Graph and wreath products of cellular automata

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Abstract

We prove that the set of subgroups of the automorphism group of a two-sided full shift is closed under graph products. We introduce the notion of an Aithful group action, and show that when A is a finite abelian group and G is a group of cellular automata whose action is Aithful, the wreath product $A \wr G$ embeds in the automorphism group of a full shift. We show that all free abelian groups and free groups admit Aithful cellular automata actions. In the one-sided case, we prove variants of these result with reasonable alphabet blow-ups.

1 Introduction

Groups generated by reversible (two-sided) cellular automata, i.e. subgroups of automorphism groups of ($\mathbb{Z}$-)full shifts, are an interesting and not very well-understood class of groups $\mathcal{G}$. It is known [14] that automorphism groups of full shifts embed in each other, so $\mathcal{G}$ does not depend on the alphabet of the full shift.

For this paper, the most relevant facts known about $\mathcal{G}$ are the following:

1. $\mathcal{G}$ is closed under countable direct sums and free products, [19]
2. $\mathcal{G}$ contains all graph groups, [14]
3. $\mathcal{G}$ contains the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$, [21]

Some other known facts are that $\mathcal{G}$ is commensurability invariant [14, Proposition 3.1], groups in $\mathcal{G}$ are residually finite (RF) [5], finitely generated (f.g.) groups in $\mathcal{G}$ have word problem in co-NP and this problem is co-NP-complete for some f.g. $G \in \mathcal{G}$ [20], the locally finite (LF) groups in $\mathcal{G}$ are exactly the RF countable ones [14], $\mathcal{G}$ has an f.g. subgroup with no free subgroups which is not virtually solvable (i.e. the Tits alternative fails) [22, 21], f.g. groups in $\mathcal{G}$ may have undecidable torsion problem [21],[1], and there is an f.g.-universal f.g. group in $\mathcal{G}$ (an f.g. group which contains a copy of every f.g. group in $\mathcal{G}$) [21].

From these facts, many additional results follow, for instance the closure properties imply that $\mathcal{G}$ contains all finite groups (first proved in [13]), thus the free product of all finite groups (first proved in [1]), thus $\mathbb{Z}_2 * \mathbb{Z}_2$, thus $\mathbb{Z}$, thus all countable free groups (first proved in [5]) and finitely-generated abelian groups.

[1]Another proof, obtained earlier, is planned for the journal version of [3].

Graph groups and commensurability invariance imply that fundamental groups of 2-manifolds are in $\mathcal{G}$ [14]. It also follows that $\mathcal{G}$ is not quasi-isometry invariant since $F_2 \times F_2$ is quasi-isometric to a non-RF group [7] and is a graph group.

To our knowledge, residual finiteness and the complexity restriction of the word problem are the only known restrictions for f.g. groups to be in $\mathcal{G}$.

Our aim in this paper is to clarify the situation in the "tame end"[2] of the complexity spectrum of $\mathcal{G}$, by clarifying the three items listed above. First, we combine the first two items, graph groups and closure under direct and free products, into a single closure property:

Theorem. *The class $\mathcal{G}$ is closed under countable graph products.*

The finite case of the construction is similar to [19], mixing what was done in the direct and free product cases. Kim and Roush proved in [14] the cases of finite graph products where the node groups are finite or infinite cyclic. Some new technical difficulties arise here, mainly since the individual generators cannot understand the global picture.

We also generalize the third item, the lamplighter group.

Theorem. *If A is a finite abelian group and $G \in \mathcal{G}$ acts Aithfully on a full shift by automorphisms, then $A \wr G \in \mathcal{G}$.*

Here, "Aithfulness" means roughly that the trace subshift (what you can see at the origin along G-orbits) has no cancellation w.r.t. the group A, although we allow a bit more freedom than this (see Definition 1). We complement this result with many examples of groups that act Aithfully by automorphisms, obtaining for example that the groups $A \wr \mathbb{Z}^d$ and $A \wr F_d$ are in $\mathcal{G}$ for any abelian A and $d \in \mathbb{N}$.

While Aithfulness is the precisely the notion needed for wreath products, in all our concrete examples we deduce Aithfulness from the stronger property of "strong faithfulness". This property is implied by having a sunny-side-up subquotient, and if G is finitely-generated it implies that some point visits some clopen set just once in a horoball (see Definition 2).

Similar constructions work in the more restricted setting of one-sided automorphism groups, though (at least without modification) we obtain somewhat weaker statements. See [4] for basic information about these groups. Let $\mathcal{G}'_n$ be the class of subgroups of $\mathrm{Aut}(\Sigma^{\mathbb{N}})$ where $|\Sigma| = n$, and let $\mathcal{G}'_\infty = \bigcup_n \mathcal{G}'_n$. We obtain the following results:

Theorem. *Any finite graph product of groups in $\mathcal{G}'_n$ is in $\mathcal{G}'_{n+1}$.*

Theorem. *If A is finite abelian and $G \in \mathcal{G}'_\infty$ is Aithful, then $A \wr G \in \mathcal{G}'_\infty$.*

Theorem. *If A is finite abelian and $n \in \mathbb{N}$, then $A \wr F_n, A \wr \mathbb{Z}^n \in \mathcal{G}'_\infty$.*

[2] "Dynamically", already f.g. subgroups of $F_2 \times F_2$ can be very wild. For example, they can have an undecidable conjugacy problem [17, 15] and can have arbitrarily badly distorted subgroups [18, 23]. However, both finite graph products and wreath products preserve polynomial-time decidability of the word problem (for the former, this is clear from the normal form [11]), which is atypical for groups of cellular automata if P $\neq$ NP. Also, the Tits alternative is preserved by a large class of graph products [2].

2 Definitions and conventions

We take $0 \in \mathbb{N}$, $\mathbb{Z}_+ = \mathbb{N} \setminus \{0\}$. Every finite set and group has the discrete topology, and the function space A^B has the compact-open topology, which is always also the product topology.

Words are 0-indexed and for a word $u \in \Sigma^*$ write u^R for its reverse $u_i = u_{|u|-1-i}$, write concatenation of words u, v as $u \cdot v$ or simply uv, write u^0 for the empty word and $u^{n+1} = uu^n$. For sets of words, $AB = \{uv \mid u \in A, v \in B\}$ and $A^{n+1} = AA^n$. Write w^* for $\{w^n \mid n \in \mathbb{N}\}$ and $A^* = \bigcup_{n \in \mathbb{N}} A^n$. A set of words $W \subset \Sigma^*$ is *mutually unbordered* if $\forall u, v \in W : uw = w'v \implies w = w' \vee |w| \geq u$. Write $u^{\mathbb{Z}}$ for the configuration x with $x_i = u_j$ where $j \equiv i \bmod n$. For $u, v \in \Sigma^*$ write

$$u \sqsubset v \iff \exists j \in \{0, \ldots, |v| - |u|\} : \forall i \in \{0, \ldots, |u| - 1\} : v_{j+i} = u_i.$$

A *(zero-dimensional discrete topological dynamical) G-system* is a pair (G, X) where G is a discrete group, X is a zero-dimensional compact metrizable space, and G acts on X by continuous maps. We denote the action of $g \in G$ on $x \in X$ by just gx. The elements $x \in X$ are called *points*. A *factor* of a system is another system (G, Y) such that there is a continuous surjection $f : X \to Y$ which intertwines the actions as $f(gx) = gf(x)$ for all $x \in X, g \in G$, and an *isomorphism* of dynamical systems is a factor map that has a factor map inverse. A *subsystem* is a closed G-invariant subset.

