# A note on canonical bases and one-based types in supersimple theories

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#### Abstract

This paper studies the CBP, a model-theoretic property first discovered by Pillay and Ziegler. We first show a general decomposition result of types of canonical bases, which one can think of as a sort of primary decomposition. This decomposition is then used to show that existentially closed difference fields of any characteristic have the CBP. We also derive consequences of the CBP, and use these results for applications to differential and difference varieties, and algebraic dynamics.

# Introduction

In [P], Anand Pillay gives a model-theoretic translation of a property enjoyed by compact complex manifolds (and proved by Frédéric Campana and by Akira Fujiki). With Martin Ziegler, he then shows in [PZ] that various algebraic structures enjoy this property (differentially closed fields of characteristic 0; existentially closed difference fields of characteristic 0). As with compact complex manifolds, their proof has as immediate consequence the dichotomy for types of rank 1 in these algebraic structures. This property will later be called the Canonical Base Property (CBP for short) by Rahim Moosa and Pillay [MP]. We will state the precise definition of the CBP later (see 1.7), as it requires several model-theoretic definitions, but here is a rough idea. Let us assume that we have good notions of independence, genericity and dimension, and let  $S \subset X \times Y$  be definable. Viewing S as a family of definable subsets  $S_x$  of Y, assume that for  $x \neq x'$  in X,  $S_x$  and  $S_{x'}$  do not have the same generics, and have finite dimension. Fix some  $a \in X$ , a generic b of  $S_a$ . The CBP then gives strong restrictions on the set  $S^b = \{x \in X \mid b \in S_x\}$ : for instance in the complex manifold case, it is Moishezon, and in the differential case it is isoconstant.

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The aim of this paper is three-fold: give reductions to prove the CBP; derive consequences of the CBP; show that existentially closed difference fields of positive characteristic have the CBP. We then give some applications of these results to differential and difference varieties.

We postpone a detailed description of the model-theoretic results of this paper to the middle of section 1 (1.15) and to the beginning of section 2, but we will now describe two of the algebraic applications. First, an algebraic consequence of Theorems 3.6 and 2.1. We work in some large existentially closed difference field  $(\mathcal{U}, \sigma)$ , of characteristic p; if p > 0, Frob denotes the map  $x \mapsto x^p$  and if p = 0, the identity map.

**Theorem 3.6**'. Let A, B be difference subfields of  $\mathcal{U}$  intersecting in C, with algebraic closures intersecting in  $C^{alg}$ , and with  $tr.deg(A/C) < \infty$ . Let  $D \subset B$  be generated over C by all tuples d such that there exist an algebraically closed difference field F containing C and free from B over C, and integers n > 0 and m such that  $d \in F(e)$  for some tuple e of elements satisfying  $\sigma^n \text{Frob}^m(x) = x$ . Then A and B are free over D.

The purely model theoretic result 2.12 yields descent results for differential and difference varieties (4.5 and 4.14). We state here a consequence in terms of algebraic dynamics:

**Theorem 4.15**. Let  $K_1, K_2$  be fields intersecting in k and with algebraic closures intersecting in  $k^{alg}$ ; for i=1,2, let  $V_i$  be an absolutely irreducible variety and  $\phi_i: V_i \to V_i$  a dominant rational map defined over  $K_i$ . Assume that  $K_2$  is a regular extension of k and that there are an integer  $r \geq 1$  and a dominant rational map  $f: V_1 \to V_2$  such that  $f \circ \phi_1 = \phi_2^{(r)} \circ f$  (where  $\phi_2^{(r)}$  denotes the map obtained by iterating r times  $\phi_2$ ). Then there is a variety  $V_0$  and a dominant rational map  $\phi_0: V_0 \to V_0$ , all defined over k, a dominant map  $g: V_2 \to V_0$  such that  $g \circ \phi_2 = \phi_0 \circ g$ , and  $\deg(\phi_0) = \deg(\phi_2)$ .

The particular way this result is stated is motivated by a question of Lucien Szpiro and Thomas Tucker concerning descent for algebraic dynamics, arising out of Northcott's theorem for dynamics over function fields. Assume that  $K_2$  is a function field over k, and that some  $limited^1$  subset S of  $V_2(K_2)$  satisfies that  $\bigcap_{j=0}^n \phi^{(j)}(S)$  is Zariski dense in  $V_2$  for every n > 0. One can then find  $(V_1, \phi_1)$ , r and f as above, so that our result applies to give a quotient  $(V_0, \phi_0)$  of  $(V_2, \phi_2)$  defined over the smaller field k and with  $\deg(\phi_0) = \deg(\phi_2)$ . Under certain hypotheses, one can even have this g be birational, see [CH1], [CH2].

This note originally contained a proof that a type analysable in terms of one-based types is one-based. However, Frank Wagner [W2] found a much nicer proof, working in a more general context, so that this part of the note disappeared. A result of independent interest, Proposition 5.2, obtained as a by-product of the study of one-based types and appearing in the appendix, tells us that if p is a type of SU-rank  $\omega^{\alpha}$  for some ordinal  $\alpha$  and with algebraically closed base of finite SU-rank, then there is a smallest algebraically closed set over which there is a type of SU-rank  $\omega^{\alpha}$  non-orthogonal to p. The condition of finite rank of the base is necessary.

The paper is organised as follows. Section 1 contains all definitions and preliminary results on supersimple theories, as well as the proof of the decomposition result Theorem 1.18. Section 2

<sup>&</sup>lt;sup>1</sup>see [CH1] for a definition

contains various results which are consequences of the CBP. Section 3 shows that if K is an existentially closed difference field of any characteristic, then Th(K) has the CBP. Section 4 contains some applications of the CBP to differential and difference varieties. Section 5 is the appendix.

Some words on the chronology of the paper and results on the CBP. It all started with the result of Pillay and Ziegler [PZ], a result inspired by a result of Campana on compact complex spaces (see [P]), and which prompted me to look at the general case. The first version of this paper, which contained only Theorem 3.6, an old version of Theorem 2.1 and Proposition 5.2, as well as the proof that a type analysable in one-based types was one-based, was written in 2002. Almost instantly the result on analysable one-based types was generalised by Wagner. The paper was submitted, but not accepted during several years. In the meantime, Moosa and Pillay, having read and believed the preprint, further investigated the CBP in [MP]. Reading their preprint alerted me to the fact that the CBP might imply other stronger properties, as suggested by the fact that compact complex analytic spaces had the UCBP. Thus the material in section 2 starting from 2.4 on, came later (end of 2008, and 2011). Independently, Prerna Juhlin ([J]) has obtained several results on theories with the CBP in her doctoral thesis (2010). Moosa studies in [Mo] variants of internality in presence of the CBP. Daniel Palacin and Wagner continue and generalise the study of the CBP in [PW]. Ehud Hrushovski ([H]) gives an example of an  $\aleph_1$ -categorical theory which does not have the CBP. This example now appears in a paper by Hrushovski, Daniel Palacín and Pillay [HPP].

### 1 Results on supersimple theories

1.1. Setting. We work in a model M (sufficiently saturated) of a complete theory T, which is supersimple and eliminates imaginaries. The results given below generalise easily to a simple theory eliminating hyperimaginaries, provided that some of the sets considered are ranked by the SU-rank.

Given (maybe infinite) tuples  $a, b \in M$ , we denote by Cb(a/b) the smallest algebraically closed subset of M over which tp(a/b) does not fork. Since our theory is supersimple, it coincides with the algebraic closure (in  $M^{eq}$ ) of the usual canonical basis Cb(a/b) of tp(a/b), and is contained in acl(b). For classical results on canonical bases and supersimple theories, see e.g. sections 3.3 and 5.1 – 5.3 of [W1]. We will use repeatedly the following consequences of our hypotheses on T:

- (1) Let  $B \subset M$ ,  $a \in M$ , and  $(a_n)_{n \in \mathbb{N}}$  a sequence of B-independent realizations of tp(a/B). Then for some m,  $\overline{\mathrm{Cb}}(a/B)$  is contained in  $\mathrm{acl}(a_1 \dots a_m)$ ; for any n,  $\mathrm{acl}(a_1 \dots a_n) \cap B \subseteq \overline{\mathrm{Cb}}(a/B)$ .
- (2) Let  $B \subset M$ ,  $a \in M$ , and  $(a_n)_{n \in \mathbb{N}}$  a sequence of B-independent realizations of tp(a/B). Let m be minimal such that  $C = \overline{\text{Cb}}(a/B) \subseteq \text{acl}(a_1 \dots a_m)$ . Then  $SU(a_1/a_2 \dots a_m) > SU(a/B)$ : otherwise  $a_1 \downarrow_{a_2 \dots a_m} C$  would imply  $C \subseteq \text{acl}(a_2 \dots a_m)$ , and contradict the minimality of m.

- (3) If A and B are algebraically closed subsets of M intersecting in C, and D is independent from AB over C, then  $\operatorname{acl}(DA) \cap \operatorname{acl}(DB) = \operatorname{acl}(DC)$  (if  $e \in \operatorname{acl}(DA) \cap \operatorname{acl}(DB)$ , then  $\overline{\operatorname{Cb}}(De/AB) \subseteq A \cap B = C$ ).
- **1.2. Internality and analysability**. In what follows, we will assume that  $\mathcal{S}$  is a set of types with algebraically closed base and which is closed under  $\operatorname{Aut}(M/\operatorname{acl}(\emptyset))$ -conjugation. Then non-orthogonality generates an equivalence relation on the regular types in  $\mathcal{S}$ . For more details, see section 3.4 of [W1].

Recall that if  $a \in M$  and  $A \subseteq M$ , then tp(a/A) is S-internal [resp., almost-S-internal] if there is some set B = acl(B) containing A and independent from a over A, and a tuple  $b_1, \ldots, b_n$  such that  $a \in dcl(Bb_1 \ldots b_n)$  [resp.  $a \in acl(Bb_1 \ldots b_n)$ ], and each  $b_i$  realises a type which is in S and has base contained in B.

tp(a/A) is S-analysable if there are  $a_1, \ldots, a_n$  such that  $acl(Aa_1 \ldots a_n) = acl(Aa)$  and each  $tp(a_i/Aa_1 \ldots a_{i-1})$  is S-internal (or equivalently, each  $tp(a_i/Aa_1 \ldots a_{i-1})$  is almost-S-internal).

- **1.3. Observations**. Let  $A = \operatorname{acl}(A) \subset M$ .
  - (1) If  $tp(a_i/A)$  is almost-S-internal for i = 1, ..., n, then so is  $tp(a_1 ... a_n/A)$ .
  - (2) If tp(a/A) is almost-S-internal, and  $b \in acl(Aa)$ , then tp(b/A) is almost-S-internal.
  - (3) If S' is a set of types which are almost-S-internal, and if p is almost-S'-internal, then p is almost-S-internal.
  - (4) Similarly for S-analysability.
  - (5) Let  $S_1$ ,  $S_2 \subset S$  be sets of types of SU-rank 1 which are closed under Aut(M)-conjugation. If all types in  $S_1$  are orthogonal to all types in  $S_2$  (denoted by  $S_1 \perp S_2$ ) and  $q_i$  is  $S_i$ -analysable for i = 1, 2, then all extensions of  $q_1$  are orthogonal to all extensions of  $q_2$ .
- **1.4. One-basedness**. Let  $S \subseteq M^k$  be A-invariant. Then S is one-based (over A) if whenever b is a tuple of elements of S, and  $B \supseteq A$  then b is independent from B over  $\operatorname{acl}(Ab) \cap \operatorname{acl}(B)$ . A type p (over A) is one-based if the set of its realisations is one-based over A. **Properties** (see [W2]).
  - (1) Let p be a type, and q a non-forking extension of p. Then p is one-based if and only if q is one-based. One-basedness is preserved under Aut(M)-conjugation.
  - (2) A union of one-based sets is one-based.
  - (3) A type analysable by one-based types is one-based.
- **1.5.** Non-orthogonality and internality. Let  $A = \operatorname{acl}(A) \subset M$ , and a a tuple in M, with  $SU(a/A) = \beta + \omega^{\alpha}$  for some  $\alpha$ ,  $\beta$ . Then there is  $B = \operatorname{acl}(B)$  containing A and independent from a over A, and b such that  $SU(b/B) = \omega^{\alpha}$  and  $b \not\downarrow_{B} a$  (see [W1], 5.1.12). Then  $C = \overline{\operatorname{Cb}}(Bb/Aa)$  is

contained in the algebraic closure of independent realisations of tp(Bb/acl(Aa)), and therefore its type over A is almost internal to the set of conjugates of tp(b/B) over A.

If tp(b/B) is one-based, then tp(C/A) is one-based (by 1.4(3)), so that acl(Bb) and C are independent over their intersection D, and therefore C = D. Since  $SU(b/B) = \omega^{\alpha}$ , a standard computation gives  $SU(C/A) = \omega^{\alpha}$ .

- **1.6. Semi-minimal analysis.** In particular every finite SU-rank type has a *semi-minimal analysis*, i.e.: given  $A = \operatorname{acl}(A)$  and a of finite SU-rank over A, there are tuples  $a_1, \ldots, a_n$  such that  $\operatorname{acl}(Aa_1 \ldots a_n) = \operatorname{acl}(Aa)$ , and for every i, either  $tp(a_i/Aa_1 \ldots a_{i-1})$  is one-based of SU-rank 1, or it is internal to the set of conjugates of some non-one-based type of SU-rank 1.
- 1.7. Definition of the CBP. Let T be a simple theory, which eliminates imaginaries and hyperimaginaries. The theory T has the CBP if whenever A and B are algebraically closed sets of finite SU-rank over their intersection, and  $A = \overline{Cb}(B/A)$ , then tp(A/B) is almost-S-internal, where S is the set of types of SU-rank 1 with algebraically closed base. [Actually, as we will see in Theorem 1.18, it suffices to take for S the set of non-one-based types of SU-rank 1 with algebraically closed base.]

One can also restrict this definition to smaller families of types: let  $\mathcal{P}$  be a family of types of finite SU-rank and with algebraically closed base. We say that  $\mathcal{P}$  has the CBP if whenever D is algebraically closed, b is a tuple of realisations of types in  $\mathcal{P}$  with base contained in D, and  $A = \overline{\text{Cb}}(Db/AD)$ , then tp(A/acl(Db)) is almost- $\mathcal{S}$ -internal, for the family  $\mathcal{S} \subset \mathcal{P}$  of types in  $\mathcal{P}$  of SU-rank 1 [and which are not one-based]. Thus Pillay and Ziegler show in [PZ] that the family of very thin types in separably closed fields of finite degree of imperfection has the CBP. See the concluding remarks at the end of section 2 for a discussion.

- **1.8. Definition**. Let p and q be types. We say that p is hereditarily orthogonal to q if every extension of p is orthogonal to q.
- **1.9. Lemma**. Let  $E, B \subset M$  be algebraically closed sets,  $b \in M$  a tuple. Assume that tp(b/B) is almost-S-internal,  $E = \overline{\mathrm{Cb}}(Bb/E)$ , and S is closed under  $\mathrm{Aut}(M/E)$ -conjugation. If  $A = \overline{\mathrm{Cb}}(B/E)$ , then tp(E/A) is almost-S-internal.

*Proof.* Let  $(B_1b_1), \ldots, (B_nb_n)$  be realisations of tp(Bb/E) which are independent over E and such that  $E \subseteq \operatorname{acl}(B_1b_1 \ldots B_nb_n)$ . Since  $B \downarrow_A E$ , we get  $B_1 \ldots B_n \downarrow_A E$ ; observation 1.3(3) then gives the result.

- **1.10. Lemma**. Let  $E, F \subset M$  be algebraically closed sets, with  $\overline{\mathrm{Cb}}(E/F) = F$ . If  $E_0 = \overline{\mathrm{Cb}}(F/E)$ , then  $F = \overline{\mathrm{Cb}}(E_0/F)$ .
- *Proof.* Let  $F_0 = \overline{\text{Cb}}(E_0/F)$ . Then  $E_0 \downarrow_{F_0} F$  and  $E \downarrow_{E_0} F$ , which imply  $E \downarrow_{E_0F_0} F$  (since  $F_0 \subseteq F$ ) and  $E \downarrow_{F_0} F$  by transitivity. Hence  $F_0 = F$ .
- **1.11. Lemma.** Let  $S_1$  and  $S_2$  be sets of types of SU-rank 1 closed under Aut(M)-conjugation, with  $S_1 \perp S_2$ . Assume that  $tp(E_i)$  is  $S_i$ -analysable for i = 1, 2, and that  $D = acl(D) \subseteq acl(E_1E_2)$ . Let  $D_i = D \cap acl(E_i)$  for i = 1, 2. Then  $D = acl(D_1D_2)$ .

