{Ext}_B^1(f^*\mathcal{P}, B)$ and the latter group vanishes, since B is connective and $f^*\mathcal{P}$ is perfect and of non-positive amplitude. Through the long exact sequence of homotopy we deduce that

$$\pi_0 \mathbb{V}(\mathcal{E}')(\text{Spec}(B) \rightarrow \text{Spec}(A)) \longrightarrow \pi_0 \mathbb{V}(\mathcal{E})(\text{Spec}(B) \rightarrow \text{Spec}(A))$$

is surjective, proving the first claim.

The second claim follows from the fact that $\mathcal{E}' \oplus_{\mathcal{E}} \mathcal{E}' \simeq \mathcal{E}' \oplus \mathcal{P}$ and Lemma 3.4. \square

Remark 3.9. — The conclusion of Lemma 3.8 can be alternatively rephrased as follows: if $\mathcal{E} \rightarrow \mathcal{E}' \rightarrow \mathcal{P}$ is a fibre sequence of quasi-coherent sheaves with \mathcal{P} perfect and of non-positive amplitude, then $\mathbb{V}_{\mathcal{X}}(\mathcal{P}) \rightarrow \mathbb{V}_{\mathcal{X}}(\mathcal{E}') \rightarrow \mathbb{V}_{\mathcal{X}}(\mathcal{E})$ is a cofibre sequence of abelian stacks over \mathcal{X} .

THEOREM 3.10. — *Let \mathcal{X} be a stack and let $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})$ be perfect to order -1 and $(-n)$ -connective, where $n \geq 0$. Then, the morphism $\mathbb{V}_{\mathcal{X}}(\mathcal{E}) \rightarrow \mathcal{X}$ is a relative n -Artin stack. If \mathcal{E} is perfect and of non-positive amplitude, then $\mathbb{V}_{\mathcal{X}}(\mathcal{E})$ is smooth.*

Proof. — By Lemma 3.5, we may assume that $\mathcal{X} \simeq \mathrm{Spec}(A)$ is affine. To declutter the notation, for $\mathcal{E} \in \mathrm{QCoh}(\mathrm{Spec}(A))$ let us denote the cone by $\mathbb{V}(\mathcal{E}) := \mathbb{V}_{\mathrm{Spec}(A)}(\mathcal{E})$.

By Lemma 3.6, we can replace \mathcal{E} by its coconnective cover. Since \mathcal{E} is perfect to order -1 and $(-n)$ -connective, it follows that $\mathcal{E}_{\leq 0} \in \mathrm{QCoh}(\mathrm{Spec}(A)) \simeq \mathcal{D}(A)$ can be represented by a complex

$$\dots \rightarrow 0 \rightarrow E_0 \rightarrow E_{-1} \rightarrow \dots \rightarrow E_{-n} \rightarrow 0 \rightarrow \dots$$

of A -modules, where E_i are finitely generated, projective for $i < 0$. If \mathcal{E} is perfect and of non-positive amplitude, then we can additionally assume that E_0 is projective as well.

Our proof will go by induction on n , the base case $n = 0$ following from Lemma 3.7. Thus, suppose that $n > 0$ and consider the cofibre sequence

$$\mathcal{E} \rightarrow E_0 \rightarrow \mathcal{P}$$

in $\mathrm{QCoh}(\mathrm{Spec}(A))$, where $\mathcal{P} \simeq [P_{-1} \rightarrow \dots \rightarrow P_{-n}]$ with P_{-1} concentrated in degree zero. Notice that \mathcal{P} is perfect and coconnective and so it follows from Lemma 3.8 that the map $\mathbb{V}(E_0) \rightarrow \mathbb{V}(\mathcal{E})$ is surjective and

$$\mathbb{V}(E_0) \times_{\mathbb{V}(\mathcal{E})} \mathbb{V}(E_0) \simeq \mathbb{V}(E_0) \times_{\mathcal{X}} \mathbb{V}(\mathcal{P}).$$

Since $\mathbb{V}(\mathcal{P}) \rightarrow \mathcal{X}$ is smooth $(n-1)$ -Artin by inductive assumption, the needed result follows. \square

COROLLARY 3.11. — *Let \mathcal{X} be an Artin and let \mathcal{E} be perfect to order -1 . Then, $\mathbb{V}_{\mathcal{X}}(\mathcal{E})$ is Artin.*

Proof. — Immediate from Theorem 3.10 and Proposition 2.6. \square

As mentioned at the beginning of the section, the abelian cone should be thought of as an analogue of the h^1/h^0 functor defined by Behrend and

Fantechi, itself inspired by Deligne’s work on Picard stacks. The following remark explores this connection further.

Remark 3.12. — A careful reader will notice quickly that h^1/h^0 is covariant, while the abelian cone functor of Definition 3.1 is contravariant. This is because the two are not really analogous, but rather dual to each other.

The *global section* functor Γ which is the higher categorical analogue of h^1/h^0 associates to any $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})$ the prestack $\Gamma(\mathcal{E})$ defined by the formula

$$(f : \mathrm{Spec}(A) \rightarrow \mathcal{X}) \longmapsto \mathrm{map}_{\mathcal{D}(A)}(A, f^*\mathcal{E})$$

One can show that the functor Γ is cocontinuous and takes values in stacks. One can then check directly that if \mathcal{X} is Deligne–Mumford stack, then the restriction $\Gamma|_{\mathrm{QCoh}(\mathcal{X})_{\leq 1}}$ coincides with the functor h^1/h^0 after suitable identifications.

When restricted to the full subcategory of almost perfect objects, the cone functor can be defined in terms of the global sections functor. That is, if \mathcal{X} is a stack, then there exists a commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{X})^{\mathrm{aperfOP}} & \xrightarrow{\underline{\mathrm{map}}_{\mathrm{QCoh}(\mathcal{X})}(-, \mathcal{O}_{\mathcal{X}})} & \mathrm{QCoh}(\mathcal{X}) \\ & \searrow \mathbb{V}_{\mathcal{X}}(-) & \downarrow \Gamma \\ & & \mathrm{Stk}/_{\mathcal{X}} \end{array}$$

where if $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})$ is almost perfect then the quasi-coherent sheaf $\underline{\mathrm{map}}_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{F}, \mathcal{O}_{\mathcal{X}})$ is defined so that we have a canonical equivalence

$$\Omega^\infty \underline{\mathrm{map}}_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{F}, \mathcal{O}_{\mathcal{X}})(\mathrm{Spec}(A) \xrightarrow{f} \mathcal{X}) \simeq \mathrm{map}_{\mathcal{D}(A)}(f^*\mathcal{F}, A),$$

as in [21, 6.5.3].

If \mathcal{X} is Deligne–Mumford, then after taking different grading conventions into account, one sees that if $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})_{[0,1]}$ has coherent homology, the stack $\Gamma(\underline{\mathrm{map}}_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{E}[-1], \mathcal{O}_{\mathcal{X}}))$ coincides with the one denoted by Behrend and Fantechi as $h^1/h^0(\mathcal{E}^\vee)$. Thus, we deduce from the above analysis that the latter coincides with our $\mathbb{V}_{\mathcal{X}}(\mathcal{E}[-1])$.

4. Adapted cosheaves

In this section we introduce the notion of an *adapted cosheaf*, which is a cosheaf which preserves pullbacks along a distinguished class of maps. Our

main result is that such cosheaves are stable under left Kan extension to a larger category.

DEFINITION 4.1. — *We will say that a class S of morphisms in an ∞ -category \mathcal{C} is a marking if*

- (1) S contains all equivalences and is stable under composition and
- (2) \mathcal{C} admits pullbacks along morphisms in S and S is stable under such pullbacks.

The usefulness of the concept of a marking is encapsulated in the following straightforward result. Recall that we say that an ∞ -category \mathcal{C} has universal coproducts if for any $D' \rightarrow D$ and a finite collection of maps $C_i \rightarrow D$, the canonical map $\bigsqcup C_i \times_D D' \rightarrow (\bigsqcup C_i) \times_D D'$ is an equivalence.

PROPOSITION 4.2. — *Let \mathcal{C} be an ∞ -category with universal coproducts and S be a marking on \mathcal{C} . Then, \mathcal{C} admits a Grothendieck topology in which a family $\{C_i \rightarrow C\}$ is covering precisely when $\bigsqcup C_i \rightarrow C$ belongs to S .*

Proof. — This is [21, A.3.2.1]. □

The category of schemes, as well as its higher categorical variants, has universal coproducts and by taking S to be an appropriate class of morphisms (such as surjective étale) we can produce the classical Grothendieck topologies in schemes (such as the étale topology).

DEFINITION 4.3. — *We will say a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ is downward closed if it has the property that if $C_0 \rightarrow C$ is a covering such that $C_i \in \mathcal{D}$ for all $i \geq 0$, where $C_i := C_0 \times_C \dots \times_C C_0$, then $C \in \mathcal{D}$ as well.*

We say a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ is generating if it is closed under coproducts, pullbacks along coverings, and the smallest downward closed subcategory which contains it is all of \mathcal{C} .

Note that since a generating subcategory is closed under coproducts and pullbacks along coverings, it inherits a topology. The definition is chosen so that the following is true.

PROPOSITION 4.4. — *Let $\mathcal{D} \subseteq \mathcal{C}$ be a generating subcategory. Then, the restriction $Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$ between sheaf ∞ -categories is an equivalence.*

Proof. — The inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ induces an adjunction $i_* \dashv i^* : Sh(\mathcal{C}) \rightleftarrows Sh(\mathcal{D})$, where i^* is the restriction and i_* is the left Kan extension. Clearly, we have $i^* \circ i_* \simeq id$.

To see that the other composite is also the identity, observe that the sheaf condition implies that for any $F \in Sh(\mathcal{C})$ the subcategory of those $C \in \mathcal{C}$ such that the counit $i_* i^* F(C) \rightarrow F(C)$ is an equivalence, is downward

closed. Since it contains all of \mathcal{D} which is generating by assumption, it must be all of \mathcal{C} . \square

The functors considered in this paper will not only be cosheaves, but will additionally have compatibility with a wider class of morphisms than just coverings. This further class will also form a marking; in practice this could be either the class of all smooth or even all flat morphisms, not necessarily surjective.

Throughout the rest of the section, which forms the technical heart of this paper, we will work in the following setup.

Notation 4.5. — By \mathcal{C} we denote an ∞ -category with universal coproducts, and we assume that we are given a pair of markings $S \subseteq T$. The word *covering* will mean a morphism in S , and we will treat \mathcal{C} as a Grothendieck site with respect to the topology determined by S .

Notation 4.6. — The symbol \mathcal{X} will denote an ∞ -topos. Following standard terminology, we will say that $F : \mathcal{C} \rightarrow \mathcal{X}$ is a *cosheaf* if the induced functor between opposite ∞ -categories

$$F^{op} : \mathcal{C}^{op} \longrightarrow \mathcal{X}^{op}$$

is a sheaf with respect to the Grothendieck topology determined by S . Equivalently, F is a cosheaf if for any $x \in \mathcal{X}$, the composite

$$\text{map}(F(-), x) : \mathcal{C}^{op} \longrightarrow \mathcal{S}$$

is a sheaf of spaces.

The purpose of the additional marking T is to single out the following class of cosheaves.

DEFINITION 4.7. — *Let T be a marking such that $S \subseteq T$. We will say that a cosheaf $F : \mathcal{C} \rightarrow \mathcal{X}$ is T -adapted if F commutes with pullbacks along morphisms in T .*

It turns out that the pullback preservation is so strong that it almost implies the cosheaf condition, as the following shows.

PROPOSITION 4.8. — *A functor $F : \mathcal{C} \rightarrow \mathcal{X}$ which commutes with pullbacks along morphisms in T is a cosheaf if and only if*

- (1) F preserves coproducts
- (2) F takes morphisms in S to effective epimorphisms.