A *subshift* is a subsystem of the *full G-shift* A^G where G acts by $gx_h = x_{hg}$. Points of subshifts are also called *configurations*. Subshifts are, up to isomorphism, the systems where the action is *expansive*, meaning there exists $\varepsilon > 0$ such that $x \neq y \implies \exists g \in X : d(gx, gy) > \varepsilon$. We use these meanings of "subshift" rather interchangeably. A system is *faithful* if $(\forall x \in x : gx = x) \implies g = e$. A subshift $X \subset \Sigma^G$ is an *SFT* if there exists a clopen set $C \subset \Sigma^G$ such that $X = \bigcap_{g \in G} gC$, and *sofic* shifts are subshifts which are factors of SFTs. Of course full shifts are sofic.

The *(one-dimensional) full shift* is the $\mathbb{Z}$-full shift $\Sigma^{\mathbb{Z}}$, and for clarity we write its action as $\sigma(x)_i = x_{i+1}$. Its shift-commuting continuous self-maps are known as *endomorphisms* or *cellular automata (CA)*, and the ones that are injective (equivalently, have a left and right inverse) are called *automorphisms* or *reversible cellular automata (RCA)*.

Reversible cellular automata form a group denoted $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$, and we denote by $\mathcal{G}$ the set of isomorphism classes of subgroups $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ (which does not depend on Σ [14]). We also call groups in $\mathcal{G}$ *groups of cellular automata*. Sometimes we talk about the natural action of a group $G \in \mathcal{G}$, this means we assume G is represented in some way as a concrete group consisting of cellular automata on some full shift $A^{\mathbb{Z}}$ and the natural action is its defining action on $A^{\mathbb{Z}}$.

By the Curtis-Hedlund-Lyndon theorem, a cellular automaton $f : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ has a *local rule* $F : \Sigma^{2r+1} \to \Sigma$ such that $f(x)_i = F(x|_{\{i-r,\ldots,i+r\}})$ for all $x \in \Sigma^{\mathbb{Z}}$. The *minimal radius* is the minimal possible r, and for this r there obviously exist $u \in \Sigma^{2r}$ and $a, b \in \Sigma$ such that either $F(au) \neq F(bu)$ or $F(ua) \neq F(ub)$.

A cellular automaton is *one-sided* if we can pick $F : \Sigma^{r+1} \to \Sigma$ such that $f(x)_i = F(x|_{\{i,i+1,\ldots,i+r\}})$ for all $x \in \Sigma^{\mathbb{Z}}$. The shift-commuting endomorphisms of $\Sigma^{\mathbb{N}}$ under the $\mathbb{N}$-action by $\sigma(x)_i = x_{i+1}$ are in an obvious one-to-one correspondence with one-sided cellular automata on $\Sigma^{\mathbb{Z}}$, and $\mathrm{Aut}(\Sigma^{\mathbb{N}})$ can be seen as the subgroup of $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ consisting of reversible cellular automata f such that both f and f^{-1} are one-sided.

Usually, we define a cellular automaton by explaining what its action is on a dense set of configurations. This is easier than giving a local rule, because in the local rule we only see what happens in individual cells, while the behavior of the group of cells is usually highly coordinated and is really just a coding (application through a suitable conjugating map) of some natural action. On the other hand, when the rule is given by describing the action, it is not automatic that it is shift-commuting (but it is if we only discuss coordinates relatively). It is also not automatic that it extends to a continuous map, and for this one should check that it is uniformly continuous. This is discussed in more detail in [19]. On the third hand, when the rule is given by describing a natural action directly, reversibity is trivial.

The identity element of an abstract group G is $e = e_G$. Groups act from the left. Conjugation in a group is $g^h = h^{-1}gh$. Write $A \Subset B$ for $A \subset B \wedge |A| < \infty$. The *free group* on n free generators is $F_n = \langle g_1, g_2, \cdots, g_n \rangle$. The cyclic group with n elements is $\mathbb{Z}_n$, usually written additively. Group elements are sometimes called *cells*, especially when $i \in \mathbb{Z}$ and working with $\mathbb{Z}$-subshifts. The groups G and H are *commensurable* if they share a finite index subgroup (up to isomorphism).

The *(restricted) wreath product* of groups A and G, which we may assume disjoint apart from the identity, is the group $A \wr G$ with the presentation

$$\langle A, G \mid \forall g, h \in G : \forall a, b \in A : (g \neq h \implies [a^g, b^h] = e) \rangle$$

where it is understood that the relations of G and A also hold.

Graph products are defined as follows: Let $\Gamma = (V, E)$ be a graph, i.e. $E \subset \{\{u, v\} \mid u, v \in V, u \neq v\}$, $V = \{1, 2, ..., n\}$ or $V = \mathbb{N}$, and let $\overline{G} = (G_1, ..., G_n)$ or $\overline{G} = (G_i)_{i \in \mathbb{N}}$ be groups, which we assume disjoint apart from sharing the identity element. Then we write $\overline{G}^{\Gamma}$ for the corresponding *graph product*

$$\overline{G}^{\Gamma} = \langle G_i \mid \forall \{i, j\} \in E : [G_i, G_j] \rangle,$$

where $[G_i, G_j] = \{[a, b] \mid a \in G_i, b \in G_j\}$, where it is understood that the relations of the groups G_i also hold. We call the groups G_i the *node groups*.

3 Closure under graph products

Theorem 1. *The class $\mathcal{G}$ is closed under countable graph products.*

Proof. Consider first the case of a finite graph product $(G_1, G_2, \ldots, G_n)^{\Gamma}$. We may assume each group G_i acts faithfully by cellular automata on $\{0, 1\}^{\mathbb{Z}}$, and that the groups G_i are disjoint apart from sharing the identity.[3] Let $B = \{0, 1\}^2$, $\mathcal{S} = B^B$, i.e. the set of functions from B to B, and pick the alphabet $\Sigma = (B \times \{1, ..., n\}) \cup \mathcal{S}$. On B we pick some abelian group structure, for example through the natural identification $B \cong \mathbb{Z}_2^2$.

We can see a word over the alphabet $B \times \{1, \ldots, n\}$ as a triple (u, v, w) where $u, v \in \{0, 1\}^*$ and $w \in \{1, \ldots, n\}^*$ in an obvious way. If $g \in G_i$, then define a

[3] Alternatively, one can use self-embeddings of $\mathrm{Aut}(\{0, 1\}^{\mathbb{Z}})$ to literally obtain disjoint groups $G_i \leq \mathrm{Aut}(\{0, 1\}^{\mathbb{Z}})$.

map $\hat{g} \in \mathrm{Aut}(\Sigma^{\mathbb{Z}})$ as follows: On a dense set, a configuration in $\Sigma^{\mathbb{Z}}$ splits into maximal finite subwords of the form (u, v, i^n) (called *segments*), with $u, v \in \{0,1\}^n$, and into symbols in $\mathcal{S}$. We define the action on these words in such a way that this extends to an automorphism, and for this we use the conveyor belt construction: words of the form (u, v, i^n) are mapped to $(y_{[0,n-1]}, y_{[n,2n-1]}, i^n)$ where $y = g((uv^R)^{\mathbb{Z}})$. Symbols in $\mathcal{S}$ are never modified, and the $\{1, ..., n\}$-components of states in $B \times \{1, ..., n\}$ are also never modified.