*Proof.* Without loss of generality, each  $E_i$  is algebraically closed. Since  $\overline{\text{Cb}}(E_1/D)$  realises an  $S_1$ -analysable type, it equals  $D_1$  and hence  $D \downarrow_{D_1} E_1$ . As  $D \subseteq \text{acl}(E_1 E_2)$ , this implies that  $tp(D/D_1)$  is  $S_2$ -analysable. Hence so is  $tp(D/D_1D_2)$ . Similarly,  $D \downarrow_{D_2} E_2$  and  $tp(D/D_1D_2)$  is  $S_1$ -analysable. Our hypothesis on the orthogonality of the members of  $S_1$  and those of  $S_2$  then implies  $D \subseteq \text{acl}(D_1D_2)$ : a type which is  $S_1$ -analysable and  $S_2$ -analysable must be algebraic.

- **1.12. Lemma**. Let S be a set of types of SU-rank 1, which is closed under Aut(M)-conjugation. Let  $B \subset F$  and A be algebraically closed sets such that tp(A) and tp(B) are almost-S-internal [resp. S-analyzable], and B is maximal contained in F with this property.
  - (1) Then acl(AB) is the maximal subset of acl(AF) whose type is almost-S-internal [resp. S-analyzable].
  - (2) Let G be independent from F. Then  $\operatorname{acl}(GB)$  is the maximal subset of  $\operatorname{acl}(GF)$  whose type over G is almost-S-internal [resp. S-analyzable].
- *Proof.* (1) Let  $d \in \operatorname{acl}(AF)$  be such that tp(d) is almost-S-internal. Then so is the type of  $\overline{\operatorname{Cb}}(Ad/F)$ ; hence  $\overline{\operatorname{Cb}}(Ad/F) \subseteq B$  and  $d \in \operatorname{acl}(AB)$ . Same proof for S-analyzable.
- (2) Let  $e \in \operatorname{acl}(GF)$  realise an almost-S-internal type over G. As  $G \downarrow F$ ,  $\overline{\operatorname{Cb}}(Ge/F)$  realises an almost-S-internal type, hence is contained in B. Hence  $Ge \downarrow_B F$ , which implies  $e \in \operatorname{acl}(GB)$ . Same proof for S-analyzable.

The following result is well-known, but for lack of a reference, we will give the proof.

**1.13. Lemma.** Let p and q be types over sets A and B respectively, and assume that  $p \not\perp q$ . Then for some integer  $\ell$  there are realisations  $a_0, \ldots, a_\ell$  of  $p, b_0, \ldots, b_\ell$  of q, such that the tuples  $a_i$  are independent over A, the tuples  $b_j$  are independent over B,

$$a_0 \dots a_\ell \downarrow_A B, \ b_0 \dots b_\ell \downarrow_B A \text{ and } a_0 \dots a_\ell \not\downarrow_{AB} b_0 \dots b_\ell.$$

*Proof.* By assumption there are some C containing A and B, and realisations a of p, b of q such that  $a \downarrow_A C$ ,  $b \downarrow_B C$  and  $a \not\downarrow_C b$ . Let  $D = \overline{\mathrm{Cb}}(a, b/C)$ . Then for some  $\ell$  there are independent realisations  $(a_i, b_i)$ ,  $i = 1, \ldots, \ell$ , of tp(a, b/C) such that  $D \subset \mathrm{acl}(ABa_1 \ldots a_\ell b_1 \ldots b_\ell)$  (by 1.1(1)); we may choose these realisations to be independent from  $(a, b) := (a_0, b_0)$  over C. Then

$$a_0 \not\downarrow ABa_1...a_\ell b_1...b_\ell b_0.$$

As  $a \downarrow_A C$ ,  $b \downarrow_B C$  and C contains AB, the tuples  $a_i$  and  $b_j$  also satisfy the required first four conditions. Transitivity of independence then implies

$$a_0 \dots a_\ell \not\downarrow_{AB} b_0 \dots b_\ell.$$

**1.14. Notation and some remarks.** Let p be a type of SU-rank 1 over some algebraically closed set, and let  $C = \operatorname{acl}(C)$ . We denote by  $\mathcal{S}(p,C)$  the smallest set of types of SU-rank 1 with algebraically closed base, which contains p and is closed under  $\operatorname{Aut}(M/C)$ -conjugation. We write  $\mathcal{S}(p)$  for  $\mathcal{S}(p,\emptyset)$ .

Let p and q be types of SU-rank 1, with algebraically closed base A and B respectively. Certainly if q is almost- $\{p\}$ -internal, then  $p \not\perp q$ . If A = B, then the converse holds:  $p \not\perp q$  iff q is almost- $\{p\}$ -internal (iff p is almost- $\{q\}$ -internal). If  $A \neq B$ , then  $p \not\perp q$  implies that q is almost-S(p,B)-internal, since any two realisations of q are in the same Aut(M/B)-orbit; but in general, q will not be almost- $\{p\}$ -internal.

In particular, if  $q \not\perp p$ , then q is almost-S(p)-internal. Hence,

either 
$$S(p) \perp S(q)$$
, or every member of  $S(p)$  is  $S(q)$ -internal

(and every member of S(q) is S(p)-internal).

In the rest of the first two sections of the paper, the letters  $\mathcal{S}$ ,  $\mathcal{S}'$ ,  $\mathcal{S}_1$ , etc. will always denote sets of SU-rank 1 types with algebraically closed base.

- 1.15. We now start towards the proof of Theorem 1.18. It will reduce the problem of showing the CBP to showing it for  $\{p\}$ -analysable types when p is a type of SU-rank 1 with algebraically closed base. This reduction is essential in the proof that existentially closed difference fields of positive characteristic have the CBP. We conclude the section with small partial results.
- **1.16. Proposition**. Let F and E be algebraically closed sets such that  $F \cap E = C$ , SU(E/C) and SU(F/C) are finite,  $\overline{Cb}(E/F) = F$  and  $\overline{Cb}(F/E) = E$ . There are non one-based types  $p_1, \ldots, p_m$  of SU-rank 1, algebraically closed sets  $E_1, \ldots, E_m, F_1, \ldots, F_m$  such that, letting  $S_i = S(p_i, C)$  for  $i = 1, \ldots, m$ ,
  - (i)  $tp(E_i/C)$  and  $tp(F_i/C)$  are  $S_i$ -analysable,  $\overline{Cb}(E_i/F_i) = F_i$  and  $\overline{Cb}(F_i/E_i) = E_i$ , and
  - (ii)  $\operatorname{acl}(E_1 \dots E_m) = E$ ,  $\operatorname{acl}(F_1 \dots F_m) = F$ .
- (iii) The sets  $E_i$  are independent over C, as well as the sets  $F_i$ .

*Proof.* Assume the result false, and take a counterexample with SU(EF/C) minimal among all possible (E, F, C), and among those, with SU(F/C) + SU(E/C) minimal.

Let  $p_1, \ldots, p_m$  be types of SU-rank 1 with algebraically closed base, which are pairwise orthogonal, such that each  $p_i$  is non-orthogonal to tp(E/C) or to tp(F/C), and such that any SU-rank 1 type which is non-orthogonal to one of tp(E/C), tp(F/C), is non-orthogonal to one of the types  $p_i$  (see section 5.2 in [W1]). We let  $S_i = S(p_i, C)$ , and  $E_i$  and  $F_i$  the maximal subsets of E and F respectively such that  $tp(E_i/C)$  and  $tp(F_i/C)$  are  $S_i$ -analysable. We need to show that no  $p_i$  is one-based, the second part of item (i), and item (ii) (item (iii) is immediate since the types  $p_i$  are pairwise orthogonal, by 1.3(5)).

Adding to the language constants symbols for the elements of C, we will assume that  $C = \emptyset$ .

We say that a set D satisfies (\*) over a set H if  $D = \operatorname{acl}(HD_1 \dots D_m)$ , where  $tp(D_i/H)$  is  $S_i$ -analysable for each i. If  $H = \operatorname{acl}(\emptyset)$ , we will simply say that D satisfies (\*). Note that by 1.11, any subset of  $\operatorname{acl}(HD)$  whose type over H is  $S_i$ -analysable will be contained in  $\operatorname{acl}(HD_i)$ .

By Lemma 1.11, an algebraically closed subset of a set satisfying (\*) over H also satisfies (\*) over H, and the algebraic closure of a union of sets satisfying (\*) over H satisfies (\*) over

H. Hence, if D satisfies (\*) over H,  $J \supseteq H$ , and  $\overline{\mathrm{Cb}}(D/J) = J$ , then J satisfies (\*) over H, as J is contained in the algebraic closure of finitely many realisations of tp(D/H).

Assume that E satisfies (\*); then  $F = \overline{\text{Cb}}(E/F)$  satisfies (\*), and therefore can be written as  $\text{acl}(F'_1, \ldots, F'_m)$ , where each  $tp(F'_i)$  is  $\mathcal{S}_i$ -analysable. Each  $F'_i$  is contained in  $F_i$ , and therefore  $\text{acl}(F_1 \ldots F_m) = F$  and  $F_i = F'_i$ . We know that F is contained in the algebraic closure of F-independent realisations of tp(E/F). Lemma 1.11 then gives us that necessarily  $F_i$  is contained in the algebraic closure of F-independent realisations of  $tp(E_i/F)$ . Then 1.1(1) implies that  $\overline{\text{Cb}}(E_i/F) \supseteq F_i$ ; the reverse inclusion holds since  $tp(\overline{\text{Cb}}(E_i/F))$  is  $\mathcal{S}_i$ -analysable. By symmetry,  $E_i = \overline{\text{Cb}}(F_i/E)$ . Furthermore, no  $p_i$  is one-based: otherwise, by 1.4(3)  $tp(E_i)$  would be one-based, whence  $E \cap F = \text{acl}(\emptyset)$  would yield  $E_i \downarrow F$ , and therefore  $E_i = F_i = \text{acl}(\emptyset)$ . This shows that if E satisfies (\*), then the conclusion of the Lemma holds. By symmetry neither E nor F satisfies (\*).

Using the semi-minimal analysis of tp(F), there is  $B = \operatorname{acl}(B) \subset F$ ,  $B \neq F$ , and a type p of SU-rank 1, such that tp(F/B) is almost- $\mathcal{S}(p)$ -internal. Note that  $B \neq \operatorname{acl}(\emptyset)$ : otherwise tp(F/C) would be  $\mathcal{S}(p)$ -internal, contradicting our assumption that F does not satisfy (\*). Let  $A = \overline{\operatorname{Cb}}(B/E)$ . Then tp(E/A) is almost- $\mathcal{S}(p)$ -internal by Lemma 1.9, so that  $A \neq \operatorname{acl}(\emptyset)$ .

#### Step 1. A satisfies (\*).

Let  $B_0 = \overline{\text{Cb}}(A/B)$ . Then  $B_0 \neq \text{acl}(\emptyset)$  (because otherwise B and A would be independent), and  $\overline{\text{Cb}}(B_0/A) = A$  by Lemma 1.10. As  $SU(B_0) < SU(F)$  and  $\text{acl}(AB_0) \subseteq \text{acl}(EF)$ , by induction hypothesis A satisfies (\*).

Thus  $A \neq E$ . Since tp(E/A) is almost-S(p)-internal, if  $B_1 = \overline{\text{Cb}}(A/F)$ , then  $tp(F/B_1)$  is almost-S(p)-internal by Lemma 1.9,  $B_1$  satisfies (\*), and  $\text{acl}(\emptyset) \neq B_1 \neq F$ . Let  $E_p$  be the largest subset of E realising an S(p)-analysable type. If p is orthogonal to every  $p_i$ , then  $E_p = \text{acl}(\emptyset)$ , by definition of the set  $\{p_1, \ldots, p_m\}$ . Otherwise, by 1.14,  $S(p) = S_i$  for some i, and therefore  $E_p = E_i$ ; we will first show in the next two steps that this case is impossible.

#### Step 2. $\operatorname{acl}(FE_p) \cap E = E_p$ .

Let  $D = \operatorname{acl}(FE_p) \cap E$ . If  $D \neq E_p$ , then, using the semi-minimal analysis of  $tp(D/E_p)$ , there is  $d \in D \setminus E_p$  with  $tp(d/E_p)$  almost- $\mathcal{S}(q)$ -internal for some type q of SU-rank 1. By maximality of  $E_p$ , we have  $\mathcal{S}(q) \perp \mathcal{S}(p)$ . Since  $tp(F/B_1)$  is almost- $\mathcal{S}(p)$ -internal, we obtain  $d \in \operatorname{acl}(B_1E_p)$ . Because  $B_1$  and  $E_p$  satisfy (\*), and  $tp(d/E_p) \perp p$ , using 1.11 we may write  $\operatorname{acl}(E_pd)$  as  $\operatorname{acl}(E_pD_1)$ , where  $tp(D_1)$  is  $\mathcal{S}(q)$ -analysable and hereditarily orthogonal to p. Thus  $D_1 \downarrow E_p$ ; because  $D_1 \subseteq \operatorname{acl}(B_1E_p)$ , we obtain  $D_1 \subseteq B_1$ ; as  $D_1 \subseteq E$ , this implies  $D_1 = \operatorname{acl}(\emptyset)$ , a contradiction.

#### Step 3. $E_p = \operatorname{acl}(\emptyset)$ .

Let  $D = \overline{\operatorname{Cb}}(E/FE_p)$ . Then  $\overline{\operatorname{Cb}}(E/D) = D$ . By Step 2,  $D \cap E = E_p$ . Moreover  $\overline{\operatorname{Cb}}(D/E) = E$ : let  $E_0 = \overline{\operatorname{Cb}}(D/E)$ ; from  $E \downarrow_D F$  we deduce  $E \downarrow_{DE_0} F$ ; since  $E \downarrow_{E_0} D$ , transitivity gives  $E \downarrow_{E_0} F$ , and therefore  $E = E_0$ . Thus, if  $E_p \neq \operatorname{acl}(\emptyset)$ , then  $SU(ED/E_p) < SU(EF)$  and by induction hypothesis (applied to D and E), E satisfies (\*) over  $E_p$ . Write  $E = \operatorname{acl}(E'_p, E''_p)$ , where  $E'_p$  satisfies (\*) over  $E_p$ ,  $tp(E'_p/E_p)$  is hereditarily orthogonal to all types in S(p), and  $tp(E''_p/E_p)$  is S(p)-analysable. Then  $tp(E''_p)$  is S(p)-analysable, so that  $E''_p = E_p$ . On the other hand, tp(E/A) is S(p)-analysable, and therefore  $E'_p \subseteq \operatorname{acl}(AE_p)$  (because  $tp(E'_p/E_p)$  is heredi-

tarily orthogonal to all members of S(p). Hence  $E = \operatorname{acl}(AE_p)$  satisfies (\*), a contradiction. By symmetry, if  $F_p$  is a subset of F whose type is S(p)-analysable, then  $F_p \subseteq \operatorname{acl}(\emptyset)$ . Step 4.  $F \subseteq \operatorname{acl}(B_1E)$ .

Let  $D = \overline{\text{Cb}}(F/B_1E)$ . Then  $B_1 \subseteq D$ , and  $tp(D/B_1)$  is almost- $\mathcal{S}(p)$ -internal, because it is contained in the algebraic closure of  $B_1$ -conjugates of F. Furthermore, we have  $\overline{\text{Cb}}(D/E) = E$ : let  $E_0 = \overline{\text{Cb}}(D/E)$ ; from  $E \downarrow_D F$  we deduce  $E \downarrow_{DE_0} F$ ; then  $E \downarrow_{E_0} D$  yields  $E \downarrow_{E_0} F$ , whence  $E_0 = E$ . We now let  $D_1 = \overline{\text{Cb}}(E/D)$ ; then  $D_1 \subseteq D \subseteq \text{acl}(B_1E)$ ,  $tp(D_1/B_1)$  is almost- $\mathcal{S}(p)$ -internal, and  $\overline{\text{Cb}}(D_1/E) = E$  (by Lemma 1.10). Since E does not satisfy (\*), our induction hypothesis implies that either  $E \cap D_1 \neq \text{acl}(\emptyset)$  or  $\text{acl}(D_1E) = \text{acl}(EF)$ .

Let us assume that  $D_1 \cap E \neq \operatorname{acl}(\emptyset)$ . Using the semi-minimal analysis of  $tp(D_1 \cap E)$ , there is  $d \in D_1 \cap E$  with tp(d) almost- $S_i$ -internal for some i. Since  $E_p = F_p = \operatorname{acl}(\emptyset)$ , we know that  $S(p) \perp S_i$ . But  $tp(D_1/B_1)$  is almost-S(p)-internal, so that  $tp(d/B_1)$  is almost-S(p)-internal, whence  $d \in B_1$ . Hence  $D_1 \cap E \subset B_1 \cap E = \operatorname{acl}(\emptyset)$ .

Hence  $\operatorname{acl}(D_1E) = \operatorname{acl}(EF)$ , which implies  $F \subseteq \operatorname{acl}(B_1E)$ .