Proof. — As a consequence of the description of sheaves of [21, A.3.3.1], F is a cosheaf if and only if it takes Čech nerves

$$\dots \rightrightarrows C \times_D C \longrightarrow C \longrightarrow D$$

of morphisms $C \rightarrow D$ into colimit diagrams in \mathcal{X} . However, since by assumption $S \subseteq T$ and F preserves pullbacks along elements of the latter, the above diagram is taken to the Čech diagram of $F(C) \rightarrow F(D)$. Thus, it is a colimit diagram precisely when $F(C) \rightarrow F(D)$ is an effective epimorphism. \square

Our main goal for this section is to prove that the property of being adapted with respect to a given marking T can be verified on a subcategory. As a consequence, we conclude that the unique extension of an adapted cosheaf is adapted. We begin with a simple lemma.

LEMMA 4.9. — *Let $C \in \mathcal{C}$ and fix a morphism $D \rightarrow C$. Then, the subcategory of those morphisms $E \rightarrow C$ such that*

$$F(D \times_C E) \simeq F(D) \times_{F(C)} F(E)$$

is a downward closed subcategory of $\mathcal{C}_{/C}$.

Proof. — Suppose that we have a covering $E_0 \rightarrow E$ such that $F(E_i \times_C D) \simeq F(E_i) \times_{F(C)} F(D)$ for all $i \geq 0$, where $E_i := E_0 \times_E \dots \times_E E_0$. Then,

$$F(E \times_C D) \simeq \varinjlim F(E_i \times_C D) \simeq \varinjlim F(E_i) \times_{F(C)} F(D) \simeq F(E) \times_{F(C)} F(D),$$

where we have twice used that F is a cosheaf and once that \mathcal{X} is an ∞ -topos, so that pullbacks therein commute with colimits. \square

THEOREM 4.10. — *Let $\mathcal{D} \subseteq \mathcal{C}$ be a generating subcategory closed under coproducts and pullbacks along coverings. Then, a cosheaf $F : \mathcal{C} \rightarrow \mathcal{X}$ is T -adapted if and only if its restriction $F|_{\mathcal{D}}$ is.*

Proof. — One direction is trivial, so instead suppose that F is a cosheaf such that $F|_{\mathcal{D}}$ is adapted, we have to show that F is adapted.

Let us say that a morphism $D \rightarrow C$ is *good* if for any other morphism $E \rightarrow C$ we have $F(E \times_C D) \simeq F(E) \times_{F(C)} F(D)$; our goal is to show that any morphism in T is good. If $D \rightarrow C$ is a morphism in \mathcal{D} which belongs to T , then since $F|_{\mathcal{D}}$ is assumed to be adapted, we deduce that the condition holds whenever $E \in \mathcal{D}$. Then, it follows from Lemma 4.9 that all such arrows are good.

Let us further say that an object C is *excellent* if any morphism $D \rightarrow C$ which belongs to T is good, we claim that any $C \in \mathcal{D}$ is excellent. By another application of Lemma 4.9 we see that the collection of good morphisms is downward closed in the ∞ -category $\mathcal{C}_{/C}^T$ of arrows $D \rightarrow C$ which belong to T , so that it is enough to verify it when we also have $D \in \mathcal{D}$, which we already did.

Lastly, we claim that the collection of excellent objects of \mathcal{C} is also downward closed; since we already verified that it contains all objects of \mathcal{D} , this will end the argument. Suppose that we have a covering $C_0 \rightarrow C$ such that all of $C_i := C_0 \times_C \dots \times_C C_0$ are excellent, we want to show that the same is true for C . We first show that F takes the Čech nerve of $C_0 \rightarrow C$ to the Čech nerve of $F(C_0) \rightarrow F(C)$, in particular, that $F(C_1) \simeq F(C_0) \times_{F(C)} F(C_0)$. Since F is a cosheaf, the image

$$\dots F(C_1) \rightrightarrows F(C_0) \longrightarrow F(C)$$

is a colimit diagram. Because \mathcal{X} is an ∞ -topos, by [19, 6.1.3.19, (iv)] to verify that the above diagram is a Čech nerve it is enough to check that the underlying simplicial object is a groupoid; in other words, that for any partition $[m] = S \cup T$ with $S \cap T = \{s\}$, the induced diagram

$$\begin{array}{ccc} F(C_m) & \longrightarrow & F(C_{|S|}) \\ \downarrow & & \downarrow \\ F(C_{|T|}) & \longrightarrow & F(C_0) \end{array}$$

is a pullback. This is clear, since C_0 is assumed to be excellent.

To check that C itself is excellent, we have to verify that an arbitrary map $D \rightarrow C$ is good; by what was said above, it is enough to check that this is the case for $D_i \rightarrow C$, where $D_i := C_i \times_C D$. Thus, we can assume that the given map $D \rightarrow C$ factors through C_0 . Then, by again invoking Lemma 4.9 we see that we only have to verify that $F(D \times_C E) \simeq F(D) \times_C F(E)$ where $E \rightarrow C$ also factors through C_0 .

To summarize, to prove that $C \in \mathcal{C}$ is excellent, it is enough to show that F preserves pullbacks of spans which can be factorized as

$$D \longrightarrow C_0 \longrightarrow C \longleftarrow C_0 \longleftarrow E,$$

where each map belongs to T . Consider the diagram

$$\begin{array}{ccccc} F(D \times_C E) & \longrightarrow & F(D \times_C E) & \longrightarrow & F(E) \\ \downarrow & & \downarrow & & \downarrow \\ F(D \times_{C_0} C_1) & \longrightarrow & F(C_1) & \longrightarrow & F(C_0) \\ \downarrow & & \downarrow & & \downarrow \\ F(D) & \longrightarrow & F(C_0) & \longrightarrow & F(C), \end{array}$$

where each of the squares except possibly the lower right one is a pullback because C_i are assumed to be excellent. Because we also verified the same about the last square, the pullback pasting lemma ends the argument. \square

We will later find ourselves in a situation where we have a natural transformation between adapted cosheaves which is particularly nice when restricted to the subcategory. It will then be useful to know that this “niceness” necessarily holds in general, as we will now verify.

DEFINITION 4.11. — *Let $F, G : \mathcal{C} \rightarrow \mathcal{X}$ be functors. Then, we say that a natural transformation $F \rightarrow G$ is T -cartesian if for any arrow $D \rightarrow C$ in T the induced diagram*

$$\begin{array}{ccc} F(D) & \longrightarrow & G(D) \\ \downarrow & & \downarrow \\ F(C) & \longrightarrow & G(C) \end{array}$$

is cartesian.

LEMMA 4.12. — *Suppose that $F, G : \mathcal{C} \rightarrow \mathcal{X}$ are coproduct-preserving functors and let $F \rightarrow G$ be T -cartesian. Then, if G is a T -adapted cosheaf, the same is true for F .*

Proof. — Suppose that we have a cospan $C_0 \rightarrow C \leftarrow D$ where the left map belongs to T , we have to check that $F(D_0) \rightarrow F(\tilde{C}) \times_{F(C)} F(D)$ is an equivalence, where $D_0 := \tilde{C} \times_C D$. Applying the cartesian property to both the source and target of this morphism, we see that this is equivalent to asking whether

$$G(D_0) \times_{G(D)} F(D) \longrightarrow G(C_0) \times_{G(C)} F(C) \times_{F(C)} F(D) \simeq G(C_0) \times_{G(C)} F(D)$$

is an equivalence. Since G is assumed to be adapted, we have $G(D_0) \simeq G(C_0) \times_{G(C)} G(D)$ and it follows that the source of the above map we can rewrite as

$$G(D_0) \times_{G(D)} F(D) \simeq G(C_0) \times_{G(C)} G(D) \times_{G(D)} F(D) \simeq G(C_0) \times_{G(C)} F(D)$$

which is what we wanted to show.

By Proposition 4.8, to finish showing that F is an adapted cosheaf, we just have to check that it takes coverings to effective epimorphisms. However, if $C_0 \rightarrow C$ is a covering, then by the cartesian property we have $F(C_0) \simeq G(C_0) \times_{G(C)} F(C)$ and we deduce that $F(C_0) \rightarrow F(C)$ is a base-change of $G(C_0) \rightarrow G(C)$, which is an effective epimorphism since G is a cosheaf. \square

PROPOSITION 4.13. — *Let $F, G : \mathcal{C} \rightarrow \mathcal{X}$ be T -adapted cosheaves, $F \rightarrow G$ be a natural transformation and let $\mathcal{D} \subseteq \mathcal{C}$ be a generating subcategory. Then, if the restriction $F|_{\mathcal{D}} \rightarrow G|_{\mathcal{D}}$ is T -cartesian, then so is $F \rightarrow G$.*

Proof. — We first claim that for any $C \in \mathcal{C}$, the subcategory of those morphisms $D \rightarrow C$ such that $F(D) \simeq G(D) \times_{G(C)} F(C)$ is a downward closed subcategory of $\mathcal{C}_{/C}^T$. To see this, let $D_0 \rightarrow C$ be a covering such that all $D_i \rightarrow C$ have this property, where $D_i := D_0 \times_D \dots \times_D D_0$. Then,

$$F(D) \simeq \varinjlim F(D_i) \simeq \varinjlim G(D_i) \times_{G(C)} F(C) \simeq G(D) \times_{G(C)} F(C),$$

where we have used that F, G are cosheaves and that colimits in \mathcal{X} commute with pullbacks. We deduce that any morphism in $D \rightarrow C$ in T with $C \in \mathcal{D}$ has the required property.

We next claim that the subcategory of those C such that for any morphism $D \rightarrow C$ in T we have $F(D) \simeq G(D) \times_{G(C)} F(C)$ is a generating subcategory of \mathcal{C} , together with what we've shown above this will finish the argument. Choose a covering $C_0 \rightarrow C$ such that C_0 has the needed property and let $D_0 := C_0 \times_C D$. Then, since F and G are adapted, we have

$$\begin{aligned} F(D) \times_{F(C)} F(C_0) &\simeq F(D_0) \simeq G(D_0) \times_{G(C_0)} F(C_0) \\ &\simeq G(D) \times_{G(C)} G(C_0) \times_{G(C_0)} F(C_0) \end{aligned}$$

and further

$$\begin{aligned} G(D) \times_{G(C)} G(C_0) \times_{G(C_0)} F(C_0) &\simeq G(D) \times_{G(C)} F(C_0) \\ &\simeq G(D) \times_{G(C)} F(C) \times_{F(C)} F(C_0). \end{aligned}$$

We deduce that we have $F(D) \simeq G(D) \times_{G(C)} F(C)$ after base-changing along $F(C_0) \rightarrow F(C)$ and since the latter is an effective epimorphism since F is a cosheaf, we deduce that this holds even before the base-change, ending the proof. \square

5. Axiomatization of the normal sheaf

In this section, we study the notion of a normal sheaf of a morphism of Artin stacks, defined in terms of the cotangent complex. Our main result is that, as a functor on relative Artin stacks, the normal sheaf is determined by its values on closed embeddings of schemes and a short list natural axioms.

Recall that if $U \hookrightarrow V$ is a closed embedding of schemes with ideal sheaf I , then the cotangent complex $L_{U/V}$ is 1-connective and $h_1(L_{U/V}) \simeq I/I^2$. The abelian cone associated to the latter quasi-coherent sheaf defines a scheme over U known as the *normal sheaf* of the embedding, denoted by $\mathcal{N}_U V := \mathbb{V}_U(I/I^2)$. In the particular case when the embedding is regular, I/I^2 is locally free and the normal sheaf is just the classical normal bundle.