If $i \neq j$, then $\hat{g}$ acts on maximal words (ua, vb, j^n) as follows, for $a, b \in \{0,1\}$: The words u and v are not modified. If there is an edge between i and j in Γ, $\hat{g}$ acts as identity. Suppose then there is no edge. If the symbols immediately to the right of the segment of the tape containing (ua, vb, j^n) are cd, then nothing is done unless $s \in \mathcal{S}$ and $d \in B \times \{i\}$. Finally, if $s \in \mathcal{S}$ and $(d, i) \in B \times \{i\}$, and if the symbol (d, i) is changed to (d', i) when $\hat{g}$ is applied in the segment on the right, then we change (a, b, j) to $((a, b) + s(d') - s(d), j)$ (using the group structure of B).

It is seen as in [19, 14] that this gives a well-defined automorphism. Briefly, the action on the segments over the subalphabet $B \times \{i\}$ mimic the periodic point action, and the modifications to the rightmost symbols of segments of the form $B \times \{j\}$ telescope to zero when the leftmost symbol of the neighboring $B \times \{i\}$ segment returns to its original value.

We can extend the action of $\hat{g}$ for $g \in \bigcup_i G_i$ in a natural way to finite words over Σ, by the same description above (so that this agrees with the action on configurations for example if we think of the finite word as being surrounded by tails entirely over $\mathcal{S}$). The maximal words of the form (u, v, i^n) which such a word splits into are again called its *segments*, and i is the *type* of the segment. As is standard, we write elements of $(G_1, \dots, G_n)^\Gamma$ as words over the alphabet $\bigcup_i G_i \setminus \{e\}$, and such a word is *reduced* if it cannot be made shorter by permuting commuting elements, joining subwords of the form $G_i G_i$ and removing occurrences of e. For a particular reduced word w we call the individual symbols in G_i its *syllables*, and we can associate to a reduced word $w \in (\bigcup_i G_i \setminus \{e\})^*$ a word $\tau(w) \in \{1, 2, \dots, n\}^*$ by only recording the types of syllables.

It is easy to check that if $g \in G_i, h \in G_j$, and $\{i, j\} \in E(\Gamma)$, then $\hat{g}$ and $\hat{h}$ commute, so $g \mapsto \hat{g}$ extends in a well-defined way to the graph product by the universal property (or the definition of the graph product). We now consider an arbitrary reduced word w and show that the corresponding automorphism is nontrivial, which proves that the action is faithful.

More precisely, we will prove the following. Let $w \in (\bigcup_i G_i \setminus \{e\})^*$ be any nonempty reduced word representing an element $g \in (G_1, \dots, G_n)^\Gamma$. Suppose either $\tau(w) = ui$, $i \in \{1, \dots, n\}$ where $\{\{k, i\} \mid k \in v\} \subset E(\Gamma)$, or we have $\tau(w) = vjui$ where $\{i, j\} \notin E(\Gamma)$ and $\{\{k, i\} \mid k \sqsubset v\} \subset E(\Gamma)$. Then there is a word $t \in \Sigma^*$ whose rightmost segment is of type i, such that some perturbation of the rightmost symbol effects some change in the leftmost symbol of $\hat{g}(t)$. In formulas, there exists $t' \in \Sigma^*$ whose rightmost segment is also of type i, with $|t| = |t'|$ and $t_{[0,|t|-2]} = t'_{[0,|t|-2]}$, such that $\hat{g}(t)_0 \neq \hat{g}(t')_0$.

The base case $\tau(w) = ui$ is obvious by the definition of the minimal radius r of a cellular automaton, by taking a word with just one suitably chosen segment of type i whose length is $r + 1$. The other case is proved by induction. Write $w = w'w''$ with $\tau(w') = vj, \tau(w'') = ui$, with the assumptions above, and let

g', g'' be the group elements corresponding to w', w''. Since vj is reduced, we can find a word t where the rightmost segment is of type i such that a suitable change in the rightmost symbol effects a change in the leftmost symbol of $\hat{g}'(t)$. Suppose the change in the first symbol happens when the rightmost symbol (a, j) of t is changed to (b, j).

As in the base case, we can find a word t' containing just one segment which is of type i, such that some change in the rightmost symbol of t' effects a change in the leftmost symbol of $\hat{g}''(t')$ from (d, i) to (d', i), for some $d, d' \in B^2$. Now recall that if we put an element $s \in \mathcal{S}$ between the words t and t', then as d changes to d', we change (a, j) to $(a + s(d') - s(d), j)$. Picking any $s \in \mathcal{S}$ such that $s(d') - s(d) = b - a$, we have $(a + s(d') - s(d), j) = (b, j)$.

Now let t'' be equal to t in all but the last coordinate. Suppose $t'_0 = (c, i)$ and assume that $\hat{g}''(t')_0 = (d, i)$ (as above). For the last symbol pick $t''_{|t''|-1} = t_{|t|-1} + s(c) - s(d)$. Now, the word $(\hat{g}'')^{-1}(t'') \cdot c \cdot t'$ has the desired property: we have

$$\hat{g}''((\hat{g}'')^{-1}(t'') \cdot c \cdot t') = t \cdot c \cdot \hat{g}''(t')$$

since the application of $\hat{g}''$ does not touch or read the last symbol of $(\hat{g}'')^{-1}(t'')$, and when t' changes to $\hat{g}''(t')$ we undo the modification in the last symbol of t we made in t''. Now, since the application of $\hat{g}''$ does not modify the segment of type j at the right end of $(\hat{g}'')^{-1}(t'')$ in other ways, a change in the rightmost symbol of $(\hat{g}'')^{-1}(t'') \cdot c \cdot t'$ changes the $\hat{g}''$-image $t \cdot c \cdot \hat{g}''(t')$ by changing the last symbol of t from (a, j) to $(a + s(d') - s(d), j) = (b, j)$. Then, an additional application of $\hat{g}'$ effects a change in the first symbol of $\hat{g}'(t)$, concluding the induction step, and thus the case of finite graph products.

Consider now the case of an infinite locally cofinite Γ, i.e. every node is neighbors with all but finitely many nodes. Pick any nontrivial alphabet Σ and for each i pick a set of words $W_i \subset \Sigma^i$ with $|W_i| = 4$, such that $\bigcup_i W_i$ is mutually unbordered. Replace the use of $B \times \{i\}$ with W_i, i.e. a segment of type i is redefined to be a maximal finite word from W_i^*, which will again be interpreted as a conveyor belt built from two binary words. We can use the same $\mathcal{S} = B^B$ (assume $\mathcal{S} \cap \Sigma = \emptyset$), and thus we use the alphabet $\Sigma \cup \mathcal{S}$.

For $g \in G_i$, the automorphism $\hat{g}$ applies the natural conveyor belt action of g in segments of type i (through uncoding the words in W_i to elements of B). If there is an element $s \in \mathcal{S}$ to the left of the segment, and a segment of type j immediately to the left of s, where $j \neq i, \{i, j\} \notin E$, then additionally permute the rightmost W_j-word of the segment of type j, as we did in the finite case. Since the graph is locally cofinite, the function is continuous, and thus we obtain an automorphism action. The proof that this gives a faithful action of the graph product is analogous to the finite case.