The proof only used the S(p)-internality of  $tp(F/B_1)$ , and we reason in the same manner with  $\overline{\mathrm{Cb}}(E/AF)$  to get  $E \subseteq \mathrm{acl}(AF)$ . Since  $E_p = F_p = \mathrm{acl}(\emptyset)$ , we know that  $S(p) \perp S_1 \cup \cdots \cup S_m$ . The final contradiction will come from the following lemma, taking  $S = S_1 \cup \cdots \cup S_m$ :

**1.17. Lemma.** Let  $A \subseteq E$  and  $B \subseteq F$  be algebraically closed sets of finite SU-rank such that  $E \cap F = \operatorname{acl}(\emptyset)$ , E and F are equi-algebraic over AB. Assume that for some set  $\mathcal S$  of types of SU-rank 1, which is closed under  $\operatorname{Aut}(M/\operatorname{acl}(\emptyset))$ -conjugation, tp(A) and tp(B) are  $\mathcal S$ -analysable. Then  $tp(E/\operatorname{acl}(\emptyset))$  and  $tp(F/\operatorname{acl}(\emptyset))$  are  $\mathcal S$ -analysable.

Proof. We may assume that A and B are maximal subsets of E and F respectively whose type are S-analysable. If E = A, then  $F \subseteq \operatorname{acl}(AB)$ , and we are done; similarly if F = B. Assume  $E \neq A$ , and let p be a type of SU-rank 1 which is non-orthogonal to tp(E/A); we then let  $E_0 \subseteq E$  and  $F_0 \subseteq F$  be maximal such that  $tp(E_0/A)$  and  $tp(F_0/B)$  are almost S(p)-internal. Then  $E_0 \neq A$  (see the discussion in 1.6) and  $p \perp S$ . If  $F_1 = \overline{\operatorname{Cb}}(E_0/F)$  and  $B_1 = \overline{\operatorname{Cb}}(A/F)$ , then  $tp(F_1/B_1)$  is almost S(p)-internal by Lemma 1.9, so that  $F_1 \subseteq F_0$  and  $F_0 \subseteq \operatorname{acl}(AF_0)$ . Similarly,  $F_0 \subseteq \operatorname{acl}(BE_0)$ .

We have therefore shown that if the conclusion of the lemma does not hold, then there is a counterexample (E, F, A, B) where tp(E/A) and tp(F/B) are almost  $\mathcal{S}(p)$ -internal for some type p of SU-rank 1 which is orthogonal to all members of  $\mathcal{S}$ . We choose such a counterexample with r = SU(B) - SU(B/A) minimal.

Let  $E_0 = \overline{\text{Cb}}(F/E)$ , and  $A_0 = A \cap E_0$ . Then  $F \subseteq \text{acl}(BE_0)$ , and  $E_0 \subseteq \text{acl}(AF)$ . Also,  $A_0$  is the maximal subset of  $E_0$  with an S-analysable type (by 1.12), whence  $A \downarrow_{A_0} E_0$ , and by transitivity  $A \downarrow_{A_0} E_0 F$ , so that  $E_0 \subseteq \text{acl}(A_0 F)$ . Since  $F \neq B$ , we have  $E_0 \neq A_0$ , so that  $E_0 \subseteq \text{acl}(E_0)$  is not  $E_0 \subseteq \text{acl}(E_0)$ . (Note that  $E_0 \subseteq \text{acl}(E_0)$ ) and  $E_0 \subseteq \text{acl}(E_0)$ , we may therefore assume that  $E_0 \subseteq \text{cb}(E_0)$ . (Note that  $E_0 \subseteq \text{cd}(E_0)$ ), so that  $E_0 \subseteq \text{cd}(E_0)$ 0 is not  $E_0 \subseteq \text{cd}(E_0)$ 1.

If r = 0, then  $B \downarrow A$ ; because tp(E/A) is orthogonal to all types in  $\mathcal{S}$ , we obtain  $B \downarrow E$ ; since tp(F/B) is almost- $\mathcal{S}(p)$ -internal and  $E = \overline{\mathrm{Cb}}(F/E)$ , we get that tp(E) is almost- $\mathcal{S}(p)$ -internal, a contradiction. Hence r > 0.

Since  $E = \overline{\operatorname{Cb}}(F/E)$ , there are E-independent realisations  $F_1, \ldots, F_s$  of tp(F/E) such that  $E \subseteq \operatorname{acl}(F_1, \ldots, F_s)$ . Let  $B_i \subset F_i$  correspond to  $B \subset F$ . Since  $tp(F_1, \ldots, F_s/B_1, \ldots, B_s)$  is orthogonal to all types in S, we necessarily have  $A \subset \operatorname{acl}(B_1, \ldots, B_s)$ . Furthermore, from  $B \downarrow_A E$ , the sets  $B_1, \ldots, B_s$  are independent over A. This implies that  $\overline{\operatorname{Cb}}(B/A) = A$  by 1.1(1). Let  $m \leq s$  be minimal such that  $A \subset \operatorname{acl}(B_1 \ldots B_m)$ . Then m > 1 and  $SU(B_m) - SU(B_m/B_1 \ldots B_{m-1}) < r$  by 1.1(2). We also have  $F_m \cap \operatorname{acl}(F_1 \ldots F_{m-1}) \subseteq F \cap E = \operatorname{acl}(\emptyset)$ , and  $E \subseteq \operatorname{acl}(B_1 \ldots B_m F_i)$  for every  $1 \leq i \leq m$ . Hence  $F_1$  and  $F_m$  are equi-algebraic over  $\operatorname{acl}(B_1 \ldots B_m)$ . The induction hypothesis applied to the quadruple  $(\operatorname{acl}(F_1B_2 \ldots B_{m-1}), F_m, \operatorname{acl}(B_1 \ldots B_{m-1}), B_m)$  gives that tp(F) is S-analysable, a contradiction.

#### **1.18.** Proposition 1.16 has the following immediate consequence:

**Theorem.** Let E, F be algebraically closed sets, and assume that  $SU(E/E \cap F)$  is finite and  $F = \overline{\text{Cb}}(E/F)$ . Then there are  $F_1, \ldots, F_m$  independent over  $E \cap F$ , types  $p_1, \ldots, p_m$  of SU-rank 1, such that each  $tp(F_i/E \cap F)$  is  $S(p_i, E \cap F)$ -analysable, and  $acl(F_1 \ldots F_m) = F$ .

*Proof.* By 1.1, we know that  $SU(F/E \cap F)$  is also finite. Replace E by E' = Cb(F/E); by Lemma 1.10,  $F = \overline{Cb}(E'/F)$ . Then apply Proposition 1.16 to E', F to get the types  $p_i$  (which are pairwise orthogonal), and the sets  $F_i$ .

- **1.19. Remark**. Let E, F be as above. Using the semi-minimal analysis of  $tp(F/E \cap F)$ , there is some  $G = \operatorname{acl}(G)$  independent from EF over  $C = E \cap F$ , and a tuple  $a \in \operatorname{acl}(GF)$  of realisations of types of SU-rank 1 over G, such that for any tuple b,  $b \not\downarrow_G F$  implies  $b \not\downarrow_G a$  (in other words: tp(a/G) dominates tp(F/G), see section 5.2 in [W1]). Then working over G, the types  $p_i$  of Theorem 1.18 can be taken to be types over G (see the proof of 1.16), and the subsets  $F_i$  of  $\operatorname{acl}(GF)$  will then realise  $\{p_i\}$ -analysable types over  $\operatorname{acl}(GC)$ . This is slightly stronger than just saying that the sets  $F_i$  realise  $\mathcal{S}(p_i)$ -analysable types.
- **1.20.** The following result is similar to Proposition 2.3. See also Theorem 1.3 in [MP].

**Proposition**. Let S be a set of types of SU-rank 1, which is closed under  $\operatorname{Aut}(M/\operatorname{acl}(\emptyset))$ -conjugation, let B and E be algebraically closed sets of finite SU-rank, and assume that tp(E/B) is S-analysable. Then so is  $tp(E/E \cap B)$ .

Proof. Without loss of generality,  $B = \overline{\text{Cb}}(E/B)$ . Let  $C = E \cap B$ , and assume that tp(E/C) is not S-analysable. Let  $D \subseteq E$  be maximal such that tp(D/C) is S-analysable. As  $B = \overline{\text{Cb}}(E/B)$ , Theorem 1.18 gives us two algebraically closed sets  $B_1$  and  $B_2$  with  $\text{acl}(B_1B_2) = B$ ,  $tp(B_1/C)$  S-analysable, and  $tp(B_2/C)$  S-analysable for some set S' of SU-rank 1 types with algebraically closed base and such that  $S \perp S$ '. Then  $\overline{\text{Cb}}(D/B) \subseteq B_1$  and  $E \downarrow_D B_1$  because  $tp(E/D) \perp S$  and  $tp(B_1/C)$  is S-analysable. If  $B_2 = C$ , then  $E \downarrow_D B$ , and the S-analysability of tp(E/DB) implies the S-analysability of tp(E/D), a contradiction.

Hence  $B_2 \neq C$ , and if  $E_2 = \overline{\text{Cb}}(B_2/E)$ , then  $E_2 \neq C$  and  $E_2$  realises an  $\mathcal{S}'$ -analysable type over C. As  $E_2 \neq C$  and  $E \cap B = C$ , we have that  $tp(E_2/B)$  is non-algebraic and  $\mathcal{S}'$ -analysable. On the other hand  $tp(E_2/B)$  is also  $\mathcal{S}$ -analysable because  $E_2 \subseteq E$ , which gives the final contradiction.

- **1.21. Corollary**. Let S be a set of types of rank 1 closed under  $\operatorname{Aut}(M/\operatorname{acl}(\emptyset))$ -conjugation, and let  $E = \operatorname{acl}(E)$  have finite SU-rank. Then there is  $A = \operatorname{acl}(A) \subseteq \operatorname{acl}(E)$  such that tp(E/A) is S-analysable, and whenever  $B = \operatorname{acl}(B)$  is such that tp(E/B) is S-analysable, then  $A \subseteq B$ . Proof. By 1.20, it is enough to show that if  $A_1, A_2$  are algebraically closed subsets of E such that  $tp(E/A_i)$  is S-analysable, then so is  $tp(E/A_1 \cap A_2)$ : but this is obvious, as  $tp(A_1/A_1 \cap A_2)$  is S-analysable, by 1.20.
- **1.22. Remark**. Let S be a set of types of rank 1 closed under  $\operatorname{Aut}(M/\operatorname{acl}(\emptyset))$ -conjugation, and let  $E = \operatorname{acl}(E)$  have finite SU-rank. Then one can find  $S' \perp S$ , closed under  $\operatorname{Aut}(M/\operatorname{acl}(\emptyset))$ -conjugation and such that  $\operatorname{tp}(E/\operatorname{acl}(\emptyset))$  is  $(S \cup S')$ -analysable. It follows that for a set  $B = \operatorname{acl}(B)$ ,  $\operatorname{tp}(E/B)$  will be hereditarily orthogonal to S' if and only if it is S-analysable. Thus the above two results can be stated in terms of hereditary orthogonality to S' instead of S-analysability.

We now state an easy lemma reducing further the problem of showing the CBP:

**1.23.** Lemma. Let S be a set of types of SU-rank 1, which is closed under  $\operatorname{Aut}(M/\operatorname{acl}(\emptyset))$ -conjugation. Assume that there are algebraically closed sets E and F whose types over  $C = E \cap F$  are S-analysable, such that  $\overline{\operatorname{Cb}}(F/E) = E$  and tp(E/C) is not almost S-internal. Then there are such sets E and F whose types over C are S-analysable in at most two steps, i.e., there is  $A \subset E$  such that tp(A/C) and tp(E/A) are almost-S-internal, and similarly for F. Furthermore,  $\overline{\operatorname{Cb}}(E/F) = F$ .

*Proof.* We take such a triple (E, F, C) with r = SU(E/C) + SU(F/C) minimal, whence  $F = \overline{\text{Cb}}(E/F)$ .

By the semi-minimal analysis of tp(F/C), there is a proper algebraically closed subset B of F such that tp(F/B) is almost-S-internal. By Lemma 1.9, if  $A = \overline{\text{Cb}}(B/E)$  then tp(E/A) is almost-S-internal. As SU(B/C) < SU(F/C), the minimality of r implies that tp(A/C) is almost-S-internal. Hence,  $A \neq C$ , E because tp(E/C) is not almost-S-internal, and tp(E/C) is S-analysable in two steps. Since F is contained in the algebraic closure of realisations of tp(E/C), tp(F/C) will also be S-analysable in two steps.

We conclude this section with a partial internality result:

**1.24. Lemma**. Let  $A \subseteq E$  and  $B \subseteq F$  be algebraically closed sets of finite SU-rank such that  $E \cap F = \operatorname{acl}(\emptyset)$ , E and F are equi-algebraic over AB. Assume that for some set S of types of SU-rank 1, which is closed under  $\operatorname{Aut}(M/\operatorname{acl}(\emptyset))$ -conjugation,  $\operatorname{tp}(A/\operatorname{acl}(\emptyset))$  and  $\operatorname{tp}(B/\operatorname{acl}(\emptyset))$  are almost-S-internal. Then  $\operatorname{tp}(E/\operatorname{acl}(\emptyset))$  and  $\operatorname{tp}(F/\operatorname{acl}(\emptyset))$  are almost-S-internal.

*Proof.* We work over  $\operatorname{acl}(\emptyset)$ . By Lemma 1.17, we already know that tp(E) and tp(F) are  $\mathcal{S}$ -analysable. Hence, reasoning as in the first paragraph of the proof of 1.17, we may assume that A, B are maximal subsets of E and F respectively which realise almost- $\mathcal{S}$ -internal-types, and that tp(E/A) and tp(F/B) are almost- $\mathcal{S}$ -internal, but neither tp(E) nor tp(F) is almost- $\mathcal{S}$ -internal. The maximality of A and B implies that  $A \downarrow_B F$  and  $E \downarrow_A B$ .

Working over some  $C = \operatorname{acl}(C)$ , independent from EF, and using 1.1(3) and 1.12, we may assume that A is the algebraic closure of a tuple of realisations of types in  $\mathcal{S}$ . We choose a

counterexample (E, F, A, B) with r = SU(A) - SU(A/B) minimal. If r = 0, then  $A \downarrow B$  so that  $A \downarrow F$ . Letting  $F_0 = \overline{\mathrm{Cb}}(E/F)$ , this implies that  $F_0$  realises an almost-S-internal-type, hence is contained in B. But  $E \downarrow_{F_0} F$  then implies  $F \subset B$ , which is absurd. So we may assume that r > 0.

Let  $(F_iB_i)_{i>0}$  be a sequence of E-independent realisations of tp(FB/E). If  $A_0 = \overline{\text{Cb}}(B/A)$ , then for some s>1, we have  $\text{acl}(B_1 \dots B_s) \supset A_0$ , and we take a minimal such s. As A is the algebraic closure of realisations of types of SU-rank 1, there is a finite tuple  $a \subset A$  such that  $a \downarrow A_0$  and  $\text{acl}(A_0a) = A$ . Then  $a \downarrow A_0B$ , which implies  $a \downarrow A_0F$  (by transitivity and because  $A \downarrow_B F$ ).

Let  $A' = \operatorname{acl}(aB_1)$ ,  $B' = \operatorname{acl}(B_2 \dots B_s)$ ,  $E' = \operatorname{acl}(A'F_1)$ ,  $F' = \operatorname{acl}(B'F_s)$ . Then  $a \downarrow A_0 F$ , and  $a \downarrow B_1 F'$ . Hence in particular,  $A' \cap F' = \operatorname{acl}(\emptyset)$  (use 1.1(3) and the fact that  $B_1 \cap B' = \operatorname{acl}(\emptyset)$ ). Moreover,  $SU(A') - SU(A'/B') = SU(B_1) - SU(B_1/B_2 \dots B_s) < r$  by 1.1(2), and E' and F' are equi-algebraic over A'B'. In order to reach a contradiction, it therefore suffices to show that  $E' \cap F' = \operatorname{acl}(\emptyset)$ : our induction hypothesis gives that  $tp(E/\operatorname{acl}(\emptyset))$  is almost-S-internal, which implies that  $tp(F_1/\operatorname{acl}(\emptyset)) = tp(F/\operatorname{acl}(\emptyset))$  is also almost-S-internal.

By Lemma 1.12, A' and B' are maximal subsets of E', F' respectively which realise almost-S-internal-types. Assume  $E' \cap F' \neq \operatorname{acl}(\emptyset)$ ; by the semi-minimal analysis of  $tp(E' \cap F'/\operatorname{acl}(\emptyset))$ , there is  $d \in E' \cap F'$  realising an almost-S-internal-type. Then  $d \in B' \cap A' \subseteq F' \cap A' = \operatorname{acl}(\emptyset)$ , which gives us the desired contradiction.