DEFINITION 5.1 ([5]). — *Let $\mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of Artin stacks. Then, its normal sheaf is defined as*

$$\mathcal{N}_{\mathcal{X}} \mathcal{Y} := \mathbb{V}_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{Y}}[-1]),$$

the abelian cone associated to the shift of the cotangent complex.

WARNING 5.2. — *The terminology normal sheaf is very much established, going back to the original paper of Behrend and Fantechi on the intrinsic normal cone [5]. It is, however, potentially confusing - we do warn the reader that the normal sheaf is not a sheaf, but rather a stack over \mathcal{X} .*

Let us give a couple of examples.

Example 5.3. — If $U \hookrightarrow V$ is a closed embedding of schemes, then $\mathcal{N}_U V$ coincides with the normal sheaf in the classical sense. To see this, note that we verified in Lemma 3.6 that the abelian cone only depends on the coconnective part of a quasi-coherent sheaf, so that we have $\mathcal{N}_U V \simeq \mathbb{V}_U((L_{U/V}[-1])_{\geq 0}) \simeq \mathbb{V}_U(I/I^2)$, where I is the ideal sheaf.

Example 5.4. — If \mathcal{X} is Deligne–Mumford, then the normal sheaf of the unique map $\mathcal{X} \rightarrow \mathrm{Spec}(k)$ coincides with the *intrinsic normal sheaf* of Behrend and Fantechi [5, 3.6]. As a particularly easy example of the latter, let us suppose that $\mathcal{X} \simeq X$ is a smooth scheme. Then, $L_X \simeq \Omega_X$ and so $\mathcal{N}_X \simeq \mathbb{V}_X(\Omega_X[-1]) \simeq \mathcal{B}T_X$; that is, there’s an equivalence between the intrinsic normal sheaf of X and the classifying stack of its tangent bundle.

Note that a priori the normal sheaf of a morphism $\mathcal{X} \rightarrow \mathcal{Y}$ of Artin stacks is a priori only a stack, and in fact it can fail to be algebraic unless we impose some finiteness conditions.

PROPOSITION 5.5. — *Let $\mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of Artin stacks which is locally of finite type. Then, $\mathcal{N}_{\mathcal{X}} \mathcal{Y}$ is Artin and moreover,*

- (1) *if $\mathcal{X} \rightarrow \mathcal{Y}$ is relative n -Artin, then $\mathcal{N}_{\mathcal{X}} \mathcal{Y} \rightarrow \mathcal{X}$ is relative $(n+1)$ -Artin,*
- (2) *if $\mathcal{X} \rightarrow \mathcal{Y}$ is smooth, then so is $\mathcal{N}_{\mathcal{X}} \mathcal{Y} \rightarrow \mathcal{X}$.*

Proof. — This follows from Theorem 3.10, Proposition 2.25 and Proposition 2.24. \square

Remark 5.6. — One can consider Proposition 5.5 as giving a quantitative reason why Behrend and Fantechi only define the normal sheaf for morphisms of Deligne–Mumford type - if $\mathcal{X} \rightarrow \mathcal{Y}$ is 1-Artin, then the correct definition makes $\mathcal{N}_{\mathcal{X}}\mathcal{Y}$ into a 2-Artin stack, forcing the introduction of higher algebraic stacks.

Remark 5.7. — If $\mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of Artin stacks which is not locally of finite type, then $\mathcal{N}_{\mathcal{X}}\mathcal{Y}$ can fail to be Artin. Nevertheless, it is always “algebraic” in the sense that it can be locally obtained by starting from a scheme and taking quotients by actions of flat group schemes; what can fail is that without finiteness these group schemes will in general not be smooth.

Our goal will be to prove that the normal cone functor is uniquely determined by a simple set of axioms. As we want to stay in the geometric context, in light of Proposition 5.5 we should introduce some finiteness conditions. To avoid repeating them over and over, let us make the following convention.

CONVENTION 5.8. — A relative Artin stack is a morphism $\mathcal{X} \rightarrow \mathcal{Y}$ of Artin stacks which is locally of finite type. We denote the ∞ -category of relative Artin stacks and commutative squares by

$$\mathrm{RelArt} := \mathrm{Fun}_{\mathrm{loc.f.t.}}(\Delta^1, \mathrm{Art}).$$

As a minor warning, note that the above notion of a relative Artin stack is more strict than the most general notion of a relative Artin stack in two different ways - we require the target to also be Artin, rather than arbitrary, and we require the morphism to be locally of finite type. For most applications, the stacks considered are finite type over a field, so that these two assumptions are trivially satisfied.

Notation 5.9. — If $\mathcal{X}' \rightarrow \mathcal{Y}'$ and $\mathcal{X} \rightarrow \mathcal{Y}$ are relative Artin stacks, then we will use the notation $(\mathcal{X}' \rightarrow \mathcal{Y}') \rightarrow (\mathcal{X} \rightarrow \mathcal{Y})$ to denote morphisms in the ∞ -category of relative Artin stacks, which are given by commutative squares

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{Y}' \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y}. \end{array}$$

This notation is introduced to lessen our need to draw complicated diagrams.

DEFINITION 5.10. — We say a morphism $(\mathcal{X}' \rightarrow \mathcal{Y}') \rightarrow (\mathcal{X} \rightarrow \mathcal{Y})$ of relative Artin stacks is smooth if both $\mathcal{X}' \rightarrow \mathcal{X}$ and $\mathcal{Y}' \rightarrow \mathcal{Y}$ are smooth. Likewise, we say it is surjective if both of those arrows are surjective.

Since one easily verifies that the ∞ -category of Artin stacks has universal coproducts, the same is true for the ∞ -category of relative Artin stacks, where limits and colimits are computed separately in the source and target. It follows that RelArt admits a unique Grothendieck topology in which covering families are given by smooth, jointly surjective maps in the sense of Definition 5.10. We will use descent with respect to this topology to prove the following result.

THEOREM 5.11. — The normal sheaf functor $\mathcal{N} : \text{RelArt} \rightarrow \text{Art}$ is determined up to a canonical natural equivalence as the unique functor subject to the following four axioms:

- (1) If $U \hookrightarrow V$ is a closed embedding of schemes, then $\mathcal{N}_U V$ coincides with the classical normal sheaf, that is, $\mathcal{N}_U V \simeq \mathbb{V}_U(I/I^2)$, where I is the ideal sheaf.
- (2) \mathcal{N} preserves coproducts; that is, for any two relative Artin stacks $\mathcal{X} \rightarrow \mathcal{Y}$ and $\mathcal{X}' \rightarrow \mathcal{Y}'$ we have $\mathcal{N}_{\mathcal{X} \sqcup \mathcal{X}'}(\mathcal{Y} \sqcup \mathcal{Y}') \simeq \mathcal{N}_{\mathcal{X}}\mathcal{Y} \sqcup \mathcal{N}_{\mathcal{X}'}\mathcal{Y}'$.
- (3) \mathcal{N} preserves smooth and smoothly surjective maps; that is, if $(\mathcal{X}' \rightarrow \mathcal{Y}') \rightarrow (\mathcal{X} \rightarrow \mathcal{Y})$ is smooth (resp. smooth surjective) map of relative Artin stacks, then the same is true for $\mathcal{N}_{\mathcal{X}'}\mathcal{Y}' \rightarrow \mathcal{N}_{\mathcal{X}}\mathcal{Y}$.
- (4) \mathcal{N} commutes with pullbacks along smooth morphisms; that is, if $(\mathcal{X}' \rightarrow \mathcal{Y}') \rightarrow (\mathcal{X} \rightarrow \mathcal{Y})$ is smooth, then $\mathcal{N}_{\mathcal{X}' \times_{\mathcal{X}} \mathcal{W}}(\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{Z}) \simeq \mathcal{N}_{\mathcal{X}'}\mathcal{Y}' \times_{\mathcal{N}_{\mathcal{X}}\mathcal{Y}} \mathcal{N}_{\mathcal{W}}\mathcal{Z}$ for any $(\mathcal{W} \rightarrow \mathcal{Z}) \rightarrow (\mathcal{X} \rightarrow \mathcal{Y})$.

Before proving uniqueness, we will first establish that the normal sheaf does have the required properties.

LEMMA 5.12. — Let $\mathcal{X} \rightarrow \mathcal{Y}$ be a relative Artin stack and suppose that $\mathcal{Y}' \rightarrow \mathcal{Y}$ is smooth. Then, $\mathcal{N}_{\mathcal{X}'}\mathcal{Y}' \simeq p^* \mathcal{N}_{\mathcal{X}}\mathcal{Y}$, where $\mathcal{X}' \simeq \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ and $p : \mathcal{X}' \rightarrow \mathcal{X}$ is the projection.

Proof. — By flat base-change for the cotangent complex, we have $L_{\mathcal{X}'/\mathcal{Y}'} \simeq p^* L_{\mathcal{X}/\mathcal{Y}}$, so that the statement follows immediately from Lemma 3.5. \square

LEMMA 5.13. — Suppose we have a composite $\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z}$ of morphisms of Artin stacks. Then

- (1) if $\mathcal{X} \rightarrow \mathcal{Y}$ is smooth, there's a cofibre sequence $\mathcal{N}_{\mathcal{X}}\mathcal{Y} \rightarrow \mathcal{N}_{\mathcal{X}}\mathcal{Z} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{N}_{\mathcal{Y}}\mathcal{Z}$

(2) if $\mathcal{Y} \rightarrow \mathcal{Z}$ is smooth, there's a cofibre sequence $\Omega_{\mathcal{X}}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{N}_{\mathcal{Y}}\mathcal{Z}) \rightarrow \mathcal{N}_{\mathcal{X}}\mathcal{Y} \rightarrow \mathcal{N}_{\mathcal{X}}\mathcal{Z}$

of abelian stacks over \mathcal{X} , where $\Omega_{\mathcal{X}}$ is the fibrewise loop space over \mathcal{X} . Moreover, in both cases the left term is smooth over \mathcal{X} and so the right map is a smooth surjection.

Proof. — Using the standard exactness properties of the cotangent complex, the first cofibre sequence from Remark 3.9 applied to $p^*L_{\mathcal{Y}/\mathcal{Z}}[-1] \rightarrow L_{\mathcal{X}/\mathcal{Z}}[-1] \rightarrow L_{\mathcal{X}/\mathcal{Y}}[-1]$, where $p : \mathcal{X} \rightarrow \mathcal{Y}$, and the second from applying it to $L_{\mathcal{X}/\mathcal{Z}}[-1] \rightarrow L_{\mathcal{X}/\mathcal{Y}} \rightarrow p^*L_{\mathcal{Y}/\mathcal{Z}}$.

To see the second claim, observe that in a cofibre sequence, the right map is always surjective, and smoothness follow from Theorem 3.10. \square

LEMMA 5.14. — *The normal sheaf functor $\mathcal{N} : \text{RelArt} \rightarrow \text{Art}$ preserves smooth and smoothly surjective maps.*

Proof. — Any morphism $(\mathcal{X}' \rightarrow \mathcal{Y}') \rightarrow (\mathcal{X} \rightarrow \mathcal{Y})$ of relative Artin stacks can be decomposed as

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{Y}' \\ \downarrow & & \downarrow \\ \mathcal{X}' & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y}; \end{array}$$

that is, into a composite of two other morphisms for which either the map on the source or on the target is the identity. If the given morphism is smooth (resp. smooth and surjective), so are the two factors, so that we can reduce to this case.