Now, consider an arbitrary Γ. In this case, we modify the previous construction further: Set $\mathcal{S} = B \times B^B \times B$, and after an element $\hat{g}$ is applied (for any $g \in G_i$), if the rightmost B-component of s was changed from d to d', as a side-effect we always change (a, b) in the leftmost B-component to $(a, b) + s'(d') - s'(d)$, where s' is the B^B-component of s. (This description does not make sense if $\hat{g}$ modifies both the leftmost and rightmost B-component, but we make sure this never happens.)

If $s \in \mathcal{S}$ has a segment of type i to the left of it, and a segment of type j to the right of it, then we "include" the rightmost B-component of s to the segment on the right (i.e. we think of it as joined in the beginning of the conveyor belt coded by the W_j^*-word) if

$$i \neq j \wedge (i > j \vee \{i, j\} \notin E).$$

We include the leftmost B-component of s to the segment on the left if

$$i \neq j \wedge (j > i \vee \{i, j\} \notin E).$$

When s has a segment on only one side, the B-symbol on the side of the segment is always included in the segment.

The inclusion of the bordermost B-symbols in the $\mathcal{S}$-symbol effectively puts a lower bound on the length of certain segments, so let us make the assumption that every nontrivial $g \in G_i$ acts on $B^{\mathbb{Z}}$ with minimal radius at least 10; this is possible because there exists a self-embedding of $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ such that all nontrivial automorphisms in the image have minimal radius at least 10.

Now again for $g \in G_i$, the map $\hat{g}$ simply applies g in segments of type i. To see that this is an automorphism, observe that a local rule can tell whether B-components of possible neighboring $\mathcal{S}$-symbols belong to the segment since when $g \in G_i$, this requires only knowing the sets W_k up to $k \leq i$. Note that as required, this never modifies both B-components of a symbol $s \in \mathcal{S}$, because that would mean the segments on both sides are of type i, and in this case neither B-component would be included in the segments on its side. By the assumption on the minimal radius of the maps $g \in G_i$, this gives a correct embedding of the graph product with an analogous proof as in the above cases. □

As a side-note, the above proof does not use the normal form theorem for graph products, so as a side-effect we prove the usual normal form theorem in the case where node groups act on a full shift.

Sometimes it is convenient to have the groups G_i act on some full shift $\Sigma^{\mathbb{Z}}$ other than $\{0,1\}^{\mathbb{Z}}$, and it is clear that one can modify the construction to use Σ^2 instead of $B = \{0,1\}^2$. The set B^B is simply replaced with $(\Sigma^2)^{\Sigma^2}$. We use this in the proof of Lemma 10. In the case of finite graph products, one can even use a different alphabet in each segment (replacing B^B by the set of functions from one square alphabet to another).

Besides graph products, another generalization of the free product is the free product with amalgamation. Here, there is also a simple normal form, and in fact using the same "segments of different types" construction it seems plausible that one can prove at least some restricted closure results for this operation.

Question 1. *If $G, H \in \mathcal{G}$, when is $G *_K H \in \mathcal{G}$?*

If we take all the node groups to be $\mathbb{Z}_2$ (resp. $\mathbb{Z}$), but instead of commutation relations we add relations of the form $(st)^m = e$, we obtain the family of Coxeter (resp. Artin) groups.

Question 2. *Which finitely-generated Coxeter (resp. Artin) groups are in $\mathcal{G}$?*

By a Theorem of Tits, all f.g. Coxeter groups are linear, and thus residually finite [8, 6]. They are automatic [?] so their word problem is decidable in polynomial time.

4 Aithful and strongly faithful actions

Recall that we restrict our systems to be discrete group actions on compact metrizable zero-dimensional spaces. This includes subshifts and the natural actions of automorphism groups of subshifts.

Definition 1. *Let X be a G-system and let A be a finite abelian group. We say the system X is* Aithful *if there exists a finite abelian group B and a continuous function $\theta : X \to \mathrm{Hom}(A, B)$ such that for any finite support map $f : G \to A$ we have*

$$(\forall x \in X : \sum_{g \in G} \theta(gx)(f(g)) = 0) \implies f = 0.$$

Recall that we consider $\mathrm{Hom}(A, B)$ with the discrete topology. While the function notation is convenient, a continuous function from a zero-dimensional compact metric space to a finite discrete set S just means a finite clopen partition of the space where the partition elements are indexed by S. Of course the definition could be applied in more general contexts, but we have not explored this, and A will always be finite abelian when we use the word Aithful.

Algebraically, when A is abelian (and written additively) maps $f : G \to A$ with finite support form an abelian group under elementwise sum. Analogously to group rings, we write this group as $A[G]$ (but there is typically no product) and write elements as (essentially) finite sums $\sum_{g \in G} f(g) \cdot g$. We can define a map $\phi = \phi_{\theta,B} : A[G] \to B^X$ by $f \mapsto (x \mapsto \sum_g \theta(gx)(f(g)))$, and this is clearly a homomorphism (for any choice of h). Aithfulness means that this homomorphism is injective for some choice of B and h.

Lemma 1. *We can always pick $B = A^d$ for some d in the definition of Aithfulness.*

Proof. The map h is determined by some finite clopen partition $P_1 \sqcup P_2 \sqcup \cdots \sqcup P_d$ of X. We can factor h as the pointwise composition $x \mapsto \theta''(x) \circ h'(x)$, where $\theta'|_{P_i} : P_i \to \mathrm{Hom}(A, A^d)$ is the constant map with image

$$a \mapsto (0, 0, ..., 0, \underset{i\text{th position}}{a}, 0, ...0) \tag{1}$$

and $\theta'' : X \to \mathrm{Hom}(A^d, B)$ is the constant map with image the homomorphism $\alpha : (0, 0, ..., 0, a, 0, ...0) \mapsto \theta(x)(a)$ for any $x \in P_i$, where a appears in the ith position. Since θ'' is a constant map and α is a homomorphism, a short calculation shows

$$\phi_{\theta,B}(f)(x) = \alpha(\phi_{\theta',A^d}(f)(x)),$$

and since we are aiming it is safe to drop θ'' and replace θ with θ'. ☐

The proof shows more: we can pick the homomorphism to be the one described in (1), with respect to some finite clopen partition.

Lemma 2. *Let X be a G-system and let A be a finite nontrivial abelian group. If the action is Aithful then there is a faithful subshift factor.*

Here, recall that group actions on compact metrizable zero-dimensional spaces are inverse limits of expansive ones, i.e. their subshift factors obtained by recording only the current partition element visited along an orbit, with respect to a finite partition.

A faithful subshift factor implies faithfulness, but not vice versa. For example the natural action of a residually finite group on its profinite completion is faithful but its subshift factors are finite.

Proof. We prove the contrapositive, so suppose no subshift factor is faithful. Let B be an abelian group and let $\theta : X \to \mathrm{Hom}(A, B)$ be a continuous. Then h factors through a subshift Y as $\theta = \theta'' \circ \theta'$ with $\theta' : X \to Y$, $\theta'' : Y \to \mathrm{Hom}(A, B)$. Since (G, Y) is not faithful, there exists $k \in G$ such that $ky = y$ for all $y \in Y$. Let $f = -a \cdot k + a \cdot e_G$. We have

$$\begin{aligned}\sum_{g \in G} \theta(gx)(f(g)) &= \sum_{g \in G} \theta''(\theta'(gx))(f(g)) \\ &= \theta''(\theta'(kx))(f(k)) + \theta''(\theta'(x))(f(e)) \\ &= \theta''(\theta'(x))(-a) + \theta''(\theta'(x))(a) = 0,\end{aligned}$$

where $\theta'(kx) = k\theta'(x) = \theta'(x)$ by the assumption on k. Thus, the map $\phi : A[G] \to A^X$ is not injective for any θ, B, contradicting Aithfulness. □

Lemma 3. *Let $G = \mathbb{Z}$, let X be a G-system, and let A be a finite nontrivial abelian group. If (G, X) has a faithful subshift factor then it is Aithful.*

Proof. We prove the contrapositive, so suppose the action is not Aithful. Consider any clopen partition $X = P_1 \sqcup \cdots \sqcup P_n$, and for $i \in \{1, \ldots, n\}$ define $\theta : X \to \mathrm{End}(A)$ by $\theta(x) = \mathrm{id}$ for $x \in P_i$, $\theta(x) = 0$ otherwise. Since the action is not Aithful, there exists nonzero $f \in A[G]$ such that

$$\sum_{g \in G} \theta(gx)(f(g)) = 0$$

for all $x \in X$.