## 2 Further properties of theories with the CBP

**Description of the results of this section**. Assumptions on M and T are as in the previous section: T is supersimple and eliminates imaginaries, M is sufficiently saturated. Most results are proved under the additional hypothesis of the CBP. We start by proving one of the main results of the paper:

**Theorem 2.1** (CBP). If E and F are algebraically closed sets of finite SU-rank over their intersection C and are such that  $E = \overline{\text{Cb}}(F/E)$ , then tp(E/C) is almost-S-internal for some family S of types of SU-rank 1.

Note that under the same hypotheses, if tp(E/F) is  $\mathcal{S}'$ -analysable for some set  $\mathcal{S}'$  of types of SU-rank 1 with algebraically closed base, then tp(E/C) is almost- $\mathcal{S}'$ -internal. We then show **Theorem 2.5** (CBP). Assume that  $E = \operatorname{acl}(E)$  has finite SU-rank, and let  $\mathcal{S}$  be a collection of types of SU-rank 1, closed under conjugation. Then there is  $A = \operatorname{acl}(A) \subseteq E$  such that tp(E/A) is almost- $\mathcal{S}$ -internal, and whenever  $B = \operatorname{acl}(B)$  is such that tp(E/B) is almost- $\mathcal{S}$ -internal, then  $B \supseteq A$ .

An immediate consequence of 2.5 is that the CBP implies the Uniform CBP (UCBP); this answers a question of Moosa and Pillay [MP]. We end the section with three results, which were proved with an eye towards geometric applications. The first result is valid in a general setting (as will be clear from the proof), and can be viewed as showing the existence of a "largest internal quotient"; the second can be viewed as showing the existence of a "maximal internal fiber", and the third one as a descent result.

**2.1. Theorem** (CBP). If E and F are algebraically closed sets of finite SU-rank over their intersection C and  $E = \overline{\text{Cb}}(F/E)$ , then tp(E/C) is almost-S-internal for some family S of types of SU-rank 1.

*Proof.* We assume the result false. By Lemma 1.23, there is a counterexample (E, F, C) with tp(E/C), tp(F/C) S-analysable in two steps, and which also satisfies  $\overline{\text{Cb}}(E/F) = F$ . By Theorem 1.18 (see also 1.19 and 1.1(3)), working over a larger set G = acl(G), we can write E as  $\text{acl}(E_1 \dots E_m)$  for some sets  $E_i$  which are independent over G, realise  $\{p_i\}$ -analysable types over G, and some  $E_i$  will not realise an almost- $\{p_i\}$ -internal type over G.

Hence, we may assume that  $S = \{p\}$  for some type p of SU-rank 1. For ease of notation we will assume that the language contains constant symbols for the elements of G.

Let  $A_0 \subset E$  and  $B \subset F$  be maximal realizing almost-S-internal types, so that  $tp(E/A_0)$  and tp(F/B) are almost-S-internal. Then  $E \downarrow_{A_0} B$  since  $\overline{\mathrm{Cb}}(B/E)$  is almost-S-internal and therefore contained in  $A_0$ , and similarly  $A_0 \downarrow_B F$ . Enlarging G (and using 1.1(3)), we may assume that  $A_0$  and B are the algebraic closures of tuples of realisations of p. Let  $A = \overline{\mathrm{Cb}}(B/E)$ . Then  $A \subseteq A_0$ , and tp(E/A) is almost-S-internal (by 1.9).

The proof is by induction on  $r = SU(B) - SU(B/A_0)$  (=  $SU(A_0) - SU(A_0/B)$ ). If r = 0, then  $A_0 \downarrow B$  and from tp(F/B) almost-S-internal and  $\overline{Cb}(F/E) = E$  we deduce that tp(E) is almost-S-internal, a contradiction. Hence r > 0.

#### **Step 1**. We may assume $A_0 = A$ .

We know that  $A_0 = \operatorname{acl}(a_0)$  for some tuple  $a_0$  of realisations of p; take  $a \subseteq a_0$  maximal independent over A and such that  $a \downarrow A$ . Then

$$\operatorname{acl}(Aa) = A_0, \quad a \downarrow AF \quad \text{and} \quad SU(B/a) - SU(B/A_0) = r.$$

Furthermore, tp(E/a) is not almost-S-internal: otherwise,  $\overline{\mathrm{Cb}}(E/Fa)$  would also be almost-S-internal, hence contained in  $\mathrm{acl}(Ba)$  by maximality of B (see 1.12(1)); from  $E \downarrow_{Ba} F$  and  $F \downarrow_{B} A_0$  we would then deduce  $E \downarrow_{B} F$ , i.e. F = B, which is absurd. We will now show that  $E \cap \mathrm{acl}(Fa) = \mathrm{acl}(a)$ . Enlarging G this will allow us to assume  $A = A_0$ .

Let  $D = E \cap \operatorname{acl}(Fa)$ . Since  $a \downarrow AF$ ,  $A_0 \cap \operatorname{acl}(Fa) = \operatorname{acl}(a)$  by 1.1(3). The set  $\overline{\operatorname{Cb}}(B/D)$  is almost-S-internal, hence contained in  $A_0 \cap \operatorname{acl}(Fa) = \operatorname{acl}(a)$ , so that  $B \downarrow D$  because  $F \downarrow a$ . From  $D \subset \operatorname{acl}(Fa)$  and the almost-S-internality of  $\operatorname{tp}(Fa/B)$  we obtain that  $\operatorname{tp}(D/B)$  is almost-S-internal, and therefore also  $\operatorname{tp}(D)$ , so that  $D \subseteq A_0 \cap \operatorname{acl}(Fa) = \operatorname{acl}(a)$ .

#### **Step 2**. We may assume $E \subseteq \operatorname{acl}(AF)$ .

By assumption, there is an algebraically closed set J containing F, such that  $J \downarrow_F E$ , and a tuple g of realisations of p such that  $E \subseteq \operatorname{acl}(Jg)$ . Then there is a subset e of g, consisting of independent tuples over AJ, and such that  $E \subseteq \operatorname{acl}(AJe)$  and  $e \downarrow AJ$ . Since  $J \supseteq F$  and  $e \downarrow J$ , we then have  $\overline{\operatorname{Cb}}(Je/Ee) = \operatorname{acl}(Ee)$  and  $\overline{\operatorname{Cb}}(Be/Ae) = \overline{\operatorname{Cb}}(Be/Ee) = \operatorname{acl}(Ae)$  (use  $\overline{\operatorname{Cb}}(J/E) = E$  and 1.1(1)).

Claim.  $acl(Ee) \cap acl(Je) = acl(e)$ .

Let  $D = \operatorname{acl}(Ee) \cap \operatorname{acl}(Je)$ . Since  $e \downarrow AJ$ , we obtain  $\operatorname{acl}(Ae) \cap \operatorname{acl}(Je) = \operatorname{acl}(e)$  by 1.1(3).

We know that if  $D_0 = \overline{\text{Cb}}(A/D)$ , then  $tp(D_0)$  is almost-S-internal; the maximal almost-S-internal subset of acl(Ee) is acl(Ae) by 1.12(1), and therefore  $D_0 \subseteq \text{acl}(Ae) \cap \text{acl}(Je) = \text{acl}(e)$ .

Hence  $A \downarrow D$ , and tp(D) is almost-S-internal (because  $D \subset acl(Ee)$  and tp(Ee/A) is almost-S-internal). Reasoning as we did for  $D_0$ , we obtain  $D \subseteq acl(Ae) \cap acl(Je) = acl(e)$ .

From  $E \downarrow_F J$ ,  $e \downarrow AJ$  and  $A \downarrow_B F$  we deduce

$$SU(A/e) = SU(A)$$
 and  $SU(A/Je) = SU(A/J) = SU(A/F) = SU(A/B)$ ,

so that

$$SU(A/e) - SU(A/Je) = SU(A) - SU(A/B) = SU(B) - SU(B/A) = r.$$

Because  $e \downarrow A$  and by maximality of A, we get  $e \downarrow E$ ; thus tp(E/e) is not almost-S-internal. As we saw above, we have  $\overline{\text{Cb}}(Je/Ee) = \text{acl}(Ee)$ . Hence, working over acl(e) and replacing F by J, we may assume  $E \subseteq \text{acl}(AF)$ .

#### Step 3. The final contradiction.

Let  $(F_nB_n)_{n\in\mathbb{N}}$  be a sequence of E-independent realisations of tp(FB/E). From  $B\downarrow_A E$ , it follows that the sets  $B_n$  are independent over A. By 1.1(1), and because  $A = \overline{\mathrm{Cb}}(B/A)$ , there is m such that  $A \subset \mathrm{acl}(B_1 \dots B_m)$ ; take the minimal such m. Then  $E \subset \mathrm{acl}(B_1 \dots B_m F_i)$  for every i, so that in particular  $F_1 \downarrow_{B_1 \dots B_m} F_m$ . On the other hand, we know that  $F_1 \cap \mathrm{acl}(B_2 \dots B_{m-1} F_m) \subseteq F \cap E = \mathrm{acl}(\emptyset)$ , and  $SU(B_1) - SU(B_1/B_2 \dots B_m) < r$  by minimality of m. We apply the induction hypothesis to  $(F_1, B_1)$  and  $(\mathrm{acl}(B_2 \dots B_{m-1} F_m), \mathrm{acl}(B_2 \dots B_m))$ : if  $J = \overline{\mathrm{Cb}}(B_2 \dots B_{m-1} F_m/F_1)$ , then  $J \not\subseteq B_1$  and tp(J) is almost-S-internal. This contradicts the maximality of B, and finishes the proof.

- **2.2.** We will now prove some more results for supersimple theories with the CBP. Note that Proposition 2.3 below implies Theorem 2.1 and is therefore equivalent to it. It was first proved by Moosa and Pillay in the stable context, see [MP].
- **2.3. Proposition** (CBP). Let B and E be algebraically closed sets, with  $SU(E) < \infty$ , and assume that tp(E/B) is almost-S-internal, for some collection S of types of SU-rank 1, which is closed under  $Aut(M/acl(\emptyset))$ -conjugation. Then  $tp(E/E \cap B)$  is almost-S-internal.

*Proof.* Let  $C = B \cap E$ , and let  $A \subseteq E$  be maximal such that tp(A/C) is almost-S-internal. If  $B_0 = \overline{\mathrm{Cb}}(E/B)$ , then  $tp(E/B_0)$  is also almost-S-internal, and we may therefore assume that  $B = \overline{\mathrm{Cb}}(E/B)$ . By Proposition 1.20, we know that tp(E/C) is S-analysable, and this implies that tp(B/C) is also S-analysable. On the other hand, by Theorem 2.1, tp(B/C) is almost-S-internal, for some collection S' of types of SU-rank 1 containing S, and these two facts imply that tp(B/C) is almost-S-internal.

Assume  $E \neq A$ . By assumption, there is some  $F = \operatorname{acl}(F)$ , independent from E over B, and such that E is equi-algebraic over F with some finite tuple of realizations of types in S. Claim.  $\operatorname{acl}(AF) \neq \operatorname{acl}(EF)$ .

Otherwise,  $A \subseteq E$  and  $E \downarrow_B F$  would imply  $E \subseteq \operatorname{acl}(AB)$ . As tp(B/C) is almost-S-internal, this would imply that also tp(E/C) is almost-S-internal, a contradiction.

We may therefore choose some  $e \in \operatorname{acl}(EF) \setminus \operatorname{acl}(AF)$  which realises a type in  $\mathcal{S}$ . Then  $E_0 = \overline{\operatorname{Cb}}(Fe/E) \not\subseteq A$ , since  $e \in \operatorname{acl}(FE_0) \setminus \operatorname{acl}(FA)$ . Note that  $E \cap F = E \cap B = C$ .

Let  $D = \operatorname{acl}(Fe) \cap E$ . Then  $D \cap F = C$ , and by Theorem 2.1  $tp(E_0/D)$  is almost-S-internal (because  $E_0 \subseteq E$  and tp(E/C) is S-analysable). If D = C, this gives us the desired contradiction, as  $E_0 \not\subseteq A$ , and A was maximal contained in E with tp(A/C) almost-S-internal.

Assume therefore that  $D \neq C$ . Then SU(D/F) = 1, because SU(e/F) = 1 and  $D \subset \operatorname{acl}(Fe)$ . If  $D \downarrow_C F$ , then SU(D/C) = 1, which implies that tp(D/C) is almost-S-internal. In that case we let  $D_0 = D$ . If  $D \downarrow_C F$ , we define  $D_0 = \overline{\operatorname{Cb}}(F/D)$ ; then  $tp(D_0/D \cap F)$  is almost-S-internal by 2.1. Hence, as  $D_0 \subseteq \operatorname{acl}(Fe)$ , and  $D_0 \not\subseteq F$ , we have that  $e \in \operatorname{acl}(FD_0)$ , and  $tp(D_0/C)$  is almost-S-internal. As  $D_0 \subseteq D \subseteq E$ , we obtain  $D_0 \subseteq A$ , whence  $e \in \operatorname{acl}(FA)$ , which gives us the desired contradiction and finishes the proof.

**2.4. Lemma** (CBP). Let  $E = \operatorname{acl}(E)$  be of finite SU-rank over some  $C = \operatorname{acl}(C)$ , and let  $\mathcal{S}$  be a collection of types of SU-rank 1, closed under  $\operatorname{Aut}(M/\operatorname{acl}(\emptyset))$ -conjugation. Assume that  $A_i = \operatorname{acl}(A_i) \subset E$ , i = 1, 2, are such that  $A_1 \cap A_2 = C$ , and  $\operatorname{tp}(E/A_i)$  is almost- $\mathcal{S}$ -internal for i = 1, 2. Then  $\operatorname{tp}(E/C)$  is almost- $\mathcal{S}$ -internal.

Proof. Let  $A \subseteq E$  be maximal such that tp(A/C) is almost-S-internal. Then  $A_1 \subseteq A$ : by hypothesis,  $tp(A_1/A_2)$  is almost-S-internal, and by 2.3,  $tp(A_1/C)$  is almost-S-internal. Reasoning similarly with  $A_2$ , we obtain that  $A_1A_2 \subseteq A$ . If  $F = \operatorname{acl}(F) \supset C$  is independent from E over C, and tp(E/F) is almost-S-internal, then so is tp(E/C), and we may therefore extend C, to a larger set over which A is equi-algebraic with a tuple of realisations of types in S (by Lemma 1.12(2), we will not lose the maximality of A). Hence, we may assume that in A there is a tuple a of realisations of types in S such that  $a \downarrow_C A_1 A_2$ , and  $A = \operatorname{acl}(CA_1 A_2 a)$ . Note that we still have  $\operatorname{acl}(A_1 a) \cap A_2 = C$ : since  $a \downarrow_C A_1 A_2$ , we know by (1.1)(3) that  $\operatorname{acl}(A_1 a) \cap \operatorname{acl}(A_2 a) = \operatorname{acl}(Ca)$ ; hence  $\operatorname{acl}(A_1 a) \cap A_2 \subseteq \operatorname{acl}(Ca) \cap A_2 = C$ . Thus, replacing  $A_1$  by  $\operatorname{acl}(A_1 a)$  we may assume that  $\operatorname{acl}(A_1 A_2) = A$ .

By assumption, for i = 1, 2, there are  $F_i = \operatorname{acl}(F_i)$  containing  $A_i$ , independent from E over  $A_i$ , and such that E is equi-algebraic over  $F_i$  with some tuple  $b_i$  of realisations of types in S. We may choose  $F_2$  independent from  $EF_1$  over  $A_2$ ; then  $F_1 \downarrow_E F_2$ , whence also  $F_1$  is independent from  $EF_2$  over  $A_1$ , and

$$C = A_1 \cap A_2 = F_1 \cap F_2;$$
  $\operatorname{acl}(F_1b_1) \cap F_2 = A_2;$   $F_1 \cap \operatorname{acl}(F_2b_2) = A_1$ 

(use  $\operatorname{acl}(F_ib_i) = \operatorname{acl}(F_iE)$ ,  $E \cap F_j = A_j$ ). For i = 1, 2, choose  $e_i \subset b_i$  maximal independent over  $F_iA$ . Then  $E \subseteq \operatorname{acl}(F_iAe_i)$ , and  $A \cap \operatorname{acl}(F_ie_i) = A_i$ . Furthermore

$$\operatorname{acl}(F_1e_1) \cap F_2 = F_1 \cap \operatorname{acl}(F_2e_2) = A_1 \cap A_2 = C.$$

Let  $D_0 = \operatorname{acl}(F_1e_1) \cap \operatorname{acl}(F_2e_2)$ . As  $D_0 \subseteq \operatorname{acl}(F_1e_1)$ ,  $tp(D_0/F_1)$  is almost-S-internal; by 2.3,  $tp(D_0/D_0 \cap F_1)$  is almost-S-internal; hence  $tp(D_0/C)$  is almost-S-internal because  $\operatorname{acl}(F_2e_2) \cap F_1 = C$ , and this implies that  $D_0 \cap E \subseteq A$ . Therefore

$$D_0 \cap E = D_0 \cap A = \operatorname{acl}(F_1 e_1) \cap \operatorname{acl}(F_2 e_2) \cap A = A_1 \cap A_2 = C.$$

Let  $D_1 = \overline{\text{Cb}}(F_1e_1/F_2e_2)$ . Then  $tp(D_1/D_0)$  is almost- $\mathcal{S}$ -internal, by 2.1. We know that  $F_1e_1$  and  $F_2e_2$  are independent over  $D_1$ , and therefore

$$F_1e_1 \downarrow_{D_1A_1A_2} F_2e_2.$$

Since  $\operatorname{acl}(A_1A_2) = A$  and  $E \subseteq \operatorname{acl}(F_iAe_i)$ , we get  $E \subseteq \operatorname{acl}(D_1A)$ . Hence  $tp(E/D_0)$  is almost-S-internal, and so is  $tp(E/D_0 \cap E)$  (by 2.3). As  $D_0 \cap E = C$ , we get the result.