The fact that $\mathcal{N}_{\mathcal{X}'}\mathcal{Y}' \rightarrow \mathcal{N}_{\mathcal{X}'}\mathcal{Y}$ is smooth and surjective is immediate from the second cofibre sequence of Lemma 5.13. The first part of the same result implies that $\mathcal{N}_{\mathcal{X}'}\mathcal{Y} \rightarrow \mathcal{X}' \times_{\mathcal{X}} \mathcal{N}_{\mathcal{X}}\mathcal{Y}$ is smooth and surjective, and so the observation that $\mathcal{X}' \times_{\mathcal{X}} \mathcal{N}_{\mathcal{X}}\mathcal{Y} \rightarrow \mathcal{N}_{\mathcal{X}}\mathcal{Y}$ is smooth (resp. smooth and surjective) whenever $\mathcal{X}' \rightarrow \mathcal{X}$ finishes the claim. \square

PROPOSITION 5.15. — *Let $(\mathcal{X}' \rightarrow \mathcal{Y}') \rightarrow (\mathcal{X} \rightarrow \mathcal{Y})$ be a smooth map of relative Artin stacks and let $(\mathcal{W} \rightarrow \mathcal{Z}) \rightarrow (\mathcal{X} \rightarrow \mathcal{Y})$ be arbitrary. Then, $\mathcal{N}_{\mathcal{X}' \times_{\mathcal{X}} \mathcal{W}}(\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{Z}) \simeq \mathcal{N}_{\mathcal{X}'}\mathcal{Y}' \times_{\mathcal{N}_{\mathcal{X}}\mathcal{Y}} \mathcal{N}_{\mathcal{W}}\mathcal{Z}$*

Proof. — Using the decomposition of a smooth morphism as in the proof of Lemma 5.14 we can assume that the given smooth map is the identity in either the source or target.

Let us tackle first the case when $\mathcal{Y}' = \mathcal{Y}$. Using the notation $\mathcal{W}' := \mathcal{W} \times_{\mathcal{X}} \mathcal{X}'$, our goal is to show that $\mathcal{N}_{\mathcal{W}'\mathcal{Z}} \simeq \mathcal{N}_{\mathcal{W}\mathcal{Z}} \times_{\mathcal{N}_{\mathcal{X}\mathcal{Y}}} \mathcal{N}_{\mathcal{X}'\mathcal{Y}}$. By Lemma 5.13, we have a cofibre sequence

$$\mathcal{N}_{\mathcal{X}'\mathcal{X}} \longrightarrow \mathcal{N}_{\mathcal{X}'\mathcal{Y}} \longrightarrow \mathcal{X}' \times_{\mathcal{X}} \mathcal{N}_{\mathcal{X}\mathcal{Y}}$$

of abelian stacks over \mathcal{X}' . Said differently, the augmented simplicial object

$$\dots \rightrightarrows \mathcal{N}_{\mathcal{X}'\mathcal{X}} \times_{\mathcal{X}'} \mathcal{N}_{\mathcal{X}'\mathcal{Y}} \rightrightarrows \mathcal{N}_{\mathcal{X}'\mathcal{Y}} \longrightarrow \mathcal{X}' \times_{\mathcal{X}} \mathcal{N}_{\mathcal{X}\mathcal{Y}}$$

determined by the action is a colimit diagram. By direct inspection, applying $- \times_{\mathcal{N}_{\mathcal{X}\mathcal{Y}}} \mathcal{N}_{\mathcal{W}\mathcal{Z}}$ to this diagram yields

$$\begin{aligned} \dots \rightrightarrows \mathcal{N}_{\mathcal{W}'\mathcal{W}} \times_{\mathcal{W}'} (\mathcal{N}_{\mathcal{X}'\mathcal{Y}} \times_{\mathcal{N}_{\mathcal{X}\mathcal{Y}}} \mathcal{N}_{\mathcal{W}\mathcal{Z}}) \\ \rightrightarrows \mathcal{N}_{\mathcal{X}'\mathcal{Y}} \times_{\mathcal{N}_{\mathcal{X}\mathcal{Y}}} \mathcal{N}_{\mathcal{W}\mathcal{Z}} \longrightarrow \mathcal{W}' \times_{\mathcal{W}} \mathcal{N}_{\mathcal{W}\mathcal{Z}}, \end{aligned}$$

where we've used that the base-change formula $\mathcal{N}_{\mathcal{W}'\mathcal{W}} \simeq \mathcal{W}' \times_{\mathcal{X}'} \mathcal{N}_{\mathcal{X}\mathcal{X}}$ of Lemma 5.12. Since taking pullbacks in stacks preserves colimits, this presents a cofibre sequence

$$\mathcal{N}_{\mathcal{W}'\mathcal{W}} \longrightarrow \mathcal{N}_{\mathcal{X}'\mathcal{Y}} \times_{\mathcal{N}_{\mathcal{X}\mathcal{Y}}} \mathcal{N}_{\mathcal{W}\mathcal{Z}} \longrightarrow \mathcal{W}' \times_{\mathcal{W}} \mathcal{N}_{\mathcal{W}\mathcal{Z}}$$

of abelian stacks on \mathcal{W}' . Since $\mathcal{N}_{\mathcal{W}'\mathcal{Z}}$ is also a middle term of such a cofibre sequence by the same argument, and these cofibre sequences are natural, we deduce that $\mathcal{N}_{\mathcal{W}'\mathcal{Z}} \simeq \mathcal{N}_{\mathcal{X}'\mathcal{Y}} \times_{\mathcal{N}_{\mathcal{X}\mathcal{Y}}} \mathcal{N}_{\mathcal{W}\mathcal{Z}}$, which is what we wanted to show.

Let us now suppose that $\mathcal{X}' = \mathcal{X}$, our goal is to show that $\mathcal{N}_{\mathcal{X}\mathcal{Y}'} \times_{\mathcal{N}_{\mathcal{X}\mathcal{Y}}} \mathcal{N}_{\mathcal{W}\mathcal{Z}} \simeq \mathcal{N}_{\mathcal{W}\mathcal{Z}'}$, where $\mathcal{Z}' := \mathcal{Z} \times_{\mathcal{Y}} \mathcal{Y}'$. Using Lemma 5.13 again we have a cofibre sequence

$$\Omega_{\mathcal{X}}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{N}_{\mathcal{Y}'\mathcal{Y}}) \longrightarrow \mathcal{N}_{\mathcal{X}\mathcal{Y}'} \longrightarrow \mathcal{N}_{\mathcal{X}\mathcal{Y}}.$$

and the same argument as before shows that by applying $- \times_{\mathcal{N}_{\mathcal{X}\mathcal{Y}}} \mathcal{N}_{\mathcal{W}\mathcal{Z}}$ we get a cofibre sequence

$$\Omega_{\mathcal{W}}(\mathcal{W} \times_{\mathcal{Z}'} \mathcal{N}_{\mathcal{Z}'\mathcal{Z}}) \longrightarrow \mathcal{N}_{\mathcal{X}\mathcal{Y}'} \times_{\mathcal{N}_{\mathcal{X}\mathcal{Y}}} \mathcal{N}_{\mathcal{W}\mathcal{Z}} \longrightarrow \mathcal{N}_{\mathcal{W}\mathcal{Z}}.$$

Since $\mathcal{N}_{\mathcal{W}\mathcal{Z}'}$ is also a middle term of a cofibre sequence of this form, this ends the argument. \square

Observe that Example 5.3, Lemma 5.14 and Proposition 5.15 taken together already verify all of the properties of the normal sheaf spelled out in the statement of Theorem 5.11. Thus, to complete the proof of the latter, we only have to check that \mathcal{N} is the *unique* functor subject to these conditions.

Note that the only part of Theorem 5.11 that specifies values of the normal sheaf without reference to anything else, is the property that $\mathcal{N}_U V \simeq$

$\mathbb{V}_U(I/I^2)$ for a closed embedding of schemes with ideal sheaf I . As this context will occur frequently, let us introduce an appropriate terminology.

DEFINITION 5.16. — *A pair is a closed embedding $U \hookrightarrow V$ of schemes. The category $\mathcal{P}\text{air}$ of pairs is a full subcategory of the ∞ -category of relative Artin stacks.*

In this language, we need to show that the normal sheaf functor is determined by its interaction with smooth maps together with its values at pairs of schemes. To show that the latter alone suffices, we will use the topology on $\mathcal{R}\text{elArt}$ determined by smooth surjections in the sense of Definition 5.10. The key is the following slightly surprising fact.

LEMMA 5.17. — *Any relative Artin stack admits a smooth surjection from a pair of schemes. In fact, the category $\mathcal{P}\text{air}$ is a generating subcategory of $\mathcal{R}\text{elArt}$ in the sense of Definition 4.3.*

Proof. — Since any n -Artin stack admits a smooth cover from a disjoint union of affine schemes whose iterated intersections are $(n-1)$ -Artin stacks, it is easy to see by induction that the ∞ -category of relative Artin stacks is generated by morphisms between such schemes. Thus, it is enough to check that the latter is generated by closed embeddings.

Let $f : X \rightarrow Y$ be a relative Artin stack where X, Y are disjoint unions of affine schemes, and let us choose one such decomposition $X \simeq \bigsqcup X_\alpha$. Since each X_α is quasi-compact, its image is contained in a finite disjoint union of affines, and hence an affine disjoint summand which we will denote by $Y_\alpha \subseteq Y$.

Let us write $X_\alpha \simeq \text{Spec}(B_\alpha)$, $Y_\alpha \simeq \text{Spec}(A_\alpha)$. By assumption that f is locally of finite type, $A_\alpha \rightarrow B_\alpha$ is finitely generated, and hence we can find a factorization

$$A_\alpha \hookrightarrow C_\alpha \twoheadrightarrow B_\alpha,$$

where $C_\alpha := A_\alpha[x_1, \dots, x_n]$ for some n . It follows that if we write $Z_\alpha := \text{Spec}(C_\alpha)$, then in the induced factorization

$$X_\alpha \hookrightarrow Z_\alpha \twoheadrightarrow Y_\alpha$$

the first arrow is a closed inclusion of schemes and the second is smooth. Writing $Z := \bigsqcup Z_\alpha$, the same is true for

$$X \longrightarrow Z \longrightarrow Y.$$

If we now let $Y' := Z \sqcup Y$, with the induced map from X the composite $Z \rightarrow Z \rightarrow Y'$, then in the factorization

$$X \longrightarrow Y' \longrightarrow Y$$

the first arrow is again a closed inclusion and the second one is now a smooth surjection.

The above factorization determines a smooth surjection $(X \hookrightarrow Y') \rightarrow (X \rightarrow Y)$ of relative Artin stacks whose source is a pair. However, the same is true for all of the iterated intersections, as they are of the form $X \hookrightarrow Y' \times_Y \dots \times_Y Y'$ and these are easily seen to be closed embeddings of schemes. We deduce that $(X \rightarrow Y)$ is in the subcategory generated by pairs, ending the argument. \square

We are now ready to prove the main result of this section.

Proof of Theorem 5.11. — We've already verified all of the required properties of the normal sheaf functor $\mathcal{N} : \text{RelArt} \rightarrow \text{Art}$, all that is left is uniqueness. As a consequence of Proposition 4.8, any functor satisfying these properties is a cosheaf with respect to the smooth topology on Artin stacks. Since by Lemma 5.17 the subcategory of pairs of schemes is generating, Proposition 4.4 implies that any such cosheaf is uniquely determined by its restriction to $\mathcal{P}\text{air}$, ending the proof. \square

6. The normal cone of a morphism of Artin stacks

In this section we generalize the construction of the normal cone of a closed embedding of schemes to any locally of finite type morphism of Artin stacks. We characterize our extension as the unique one satisfying certain natural axioms and verify that in the case of a morphism of Deligne–Mumford type, our construction agrees with that of Behrend and Fantechi.

Recall that if $U \hookrightarrow V$ is a closed embedding of schemes with ideal sheaf I , then the *normal cone* $\mathcal{C}_U V := \text{Spec}_U(\bigoplus I^k/I^{k+1})$ is the relative spectrum of the associated graded \mathcal{O}_V -algebra. The construction of the normal cone is fundamental in intersection theory [8].