Let $g' = \max\{g \in \mathbb{Z} \mid f(g) \neq 0_A\}$ in the usual ordering of $\mathbb{Z}$. Then for all $x \in X$ we have $\sum_{g<g'} \theta(gx)(f(g)) = -\theta(g'x)(f(g'))$ for all x. We have to have $g'x \in P_i$ whenever $\sum_{g<g'} \theta(gx)(f(g)) \neq 0$ (though this may not be sufficient), and have to have $g'x \notin P_i$ whenever $\sum_{g<g'} \theta(gx)(f(g)) = 0$, so whether $g'x \in P_i$ can be deduced from $\sum_{g<g'} \theta(gx)(f(g))$, and thus from the set of $g < g'$ such that $f(g) \neq 0$ and $gx \in P_i$.

This is easily seen to imply that the set of all $g \in G$ such that $g \cdot x \in P_i$ forms a finite union of arithmetic progressions with a bounded period (over all of X). Since this happens for all $i = 1, \ldots, n$, the subshift factor given by the partition is uniformly periodic, thus not faithful. □

In particular, for infinite $\mathbb{Z}$-subshifts, as well as faithful $\mathbb{Z}$-actions commuting with subshifts (on any group), faithfulness is equivalent to Aithfulness for any nontrivial A (for the latter class this follows from e.g. [16]).

Question 3. *For which pairs (G, A) does there exist a faithful G-subshift which is not Aithful?*

Apart from $G = \mathbb{Z}$, we do not know the answer for any (G, A). We next show that the Ledrappier subshift is Aithful $A = \mathbb{Z}_2$, even though (by definition) it admits a cancelling pattern over $\mathbb{Z}_2$. (In fact, it is Aithful for all nontrivial A.)

Example 1: Consider the Ledrappier subshift

$$X = \{x \in \mathbb{Z}_2^{\mathbb{Z}^2} \mid \forall v \in \mathbb{Z}^2 : x_v + x_{v+(0,1)} + x_{v+(1,0)} = 0\},$$

and let $A = \mathbb{Z}_2$. Let $\theta : X \to \mathrm{Hom}(\mathbb{Z}_2, B)$ be any continuous function for B an abelian group. We may assume $B = \mathbb{Z}_2^d$ for some d, and then $|\mathrm{Hom}(\mathbb{Z}_2, B)| = 2^d$. The map θ is then determined by d and by a partition $(C_v)_{v \in B}$.

If the partition depends only on the symbol at $(0,0)$, then θ never proves $\mathbb{Z}_2$ithfulness: take

$$f = 1_{\mathbb{Z}_2} \cdot (0,0) + 1_{\mathbb{Z}_2} \cdot (1,0) + 1_{\mathbb{Z}_2} \cdot (0,2) + 1_{\mathbb{Z}_2} \cdot (1,1),$$

(where $1_{\mathbb{Z}_2}$ is the generator of $\mathbb{Z}_2$) and consider the sum

$$\sum_{v \in \mathbb{Z}^2} \theta(vx)(f(v)).$$

By assumption, $\theta(vx)$ only depends on x_v, and adds a particular vector to B depending on its value. In any single coordinate of B, we either add 1 in any case (in which case the sum is 0 since f has support of size four), never add 1 (a trivial case), or we add 1 when $x_v = a$ for a particular $a \in \mathbb{Z}_2$. This amounts to counting the parity of the number of 0s or 1s in $x|_{\{(0,0),(1,0),(0,2),(1,1)\}}$, and a short calculation shows that this is always even.

Setting $d = 1$, $C = \{x \in X \mid (x_{(0,0)}, x_{(-1,0)}) = (0,1)\}$, and letting $\theta|_C = \mathrm{id}, \theta|_{X \setminus C} = 0$, on the other hand, proves $\mathbb{Z}_2$ithfulness: consider any nonzero $f \in \mathbb{Z}_2[\mathbb{Z}^2]$. The sum $\sum_{v \in \mathbb{Z}^2} \theta(vx)(f(v))$ now amounts to counting (modulo 2) how many times C is seen on the support of f, i.e. how many 0s there are on the support of f, such that the symbol on the left is 1.

Let $i \in \mathbb{Z}$ be the leftmost x-coordinate that appears in the support of f. There is a configuration in the Ledrappier subshift such that on the ith column, there is exactly one occurrence of 0, and we can align it with one of the elements in the support of f to obtain a configuration x. There is also a configuration $y \in X$ satisfying $y_{(i-1,j)} = 1$ for all $j \in \mathbb{Z}$, and any such configuration satisfies $y_{(i+k,j)} = 0$ for all $k \geq 0, j \in \mathbb{Z}$. Clearly the number of times C is entered by x and $x + y$ in the support of f differs by exactly one, so one of these numbers is odd, and this configuration proves $\mathbb{Z}_2$ithfulness. ○

Most of our examples of Aithful actions, in particular all our cellular automata actions, come from the following stronger property.[4]

Definition 2. *An action (G, X) is* strongly faithful *if there is a clopen set $C \subset X$ such that for all $\emptyset \neq F \Subset G$ there exists $x \in X$ such that $\exists! g \in F : gx \in C$.*

Lemma 4. *Every strongly faithful action is faithful and Aithful for all A.*

Proof. Assume a G-system X is strongly faithful. Since Aithfulness implies faithfulness (and even a faithful subshift factor), it suffices to prove the latter claim, but we give a direct proof of faithfulness: let $g \in G \setminus \{e\}$ and take $F = \{e, g\}$. By strong faithfulness there exists x such that exactly one of x, gx is in C. In particular $x \neq gx$ and the action is faithful.

For Aithfulness, let A be any finite abelian group. We need to show the action is Aithful. Define $\theta : X \to \mathrm{End}(A)$ by $\theta^{-1}(\mathrm{id}_A) = C$, and $\theta(x) = 0$ for

[4] It is possible that also the Ledrappier subshift has this property, but we have no proof.

$x \notin C$. Suppose $f \in A[G]$ and that $\sum_{g \in G} \theta(gx)(f(g)) = 0$ for all $x \in X$. Let $F \Subset G$ be the support of f. If $F \neq \emptyset$, let $x \in X$ be given by strong faithfulness, so there is a unique $g' \in F$ with $g'x \in C$. By the choice of θ we then have $\sum_{g \in G} \theta(gx)(f(g)) = \theta(g'x)(f(g')) = f(g') \neq 0$, a contradiction. Thus $F = \emptyset$, i.e $f = 0$, thus the action is Aithful. □

The following is proved in a straightforward fashion from the definitions.