**2.5. Theorem** (CBP). Assume that  $E = \operatorname{acl}(E)$  has finite SU-rank, and let  $\mathcal{S}$  be a collection of types of SU-rank 1, closed under  $\operatorname{Aut}(M/\operatorname{acl}(\emptyset))$ -conjugation. Then there is  $A = \operatorname{acl}(A) \subseteq E$  such that tp(E/A) is almost- $\mathcal{S}$ -internal, and whenever  $B = \operatorname{acl}(B)$  is such that tp(E/B) is almost- $\mathcal{S}$ -internal, then  $B \supseteq A$ .

*Proof.* This follows immediately from Proposition 2.3 and Lemma 2.4.

**2.6. Theorem** (CBP). Let  $B = \overline{\text{Cb}}(A/B)$ , where A = acl(A) has finite SU-rank, and let  $\mathcal{S}$  be a collection of types of SU-rank 1, closed under  $\text{Aut}(M/\text{acl}(\emptyset))$ -conjugation, and such that tp(B/A) is almost- $\mathcal{S}$ -internal. If C = acl(C) is such that tp(A/C) is almost- $\mathcal{S}$ -internal, then so is tp(AB/C). That is, T has the UCBP.

*Proof.* Let  $D = \overline{\text{Cb}}(B/A)$ . Then tp(D/B) is almost-S-internal, and  $B = \overline{\text{Cb}}(D/B)$  (by Lemma 1.10). As  $D \subseteq A$ , tp(D/C) is almost-S-internal, and by 2.4, so is  $tp(D/B \cap C)$ ; this implies that tp(B/C) is also almost-S-internal, since B is contained in the algebraic closure of realisations of the almost-S-internal-type  $tp(D/B \cap C)$  (see 1.1(1)).

**2.7. Proposition** (CBP). Let G be a group of finite SU-rank, let p be a type (over  $\emptyset$ ) realised by  $a \in G$ , and let H = Stab(p) be the left stabilizer of p. If d is the code of  $H \cdot a$ , then tp(d) is almost-S-internal, where S is the collection of non-locally modular types of SU-rank 1 and with algebraically closed base.

*Proof.* The proof is essentially identical to the one given in [PZ], Corollary 3.11, where it was done in the stable case. Let  $c \in G$  be a generic of G over a, and let  $D = \overline{\mathrm{Cb}}(c/e)$ , where  $e = a \cdot c$ .

We will first show that  $d \in \operatorname{acl}(Dc)$ . By genericity of c, we know that  $e \downarrow a$ , and therefore  $a \downarrow De$ . The set D has the following property: if  $e_1, e_2$  are D-independent realisations of tp(e/D), then there is c' independent from  $e_1e_2$  over D, and such that  $tp(c'e_i/D) = tp(ce/D)$  for i = 1, 2. If  $a_i = e_i \cdot c'^{-1}$ , then  $e_1 \cdot e_2^{-1} = a_1 \cdot a_2^{-1}$ , and  $a_1, a_2$  realise p. We then deduce successively the following relations:

$$c' \downarrow_D e_1 e_2;$$
  $c' \downarrow_D (e_1 \cdot e_2^{-1}) e_2;$   $c' \downarrow_{De_2} e_1 \cdot e_2^{-1};$   $a_2 \downarrow_{De_2} e_1 \cdot e_2^{-1};$  since  $a_2 \downarrow_D e_2$ , transitivity implies  $a_2 \downarrow_D e_1 \cdot e_2^{-1}.$ 

As both  $a_1$  and  $a_2$  realise p, and  $a_1 = e_1 \cdot e_2^{-1} \cdot a_2$ , we get that  $e_1 \cdot e_2^{-1} \in H$ . So we have shown that if  $e_1$ ,  $e_2$  are any D-independent realisations of tp(e/D), then  $e_1 \cdot e_2^{-1} \in H$ . Hence, if  $e_1$  and  $e_2$  realise tp(e/D), then  $e_1 \cdot e_2^{-1} \in H$ .

If  $\tau \in \text{Aut}(M/Dc)$ , then  $\tau(e) \cdot e^{-1} \in H$ , and  $\tau(a) = \tau(e) \cdot e^{-1} \cdot a \in H \cdot a$ . This shows that  $d \in \text{acl}(Dc)$ .

By the CBP, we know that  $tp(D/\operatorname{acl}(c))$  is almost- $\mathcal{S}$ -internal, and therefore so is  $tp(d/\operatorname{acl}(c))$ . But on the other hand, we know that  $d \in \operatorname{acl}(a)$  and  $a \downarrow c$ : hence  $d \downarrow c$  and tp(d) is almost- $\mathcal{S}$ -internal. **2.8. Remark/Corollary** (CBP). Let G be a group of finite SU-rank, and p a type over  $\emptyset$ , realised by  $a \in G$ . Let  $b \in dcl(a)$  be maximal realising an almost-S-internal-type, and let  $S = \{g \in G \mid tp(g \cdot a/b) = tp(a/b)\}$ , and let N be the subgroup of G generated by S. Then  $N \subseteq H$ , where H is the left stabiliser of p.

*Proof.* If  $\pi: G \to H\backslash G$  is the natural projection, then we know that  $H \cdot a$  is coded by  $\pi(a)$ . By 2.7,  $tp(\pi(a))$  is almost-S-internal, and therefore  $\pi(a) \in dcl(b)$ . By definition of b, tp(a'/b) = tp(a/b) implies  $a' \in H \cdot a$  and  $a' \cdot a^{-1} \in S$ , which gives the result.

**2.9.** The next results allow us in many cases to pass from the algebraic closure of a set to the set itself. In geometric situations, it will allow us to replace correspondences by rational maps. The delicate point is that in general, if  $B = \operatorname{acl}(B) \subset \operatorname{acl}(A)$  and  $B_0 = B \cap A$ , it may happen that  $B \neq \operatorname{acl}(B_0)$ . The first result, Observation 2.10, does not need the CBP hypothesis.

In what follows, we work over  $\emptyset$ , and have a set  $\mathcal{S}$  of SU-rank 1 types with algebraically closed base, and which is closed under  $\operatorname{Aut}(M)$ -conjugation.

**2.10. Observation**. Let a a tuple, let  $B = \operatorname{acl}(B)$  be maximal contained in  $\operatorname{acl}(a)$  and such that tp(B) is almost-S-internal. Let  $B_0 = \operatorname{dcl}(a) \cap B$ ; then  $\operatorname{acl}(B_0) = B$ .

*Proof.* Let  $b \in B$  be such that  $B = \operatorname{acl}(B)$ , and let b' be a conjugate of b over  $\operatorname{dcl}(a)$ . Then tp(b') = tp(b), and therefore tp(b') is almost-S-internal. Hence, if c is a tuple encoding the set of conjugates of b over  $\operatorname{dcl}(a)$ , then  $c \in \operatorname{dcl}(a)$ , and tp(c) is almost-S-internal, so that  $c \in B_0$ . As  $b \in \operatorname{acl}(c)$ , we get  $\operatorname{acl}(c) = B$ .

**2.11. Proposition** (CBP). Let a be a tuple of finite SU-rank, let  $B = \operatorname{acl}(B)$  be such that tp(a/B) is almost-S-internal. If  $B_0 = B \cap \operatorname{dcl}(a)$ , then  $tp(a/B_0)$  is almost-S-internal.

Proof. We may assume that B is minimal algebraically closed such that tp(a/B) is almost-S-internal. Choose a tuple  $b \in B$  such that  $B = \operatorname{acl}(b)$ . If b' is a conjugate of b over  $\operatorname{dcl}(a)$ , then tp(a,b) = tp(a,b'), and therefore tp(a/b') is also almost-S-internal. The minimality of B (and 2.4) implies that  $\operatorname{acl}(b') = \operatorname{acl}(b)$ . Hence, if c is a tuple encoding the set of conjugates of b over  $\operatorname{dcl}(a)$ , then  $\operatorname{acl}(c) = \operatorname{acl}(b)$ ; as  $c \in B \cap \operatorname{dcl}(a) = B_0$ , we get  $B = \operatorname{acl}(B_0)$ .

- **2.12. Proposition** (CBP). Let  $a_1, a_2, b_1, b_2$  be tuples of finite SU-rank and assume that
  - $tp(b_2)$  is almost-S-internal,
  - $\operatorname{acl}(b_1) \cap \operatorname{acl}(b_2) = \operatorname{acl}(\emptyset),$
  - $a_1 \downarrow_{b_1} b_2$  and  $a_2 \downarrow_{b_2} b_1$ ,
  - $a_2 \in \operatorname{acl}(a_1b_1b_2)$ .

Then there is  $e \subset \operatorname{dcl}(a_2b_2)$  such that  $tp(a_2/e)$  is almost- $\mathcal{S}$ -internal and  $e \downarrow b_2$ . In particular, if  $tp(a_2/b_2)$  is hereditarily orthogonal to all types in  $\mathcal{S}$ , then  $a_2 \in \operatorname{acl}(eb_2)$ .

Proof. If  $C = \overline{\text{Cb}}(a_1b_1/a_2b_2)$ , then  $a_2 \in \text{acl}(Cb_2)$ . Let  $D = \text{acl}(a_1b_1) \cap \text{acl}(a_2b_2)$ . As  $D \subset \text{acl}(a_ib_i)$  for i = 1, 2, we have  $D \downarrow_{b_1} b_2$  and  $D \downarrow_{b_2} b_1$ . Hence  $D \downarrow_{b_1} b_2$  because  $\text{acl}(b_1) \cap \text{acl}(b_2) = \text{acl}(\emptyset)$ . Furthermore, we know by Theorem 1.18 that there is a set S' of SU-rank 1 types orthogonal to all

members of S and such that tp(C/D) is almost- $(S \cup S')$ -internal. We may write C as  $\operatorname{acl}(c_1c_2)$  where  $tp(c_1/D)$  is almost-S-internal, and  $tp(c_2/D)$  is almost-S-internal. Then  $\operatorname{acl}(c_2D) \downarrow b_2$  because  $tp(c_2/D)$  is hereditarily orthogonal to all members of S and  $tp(b_2)$  is almost-S-internal. Furthermore, as  $a_2 \in \operatorname{acl}(Dc_1c_2b_2)$ , it follows that  $tp(a_2/\operatorname{acl}(Dc_2))$  is almost-S-internal. Now,  $Dc_2 \subseteq \operatorname{acl}(a_2b_2)$ , and Proposition 2.11 implies that if  $e = \operatorname{acl}(Dc_2) \cap \operatorname{dcl}(a_2b_2)$ , then  $tp(a_2/e)$  is almost-S-internal.

The last assertion is clear:  $tp(a_2/e)$  almost-S-internal implies  $tp(a_2/eb_2)$  almost-S-internal, and our assumption of hereditary orthogonality implies that  $a_2 \in acl(eb_2)$ .

**2.13. Concluding remarks.** Inspection of the proofs shows that our assumption of supersimplicity on the ambient theory is unnecessary, as long as one restricts one's attention to types ranked by the SU-rank, and the relevant hyperimaginaries and imaginaries are eliminated. Thus, the results of section 1 do apply to types of finite U-rank in separably closed fields of finite degree of imperfection. It is unknown whether this family of types enjoys the CBP, we will explain now what one needs to prove. Let K be a separably closed field of characteristic p>0 and finite (positive) degree of imperfection. It follows from results of Margit Messmer, Hrushovski and Françoise Delon (see e.g. [Bo]), that a type of finite U-rank which is not one-based is non-orthogonal to the generic type q of  $\bigcap_n K^{p^n}$ . By 1.18, it is therefore enough to show that the family of all  $\{q\}$ -analysable types has the CBP. A partial result in this direction is obtained by Pillay and Ziegler in [PZ]: they show that the family of very thin types has the CBP. Thus, the results of section 2 apply for the family  $\mathcal P$  of very thin types. Unfortunately, Pillay and Ziegler also give an example of a  $\{q\}$ -analysable type (of U-rank 2) which is not very thin.

The result of Pillay and Ziegler on types in differentially closed fields of characteristic 0 is stronger than the CBP: indeed, if Cb denotes the usual canonical base, then they show that given two tuples a and b of finite rank such that b = Cb(a/b), then tp(b/a) is internal to the constants. It would be interesting to know whether this implies that tp(b/C) is also internal to the constants (as opposed to almost-internal to the constants), under some reasonable conditions on a, b, and with  $C = \text{acl}(a) \cap \text{acl}(b)$ , or even  $C = \text{dcl}(a) \cap \text{dcl}(b)$ .

# 3 Existentially closed difference fields have the CBP

Recall that a difference field is a field with a distinguished endomorphism (usually denoted by  $\sigma$ ), which we study in the language of rings augmented by a symbol for  $\sigma$ . A difference field K is inversive if  $\sigma(K) = K$ . We refer to [C] for basic algebraic results on difference fields, and to [CH] for basic model-theoretic results. Any completion of the theory ACFA of existentially closed difference fields is supersimple and eliminates imaginaries. Moreover, if K is an existentially closed difference field, and  $A \subseteq K$ , then  $\operatorname{acl}(A)$  is the smallest algebraically closed subfield B of K satisfying  $\sigma(B) = B$  and containing A. Independence of algebraically closed sets coincides with independence in the sense of the theory of algebraically closed fields, i.e., if  $C \subseteq A$ , B are algebraically closed difference subfields of K, then A and B are independent over C if and only if A and B are linearly disjoint over C.

As an immediate corollary of the results of Pillay and Ziegler and of Proposition 2.1, we then obtain

- **3.1. Proposition**. Let (K, D) [resp.  $(K, \sigma)$ ] be a differentially closed field [resp., an existentially closed difference field] of characteristic 0. Let  $C \subseteq A, B$  be algebraically closed differential [resp. difference] subfields of K, with  $SU(B/C) < \omega$ . Assume that  $A = \overline{\text{Cb}}(B/A)$ . Then  $tp(A/A \cap B)$  is almost internal to Dx = 0 [resp.  $\sigma(x) = x$ ].
- **3.2. Notation**. We denote by  $A^{alg}$  the field-theoretic algebraic closure of a field A, and by  $A^s$  its separable closure. If  $\sigma(E) = E$  is a difference subfield of the inversive difference field K, and a is a tuple of elements of K, then  $E(a)_{\sigma}$  denotes the (inversive) difference subfield  $E(\sigma^i(a) \mid i \in \mathbb{Z})$  of K. If  $\tau$  is an automorphism of K, we denote by  $Fix(\tau)$  the subfield of K consisting of elements fixed by  $\tau$ . We denote by Frob the Frobenius map  $x \mapsto x^p$ .
- **3.3.** p-bases and degree of imperfection. For details and proofs, see [B] §13. Let  $E \subseteq K \subseteq L \subseteq K^{alg}$  be fields of characteristic p > 0, with E perfect and  $tr.deg(K/E) = d < \infty$ . Then  $[K:K^p] = p^e$  for some  $e \le d$ , and there is an e-tuple c of elements of K such that  $K = K^p[c]$ . Such a tuple is called a p-basis of K and its elements are algebraically independent over E. Moreover, if e = d, then c is a separating transcendence basis of K over E, i.e.,  $K \subseteq E(c)^s$ . The integer e is called the degree of imperfection of K.

We also have:  $[L:L^p]$  divides  $p^e$ , and  $[L:L^p]=p^e$  if  $L\subseteq K^s$  or if  $[L:K]<\infty$ .

**3.4. Lemma.** Let  $(K, \sigma)$  be an existentially closed difference field of characteristic p > 0, let  $E = \operatorname{acl}(E) \subset K$ , a a finite tuple in K, and assume that tp(a/E) is  $\operatorname{Fix}(\sigma)$ -analysable. Then there is a finite tuple b such that  $E(a)_{\sigma} = E(b)_{\sigma}$ , and  $\sigma(b), \sigma^{-1}(b) \in E(b)^{s}$ .