Notation 6.1. — We will use the symbol \mathcal{C} to denote the normal cone of a morphism, rather than \mathbb{V} , which we reserve for the abelian cone associated to a quasi-coherent sheaf introduced in Definition 3.1. The normal cone is usually not abelian.

The normal cone is intimately related to the normal sheaf discussed in the previous chapter; the graded algebra $\bigoplus I^k/I^{k+1}$ is generated in degree 1, and it follows that there is a canonical closed embedding $\mathcal{C}_U V \hookrightarrow \mathcal{N}_U V$ into the normal sheaf. If $U \hookrightarrow V$ is regular, this embedding is an isomorphism, and so one can consider the normal cone as a measure of non-smoothness.

In Theorem 5.11 we proved that the natural extension of the notion of a normal sheaf using the theory of the cotangent complex can be characterized uniquely by simple axioms. We will now prove that an analogous extension can also be constructed for the normal cone.

THEOREM 6.2. — *There exists a unique functor $\mathcal{C} : \text{RelArt} \rightarrow \text{Art}$ on the ∞ -category of relative Artin stacks, called the normal cone, such that:*

- (1) *if $U \hookrightarrow V$ is a closed embedding of schemes, then $\mathcal{C}_U V$ coincides with the classical normal cone, that is, $\mathcal{C}_U V \simeq \text{Spec}_U(\bigoplus I^k/I^{k+1})$, where I is the ideal sheaf;*
- (2) *\mathcal{C} preserves coproducts;*
- (3) *\mathcal{C} preserves smooth and smoothly surjective maps;*
- (4) *\mathcal{C} commutes with pullbacks along smooth morphisms of relative Artin stacks.*

Note that the content of the above result is slightly different than that of Theorem 5.11 which concerned the normal sheaf, as in the latter case we constructed the functor a priori. In the case of the normal cone, the existence of this functor is part of the result. Nevertheless, the normal cone is strongly related to the normal sheaf, as the following shows.

THEOREM 6.3. — *There is a unique natural transformation $\mathcal{C} \rightarrow \mathcal{N}$ of functors on relative Artin stacks which for every pair $U \hookrightarrow V$ of schemes with ideal sheaf I coincides with the natural morphism $\mathcal{C}_U V \rightarrow \mathcal{N}_U V$ induced by the surjection $\text{Sym}(I/I^2) \rightarrow \bigoplus I^k/I^{k+1}$.*

Moreover, for any relative Artin stack $\mathcal{X} \rightarrow \mathcal{Y}$

- (1) *$\mathcal{C}_{\mathcal{X}} \mathcal{Y} \rightarrow \mathcal{N}_{\mathcal{X}} \mathcal{Y}$ is a closed embedding and*
- (2) *for any smooth morphism $(\tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{Y}}) \rightarrow (\mathcal{X} \rightarrow \mathcal{Y})$ of relative Artin stacks the induced square*

$$\begin{array}{ccc}
 \mathcal{C}_{\tilde{\mathcal{X}}} \tilde{\mathcal{Y}} & \longrightarrow & \mathcal{N}_{\tilde{\mathcal{X}}} \tilde{\mathcal{Y}} \\
 \downarrow & & \downarrow \\
 \mathcal{C}_{\mathcal{X}} \mathcal{Y} & \longrightarrow & \mathcal{N}_{\mathcal{X}} \mathcal{Y}
 \end{array}$$

is cartesian.

The proofs of these two results are intimately related. Observe that Theorem 6.2 implies that \mathcal{C} is a cosheaf adapted to the class of smooth maps in the sense of Definition 4.7. Since we've shown before in Theorem 4.10 that an adapted cosheaf can be uniquely extended from a generating subcategory, it is enough to verify that the classical normal cone of a closed embedding of schemes has the required properties.

The latter is a problem in commutative algebra which can be tackled directly, but we will not do so. Instead, we verify that the cartesian property of Theorem 6.3 holds for smooth morphisms between pairs of schemes, the other properties will then follow from what we've already proven about the normal sheaf.

Before proceeding with the proofs, let us first show that expected properties of the normal cone follow from the above axiomatics.

LEMMA 6.4. — *For any $\mathcal{X} \rightarrow \mathcal{Y}$, the normal cone $\mathcal{C}_{\mathcal{X}}\mathcal{Y}$ is canonically a pointed stack over \mathcal{X} .*

Proof. — Observe that both \mathcal{C} and the “source” functor $s(\mathcal{X} \rightarrow \mathcal{Y}) = \mathcal{X}$ are adapted cosheaves on relative Artin stacks. It follows that any natural transformations between them defined for closed embeddings of affine schemes extend uniquely to all of $\mathcal{R}elArt$. In particular, for any $\mathcal{X} \rightarrow \mathcal{Y}$ there's a canonical projection $\mathcal{C}_{\mathcal{X}}\mathcal{Y} \rightarrow \mathcal{X}$, and this projection admits a canonical section, because that is the case in the classical setting. \square

Remark 6.5. — The classical normal cone $C_X Y$ of a closed embedding $X \hookrightarrow Y$ of schemes has more structure than just being pointed, namely, it is also equipped with an action of \mathbb{A}^1 induced by the grading of $\bigoplus I^k/I^{k+1}$.

A slight variation in the arguments we give shows that the same is true for the normal cone for an arbitrary morphisms of Artin stacks. Namely, instead of considering \mathcal{C} as an adapted cosheaf valued in Artin stacks, one should consider it as valued in the ∞ -category of morphisms $\mathcal{M} \rightarrow \mathcal{X}$ where \mathcal{M} is a pointed \mathcal{X} -Artin stack equipped with an \mathbb{A}^1 -action.

LEMMA 6.6. — *The normal cone of the identity is trivial; that is, for any \mathcal{X} we have $\mathcal{C}_{\mathcal{X}}\mathcal{X} \simeq \mathcal{X}$.*

Proof. — Since \mathcal{C} is a cosheaf on relative Artin stacks, it is easy to see that the subcategory of those Artin stacks which satisfy the above condition is downward closed. Since it contains all affine schemes, we deduce that it must be all of Art . \square

PROPOSITION 6.7 (Smooth base-change). — *The normal cone satisfies smooth base-change. That is, for any $\mathcal{X} \rightarrow \mathcal{Y}$ and smooth $\mathcal{Y}' \rightarrow \mathcal{Y}$ we have $\mathcal{C}_{\mathcal{X}'}\mathcal{Y}' \simeq \mathcal{X}' \times_{\mathcal{X}} \mathcal{C}_{\mathcal{X}}\mathcal{Y}$, where $\mathcal{X}' \simeq \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$.*

Proof. — Keeping in mind Lemma 6.6, this is immediate from applying the pullback axiom to the span $(\mathcal{X} \rightarrow \mathcal{Y}) \rightarrow (\mathcal{Y} \rightarrow \mathcal{Y}) \leftarrow (\mathcal{Y}' \rightarrow \mathcal{Y}')$. \square

PROPOSITION 6.8 (Étale invariance). — *Suppose we have a morphism $(\tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{Y}}) \rightarrow (\mathcal{X} \rightarrow \mathcal{Y})$ of relative Artin stacks which is étale on both source and target. Then, $\mathcal{C}_{\tilde{\mathcal{X}}} \tilde{\mathcal{Y}} \simeq \tilde{\mathcal{X}} \times_{\mathcal{X}} \mathcal{C}_{\mathcal{X}}\mathcal{Y}$.*

Proof. — Since $(\tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{Y}}) \rightarrow (\mathcal{X} \rightarrow \mathcal{Y})$ is smooth, this is immediate from the cartesian square of Theorem 6.3 and the étale invariance of the normal sheaf. \square

We can also verify that in the Deligne–Mumford case, our construction recovers the intrinsic normal cone of Behrend and Fantechi.

THEOREM 6.9. — *Let $\mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of 1-Artin stacks of finite type which is a relative Deligne–Mumford stack. Then, the normal cone $\mathcal{C}_{\mathcal{X}}\mathcal{Y}$ is equivalent to the intrinsic normal cone of Behrend–Fantechi.*

Proof. — Since both the Behrend–Fantechi intrinsic normal cone and the normal cone of Theorem 6.2 satisfy smooth base-change and are étale-invariant, the latter by Proposition 6.7 and Proposition 6.8, we can assume that we have a morphism $X \rightarrow Y$ of schemes.

We can lift the given morphism to a closed embedding $i : X \hookrightarrow M$ such that $M \rightarrow Y$ is smooth and surjective. Observe that this then defines a smooth surjection

$$(X \hookrightarrow M) \longrightarrow (X \longrightarrow Y)$$

of relative Artin stacks. It follows from Theorem 6.3 that we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{C}_X M & \longrightarrow & \mathcal{C}_X Y \\ \downarrow & & \downarrow \\ \mathcal{N}_X M & \longrightarrow & \mathcal{N}_X Y, \end{array}$$

which is precisely how Behrend–Fantechi defined the intrinsic normal cone [5, 3.10]. \square

Remark 6.10. — Note that even in the classical Deligne–Mumford case, Theorem 6.2 clarifies the construction of Behrend and Fantechi by showing that it is the only extension of the normal cone of a closed embedding of schemes that preserves certain natural properties.

The rest of this chapter will be devoted to the proofs of Theorem 6.2 and Theorem 6.3; as explained above, the main step is to establish that the classical normal cone has the required properties when restricted to the category of pairs $U \hookrightarrow V$ of schemes.

In more detail, we need to prove that the normal cone functor $\mathcal{C} : \text{Pair} \rightarrow \text{Art}$ preserves smooth and smoothly surjective morphisms, and commutes with pullbacks along smooth morphisms. To do so, it will be convenient to introduce some temporary terminology.

DEFINITION 6.11. — We say a morphism $f : (N \hookrightarrow M) \rightarrow (X \hookrightarrow Y)$ of pairs is good if the induced diagram

$$\begin{array}{ccc} \mathcal{C}_N M & \longrightarrow & \mathcal{N}_N M \\ \downarrow & & \downarrow \\ \mathcal{C}_X Y & \longrightarrow & \mathcal{N}_X Y \end{array}$$

between the normal cones and normal sheaves is cartesian.

Our goal is to prove that an arbitrary smooth morphisms of pairs; that is, one that is smooth on both source and target, is good. As an easy example, any flat cartesian morphism of pairs is good, as an easy consequence of the flat base-change for the normal cone and the normal sheaf.

Remark 6.12. — It is not true that every morphism of pairs of schemes is good in the sense of Definition 6.11. As an example, let $L \subseteq \mathbb{A}^2$ be the union of the coordinate axes and let $\text{Spec}(k) \hookrightarrow L$ be the inclusion of the origin. Then, one can verify that the obvious morphism $(\text{Spec}(k) \hookrightarrow L) \rightarrow (\text{Spec}(k) \rightarrow \mathbb{A}^2)$ induces an isomorphism between normal bundles, but not between the normal cones.

LEMMA 6.13. — Any morphism $(\tilde{X} \hookrightarrow \tilde{Y}) \rightarrow (X \hookrightarrow Y)$ of pairs of schemes which is étale on both source and target is good.

Proof. — Since both the normal sheaf and normal cone satisfy flat base-change, we can replace Y by the spectrum of the strict henselization of the local ring at each of its points; then, both X and Y will be of this form. It follows that \tilde{X} is a disjoint union of copies of X mapping onto it isomorphically, and likewise for \tilde{Y} . The claim then follows. \square

LEMMA 6.14. — Suppose we have a morphism $f : (N \hookrightarrow M) \rightarrow (X \hookrightarrow Y)$ of pairs of schemes. Then, f is good if and only if for each $n \in N$ there exists an open neighbourhood $U \subseteq M$ of n such that the restriction $(U \cap N \hookrightarrow U) \rightarrow (X \hookrightarrow Y)$ is good.