Lemma 5. *If (G, X) is strongly faithful (resp. Aithful) then so is (H, X) for any $H \leq G$.*

We sandwich strong faithfulness between two other properties, which may clarify it. Let $X_{\leq 1} = \{x \in \{0,1\}^G \mid \sum x \leq 1\}$ be the *sunny-side-up* subshift on G. Say a system is a *subquotient* of another if it is a factor of a subsystem. A *horoball* in a finitely-generated group G is a limit of balls with radius tending to infinity, with respect to a fixed generating set (see [10]; here we include the denegerate horoball G, and one should take limits with right translates of balls $B_r g$).

Lemma 6. *If (G, X) admits a sunny-side-up subquotient, then it is strongly faithful. If (G, X) is strongly faithful and G is generated by the finite set S, then there exists a clopen set C, an S-horoball H and $x \in X$ such that $hx \in C$ for a unique element $h \in H$.*

Proof. For the first claim, if $Y \subset X$ has a sunny-side-up factor $\phi : Y \to X_{\leq 1}$. Observe that $[1] = \{x \in X_{\leq 1} x_e = 1\}$ is clopen in $X_{\leq 1}$ so $\phi^{-1}([1])$ is clopen in Y, so by basic topology there is a clopen set $C \subset X$ such that $C \cap Y = \phi^{-1}([1])$. Let $y \in Y$ be any element of $C \cap Y$. Then we can use C and translates of y to satisfy strong faithfulness.

For the latter claim, under strong faithfulness for any finite $F \Subset G$ there exist $x \in G$ and $g \in F$ such that

$$\{h | hx \in C\} \cap F = \{g\}$$

This implies $\{hg^{-1} | hx \in C\} \cap Fg^{-1} = \{e\}$ implies

$$\{h | hgx \in C\} \cap Fg^{-1} = \{e\}$$

and thus letting $y(F) = gx$ and $t(F) = Fg^{-1}$ we have $y(F) \in C$ and $\forall h \in t(F) \setminus \{e\} : gy(F) \notin C$.

Let P be the set of pairs (H, z) such that $e \in H$ and

$$\forall h \in H : (hz \in C \iff h = e).$$

Clearly this set is closed in $2^G \times X$ since C is clopen. In particular applying the observation of the previous paragraph to balls with respect to the generating set S, one obtains an S-horoball H and $z \in C$ such that the H-orbit of z does not revisit C. □

Of course one can apply this argument to finite sets other than balls.

Remark 1. In the case of $\mathbb{Z}^2$, in a precise sense an "optimal" sequence of sets to apply it to are discretized balls whose boundaries eventually contain arithmetic progressions in every rational direction, namely this forces t to give a set $H \ni \{(0,0)\}$ in the limit which contains the predecessor set N of $(0,0)$ in some translation-invariant total order of $\mathbb{Z}^2$ – this is the best we can do since t could plausibly always force translates inside any such set N. ○

Example 2: There is a strongly faithful subshift without a sunny-side-up subquotient. For example the Cantor's dust $\mathbb{Z}^2$ subshift [20, Figure 1c] has this property. On the other hand, consider any $\mathbb{Z}^2$-subshift obtained as the orbit closure of a discretization of a rational line in direction $\vec{v}$ (see [20, Figure 1b], but consider a rational continuation). Such a subshift is not strongly faithful (as it is not even faithful), but for any generating set whose convex hull has no edge in direction $\vec{v}$ (up to orientation), there are points that visit [1] (the clopen set of drawings that hit the origin) just once in a horoball. ○

5 Strongly faithful actions by cellular automata

Lemma 7. *If G admits a strongly faithful (resp. Aithful) action by automorphisms on some full shift $(\Sigma^{\mathbb{Z}}, \sigma)$, then it admits a strongly faithful (resp. Aithful) action by automorphisms on any uncountable sofic $\mathbb{Z}$-shift.*

In particular, the class of groups admitting strongly faithful (resp. Aithful) actions on the full shift $\Sigma^{\mathbb{Z}}$ does not depend on the choice of nontrivial alphabet Σ. We include sofic shifts here, but other than this lemma we stick to full shifts in all statements.

Proof. It is straightforward to check that strong faithfulness (resp. Aithfulness) is preserved under the conveyor belt simulation of [19] (or the construction of [14]): to determine the clopen set C' for the simulating action from the clopen set C for the original action (resp. a map θ' from a map θ), read a part of the simulated configuration along the conveyor belts, taking the contents of the top track at the origin as the starting point, and checks whether it is in C (resp. which partition element it lies in the definition of h). Considering configurations where the simulated configurations are $\mathbb{Z}$-shaped (i.e. the conveyor belts do not wrap around), strong faithfulness (resp. Aithfulness) of G implies the same property for H. □

We mention the following closure property out of general interest.

Lemma 8. *If G admits a strongly faithful action by automorphisms on some full shift $(\Sigma^{\mathbb{Z}}, \sigma)$ and H is commensurable to G, then H admits a strongly faithful action on any full shift.*

Proof. By Lemma 5, it is enough to show that the class of groups acting strongly faithfully is closed under passing to finite index supergroups. Suppose first G is infinite. The result is then straightforward from the proof of closure under finite extensions in [14]: If $[H : G] = n$, we construct the induced representation of H on $(\Sigma^n)^{\mathbb{Z}}$ from a set of left coset representatives. Let C be the clopen set for strong faithfulness of the G-action, and let C' to be the clopen set of configurations where the coordinate corresponding to the trivial coset contains a configuration from C.

Now consider nonempty $F \Subset H$, w.l.o.g. suppose $e_G \in F$ so $F \cap G$ is a nonempty subset of G and apply strong faithfulness to obtain a configuration x such that there is a unique element $h \in F \cap G$ such that $hx \in C$. Put the configuration x in the coordinate corresponding to the trivial coset, and in other coordinates put a configuration y whose G-orbit does not enter C (which is possible when G is infinite). Let $z \in (\Sigma^n)^{\mathbb{Z}}$ be the resulting configuration.

Now, observe that the H-orbit of z contains only configurations where the coordinates contain elements from Gx and Gy, and the coordinate of gz corresponding to the trivial coset contains an element of Gx if and only if $g \in G$. Now hz is in C' by the definition of the induced action. On the other hand, $gz \in C'$ implies that the trivial coset coordinate of gz contains an element in C, thus an element of Gx, thus $g \in G$. But $g \in F \cap G$ implies $g = h$, thus h is the only element of F mapping z into C'.

To cover finite extensions of finite groups G, it is necessary and sufficient to prove that all finite groups act strongly faithfully, and thus it is enough to show this for the symmetric groups. The cellwise coordinate permutation action on $(\Sigma^n)^{\mathbb{Z}}$ gives a strongly faithful action of the symmetric group $\mathrm{Sym}(n)$. ☐

Definition 3. *Let G be a group of cellular automata. We say G has* property P *if for all $g \in G \setminus \{e_G\}$ acting with minimal radius r, the following holds: there exist $u \in \Sigma^{2r}$ and $a, b, c, d \in \Sigma$ such that $g(auc) \neq g(bud)$, while for all $h \in G \setminus \{g\}$ we have $h((auc)^{\mathbb{Z}})|_{\{r,r+1\}} = h((bud)^{\mathbb{Z}})|_{\{r,r+1\}}$.*

In the above definition, auc is a word of length $2r+2$, and $g(auc)$ and $g(bud)$ refer to the application of the local rule of g to the prefix and suffix of length $2r+1$ to obtain a word of length 2. The definition is not particularly natural from a dynamical point of view, it is simply what we encounter in the following proof, and is sufficient for our purposes.