*Proof.* We will show that if  $d = tr.deg(E(a)_{\sigma}/E)$ , then  $[E(a)_{\sigma} : E(a^{p})_{\sigma}] = p^{d}$  and  $d < \infty$ . This will yield the result: let c be a p-basis of  $E(a)_{\sigma}$ . Then  $E(a)_{\sigma} \subseteq E(c)^{s}$ , and therefore  $E(a)_{\sigma} = E(c,a)_{\sigma} \subseteq E(c,a)^{s}$ .

The proof is by induction on the length of a semi-minimal analysis in  $\operatorname{Fix}(\sigma)$  of tp(a/E). Assume first that tp(a/E) is almost- $\operatorname{Fix}(\sigma)$ -internal. Let  $F = \operatorname{acl}(F)$  be independent from a over E, and such that a is equi-algebraic over F with some finite tuple b of  $\operatorname{Fix}(\sigma)$ . We may assume that  $a \in F(b)^s$  (we replace b by  $b^{1/p^n}$  if necessary). From  $\sigma(b) = b$ , we deduce that  $F(a)_{\sigma} \subseteq F(b)^s$ , and therefore

$$p^d \ge [F(a)_{\sigma} : F(a^p)_{\sigma}] \ge [F(b) : F(b^p)] = p^d.$$

As F was linearly disjoint from  $E(a)_{\sigma}$  over E, this shows  $[E(a)_{\sigma}: E(a^p)_{\sigma}] = p^d$ , with  $d < \infty$ . For the general case, choose  $a_1, \ldots, a_n \in \operatorname{acl}(Ea)$  such that  $a \in E(a_1, \ldots, a_n)_{\sigma}$ , and for every i,  $tp(a_i/\operatorname{acl}(Ea_1, \ldots, a_{i-1}))$  is almost-Fix $(\sigma)$ -internal. Let  $F_i = \operatorname{acl}(Ea_1, \ldots, a_i)$  for  $i = 1, \ldots, n$ . By reverse induction, we may enlarge  $a_n, \ldots, a_1$  so that for every i < n:

- (a)  $a_{i+1}$  contains a p-basis of  $F_i(a_{i+1})_{\sigma}$  and a transcendence basis of  $F_i(a_{i+1})_{\sigma}$  over  $F_i$ .
- (b) The  $\sigma$ -ideal of difference equations satisfied by  $a_{i+1}$  over  $F_i$  is generated by the difference equations satisfied by  $a_{i+1}$  over  $E_i = E(a_1, \ldots, a_i)_{\sigma}$  (this is possible, since this  $\sigma$ -ideal is finitely generated as a  $\sigma$ -ideal, see e.g. [C]).

Condition (b) then implies that for every i < n,  $E_i(a_{i+1})_{\sigma}$  and  $F_i$  are linearly disjoint over  $E_i$ . By the first case and (a),  $F_i(a_{i+1})_{\sigma} \subseteq F_i(a_{i+1})^s$ , and the linear disjointness of  $F_i$  and  $E_i(a_{i+1})_{\sigma}$  over  $E_i$  then implies that  $E_i(a_{i+1})_{\sigma} \subseteq E_i(a_{i+1})^s$ , so that  $E(a)_{\sigma} \subseteq E(a_1, \ldots, a_n)_{\sigma} \subseteq E(a_1, \ldots, a_n)^s$ . Then  $[E(a_1, \ldots, a_n) : E(a_1^p, \ldots, a_n^p)] = p^d$  where  $d = tr.deg(E(a_1, \ldots, a_n)/E) < \infty$ . Reasoning as in the first case, we deduce  $[E(a)_{\sigma} : E(a^p)_{\sigma}] = p^d$ .

**3.5.** Definition of S. Let  $(K, \sigma)$  be an existentially closed difference field of characteristic p > 0. In this paragraph we give a description of the classes S(q), q a non-one-based type of SU-rank 1.

Let I be the set of pairs  $(n, m) \in \mathbb{N}^{>0} \times \mathbb{Z}$ , with (n, m) = 1 if  $m \neq 0$  and n = 1 if m = 0. For each pair  $(n, m) \in I$ , choose a non-algebraic type  $q_{n,m}$  (over  $\mathbb{F}_p^{alg}$ ) containing the formula  $\sigma^n(x^{p^m}) = x$ , and let  $\mathcal{S}_{n,m} = \mathcal{S}(q_{n,m})$ . Then  $q_{n,m}$  is not one-based.

By (7.1)(1) in [CHP],  $SU(\sigma^n(x^{p^m}) = x) = 1$ ; as the formula  $\sigma^n(x^{p^m}) = x$  defines a subfield of K, this implies that any two non-algebraic types containing this formula are non-orthogonal. This observation, together with the main result of [CHP] (see the Theorem in section 6), shows that any type of SU-rank 1 which is not one-based is non-orthogonal to some  $q_{n,m}$ . We define  $S = \bigcup S_{n,m}$ .

We will now show that if  $(n, m) \neq (n', m')$  are in I, then  $S_{m,n} \cap S_{m',n'} = \emptyset$ .

Indeed, let  $F = \operatorname{acl}(F)$ , and  $a, b \in K \setminus F$  with  $\sigma^n(a^{p^m}) = a$ ,  $\sigma^{n'}(b^{p^{m'}}) = b$ , and assume that a, b are equi-algebraic over F. Then clearly  $n = n' = tr.deg(F(a)_{\sigma}/F) = tr.deg(F(b)_{\sigma}/F)$ . Taking a  $p^\ell$ -power of b, we may assume that  $b \in F(a, \ldots, \sigma^{n-1}(a))^s$ . Let  $\tau = \sigma^n \operatorname{Frob}^m$ . Then  $F(a, \ldots, \sigma^{n-1}(a))^s$  is closed under  $\tau$  and  $\tau^{-1}$  (because  $\tau \sigma^i = \sigma^i \tau$  and  $\tau(a) = a$ ), and has degree of imperfection n. On the other hand, if  $m \neq m'$ , then the closure under  $\tau$  and  $\tau^{-1}$  of F(b) is perfect because  $\tau(b) = b^{p^{m-m'}}$ . This contradicts  $b \in F(a, \ldots, \sigma^{n-1}(a))^s$ .

**3.6. Theorem.** Let  $(K, \sigma)$  be an existentially closed difference field of characteristic p > 0, let  $C \subseteq A, B$  be algebraically closed difference fields, with  $SU(B/C) < \omega$ . Assume that  $\overline{\mathrm{Cb}}(B/A) = A$ . Then  $tp(A/A \cap B)$  is almost- $\mathcal{S}$ -internal.

*Proof.* By Theorem 2.1, it suffices to show that whenever A and B satisfy the hypotheses of the theorem, then tp(A/B) is almost-S-internal. Fix such A, B, with  $C = A \cap B$ . We may assume  $B = \overline{\text{Cb}}(A/B)$ ; observe that by 1.1(1),  $A = \overline{\text{Cb}}(B/A)$  implies  $SU(A/C) < \omega$ .

By Proposition 1.16 and the discussion in 3.5, we already know that  $A = \operatorname{acl}(A_1 \dots A_j)$ , where each  $tp(A_i/C)$  is  $\mathcal{S}_{n,m}$ -analysable for some  $(n,m) \in I$ , and  $B = \operatorname{acl}(B_1 \dots B_j)$ , where  $B_i = \overline{\operatorname{Cb}}(A_i/B)$ ,  $A_i = \overline{\operatorname{Cb}}(B_i/A)$ . If there is a counterexample to our assertion, then there is one where tp(A/C) and tp(B/C) are  $\mathcal{S}_{n,m}$ -analysable for some  $(n,m) \in I$ , and this is what we will assume. We will also assume that K is sufficiently saturated.

Let  $\tau = \sigma^n \text{Frob}^m$ . Let b be a (finite) tuple of elements of B such that  $B = C(b)^{alg}$ . Then A is the smallest algebraically closed field containing C and the field of definition of the algebraic locus of b over A.

We now work in the difference field  $(K, \tau)$ , which is a reduct of  $(K, \sigma)$ , and is also a model of ACFA by Corollary 1.12(1) in [CH]. In the reduct  $(K, \tau)$  we also have  $A = \overline{\text{Cb}}(Cb/A)$ . By Lemma 3.4, we may assume that  $\tau(b)$  and  $\tau^{-1}(b)$  are in  $C(b)^s$ . Hence, there are varieties V, W

defined over C, with generics b and  $(b, \tau(b))$  respectively, and with  $W \subseteq V \times \tau(V)$ , and such that the projection maps  $W \to V$  and  $W \to \tau(V)$  are separable and generically finite. These maps therefore induce isomorphisms between the jetspaces  $J_{(b,\tau(b))}^k(W)$  and  $J_b^k(V)$ ,  $J_{\tau(b)}^k(\tau(V))$  for every k > 0. The proof of Pillay and Ziegler then goes through (see chapter 3 of [PZ]), and shows that tp(A/B) is almost-Fix( $\tau$ )-internal (in  $(K,\tau)$ ). Hence there is  $M = \tau(M)^{alg} \supseteq B$ , linearly disjoint from AB over B, and some tuple  $a \in \text{Fix}(\tau)$  such that  $A \subseteq M(a)^{alg}$ . Since the elements of a have SU-rank 1 in the difference field  $(K,\tau)$ , we may assume that a and A are equi-algebraic over M.

If n=1, then  $M=\sigma(M)^{alg}$ , and we are done. Assume that n>1; then M is closed under  $\sigma^n$  and  $\sigma^{-n}$ , but not necessarily under  $\sigma, \sigma^{-1}$ . We need to show that there is a difference field  $(N,\sigma)$  extending  $(B,\sigma)$ , containing M and linearly disjoint from AM over M, and such that  $(N,\sigma^n)$  extends  $(M,\tau\operatorname{Frob}^{-m})$ . This is done as in [CH], Lemma 1.12. The saturation of  $(K,\sigma)$  then implies that K contains (a copy of)  $(N,\sigma)$ , and shows that tp(A/B) is almost-Fix $(\tau)$ -internal.

**Theorem 3.6**'. Let A, B be difference subfields of  $\mathcal{U}$  intersecting in C, such that  $A^{alg} \cap B^{alg} = C^{alg}$  and  $tr.deg(A/C) < \infty$ . Let  $D \subset B$  be generated over C by all tuples d such that there exist an algebraically closed difference field F containing C and free from B over C, and integers n > 0 and m such that  $d \in F(e)$  for some tuple e of elements satisfying  $\sigma^n \operatorname{Frob}^m(x) = x$ . Then A and B are free over D.

*Proof.* When A and B are algebraically closed, this is a direct consequence of 3.6 and 2.1: we know that  $\overline{\text{Cb}}(A/B)$  realises a type over  $A \cap B$  which is almost-S-internal, where S is the family of SU-rank 1 types realised in some  $\text{Fix}(\tau)$ . Hence  $\overline{\text{Cb}}(A/B)$  is contained in  $D^{alg}$ , which implies that A and B are free over D.

Assume now that A and B are not algebraically closed, and work over their intersection C. Again, we know that  $\overline{\mathrm{Cb}}(A/B)$  realises a type over  $C^{alg}$  which is almost-S-internal. Hence  $\overline{\mathrm{Cb}}(A/B)$  is contained in the maximal subset  $D_0$  of  $\mathrm{acl}(A)$  which realises an almost-S-internal-type over  $C^{alg}$ . By Remark 4.11, we have  $D_0 = D^{alg}$ , which gives the result.

# 4 Applications of the CBP to differential and difference varieties

**4.1. Differential fields.** We will now apply some of the results of section 2 to the study of (affine) differential varieties. For an introduction to the model theory of differential fields of characteristic 0, see e.g. [M].

Known facts. We work in some large differentially closed field  $(\mathcal{U}, \delta)$  of characteristic 0. In analogy with the Zariski topology, we define the *Kolchin topology* on each cartesian power  $\mathcal{U}^n$ , as the topology with basic closed sets the zero-sets of differential polynomials, which are called *Kolchin closed sets*. This topology is Noetherian. A differential (affine) variety V is an irreducible Kolchin closed set.

If  $A \subset \mathcal{U}$  is a differential field, then  $A = \operatorname{dcl}(A)$  and  $\operatorname{acl}(A) = A^{alg}$ . The theory of differentially closed fields of characteristic 0 eliminates quantifiers and imaginaries.

**4.2.** Since our results concern differential fields, we first define the analogues of function fields and birational morphisms. The definitions are straightforward.

If a differential variety V is defined over the differential field K, we define the coordinate ring  $K[V]_D$  and function field  $K(V)_D$  of V as follows: let  $K[\bar{X}]_D$  be the ring of differential polynomials in  $\bar{X} = (X_1, \ldots, X_n)$ , and I the ideal of differential polynomials vanishing on V. Then

$$K[V]_D = K[\bar{X}]_D/I, \qquad K(V)_D = \operatorname{Frac}(K[V]_D).$$

A differential variety V has finite order if the transcendence degree of  $K(V)_D$  over K is finite. If V, W are differential varieties, a differential-rational map  $f: V \to W$  is simply a map whose coordinate functions are given by elements of  $K(V)_D$ ; it is therefore defined on some Kolchinopen subset U of V. If f(U) is dense in W for the Kolchin topology, then we will say that f is dominant, and the map f induces a K-embedding of  $K(W)_D$  into  $K(V)_D$ . Conversely, any K-embedding of  $K(W)_D$  into  $K(V)_D$  is induced by some dominant differential-rational  $f: V \to W$ . A finite cover of V is a dominant differential-rational map  $f: W \to V$  such that the generic fiber of f is finite. It corresponds to a finite algebraic extension  $K(W)_D$  of  $K(V)_D$ .

The constant field is  $C = \{x \in \mathcal{U} \mid Dx = 0\}$ . Any non-one-based type is non-orthogonal to the generic type of C. We let S be this generic type (over  $acl(\emptyset)$ ).

If  $A \subset \mathcal{U}$ , then  $K(A)_D$  denotes the differential field generated by A over K. If a is a finite tuple, then a is a generic of the differential variety V over K if  $a \in V$  and the specialisation map  $K[V]_D \to K(a)_D$  is injective.

We say that a differential variety V is C-internal<sup>2</sup> if there is a birational  $f: V \to \overline{W}(C)$  for some algebraic variety  $\overline{W}$ . We say that V is almost-C-internal if it is a finite cover of a C-internal differential variety. This is equivalent to: if a is a generic of V over K, then tp(a/K) is almost-S-internal.

**4.3.** Maximal almost- $\mathcal{C}$ -internal quotient. Let V be a differential variety of finite order defined over the differential subfield K of  $\mathcal{U}$ . Then V has a maximal almost- $\mathcal{C}$ -internal quotient  $V^{\#}$ , i.e.,  $V^{\#}$  is almost- $\mathcal{C}$ -internal, and if  $\pi$  is a dominant differential-rational map from V to an almost- $\mathcal{C}$ -internal variety  $V_1$ , then  $\pi$  factors through  $V^{\#}$ . Furthermore, if  $f:W\to V$  is a finite cover of V, then there is a generically finite map  $W^{\#}\to V^{\#}$ .

*Proof.* Let K be a differential field over which everything is defined. Translated into terms of elements, this becomes: let a be a generic of V over K, let  $A = \operatorname{acl}(A)$  be the maximal subfield of  $\operatorname{acl}(Ka)$  whose type over K is almost- $\mathcal{S}$ -internal, and let  $A_0 = A \cap K(a)_D$ . Then  $A_0$  is finitely generated over K (as a differential field or as a field), say by a tuple b, and we let  $V^\#$  be the differential variety with generic b over K, and  $V \to V^\#$  the birational map dual to the inclusion  $K(b)_D \to K(a)_D$ . Note that this defines  $V^\#$  uniquely up to a differential birational correspondence, and by definition,  $V^\#$  is almost- $\mathcal{C}$ -internal.

<sup>&</sup>lt;sup>2</sup>Some authors say that V is *iso-constant*.

Assume that  $\pi: V \to V_1$  is dominant differential-rational, and let  $c = \pi(a)$ . The almost- $\mathcal{C}$ -internality of  $V_1$  is equivalent to the almost- $\mathcal{C}$ -internality of tp(c/K), and this implies that  $c \in A_0 = K(b)_D$ , and shows that the map  $\pi$  factors through  $V^{\#}$ .

For the last assertion, let  $g:W\to V$  be a finite cover of V, let c be a generic of W such that g(c)=a, and let  $A_1=A\cap K(c)_D$ . As  $c\in K(a)_D^{alg}$ , we know that  $A_1$  is the maximal subfield of  $K(c)_D$  which realises an almost- $\mathcal C$ -internal type over K, i.e., we can take  $W^\#$  to be the differential variety of which a generator of  $A_1$  over K is a generic. We clearly have  $A_0\subseteq A_1\subseteq A$ , and we need to show that this extension is algebraic: but 2.10 tells us that  $A=A_0^{alg}$ .