Proof. — Observe that for any such neighbourhood we have $\mathcal{C}_{U \cap N} U \simeq \mathcal{C}_N M \times_N (U \cap N)$ and likewise for the normal sheaf, and that as $n \in N$ varies the open sets $U \cap N$ form a covering of N . Then, since $\mathcal{C}_N M \rightarrow \mathcal{C}_X Y \times_{\mathcal{N}_X Y} \mathcal{N}_N M$ is a morphism of N -schemes, the claim is then equivalent to saying that is in isomorphism if and only if the same is true for the maps

$$\begin{aligned} \mathcal{C}_{U \cap N} M &\simeq \mathcal{C}_N M \times_N (U \cap N) \\ &\longrightarrow \mathcal{C}_X Y \times_{\mathcal{N}_X Y} \mathcal{N}_N M \times_N (U \cap N) \simeq \mathcal{C}_X Y \times_{\mathcal{N}_X Y} \mathcal{N}_{U \cap N} U. \quad \square \end{aligned}$$

LEMMA 6.15. — *Suppose we have a morphism of pairs of schemes of the form*

$$\begin{array}{ccc} X & \hookrightarrow & Y \times \mathbb{A}^n \\ \downarrow & & \downarrow \\ X & \hookrightarrow & Y, \end{array}$$

where the left vertical arrow is the identity, the right one is the projection, and the upper horizontal arrow is the composite $X \hookrightarrow Y \times \{0\} \hookrightarrow Y \times \mathbb{A}^n$ of the lower one with the natural inclusion. Then, any such morphism is good.

Proof. — In this case, one can compute directly that $\mathcal{C}_X(Y \times \mathbb{A}^n) \simeq (\mathcal{C}_X Y) \times \mathbb{A}^n$, see [3, 3.5.1], and since an analogous formula holds for the normal sheaf, the claim follows. \square

PROPOSITION 6.16. — *Any smooth morphism $(N \hookrightarrow M) \rightarrow (X \hookrightarrow Y)$ of pairs of schemes is good. In other words, for any such morphism we have $\mathcal{C}_N M \simeq \mathcal{C}_X Y \times_{\mathcal{N}_X Y} \mathcal{N}_N M$.*

Proof. — Observe that the induced morphism $N \hookrightarrow X \times_Y M$ is a closed embedding of smooth X -schemes. It follows that by picking $n \in N$ and choosing a smaller affine neighbourhood of its image in M , which we can do by Lemma 6.14, we can assume that there are regular functions $(g_i)_{1 \leq i \leq n}$ on $X \times_Y M$ and an $m \leq n$ such that the resulting diagram

$$\begin{array}{ccc} N & \hookrightarrow & X \times_Y M \\ \downarrow & & \downarrow \\ X \times \mathbb{A}^m & \hookrightarrow & X \times \mathbb{A}^n, \end{array}$$

where the bottom arrow is the natural inclusion, is cartesian and with vertical arrows étale [24, Exp.II, Prop 4.9]. By lifting those regular functions to all of M , we can extend the right vertical arrow to a morphism $M \rightarrow Y \times \mathbb{A}^n$ and by making M smaller if necessary we can assume that the latter is étale as well. We can then consider the larger diagram

$$\begin{array}{ccc} N & \hookrightarrow & M \\ \downarrow & & \downarrow \\ X \times \mathbb{A}^m & \hookrightarrow & Y \times \mathbb{A}^n \\ \downarrow & & \downarrow \\ X \times \mathbb{A}^m & \hookrightarrow & Y \times \mathbb{A}^m \\ \downarrow & & \downarrow \\ X & \hookrightarrow & Y, \end{array}$$

where the map $Y \times \mathbb{A}^n \rightarrow Y \times \mathbb{A}^m$ is the obvious projection. Out of the three squares stacked on top of each other, the bottom one is good because it is smooth cartesian and the top one by Lemma 6.13. Since the middle square is good by Lemma 6.15 and composition of good squares is good by the pullback pasting lemma, we are done. \square

We are now ready to give proofs of the two main results of this chapter.

Proof of Theorem 6.2 and Theorem 6.3. — We first claim that $\mathcal{C} : \text{Pair} \rightarrow \text{Art}$ is an cosheaf on the site of pairs of schemes adapted to the class of smooth maps. We have a natural transformation $\mathcal{C} \rightarrow \mathcal{N}$, which as we verified in Proposition 6.16 is smooth-cartesian in the sense of Definition 4.11. It then follows from Lemma 4.12 that \mathcal{C} is adapted, because this is true for the normal sheaf as a consequence of Theorem 5.11.

We thus deduce from Proposition 4.4 and Lemma 5.17 that the normal cone functor uniquely extends to a Stk -valued cosheaf on all relative Artin stacks, and moreover that this cosheaf is also adapted by Theorem 4.10. We will now show that for any relative Artin stack $\mathcal{X} \rightarrow \mathcal{Y}$, the stack $\mathcal{C}_\mathcal{X}\mathcal{Y}$ is in fact Artin.

We claim that the subcategory of those relative Artin stacks for which the normal cone is Artin is downward closed, since we know it contains all pairs of schemes, this will imply the claim. Suppose that $(\mathcal{X}_0 \rightarrow \mathcal{Y}_0) \rightarrow (\mathcal{X} \rightarrow \mathcal{Y})$ is a smooth surjection such that $\mathcal{C}_{\mathcal{X}_k}\mathcal{Y}_k$ is Artin, where $\mathcal{X}_k := \mathcal{X}_0 \times_{\mathcal{X}} \dots \times_{\mathcal{X}} \mathcal{X}_0$ and likewise for \mathcal{Y} . Since \mathcal{C} is an adapted cosheaf, we see that the diagram

$$\dots \rightrightarrows \mathcal{C}_{\mathcal{X}_1}\mathcal{Y}_1 \rightrightarrows \mathcal{C}_{\mathcal{X}_0}\mathcal{Y}_0 \longrightarrow \mathcal{C}_\mathcal{X}\mathcal{Y}$$

is an effective groupoid in the ∞ -topos Stk . Thus, $\mathcal{C}_\mathcal{X}\mathcal{Y}$ admits a smooth relatively Artin surjection from an Artin stack, and it follows that it itself must be Artin, see [2, 4.30].

We have a natural transformation $\mathcal{C} \rightarrow \mathcal{N}$ defined on the category of pairs of schemes, and since both the source and target are cosheaves, it follows from another application of Proposition 4.4 that this natural transformation uniquely extends to one defined on all relative Artin stacks.

Since this natural transformation yields cartesian squares when applied to any smooth morphism of pairs of schemes, as we verified in Proposition 6.16, it follows formally through Proposition 4.13 that it has this property for any smooth morphism of relative Artin stacks. It follows from this that \mathcal{C} preserves smooth and smoothly surjective morphisms, finishing the proof of Theorem 6.2.

Since we already constructed the natural transformation $\mathcal{C} \rightarrow \mathcal{N}$ and we checked that it is smooth-cartesian, to prove Theorem 6.3 we're only

left with checking that for any relative Artin stack $X \rightarrow Y$, the resulting morphism $\mathcal{C}_X Y \rightarrow \mathcal{N}_X Y$ is a closed embedding. Choose a smooth surjection $(X \hookrightarrow Y) \rightarrow (X \rightarrow Y)$ whose source is a pair of schemes, it follows that the diagram

$$\begin{array}{ccc} \mathcal{C}_X Y & \longrightarrow & \mathcal{N}_X Y \\ \downarrow & & \downarrow \\ \mathcal{C}_x Y & \longrightarrow & \mathcal{N}_x Y \end{array}$$

is cartesian and that both vertical arrows are smooth surjections. Since it is clear that the top horizontal arrow is a closed embedding, we deduce that the same is true for the bottom one, ending the proof. \square

7. The deformation space

Deforming a closed embedding of schemes $X \hookrightarrow Y$ into the zero section imbedding of X into $\mathcal{C}_X Y$ is a fundamental procedure in intersection theory, known as the deformation to the normal cone. In this section we will generalize this construction to any locally of finite type morphism of Artin stacks.

Recall that for a closed embedding $X \hookrightarrow Y$ of schemes, the deformation $M_X^\circ Y$ is a flat scheme over \mathbb{P}^1 which fits into a commutative diagram

$$\begin{array}{ccc} X \times \mathbb{P}^1 & \hookrightarrow & M_X^\circ Y \\ & \searrow & \swarrow \\ & \mathbb{P}^1 & \end{array}$$

such that

- (1) over $\mathbb{A}^1 \simeq \mathbb{P}^1 - \{\infty\}$ the horizontal arrow is isomorphic to $X \times \mathbb{A}^1 \hookrightarrow Y \times \mathbb{A}^1$ and
- (2) over $\{\infty\}$, the horizontal arrow is isomorphic to $X \hookrightarrow \mathcal{C}_X Y$.

Explicitly, $M_X^\circ Y$ can be constructed as the difference

$$M_X^\circ Y := \text{Bl}_{X \times \{\infty\}} Y \times \mathbb{P}^1 - \text{Bl}_{X \times \{\infty\}} Y \times \{\infty\}$$

between two blow-ups along $X \times \{\infty\}$. Alternatively, if $X \hookrightarrow Y \simeq \text{Spec}(A)$ is defined by ideal I , then the restriction of $M_X^\circ Y$ to $\mathbb{P}^1 - \{0\} \simeq \mathbb{A}^1 \simeq \text{Spec}(k[t])$ is isomorphic to the spectrum of the Rees algebra $R(A, I) := \bigoplus_{k \in \mathbb{Z}} I^k t^{-k} \subseteq A[t, t^{-1}]$.

LEMMA 7.1. — *The deformation space functor $M^\circ : \mathcal{P}\text{air} \rightarrow \mathcal{A}\text{rt}/_{\mathbb{P}^1}$ preserves coproducts, smooth morphisms, smooth surjections and commutes with pullbacks along smooth maps. In particular, it is a cosheaf adapted to the class of smooth maps.*

Proof. — Since for any pair $X \hookrightarrow Y$ the deformation space $M_X^\circ Y$ is flat over \mathbb{P}^1 , it suffices to check all of the claims fibrewise. This is clear, since $M_X^\circ Y \times_{\mathbb{P}^1} \{t\} \simeq Y$ for $t \neq \infty$, $M_X^\circ Y \times_{\mathbb{P}^1} \{\infty\} \simeq \mathcal{C}_X Y$ and both of the right hand sides have these properties, the latter by Theorem 6.2. \square

THEOREM 7.2. — *For any relative Artin stack $\mathcal{X} \rightarrow \mathcal{Y}$ there exists an Artin stack $M_{\mathcal{X}}^\circ \mathcal{Y}$ which fits into a commutative diagram*

$$\begin{array}{ccc} \mathcal{X} \times \mathbb{P}^1 & \hookrightarrow & M_{\mathcal{X}}^\circ \mathcal{Y} \\ & \searrow & \swarrow \\ & \mathbb{P}^1 & \end{array}$$

where both vertical arrows are flat and such that

- (1) over $\mathbb{A}^1 \simeq \mathbb{P}^1 - \{\infty\}$, the horizontal arrow is isomorphic to $\mathcal{X} \times \mathbb{A}^1 \hookrightarrow \mathcal{Y} \times \mathbb{A}^1$ and
- (2) over $\{\infty\}$, the horizontal arrow is isomorphic to $\mathcal{X} \hookrightarrow \mathcal{C}_{\mathcal{X}} \mathcal{Y}$.