Lemma 9. *Let G be a group of cellular automata with property P and let $n \in \mathbb{N}$. Then there is an isomorphic group of cellular automata with property P (possibly on a larger alphabet) such that every element $g \in G \setminus \{e\}$ has minimal radius at least r.*

Proof. The property P is preserved in the conveyor belt construction, at least when it is performed using suitable mutually unbordered words. Namely, if G acts on $\Sigma^{\mathbb{Z}}$, pick alphabet $\Sigma^2 \cup \{\#\}$ and act on maximal subwords of the form $\Sigma^2 \#^r$, wrapped into conveyor belts. To ensure a minimal radius $\geq r$ for all nontrivial elements, we do not act on length-2 conveyor belts consisting of a single element of Σ^2. It is easy to see that property P is inherited from G: Write aub and $bu^R a$ on the top and bottom track of a conveyor belt, respectively (separated by $\#^r$-runs), to obtain a word w; then do the same for cud to obtain w'. Looking at the top track we see $g(w) \neq g(w')$, while h sees the same periodic point it sees in the original action. ☐

Lemma 10. *Suppose that finitely many groups $(G_i)_{i \in I}$ each act strongly faithfully on a one-dimensional full shift, and all have property P. Then their free product acts strongly faithfully by automorphisms on any full shift.*

Proof. Suppose $I = \{0, 1, \dots, \ell\}$. By Lemma 7, it is enough to show the free product acts strongly faithfully on some full shift. We consider the construction from the finite case of Theorem 1, and assume the groups G_i all have property P. By the previous lemma, we may assume all $g \in G_i \setminus \{e\}$ have minimal radius at least 10.

If $F \Subset G = G_0 * G_1 * \cdots * G_\ell$, take $g \in F$ with a maximal number of syllables among elements of F, and as in the proof of Theorem 1 construct a configuration where g sends a bit of information to the origin (the cell $0 \in \mathbb{Z}$), in the sense that the leftmost difference between two words t, t' moves from the right end to the left, one segment at a time.

If we use words given by property P in this construction, then no $h \in F \setminus \{g\}$ has time to send information to the origin: by construction the leftmost difference between t, t' can move at most one segment per applied syllable, so at the first difference between g and h (counted from the right), h fails to move the difference to the next syllable, and can never catch up. The lower bound on minimal radius ensures that radius 0 maps do not cause a problem, noting that $10 > 0$. □

Theorem 2. *The free product of any finite family of finite groups acts strongly faithfully by automorphisms on any full shift.*

Proof. It is enough to construct an action of an arbitrary finite group G with the property from the previous theorem. Let $|G| \in \{n, n-1\}$ for n even, take alphabet $\{0, 1, 2, 3, 4\}$ and let G act on the words $w_i = 10^{2i}20^{2n-2i-1}3$ by the left regular action, identifying w_i with the ith group element, under an arbitrary identification (starting with w_0, and ignoring $w_{|G|}$ if $n = |G| + 1$).

It is easy to see that property P holds: by changing 3 to 4 we can "cancel" the application of $g \in G \setminus \{e_G\}$, so all that is needed is to pick w_i so that the action of $g \in G$, if not cancelled, changes w_i to $w_{n/2}$. Since the action is free, no other $h \in G$ changes i to the same value (or a neighboring value, because the 1s are separated by distance two), whether or not it is cancelled. □

We believe it should be possible to generalize this to infinite free products with a bit of work. The fact finite free products act faithfully on full shifts was first proved in [1].

Corollary 1. *Every free group acts strongly faithfully by automorphisms on any full shift.*

Proof. The free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ contains all free groups, and by Lemma 5 a subaction of a strongly faithful action is strongly faithful. □

It seems that typically in concrete cases of the graph product construction of Theorem 1, we obtain a strongly faithful action, and we do not have examples where an Aithful action is not obtained. Nevertheless, we do not know whether the set of groups acting strongly faithfully (or Aithfully) by full shift automorphisms is closed under free product. We also do not know if the set of groups acting strongly faithfully by automorphisms of a full shift is closed under $G \mapsto \mathbb{Z}_2 \wr G$, though in this case we know that the construction given in the following section usually does not give an Aithful action.

Lemma 11. *The class of groups acting strongly faithfully by automorphisms on a full shift is closed under finite direct products.*

Proof. The product action of strongly faithful actions is obviously strongly faithful, and can be seen as an automorphism action on a larger full shift. □

It does not seem obvious that the product action of a finite family of Aithful actions is Aithful, as the assumption only tells us something about polynomials with a product decomposition.

We do not know whether either class (strongly faithful or Aithful CA actions) are closed under infinite direct sums, indeed for infinite sums we used a different construction in the proof of Theorem 1, and the following remark outlines the problem with this approach. It also shows that Aithfulness fails in this construction even for product polynomials.

Remark 2. The direct sum case of Theorem 1 does not typically give strongly faithful (resp. Aithful) actions, at least if the clopen set C (resp. function h) looks at just one coordinate. Perform the construction for $G \times G'$. It turns out the action is not Aithfulness no matter how G and G' act. To see this, take $f = (e_G - g)(e_{G'} - g') \in \mathbb{Z}_2[G \times H]$. Here we suggestively use additive notation and standard conventions and notation for the grofcoup ring structure, more precisely the element is

$$f = 1_{\mathbb{Z}_2} \cdot (e, e) + 1_{\mathbb{Z}_2} \cdot (g, e) + 1_{\mathbb{Z}_2} \cdot (e, g') + 1_{\mathbb{Z}_2} \cdot (g, g').$$

Observe now that if the segment at the origin of a configuration x is "of type G" (the case G' being symmetric), then we have

$$\sum_{(k,k') \in G \times G'} \theta((k, k')x)(f(k, k')) = \theta(x)(f(e, e)) - \theta((g, e)x)(f(g, e)) + \\ \theta((g, g')x)(f(g, g')) - \theta((e, g')x)(f(e, g')) = 0,$$

if θ only looks at the central coordinate: we have

$$\theta(x)(f(e, e)) = \theta((e, g')x)(f(e, g'))$$

and

$$\theta((g, e)x)(f(g, e)) = \theta((g, g')x)(f(g, g')),$$

since $f((e, e) = f((e, g')), f((g, e) = f((g, g')$ by our choice of f and since the action of (e, g') is not visible at the origin. Similarly, strong faithfulness is contradicted for any C that only looks at the central coordinate. ○

6 Wreath products

Theorem 3. *Let A be a finite abelian group and suppose $G \leq \mathrm{Aut}(\Sigma^{\mathbb{Z}})$ acts Aithfully. Then $A \wr G \in \mathcal{G}$.*

Proof. Suppose G and A are disjoint. Let $\theta : \Sigma^{\mathbb{Z}} \to \mathrm{Hom}(A, B)$ be the proof of Aithfulness. We pick the alphabet $\Sigma \times B$, and to $g \in G$ associate $\hat{g} \in \mathrm{Aut}((\Sigma \times B)^{\mathbb{Z}})$ by $\hat{g}(x, y) = (gx, y)$; clearly $g \mapsto \hat{g}$ is a homomorphism. To $a \in A$ associate $\hat{a}$ by

$$\hat{a}(x, y)_i = (x_i, \theta(\sigma^i(x))(a) + y_i)$$

The map $a \mapsto \hat{a} : A \to \mathrm{Aut}((\Sigma \times B)^{\mathbb{Z}})$ is a homomorphism because

$$\begin{aligned} \widehat{(a+b)}(x, y)_i &= (x_i, \theta(\sigma^i(x))(a + b) + y_i) \\ &= (x_i, \theta(\sigma^i(x))(a) + \theta(\sigma^i(x))(b) + y_i) \\ &= \hat{b}(\hat{a}(x, y))_i. \end{aligned}$$

We have
$$\hat{a}^{\hat{g}}(x,y)_i = (x_i, \theta(\sigma^i(gx))(a) + y_i),$$
from which it is clear that any $\hat{a}^{\hat{g}}$ and $\hat{b}^{\hat{g}'}$ commute for $a, b \in A, g, g' \in G$. These are the standard relations of the wreath product, and thus we have obtained an action of the wreath product.