**4.4. Maximal almost-** $\mathcal{C}$ **-internal fiber**. Let V be a differential variety of finite order defined over the differential subfield K of  $\mathcal{U}$ . Then V has a smallest quotient  $V^{\flat}$ , with generic fiber an almost- $\mathcal{C}$ -internal differential variety.<sup>3</sup> Furthermore, if  $f: W \to V$  is a finite cover of V, then there is a generically finite map  $W^{\flat} \to V^{\flat}$ .

*Proof.* The translation in terms of differential extensions is similar to the one done in 4.3, and reduces the problem to the following:

Let  $B = \operatorname{acl}(B)$  be minimal such that tp(a/B) is almost-S-internal (cf Theorem 2.5 for the existence), and let  $B_0 = B \cap K(a)_D$ . Then  $B_0^{alg} = B$  and  $tp(a/B_0)$  is almost-S-internal. But this last statement is given by (the proof of) 2.11.

**4.5. Descent result**. For i=1,2, let  $V_i$  be a differential variety of finite order defined over the differential subfield  $K_i$ , of  $\mathcal{U}$ , and let  $k=K_1\cap K_2$ . Assume that  $K_1^{alg}\cap K_2^{alg}=k^{alg}$ , that  $K_2$  is a regular extension of k, that there is a differential rational dominant map  $f:V_1\to V_2$  defined over  $(K_1K_2)^{alg}$ , and that  $tp(K_2/k)$  is almost- $\mathcal{S}$ -internal. Then there is a differential variety  $V_3$  defined over k, and a dominant differential rational map  $g:V_2\to V_3$  such that the generic fiber of g is almost- $\mathcal{C}$ -internal.

Proof. Use Proposition 2.12 with  $dcl(\emptyset) = k$ ,  $b_i = K_i$ , and  $a_i$  a generic of  $V_i$  over  $K_1K_2$ ,  $a_2 = f(a_1)$  to get  $e \in K_2(a_2)_D$  such that  $e \downarrow_k K_2$  and  $tp(a_2/e)$  is almost- $\mathcal{S}$ -internal. Since the property of almost- $\mathcal{S}$ -internality only depends on  $tp(a_2/e)$ , we may take for e a finite tuple. Our hypothesis on the extension  $K_2$  of K implies that  $k(e)_D$  is a regular extension of k. If  $V_3$  is the differential locus of e over k, and  $g: V_2 \to V_3$  is the dominant map induced by the inclusion  $K_2(e)_D \subset K_2(a_2)_D$ , then the generic fiber of g realises an almost- $\mathcal{S}$ -internal-type (over  $K_2(e)_D$  or  $k(e)_D$ ).

**4.6.** Difference fields. In the same vein, we now apply the results of section 2 to the study of (affine) difference varieties. Again, we have to define the analogues of function fields and birational morphisms. The definitions are straightforward.

We work in some large existentially closed difference field  $\mathcal{U}$ . In analogy with the Zariski topology, we define the  $\sigma$ -topology on each cartesian power  $\mathcal{U}^n$ , as the topology with basic closed sets the zero-sets of difference polynomials, which are called  $\sigma$ -closed sets. This topology is Noetherian. A difference (affine) variety is an irreducible  $\sigma$ -closed set, and if this variety

<sup>&</sup>lt;sup>3</sup>In otherwords, if  $\pi: V \to V_1$  is dominant with generic fiber almost- $\mathcal{C}$ -internal, then  $V^{\flat}$  is a quotient of  $V_1$ 

is defined over the difference field K, we define its coordinate ring  $K[V]_{\sigma+}$  and function field  $K(V)_{\sigma+}$  as follows: let  $K[\bar{X}]_{\sigma}$  be the ring of difference polynomials in  $\bar{X}=(X_1,\ldots,X_n)$ , and I the ideal of difference polynomials vanishing on V. Then

$$K[V]_{\sigma+} = K[\bar{X}]_{\sigma}/I, \qquad K(V)_{\sigma+} = \operatorname{Frac}(K[V]_{\sigma+}).$$

The order of a difference variety V is the transcendence degree of  $K(V)_{\sigma+}$  over K. If V, W are difference varieties, a  $\sigma$ -rational map  $f: V \to W$  is simply a map whose coordinate functions are given by elements of  $K(V)_{\sigma+}$ ; it is therefore defined on some  $\sigma$ -open subset U of V. If f(U) is dense in W for the  $\sigma$ -topology, then we will say that f is dominant, and the map f induces a K-embedding of  $K(W)_{\sigma+}$  into  $K(V)_{\sigma+}$ . Conversely, any K-embedding of  $K(W)_{\sigma+}$  into  $K(V)_{\sigma+}$  is induced by some dominant  $\sigma$ -rational map  $f: V \to W$ . A finite cover of V is a dominant  $\sigma$ -rational map  $f: W \to V$  such that the generic fiber of f is finite. It corresponds to a finite algebraic extension  $K(W)_{\sigma+}$  of  $K(V)_{\sigma+}$ .

If a is a tuple in  $\mathcal{U}$ , we let  $K(a)_{\sigma+} = K(\sigma^i(a) \mid i \geq 0)$ ; if  $\sigma(K) = K$ , then  $K(a)_{\sigma} = K(\sigma^i(a) \mid i \in \mathbb{Z})$  as in section 3. We say that a tuple a is a generic of the difference variety V over K if  $a \in V$  and the natural specialisation map  $K[V]_{\sigma+} \to K(a)_{\sigma+}$  is injective.

We will often use the following result (see [C], 5.23.18): if  $K \subset L \subset M$  are difference fields, with M finitely generated over K (as a difference field), then L is finitely generated over K.

**4.7. Internality**. The definable closure of a difference field K, dcl(K), is usually much larger than the perfect closure of K. The notion of internality to  $Fix(\sigma)$  therefore does not have a natural geometric interpretation. The right notion to consider is the one of qf-internality: one replaces dcl by "difference field generated by".

**Definition.** Let K be a difference field, a a tuple in  $\mathcal{U}$ , such that  $K(a)_{\sigma}/K$  is regular, let V be the difference locus of a over K (i.e., the smallest  $\sigma$ -closed set containing a and defined over K), and let  $\mathcal{S}$  be a set of types with algebraically closed base, which is closed under conjugation by  $\operatorname{Aut}(\mathcal{U}/K)$ . We say that tp(a/K) is qf-internal to  $\mathcal{S}$ , or qf- $\mathcal{S}$ -internal, if for some  $L = \operatorname{acl}(L)$  containing K and free from  $K(a)_{\sigma}$  over K, and some tuple b of realisations of types in  $\mathcal{S}$ ,  $a \in L(b)_{\sigma}$ . In that case, we also say that the extension  $K(a)_{\sigma}/K$ , and the difference variety V are qf-internal to  $\mathcal{S}$ , or qf- $\mathcal{S}$ -internal. (For the difference variety, we should really speak of "generic" qf-internality). And similarly we will speak of almost- $\mathcal{S}$ -internal extensions, and almost- $\mathcal{S}$ -internal difference varieties. Let  $\tau = \sigma^n \operatorname{Frob}^m$  for some  $(m, n) \in I$  (see 3.5). If  $\mathcal{S}$  consists of all types realised in  $\operatorname{Fix}(\tau)$ , then we will also speak of qf- $\operatorname{Fix}(\tau)$ -internality, or qf-internality to  $\operatorname{Fix}(\tau)$ .

**4.8. Internality to fixed fields.** Let  $\tau = \sigma^n \operatorname{Frob}^m$  for some  $(m,n) \in I$ , and assume that tp(a/K) is qf-internal to  $\operatorname{Fix}(\tau)$ . Then one can find L and b as above, such that  $L(a)_{\sigma} = L(b)_{\sigma}$ : take b such that  $L(a)_{\sigma} \cap \operatorname{Fix}(\tau) = (\operatorname{Fix}(\tau) \cap L)(b)_{\sigma}$ ; since  $L(a)_{\sigma}$  and  $\operatorname{Fix}(\tau)$  are linearly disjoint over their intersection, it follows that  $L(a)_{\sigma}$  and  $L\operatorname{Fix}(\tau)$  are linearly disjoint over  $L(b)_{\sigma}$ , and therefore  $L(a)_{\sigma} = L(b)_{\sigma}$ . Note that if  $m \geq 0$ , then  $L(b)_{\sigma} = L(b)_{\sigma+1}$  and therefore also  $L(a)_{\sigma+1} = L(a)_{\sigma}$ . If m < 0, then  $L(b)_{\sigma}$  is the perfect hull of  $L(b)_{\sigma+1}$ , and this implies that, choosing b so

that  $(L \cap \text{Fix}(\tau))(b)_{\sigma+} = L(a)_{\sigma+} \cap \text{Fix}(\tau)$ , we have  $L(a)_{\sigma+} \supseteq L(b)_{\sigma+} \supseteq L(\sigma^{j}(a))_{\sigma+}$  for some  $j \ge 0$ . If  $\bar{W}$  is the algebraic locus of b over L, then there is a purely inseparable map  $\pi$  such that  $\pi(V)$  is  $\sigma$ -birationally isomorphic (over L) to  $\bar{W}(\text{Fix}(\tau))$ , the difference variety defined by  $x \in \bar{W} \wedge \tau(x) = x$ .

- **4.9. Facts.** Let  $\tau = \sigma^m \operatorname{Frob}^n$  for some  $(m, n) \in I$ , let  $\ell \geq 1$  be an integer. Let K be a difference subfield of  $\mathcal{U}$ , and K' a difference field isomorphic to K by an isomorphism  $\varphi_0$ , and  $\mathcal{U}'$  an existentially closed difference field containing K'. We will work in the  $\sigma^{\ell}$ -difference field  $\mathcal{U}[\ell] = (\mathcal{U}, \sigma^{\ell})$ , and denote by  $qftp(-)[\ell]$ ,  $tp(-)[\ell]$ ,  $acl_{\sigma^{\ell}}$  the quantifier-free types, types, and algebraic closure respectively, with superscript  $\mathcal{U}$  or  $\mathcal{U}'$  if necessary. We will use the following results:
  - (1) ([CHP], 1.12) Assume that  $a \in K^{alg}$ , and let a' be a field-conjugate of a over K. Then qftp(a/K)[m] = qftp(a'/K)[m] for some  $m \ge 1$ .
  - (2) ([CH2], 2.9) Let  $a \in \mathcal{U}$ ,  $a' \in \mathcal{U}$ , and assume that there is an isomorphism of  $\sigma^{\ell}$ -difference fields between  $K(a)_{\sigma^{\ell}}$  and  $K'(a')_{\sigma^{\ell}}$  which extends  $\varphi_0$  and sends a to a'. Then  $tp^{\mathcal{U}}(a/K)$  is qf-Fix $(\tau)$ -internal if and only if  $tp^{\mathcal{U}'}(a'/K')[\ell]$  is qf-Fix $(\tau^{\ell})$ -internal.
  - (3) ([CH2], 2.11). Let a and a' be as in (2). Then  $tp^{\mathcal{U}}(a/K)$  is one-based if and only if  $tp^{\mathcal{U}'}(a'/K')[\ell]$  is one-based.

Conditions on the set S. We fix a set S of types of SU-rank 1 with algebraically closed base, which is closed under  $\operatorname{Aut}(\mathcal{U})$ -conjugation. If  $p \in S$  is not one-based, then for some  $\tau$  as above, p is non-orthogonal to any non-algebraic type realised in  $\operatorname{Fix}(\tau)$ . If S consists only of non-one-based types, then we do not impose any additional condition.

If S contains some one-based type, for convenience we will impose that S contains all one-based types of SU-rank 1. By abuse of language, we will speak about almost-S-internality even when working in  $\mathcal{U}[\ell]$ .

**4.10.** Maximal almost-S-internal quotient of a variety. Let V be a difference variety of finite order defined over the difference subfield K of  $\mathcal{U}$ , and S as above. Then V has a maximal almost-S-internal quotient  $V^{\#}$ . Furthermore, if W is a finite cover of V, then  $W^{\#}$  is a finite cover of  $V^{\#}$  via a map  $\sigma^{-n}f$  for some integer n and tuple f of rational difference functions on  $W^{\#}$ .

*Proof.* Let a be a generic of V over K, and let  $A = \operatorname{acl}(A) \subseteq \operatorname{acl}(Ka)$  be maximal realising an almost-S-internal-type over K. Let  $A_0 = A \cap K(a)_{\sigma+}$  and let b be a finite tuple such that  $A_0 = K(b)_{\sigma+}$ . Then tp(b/K) is almost-S-internal, which translates into: if  $V^{\#}$  is the difference variety of which b is a generic, then  $V^{\#}$  is almost-S-internal, is a quotient of V by a difference rational map, and is maximal such (up to birational difference equivalence). This is immediate observing that  $K(a)_{\sigma+} = K(V)_{\sigma+}$ , and  $K(b)_{\sigma+} = K(V^{\#})_{\sigma+}$ .

<sup>&</sup>lt;sup>4</sup>Because Fix( $\tau$ ) is stably embedded (see [CHP] 7.1), it follows that  $\bar{W}$  is defined over  $L \cap \text{Fix}(\tau)$ . If  $m \geq 0$  then j = 0 and one does not need the map  $\pi$ .

As in the proof of 4.3, the statement about W and  $W^{\#}$  reduces to showing that  $A = A_0^{alg}$ . First note that because b realises an almost-S-internal type over K, we have  $K(b)_{\sigma} \subset A_0^{alg}$ .

As in 4.3, we argue that if  $c \in A$  and c' is a field conjugate of c over  $K(b)_{\sigma}$ , then tp(c'/K) is almost-S-internal because  $c' \in A$ . Hence if c is such that  $A = K(c)^{alg}$ , then c and c' are equi-algebraic over K; it then follows that the code d of the set of field conjugates of c over  $K(b)_{\sigma}$  is equi-algebraic with c over K, and therefore that  $A = A_0^{alg}$ : if the characteristic is 0, then  $d \in K(b)_{\sigma}$ , and if the characteristic is p > 0, some  $p^m$ -power of d is in  $K(b)_{\sigma}$ .

**4.11. Remark**. A similar statement could be obtained with maximal qf-S-internal quotients instead: replace  $A_0$  by its maximal subset  $A_1$  realising a qf-S-internal type over K; then  $A_0/A_1$  is algebraic.

Observe also the following direct consequence of the proof of 4.10: let a be a tuple in  $\mathcal{U}$ , K a difference subfield of  $\mathcal{U}$  and A the maximal subset of  $\operatorname{acl}(Ka)$  realising an almost- $\mathcal{S}$ -internal type over  $K^{alg}$ . If  $A = A_0 \cap K(a)_{\sigma+}$ , then  $A = A_0^{alg}$ .

**4.12.** Maximal almost-S-internal fiber of a variety. Let V be a difference variety of finite order defined over the difference field K. Then, up to composition with a power of Frobenius, V has a unique minimal  $\sigma$ -rational quotient  $V^{\flat}$  with the property that the generic fiber of the quotient map is irreducible and almost-S-internal. Furthermore, if W is a finite cover of V, then  $W^{\flat}$  is a finite cover of  $V^{\flat}$  via a map f, for some integer n and tuple f of rational difference functions on  $W^{\flat}$ .

*Proof.* Let a be a generic of V over K, and  $A = \operatorname{acl}(A) \subset \operatorname{acl}(Ka)$  be minimal such that tp(a/A) is almost-S-internal, let  $A_0 = A \cap K(a)_{\sigma+}$ , and let c be a finite tuple such that  $A_0 = K(c)_{\sigma+}$ . We now let  $V^{\flat}$  be the difference variety defined over K of which c is a generic and  $f: V \to V^{\flat}$  the map induced by the inclusion  $K(c)_{\sigma+} \subseteq K(a)_{\sigma+}$ .

As in the proof of 4.4, the assertion about W and  $W^{\flat}$  reduces to showing that  $A = A_0^{alg}$ . Let  $b \in A_0^s$  be such that  $A = K(b)^{alg}$ , and let  $b_2, \ldots, b_m$  be the field-conjugates of  $b = b_1$  over  $K(a)_{\sigma}$ . By 4.9, there is  $\ell \geq 1$  such that, for each  $i \geq 2$ , there is a  $\sigma^{\ell}$ - $K(a)_{\sigma}$ -isomorphism  $f_i : K(a)_{\sigma}(b)_{\sigma^{\ell}} \to K(a)_{\sigma}(b_i)_{\sigma^{\ell}}$  sending b to  $b_i$ . Since  $\sigma(b) \in K(b)^{alg}$ , we know that  $qftp(a,\ldots,\sigma^{\ell-1}(a)/K(b)_{\sigma^{\ell}})[\ell]$  is almost- $\mathcal{S}$ -internal, and therefore so are the types  $tp(a,\ldots,\sigma^{\ell-1}(a)/K(\sigma^j(b_i)_{\sigma^{\ell}}))[\ell]$  for  $0 \leq j < \ell$  (it is clear for j=0; then apply powers of  $\sigma$  to get the result for the other values of j). Letting  $B = \bigcap_{i=1}^m \bigcap_{j=0}^{\ell-1} \operatorname{acl}_{\sigma^{\ell}}(K\sigma^j(b_i))$  and noting that  $B = \sigma(B)$ , 4.9 and 2.4 imply that tp(a/B) is almost- $\mathcal{S}$ -internal. The minimality of A and the fact that  $b_1 \in A$  now imply A = B. It follows that all tuples  $b_i$  belong to A, since  $tr.deg(K(b_i)/K) = tr.deg(K(b)/K)$ . Hence, if d is the tuple encoding the set  $\{b_1,\ldots,b_m\}$ , then  $K(d)^{alg} = K(b)^{alg}$  and  $tp(a/K(d)_{\sigma})$  is almost- $\mathcal{S}$ -internal. For some  $n, m \geq 0$  we then have  $\sigma^n(d^{p^m}) \in K(a)_{\sigma^+}$ , which shows  $A = A_0^{alg}$ .