Proof. — Since M° is an adapted cosheaf on the site of pairs by Lemma 7.1, it extends uniquely to an adapted cosheaf on all of $\mathcal{R}\text{elArt}$ by Theorem 4.10. It is easy to see that the formula $\mathcal{X} \mapsto \mathcal{X} \times \mathbb{P}^1$ also yields an adapted cosheaf, and so the natural transformation between the two defined for pairs also extends uniquely.

To see that $M_{\mathcal{X}}^\circ \mathcal{Y} \rightarrow \mathbb{P}^1$ is flat, choose a smooth surjection $(X \hookrightarrow Y) \rightarrow (\mathcal{X} \rightarrow \mathcal{Y})$ of relative Artin stacks whose source is a closed immersion of schemes. It then follows from Lemma 7.1, that $M_X^\circ Y \rightarrow M_{\mathcal{X}}^\circ \mathcal{Y}$ is a smooth surjection, and since the composite $M_X^\circ Y \rightarrow \mathbb{P}^1$ is flat, we deduce the same is true for $M_{\mathcal{X}}^\circ \mathcal{Y} \rightarrow \mathbb{P}^1$.

To deduce the two properties, observe that $\mathcal{Y} \times \mathbb{A}^1 \rightarrow M_{\mathcal{X}}^\circ \mathcal{Y}|_{\mathbb{A}^1}$ and $\mathcal{C}_{\mathcal{X}} \mathcal{Y} \rightarrow M_{\mathcal{X}}^\circ \mathcal{Y}|_{\{\infty\}}$ are natural transformations of adapted cosheaves on $\mathcal{R}\text{elArt}$ which restrict to isomorphisms for closed embeddings of schemes, and so must be equivalences in general. \square

The existence of the deformation space has the following important consequence, which in practice allows one to deduce many properties of the normal cone automatically.

COROLLARY 7.3. — *Let P be a property of Artin stacks which is stable under flat deformation over an affine base. Then, for any relative Artin stack $\mathcal{X} \rightarrow \mathcal{Y}$, \mathcal{Y} has property P if and only if the normal cone $\mathcal{C}_{\mathcal{X}}\mathcal{Y}$ has property P .*

Proof. — This is immediate from Theorem 7.2. □

8. The virtual fundamental class

In this chapter we introduce the notion of a perfect obstruction theory generalizing the classical one due to Behrend and Fantechi, and show that obstruction theories correspond to closed immersions under the abelian cone functor. We then specialize to the case of an 1-Artin stacks, where we have access to Chow groups, and construct the virtual fundamental class in the presence of global resolutions. Finally, we give a few examples of moduli stacks to which these methods apply.

DEFINITION 8.1. — *Let $\mathcal{X} \rightarrow \mathcal{Y}$ be relative Artin stack and $\varphi : \mathcal{E} \rightarrow L_{\mathcal{X}/\mathcal{Y}}[-1]$ be a morphism in $\mathrm{QCoh}(\mathcal{X})$. We say that φ is an obstruction theory if*

- (1) *The homomorphism $h_0(\varphi)$ is surjective*
- (2) *The homomorphism $h_i(\varphi)$ is an isomorphism for $i \leq -1$.*

We say that an obstruction theory is perfect if \mathcal{E} is perfect of non-positive amplitude.

Keeping in mind that we use the homological grading convention, it is easy to see that if \mathcal{X} is Deligne–Mumford, our definition coincides up to a shift with the one given by Behrend and Fantechi in [5]. In this case, $L_{\mathcal{X}}$ will be in fact connective.

Informally, a perfect obstruction theory can be thought of as a “shadow” of a quasi-smooth derived enhancement, see Example 8.14 for more detail.

The abelian cone functor of Definition 3.1 provides us with a bridge from algebraic objects, namely quasi-coherent sheaves, to objects of geometric nature, namely abelian Artin stacks over \mathcal{X} . We will now show what the condition of being an obstruction theory translates to in geometry, generalizing Behrend and Fantechi’s criterion in the Deligne–Mumford case.

PROPOSITION 8.2. — *Let \mathcal{X} be an Artin stack and let $\varphi : \mathcal{E} \rightarrow \mathcal{F}$ be a morphism of bounded below quasi-coherent sheaves. Then, $\mathbb{V}_{\mathcal{X}}(\mathcal{F}) \rightarrow \mathbb{V}_{\mathcal{X}}(\mathcal{E})$ is*

- (1) *affine if and only if $h_{-1}(\varphi)$ is surjective and $h_i(\varphi)$ is an isomorphism for $i \leq -2$ and*
- (2) *a closed immersion if and only if $h_0(\varphi)$ is surjective and $h_i(\varphi)$ is an isomorphism for $i \leq -1$.*

Proof. — As formation of the abelian cone commutes with arbitrary base-change by Lemma 3.5, we can assume that $\mathcal{X} \simeq \mathrm{Spec}(A)$ is affine by replacing it by a smooth atlas. Before proceeding, let us observe that both of the homological conditions can be rephrased as saying that the cofibre of $\mathcal{E} \rightarrow \mathcal{F}$ is respectively, 0- and 1-connective.

Since both the above homological conditions and the abelian cone only depend on the coconnective truncations, the latter as a consequence of Lemma 3.6, we can assume that \mathcal{E} and \mathcal{F} are coconnective. In this case, \mathcal{E} can be represented by a non-positively graded chain complex

$$0 \longrightarrow E_0 \longrightarrow E_{-1} \longrightarrow \dots$$

of A -modules, and since \mathcal{E} is bounded below we can assume that E_i are free for $i < 0$ and eventually vanish.

In this case, we see from filtering \mathcal{E} using the truncations of the given chain complex that for any A -module M , the induced map $\mathrm{Ext}_A^0(E_0, M) \rightarrow \mathrm{Ext}_A^0(\mathcal{E}, M)$ is surjective. Thus, the morphism $\mathbb{V}_{\mathrm{Spec}(A)}(E_0) \rightarrow \mathbb{V}_{\mathrm{Spec}(A)}(\mathcal{E})$ is a surjection of stacks. It follows that the given map between abelian cones is affine or a closed immersion if and only if this is true for the base-change $\mathbb{V}_{\mathrm{Spec}(A)}(E_0) \times_{\mathbb{V}_{\mathrm{Spec}(A)}(\mathcal{E})} \mathbb{V}_{\mathrm{Spec}(A)}(\mathcal{F}) \rightarrow \mathbb{V}_{\mathrm{Spec}(A)}(E_0)$.

Since the abelian cone takes colimits to limits, the above base-change can be identified with the map induced by $E_0 \rightarrow \mathcal{F} \oplus_{\mathcal{E}} E_0$. Since this map has the same cofibre as $\mathcal{E} \rightarrow \mathcal{F}$, we see that by replacing \mathcal{E} by E_0 and \mathcal{F} by the pushout, we can assume that \mathcal{E} is an A -module.

If \mathcal{E} is discrete, then $\mathbb{V}_{\mathrm{Spec}(A)}(\mathcal{E})$ is affine and we see that the morphism between cones is affine if only if $\mathbb{V}_{\mathrm{Spec}(A)}(\mathcal{F})$ is affine. As a consequence of Lemma 3.7, this happens precisely when \mathcal{F} is connective, which is equivalent to the first homological condition, as $h_i(\mathcal{E}) = 0$ for $i < 0$.

To see that the second homological condition controls whether the morphism is a closed immersion, observe that in the case above when both \mathcal{E} and \mathcal{F} are 0-connective, the map between cones can be identified with $\mathrm{Spec}(\mathrm{Sym}_A(h_0(\mathcal{F}))) \rightarrow \mathrm{Spec}(\mathrm{Sym}_A(h_0(\mathcal{E})))$. This is clearly a closed immersion if and only if $h_0(\mathcal{E}) \rightarrow h_0(\mathcal{F})$ is a surjection, ending the proof. \square

COROLLARY 8.3. — *Let $\mathcal{X} \rightarrow \mathcal{Y}$ be a relative Artin stack. Then, $\varphi : \mathcal{E} \rightarrow L_{\mathcal{X}/\mathcal{Y}}[-1]$ is an obstruction theory if and only if \mathcal{E} is bounded below and $\mathcal{N}_{\mathcal{X}}\mathcal{Y} \rightarrow \mathbb{V}_{\mathcal{X}}(\mathcal{E})$ is a closed immersion.*

Proof. — This is immediate from Proposition 8.2. □

Recall that if $\mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of Artin stacks, then the cotangent complex $L_{\mathcal{X}/\mathcal{Y}}$ controls the deformation theory in the sense that for any A -valued point $\eta : \text{Spec}(A) \rightarrow \mathcal{X}$, an A -module M , and \tilde{A} a square-zero extension of A by M and a diagram

$$\begin{array}{ccc}
 \text{Spec}(A) & \longrightarrow & \text{Spec}(\tilde{A}) \\
 \eta \downarrow & \swarrow \text{dotted} & \downarrow \\
 \mathcal{X} & \longrightarrow & \mathcal{Y}
 \end{array}$$

(♠)

the extension denoted above by the dotted arrow exists if and only if the associated obstruction in $\text{Ext}_A^1(\eta^*L_{\mathcal{X}/\mathcal{Y}}, M)$ vanishes. More generally, the cotangent complex has this property also for square-zero extensions of derived rings, by which it is then determined uniquely, see the discussion preceding Definition 2.21.

This uniqueness does not hold if we consider only discrete rings, in fact we will now prove a minor generalization of a criterion of Behrend and Fantechi which tells us that a morphism $\mathcal{E}[1] \rightarrow L_{\mathcal{X}/\mathcal{Y}}$ is a shift of an obstruction theory if and only if $\mathcal{E}[1]$ also controls the deformation theory of affine schemes mapping into \mathcal{X} .

PROPOSITION 8.4. — *A morphism $\mathcal{E}[1] \rightarrow L_{\mathcal{X}/\mathcal{Y}}$ is a shift of an obstruction theory if and only if for any point $\eta : \text{Spec}(A) \rightarrow \mathcal{X}$ and any A -module M , the induced morphism*

$$\text{Ext}_A^i(\eta^*L_{\mathcal{X}/\mathcal{Y}}, M) \longrightarrow \text{Ext}_A^i(\eta^*\mathcal{E}[1], M)$$

is injective for $i = 1$ and an isomorphism for $i \leq 0$.

Proof. — Since a morphism $\mathcal{E}[1] \rightarrow L_{\mathcal{X}/\mathcal{Y}}$ is a shift of an obstruction theory if and only if its cofibre is 2-connective, the statement is equivalent to saying that $\mathcal{C} \in \text{QCoh}(\mathcal{X})$ is 2-connective if and only if $\text{Ext}^k(\eta^*\mathcal{C}, M) = 0$ for any η, M as above and $k \leq 1$.

This is clear, since \mathcal{C} is 2-connective if and only if $\eta^*\mathcal{C}$ is 2-connective for all η , and that's equivalent to saying that $\text{Ext}^k(\eta^*\mathcal{C}, \mathcal{M}) = 0$ for any $k \geq 1$ and $\mathcal{M} \in \text{QCoh}(\text{Spec}(A))_{\leq 0}$. Since the latter ∞ -category is generated under limits by A -modules, it is enough to check this condition in this case, ending the argument. □

COROLLARY 8.5. — *Let $\phi : \mathcal{E}[1] \rightarrow L_{\mathcal{X}/\mathcal{Y}}$ be a morphism of quasi-coherent sheaves. Then, ϕ is a shift of an obstruction theory if and only if for any diagram of the form (\spadesuit)*

- (1) *the dotted arrow exists if and only if the associated obstruction in $\mathrm{Ext}^1(\eta^*\mathcal{E}, M)$ vanishes, and if this is the case then*
- (2) *the space of such dotted arrows is equivalent to $\mathrm{map}_{\mathrm{QCoh}(A)}(\eta^*\mathcal{E}, M)$, in particular their homotopy classes form an $\mathrm{Ext}_A^0(\eta^*\mathcal{E}, M)$ -torsor.*

Proof. — It is clear that $L_{\mathcal{X}/\mathcal{Y}}$ has this property, and the statement is then an immediate consequence of Proposition 8.4. \square

We now move on to the construction of the virtual fundamental class. Let us now restrict to the case where $\mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of finite type 1-Artin stacks, where we have access to the Chow groups as constructed by Kresch [16].