- **4.13. Remark**. The proof gives the following: let a be a tuple in  $\mathcal{U}$ , K a difference subfield of  $\mathcal{U}$  and A an algebraically closed difference subfield of  $\operatorname{acl}(Ka)$  such that  $\operatorname{tp}(a/A)$  is almost- $\mathcal{S}$ -internal. If  $A_0 = A \cap K(a)_{\sigma+}$  then  $\operatorname{tp}(a/A_0)$  is almost- $\mathcal{S}$ -internal.
- **4.14. Descent of difference varieties.** The main application of our results are given by Theorems 4.14 and 4.15. Theorem 4.15 is an almost optimal generalisation of Theorem 3.3 of [CH2].

**Theorem.** Let  $K_i$ , i = 1, 2, be difference subfields of  $\mathcal{U}$  with intersection k, and  $V_i$  difference varieties of finite order defined over  $K_i$ , and assume that  $k^{alg} = K_1^{alg} \cap K_2^{alg}$ . Assume that there is a  $\sigma$ -rational dominant  $f: V_1 \to V_2$  defined over  $(K_1K_2)^{alg}$ , that  $tp(K_2/k)$  is almost- $\mathcal{S}$ -internal and that  $K_2$  is a regular extension of k. Then there is a dominant map  $g: V_2 \to V_3$ , with  $V_3$  a difference variety defined over k, such that the generic fiber of g is almost- $\mathcal{S}$ -internal.

Proof. Let  $a_1$  be a generic of  $V_1$  over  $K_1K_2$ , and  $a_2 = f(a_1)$ . Letting  $b_1 = K_1$  and  $b_2 = K_2$ , applying 2.12 and using 4.12, we obtain  $e \in K_2(a_2)_{\sigma}$  such that  $k(e)_{\sigma}$  and  $K_2$  are free over k, and  $tp(a_2/k(e)_{\sigma})$  is almost- $\mathcal{S}$ -internal. Moreover,  $k(e)_{\sigma}$ , being a subfield of  $K_2(a_2)_{\sigma}$ , is a regular extension of k and is therefore linearly disjoint from  $K_2$  over k; we may assume that  $k(e)_{\sigma+} = K_2(a_2)_{\sigma+} \cap k(e)_{\sigma}^{alg}$ . If  $V_3$  is the difference variety of which e is a generic (over  $K_2$ ), then  $V_3$  is defined over k, and the inclusion  $K_2(e)_{\sigma+} \subset K_2(a_2)_{\sigma+}$  gives a dominant rational difference map  $g: V_2 \to V_3$  (defined over  $K_2$ ) such that  $g(a_2) = e$ , and with generic fiber almost- $\mathcal{S}$ -internal.

**4.15. Theorem.** Let  $K_1, K_2$  be fields intersecting in k and with algebraic closures intersecting in  $k^{alg}$ ; for i=1,2, let  $V_i$  be an absolutely irreducible variety and  $\phi_i: V_i \to V_i$  a dominant rational map defined over  $K_i$ . Assume that  $K_2$  is a regular extension of k and that there are an integer  $r \geq 1$  and a dominant rational map  $f: V_1 \to V_2$  such that  $f \circ \phi_1 = \phi_2^{(r)} \circ f$ , where  $\phi_2^{(r)}$  denotes the function obtained by iterating r times  $\phi_2$ . Then there is a variety  $V_0$  and a dominant rational map  $\phi_0: V_0 \to V_0$ , all defined over k, a dominant map  $g: V_2 \to V_0$  such that  $g \circ \phi_2 = \phi_0 \circ g$ , and  $\deg(\phi_0) = \deg(\phi_2)$ .

*Proof.* Observe that the rational map f will be defined over  $(K_1K_2)^{alg}$ , because this is a statement about algebraic varieties and rational morphisms.

Let  $a_1$  be a generic of  $V_1$  over  $K_1K_2$ , and let  $a_2 = f(a_1)$ . Then  $a_2$  is a generic of  $V_2$  over  $K_1K_2$ . We fix an existentially closed difference field  $(\mathcal{U}, \sigma)$  containing  $K_2(a_2)$  and such that  $\sigma$  is the identity on  $K_2$  and  $\sigma(a_2) = \phi_2(a_2)$ . We fix another existentially closed field  $(\mathcal{U}', \tau)$  containing  $K_1K_2(a_1)$ , such that  $\tau$  is the identity on  $K_1K_2$ , and  $\tau(a_1) = \phi_1(a_1)$ . Note that  $\tau$  and  $\sigma^r$  agree on  $K_2(a_2)$ . By abuse of notation, we let  $\mathcal{S}$  denote the set of non-algebraic types of rank 1 realised in  $Fix(\sigma)$  when working in  $\mathcal{U}$ , in  $Fix(\tau)$  when working in  $\mathcal{U}'$ , and in  $Fix(\sigma^r)$  when working in  $\mathcal{U}[r]$ .

Working in  $\mathcal{U}'$ , by 2.12, there is an algebraically closed  $\tau$ -difference field E contained in  $K_2(a_2)^{alg}$  and free from  $K_2$  over k, such that  $tp^{\mathcal{U}',\tau}(K_2a_2/E)$  is almost- $\mathcal{S}$ -internal. By 4.13 and 4.9, we obtain that  $tp^{\mathcal{U}}(K_2a_2/E \cap K_2(a_2))[r]$  is almost- $\mathcal{S}$ -internal, and therefore so is  $tp^{\mathcal{U}}(K_2a_2/E)[r]$ . Applying  $\sigma^i$  for  $i \geq 0$ , we get that  $tp(K_2\sigma^i(a_2)/\sigma^i(E))[r]$  is almost- $\mathcal{S}$ -internal, and because  $a_2 \in K_2(\sigma^i(a_2))^{alg}$  so is  $tp(a_2/\sigma^i(E))[r]$ .

Observe now that because  $E = (E \cap K_2(a_2))^{alg}$  and  $\tau$  agrees with  $\sigma^r$  on  $K_2(a_2)$ , we have  $\sigma^r(E) = E$ . Hence, by Lemma 2.4, we may replace E by  $\bigcap_i \sigma^i(E)$  and assume that  $\sigma(E) = E$ . We now reason as in 4.12 to show that if  $a_3$  is such that  $K_2(a_2) \cap E = k(a_3)$ , then  $tp(a_2/k(a_3)_{\sigma})$  is almost-S-internal. Note that as  $K_2(a_2)$  and E are closed under  $\sigma$ , so is  $k(a_3)$ . Hence,  $\sigma(a_3) \in k(a_3)$ . As  $K_2$  is a regular extension of k, and  $k(a_3) \subset E$ , it follows that  $k(a_3)_{\sigma}$  and  $K_2$  are linearly disjoint over k. Letting  $V_0$  be the algebraic locus of  $a_3$  over k, and  $\phi_0$  the rational endomorphism of  $V_0$  such that  $\sigma(a_3) = \phi_0(a_3)$ , we get the desired  $(V_0, \phi_0)$ . The rational map g

is the one given by the inclusion  $K_2(a_3) \subset K_2(a_2)$ .

Remains the assertion about the degrees of the maps. By Lemma 1.11 of [CH2], we have  $1 = \operatorname{ld}(a_2/K_2(a_3)_{\sigma}) = \operatorname{ild}(a_2/K_2(a_3)_{\sigma})$ , which implies  $\operatorname{deg}(\phi_2) = \operatorname{deg}(\phi_0)$  and finishes the proof.

#### 4.16. Remarks.

- (1) As stated, the theorem says nothing when  $deg(\phi_2) = 1$ , since one can take  $V_0$  of dimension 0.
- (2) The assertion on the degrees of the map  $\phi_2$  and  $\phi_0$  is weaker than the assertion that  $tp(a_2/k(a_3)_{\sigma})$  is almost-S-internal. Note that for instance if  $c \in K_2$  is a finite tuple which generates over k the field of definition of  $(V_2, \phi_2)$ , then  $c \in k(a_2)_{\sigma}$ , and therefore  $tp(c, a_2/k(a_3)_{\sigma})$  is almost-S-internal. This should have consequences on the data  $(V_2, \phi_2)$ .
- (3) One can in fact show that the generic fiber of g is qf-Fix( $\sigma$ )-internal. The proof goes as follows: we know that there is some  $a_4 \in K_2(a_2)$  such that  $tp(a_4/k(a_3))$  is qf-Fix( $\sigma$ )-internal, and  $a_2 \in K_2(a_4)^{alg}$ . Observe that because  $\mathrm{ld}(a_2/K_2) = \mathrm{ld}(a_4/K_2)(=1)$ , the field  $K_2(a_2)_{\sigma}$  is a finite extension of  $K_2(a_4)_{\sigma}$ . Let L be a difference field containing  $k(a_3)_{\sigma}$ , linearly disjoint from  $K_2(a_2)_{\sigma}$  over  $k(a_3)_{\sigma}$ , and such that  $L(a_4)_{\sigma} = L(b)$  for some tuple b in Fix( $\sigma$ ). Enlarging L if necessary, we will assume that L is algebraically closed and that Fix( $\sigma$ )  $\cap$  L has absolute Galois group isomorphic to  $\hat{\mathbb{Z}}$ , so that Fix( $\sigma$ )L contains the algebraic closure of Fix( $\sigma$ ). It then follows by Lemma 4.2 of [CHS] that  $L(a_2) \subset L$ Fix( $\sigma$ ), which shows that  $tp(a_2/k(a_3))$  is qf-Fix( $\sigma$ )-internal.

# 5 Appendix

**5.1. Proposition**. Let E and B be algebraically closed subsets of M, b a tuple in M. Assume that  $SU(B/B \cap E) < \omega$ , that tp(b/B) is one-based, and that  $B \cap E = \operatorname{acl}(Bb) \cap E$ . Then b is independent from E over B.

*Proof.* Assume the result false, and take a counterexample with  $r = SU(B/B \cap E) - SU(B/E)$  minimal among all such (B, E, b). We may assume that  $B \cap E = \operatorname{acl}(\emptyset)$ , and  $E = \overline{\operatorname{Cb}}(Bb/E)$ . Since tp(b/B) is one-based,  $\operatorname{acl}(Bb) \cap \operatorname{acl}(BE) \neq B$ , and we may therefore assume that  $b \in \operatorname{acl}(BE)$ .

If r = 0, then  $B \downarrow E$ , so that  $\overline{\text{Cb}}(Bb/E)$  realises a one-based type over  $B \cap E$  (by 1.9 with S the set of one-based types with algebraically closed base), and therefore  $Bb \downarrow E$ . This contradicts  $b \in \text{acl}(BE)$ . Hence r > 0.

Let  $A = \overline{\text{Cb}}(B/E)$ . We may then assume that E and b are equi-algebraic over AB: by 1.9 tp(E/A) is one-based, and if  $D = \operatorname{acl}(ABb) \cap E$ , then  $b \in \operatorname{acl}(BD)$ . Replace E by D. Reasoning as in Step 3 of 2.1, there is  $m \geq 2$ , and E-independent realisations  $(B_1b_1), \ldots, (B_mb_m)$  of tp(Bb/E) with  $A \subset \operatorname{acl}(B_1 \ldots B_m)$  and  $SU(B_m/B_1 \ldots B_{m-1}) > SU(B/A)$ . The induction hypothesis implies  $b_1 \downarrow_{B_1} B_2 \ldots B_m$  (since  $\operatorname{acl}(B_1b_1) \cap \operatorname{acl}(B_2 \ldots B_m) \subseteq \operatorname{acl}(Bb) \cap E = \operatorname{acl}(\emptyset)$ ). Similarly  $b_m \downarrow_{B_m} B_1 \ldots B_{m-1}$ , and therefore  $B_1 \downarrow_{B_2 \ldots B_m} b_m$ .

If E = A, then  $b \in \operatorname{acl}(AB)$ , and  $b_1 \in \operatorname{acl}(B_1 \dots B_m)$ ; by the above we get  $b_1 \in B_1$  which is absurd.

If  $E \neq A$ , then  $E \nsubseteq \operatorname{acl}(B_1 \dots B_m)$  because  $E \downarrow_A B$ , and each  $b_i$  is equi-algebraic with E over  $B_1 \dots B_m$ . Hence  $b_1$  and  $b_m$  are equialgebraic over  $B_1 \dots B_m$ . However  $SU(B_1/B_2 \dots B_m b_m) = SU(B_1/B_2 \dots B_m) > SU(B/E)$ . The induction hypothesis, together with the fact that  $\operatorname{acl}(B_1b_1) \cap \operatorname{acl}(B_2 \dots B_m b_m) \subseteq \operatorname{acl}(B_1b_1) \cap E = \operatorname{acl}(\emptyset)$ , gives  $b_1 \downarrow_{B_1} B_2 \dots B_m b_m$ , a contradiction.

**Remark**. This result does not hold when  $SU(B/B\cap E)$  is infinite. Here is a counterexample for T a completion of ACFA in characteristic 0. Let a,b,c be generics and independent over  $\mathbb{Q}^{alg}$ , and consider d=ac+b, and  $e=\sigma(b)-b^2$ . Then  $\overline{\mathrm{Cb}}(c,d/a,b)=\mathrm{acl}(\mathbb{Q},a,b)$ . Moreover, tp(b/e) is one-based (by example 6.1 of [CH]) and has SU-rank 1. One also has  $\mathrm{acl}(a,b)\cap\mathrm{acl}(c,d)=\mathbb{Q}^{alg}=\mathrm{acl}(\emptyset)$ . Take for (B,b,E) the triple  $(\mathbb{Q}(a,e)^{alg}_{\sigma},b,\mathbb{Q}(c,d)^{alg}_{\sigma})$ .

**5.2. Proposition**. Let tp(a/A) be a one-based type of SU-rank  $\omega^{\alpha}$  for some ordinal  $\alpha$ , with  $SU(A) < \omega$ ,  $A = \operatorname{acl}(A)$ , and consider the class  $\mathcal{P}$  of all types of SU-rank  $\omega^{\alpha}$  with algebraically closed base, which are non-orthogonal to tp(a/A). Then  $\mathcal{P}$  contains a type q whose base C is contained in all bases of elements of  $\mathcal{P}$ . If tp(a/A) has SU-rank 1 and is trivial, then there is c such that SU(c/C) = 1 and  $a \in \operatorname{acl}(Ac)$ .

Proof. Assume that  $tp(b/B) \in \mathcal{P}$ . Moving a, we may assume that  $a \downarrow_A B$ . By Lemma 1.13, there are realisations  $a_1, \ldots, a_n$  of tp(a/A) which are independent from Ba over A, and realisations  $b_1, \ldots, b_m$  of tp(b/B) which are independent from A over B, such that  $SU(a/ABa_1 \ldots a_nb_1 \ldots b_m) < \omega^{\alpha}$ . Choose such m, n minimal. Then  $SU(a_1, \ldots, a_n, b_1, \ldots, b_m/AB) = \omega^{\alpha}(n+m)$ , and  $acl(Aa_1 \ldots a_n) \cap acl(Bb_1 \ldots b_m) = A \cap B = C$ . By Proposition 5.1, we know that  $acl(Aaa_1 \ldots a_n) \cap acl(Bb_1 \ldots b_m)$  contains some element  $d \notin C$ . The usual routine arguments then give  $tp(d/C) \not\perp tp(a/A)$  and  $SU(d/C) = \omega^{\alpha}$ .

Let  $p_1, p_2 \in \mathcal{P}$ , with bases  $A_1, A_2$  contained in A. Because  $SU(p) = \omega^{\alpha}$ , the type p has weight 1. Hence the inclusions  $A_1, A_2 \subseteq A$  and the non-orthogonality of  $p_1, p_2$  to p imply  $p_1 \not\perp p_2$ .

Thus the set of bases of types in  $\mathcal{P}$  is closed under intersection, and has a smallest element, since one cannot have an infinite decreasing sequence of algebraically closed sets of finite SUrank.

The last assertion follows immediately from triviality, as non-orthogonality then implies non-almost-orthogonality.

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