DEFINITION 8.6. — *If \mathcal{E} is a perfect obstruction theory, then a global resolution is a morphism $\mathcal{E} \rightarrow E$ injective on h_0 such that $E \in \mathrm{QCoh}(\mathcal{X})^\heartsuit$ is a locally free sheaf of finite rank.*

CONSTRUCTION 8.7. — *Suppose that we have a morphism of 1-Artin stacks $\mathcal{X} \rightarrow \mathcal{Y}$ with target purely of dimension r , in which case the same is true for the normal cone $\mathcal{C}_{\mathcal{X}}\mathcal{Y}$ as a consequence of Corollary 7.3.*

If $\mathcal{E} \rightarrow E$ is a global resolution, then Lemma 3.8 implies that $\mathbb{V}_{\mathcal{X}}(E) \rightarrow \mathbb{V}_{\mathcal{X}}(\mathcal{E})$ is a smooth surjection, and since the source is a vector bundle it forms a smooth atlas for $\mathbb{V}_{\mathcal{X}}(\mathcal{E})$ relative to \mathcal{X} . We can then consider the pullback diagram

$$\begin{array}{ccc} \mathbb{V}_{\mathcal{X}}(E) \times_{\mathbb{V}_{\mathcal{X}}(\mathcal{E})} \mathcal{C}_{\mathcal{X}}\mathcal{Y} & \longrightarrow & \mathbb{V}_{\mathcal{X}}(E) \\ \downarrow & & \downarrow \\ \mathcal{C}_{\mathcal{X}}\mathcal{Y} & \longrightarrow & \mathbb{V}_{\mathcal{X}}(\mathcal{E}), \end{array}$$

where the bottom map is the composite $\mathcal{C}_{\mathcal{X}}\mathcal{Y} \hookrightarrow \mathcal{N}_{\mathcal{X}}\mathcal{Y} \simeq \mathbb{V}_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{Y}}[-1]) \hookrightarrow \mathbb{V}_{\mathcal{X}}(\mathcal{E})$, which is a closed embedding as a consequence of Corollary 8.3.

In the setting of Construction 8.7, we can now define the virtual fundamental class.

DEFINITION 8.8. — *Let $\mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of 1-Artin stacks as above. Then, the virtual fundamental class associated to a perfect obstruction theory $\mathcal{E} \rightarrow L_{\mathcal{X}/\mathcal{Y}}[-1]$ which admits a global resolution E is given by*

$$[\mathcal{X} \rightarrow \mathcal{Y}, \mathcal{E}]^{vir} := 0^![\mathbb{V}_{\mathcal{X}}(E) \times_{\mathbb{V}_{\mathcal{X}}(\mathcal{E})} \mathcal{C}_{\mathcal{X}}\mathcal{Y}] \in \mathrm{CH}_{r-\chi(\mathcal{E})}(\mathcal{X})$$

where $0 : \mathcal{X} \rightarrow \mathcal{C}_{\mathcal{X}}(E)$ is the zero section.

A priori our construction of the virtual class depends on the choice of global resolution E we used to define it, we will now show that it is in fact canonically attached to the perfect obstruction theory \mathcal{E} .

PROPOSITION 8.9. — *The virtual fundamental class $[\mathcal{X} \rightarrow \mathcal{Y}, \mathcal{E}]^{vir}$ is independent of the choice of a global resolution of a perfect obstruction theory \mathcal{E} .*

Proof. — If $\mathcal{E} \rightarrow E$, $\mathcal{E} \rightarrow F$ are global resolutions, then it is easy to see that the same is true for $\mathcal{E} \rightarrow E \oplus F$. Thus, it is enough to check that the virtual fundamental class constructed using E coincides with that of $E \oplus F$. We have a commutative diagram

$$\begin{CD} \mathbb{V}_{\mathcal{X}}(E \oplus F) \times_{\mathbb{V}_{\mathcal{X}}(\mathcal{E})} \mathcal{C}_{\mathcal{X}}\mathcal{Y} @>>> \mathbb{V}_{\mathcal{X}}(E \oplus F) \\ @VVV @VVpV \\ \mathbb{V}_{\mathcal{X}}(E) \times_{\mathbb{V}_{\mathcal{X}}(\mathcal{E})} \mathcal{C}_{\mathcal{X}}\mathcal{Y} @>>> \mathbb{V}_{\mathcal{X}}(E), \end{CD}$$

and so $p^*[\mathbb{V}_{\mathcal{X}}(E) \times_{\mathbb{V}_{\mathcal{X}}(\mathcal{E})} \mathcal{C}_{\mathcal{X}}\mathcal{Y}] = [\mathbb{V}_{\mathcal{X}}(E \oplus F) \times_{\mathbb{V}_{\mathcal{X}}(\mathcal{E})} \mathcal{C}_{\mathcal{X}}\mathcal{Y}]$ as elements of $\text{CH}(\mathbb{V}_{\mathcal{X}}(E \oplus F))$. Then, the needed equality is obtained by intersecting with the zero sections of E and $E \oplus F$, since $\pi_{E \oplus F} \simeq p \circ \pi_E$ implies $0^!_{E \oplus F} \circ p^* \simeq 0^!_E$. \square

Remark 8.10. — Note that the only reason we restricted to 1-Artin stacks is that we needed a suitably well-behaved theory of Chow groups. It is clear that the above formula gives a fundamental class associated to any suitable homology theory of relative Artin stacks. In particular,

$$0^*[\mathcal{O}_{\mathcal{C}_{\mathcal{X}}}] \in K_0(\text{Coh}(\mathcal{X})),$$

where $0 : \mathcal{X} \rightarrow \mathbb{V}_{\mathcal{X}}(\mathcal{E})$ is the zero section, defines a *virtual fundamental class in G-theory* of a finite type Artin stack \mathcal{X} equipped with a choice of a perfect obstruction theory.

We now give a few examples of applications of our constructions.

Example 8.11 (*Intersection theory*). — Suppose we have a cartesian diagram

$$\begin{CD} \mathcal{W} @>j>> \mathcal{X} \\ @VgVV @VVfV \\ \mathcal{Y} @>i>> \mathcal{Z} \end{CD}$$

of 1-Artin stacks such that \mathcal{X} and \mathcal{Z} are smooth, \mathcal{X} has the resolution property and i is a regular closed embedding. Consider the cofibre sequence

$$g^*L_{\mathcal{Y}/\mathcal{Z}}[-1] \longrightarrow j^*L_{\mathcal{X}} \longrightarrow E.$$

where the left map is induced by the morphisms $L_{\mathcal{Y}/\mathcal{Z}}[-1] \rightarrow i^*L_{\mathcal{Y}}$ and $f^*L_{\mathcal{Z}} \rightarrow L_{\mathcal{X}}$. Since \mathcal{X} is smooth, $L_{\mathcal{X}}$ has perfect amplitude in $[0, -1]$ and since i is regular, $L_{\mathcal{Y}/\mathcal{Z}}$ is equivalent to $\mathcal{J}/\mathcal{J}^2[1]$. It follows that E is perfect, and one easily observes that the induced morphism

$$E \longrightarrow L_{\mathcal{W}}$$

is in fact a perfect obstruction theory. We deduce that \mathcal{W} admits a virtual fundamental class.

In the case of schemes, the resulting class coincides with $i^![\mathcal{X}]$ in the classical sense, as observed by Behrend and Fantechi [5, 6.1]. Thus, the above construction can be thought of as generalizing Fulton’s construction to the setting of Artin stacks, recovering Kresch’s Gysin maps.

Example 8.12 (Twisted stable maps). — Let \mathcal{X} be a finitely presented, proper, smooth, tame 1-Artin stack with finite inertia. Moreover, suppose that \mathcal{X} has the resolution property, that the coarse moduli space of \mathcal{X} is projective, and that we have fixed an element $\beta \in \text{CH}_1^{\text{num}}(X)$.

In this context, one can show that the canonical morphism

$$\mathcal{K}_{g,n}(\mathcal{X}, \beta) \longrightarrow \mathfrak{M}_{g,n}^{\text{tw}}$$

from the moduli stack of twisted stable maps to the moduli stack of twisted curves has a perfect obstruction theory which admits a global resolution. To do so, one considers the diagram

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \overline{\mathcal{C}} & \xrightarrow{\overline{\psi}} & \mathcal{X} \\ \pi \downarrow & & \downarrow \overline{\pi} & & \\ \mathcal{K}_{g,n}(\mathcal{X}, \beta) & \xrightarrow{\iota} & \underline{\text{Hom}}_{\mathfrak{M}_{g,n}^{\text{tw}}}(\mathcal{C}, \mathcal{X}) & & \\ & \searrow \overline{\varphi} & \downarrow \varphi & & \\ & & \mathfrak{M}_{g,n}^{\text{tw}} & & \end{array}$$

where \mathcal{C} is the universal twisted curve and $\overline{\mathcal{C}} := \mathcal{C} \times_{\mathfrak{M}_{g,n}^{\text{tw}}} \underline{\text{Hom}}_{\mathfrak{M}_{g,n}^{\text{tw}}}(\mathcal{C}, \mathcal{X})$. Then, by Grothendieck duality there is a canonical morphism

$$\overline{\pi}_*(\overline{\psi}^*L_{\mathcal{X}} \otimes \omega_{\overline{\pi}})|_{\mathcal{K}_{g,n}(\mathcal{X}, \beta)} \longrightarrow L_{\varphi}[-1]$$

which is shown to be a perfect obstruction theory using Corollary 8.5. Moreover, this obstruction theory has a global resolution due to the fact

that \mathcal{X} has the resolution property and we deduce that the stack $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ admits a relative virtual fundamental class. In future work, will prove that this class satisfies the Gromov–Witten axioms.

Example 8.13 (Quantum K-theory). — In the context of the previous example, one can instead consider the fundamental class in K -theory discussed in Remark 8.10. This class is related to quantum K -theory in the sense of Lee [17], which we hope to revisit in future work.

Example 8.14 (Quasi-smooth derived stacks). — Let $\mathcal{X} \rightarrow \mathcal{Y}$ be a quasi-smooth morphism of derived 1-stacks [13, 27]. If $\iota : \mathcal{X}^{cl} \hookrightarrow \mathcal{X}$ denotes the inclusion of the underlying classical stack, then the canonical morphism

$$i^* L_{\mathcal{X}/\mathcal{Y}}[-1] \longrightarrow L_{\mathcal{X}^{cl}/\mathcal{Y}^{cl}}[-1]$$

can be shown to be a perfect obstruction theory on \mathcal{X}^{cl} using the connectivity estimates given in [21, I.1.2.5.6]. If $i^* L_{\mathcal{X}/\mathcal{Y}}[-1]$ has a global resolution, which is always the case if \mathcal{X}^{cl} has the resolution property, it follows that we have a virtual fundamental class $[\mathcal{X}^{cl} \rightarrow \mathcal{Y}^{cl}, i^* L_{\mathcal{X}/\mathcal{Y}}]^{vir}$.

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Dhyan ARANHA
Universität Duisburg-Essen
Essen (Germany)
dhyan.aranha@gmail.com
Piotr PSTRĄGOWSKI
Institute for Advanced Study
Princeton (USA)
pstragowski.piotr@gmail.com