We need to show the faithfulness of this action. For this, suppose $w \in A \wr G$ and $\hat{w} = \mathrm{id}_{(\Sigma \times B)^{\mathbb{Z}}}$. Clearly the natural homomorphism from $A \wr G$ to G maps w to the identity, and we can write $w = \prod_{j=1}^{n} a_j^{g_j}$ for some n, and some $g_j \in G, a_j \in A$, where we may assume $g_j \neq g_j$ for $i \neq j$ and $a_i \neq 0$ for all i. By the formula for conjugates $\hat{a}^{\hat{g}}$, we have
$$\hat{w}(x,y)_i = (x_i, \sum_{j=1}^{n} \theta(\sigma^i(g_j x))(a_j) + y_i) = (x_i, y_i)$$
for all x, y, so $\sum_{j=1}^{n} \theta(\sigma^i(g_j x))(a_j) = 0$ for all $x \in \Sigma^{\mathbb{Z}}$ and $i \in \mathbb{Z}$. Setting $i =$ we have
$$\sum_{j=1}^{n} \theta(g_j x)(a_j) = \sum_{g \in G} \theta(gx)(f(g)) = 0$$
where $f = \sum_{j=1}^{n} a_j \cdot g_j$, so by Aithfulness $f = 0$, meaning $w = \mathrm{id}$. Thus $w \mapsto \hat{w}$ is injective. □

Combining with the results of the previous section, we obtain that for any finite abelian group A and any $n \in \mathbb{N}$, we have $A \wr F_n \in \mathcal{G}$ and $A \wr \mathbb{Z}^n \in \mathcal{G}$. It is plausible that A can be generalized to other (necessarily abelian) groups.

Question 4. *Is $\mathbb{Z} \wr \mathbb{Z}$ is in $\mathcal{G}$?*

The proof that (for instance) $\mathbb{Z}_2 \wr \mathbb{Z}$, requires that the action of $\mathbb{Z}$ is Aithful, and the action of $\mathbb{Z}_2 \wr \mathbb{Z}$ we obtain is typically not Aithful. This suggests the following candidate group to study. Note that it is residually finite by [12, Theorem 3.1] and has word problem solvable in polynomial time.

Question 5. *Is $\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr \mathbb{Z})$ is in $\mathcal{G}$?*

7 The one-sided case

Theorem 4. *For finite Γ and $G_i \in \mathcal{G}'_{n_i}$ we have $(G_1, G_2, \ldots, G_k)^\Gamma \in \mathcal{G}'_{\max n_i + 1}$.*

Proof. The construction for finite Γ in the proof of Theorem 1 works almost directly, the main difference being that we replace the conveyor belts with words. We see $\mathrm{Aut}(\Sigma^{\mathbb{N}})$ as the subgroup of $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ containing those f such that both f and f^{-1} are one-sided as automorphisms of $\Sigma^{\mathbb{Z}}$.

First, we give a construction with a large alphabet. Suppose $|B_i| = n_i$ with $B_i \cap B_j = \emptyset$ for $i \neq j$ and let $\Sigma = \mathcal{S} \sqcup \bigsqcup_i B_i$ where $\mathcal{S}$ is the disjoint union of sets of functions $B_i^{B_j}$. If $g \in G_i$, on segments of type i (maximal words over the subalphabet B_i) $\hat{g}$ applies the natural action, interpreting the last symbol as a constant tail: the segment contents $ua \in B_i^\ell, a \in B_i$, is replaced by $vb \in B_i^\ell, b \in B_i$, where $g(ua^\infty) = vb^\infty$.

On segments of type j, $j \neq i$, we modify the last symbol as in Theorem 1 when the leftmost symbol of a segment of type i is changed: if c is changed to d when $\hat{g}$ is applied, then in the segment of type j on the left we change a to $a + s(d) - s(c)$. It is clear that $\hat{g}$ is a one-sided automorphism, and the proof of correctness is analogous to that in Theorem 1.

Now, let us optimize the alphabet size. First, we need not actually have the individual symbols know the type of the segment they belong to, as long as they can tell their type by looking to the right. Thus, we can take an alphabet of size $\max_i n_i + 1$, use the same symbols for all segments, and use the one extra symbol, say $\#$, to denote the type of the segment, representing $w \in B_i^*$ by $w_0 \#^i w_1 \#^i \cdots \#^i w_{|w|-1} \#^i$.

Similarly, the large set $\mathcal{S}$ can be replaced by words of the form $\#^{mk}$ for $m = 1, 2, \ldots, |\mathcal{S}|$, since $i \in \{0, 1, \ldots, k-1\}$ and m can be deduced from $\#^i \#^{mk}$ (interpreting $m = 0$ as the lack of an $\mathcal{S}$-symbol). □

Apart from some trivial cases, we do not know if increasing the alphabet is necessary, nor whether it is possible to do infinite products.

Remark 3. The radii of the automorphisms are rather massive in this construction. If desired, this can be avoided by using two special symbols instead of one, and coding the lengths of the $\#^*$-runs in binary, giving embeddings of graph products of groups $G_i \in \mathcal{G}'_{n_i}$ in $\mathcal{G}'_{\max n_i + 2}$ with more reasonable radii. With a bit more work, one can optimize both the alphabet size and the radii simultaneously, by using the basic size $\max n_i$ alphabet also for coding the lengths of runs. ○

Theorem 5. *Every right-angled Coxeter and Artin group is in $\mathcal{G}'_3$.*

Proof. We have $\mathbb{Z}_2 \in \mathcal{G}'_2$, since $f(x)_i = 1 - x_i$ defines an automorphism of the binary full shift. If Γ is a graph, then $(\mathbb{Z}_2, \mathbb{Z}_2, \ldots, \mathbb{Z}_2)^\Gamma$ is, by definition, just the right-angled Coxeter group $C(\Gamma)$. The previous theorem then shows that that $\mathrm{Aut}(\{0,1,2\}^\mathbb{Z})$ contains every right-angled Coxeter group. Right-angled Artin groups are clearly subgroups of right-angled Coxeter groups (see [9] for a finite-index embedding). □

Question 6. *Which Coxeter groups (or Artin groups) are in $\mathcal{G}'_n$ for each n?*

There is a bound on orders of finite-order elements [4], which forbids some Coxeter groups, but it is likely that there are additional restrictions.

Example 3: As RACGs and RAAGs are important families of groups, we explain the construction explicitly, with an ad hoc optimization of the set $\mathcal{S}$. Let $g \in G_i$ where G_i is the ith copy of $\mathbb{Z}_2$. Then $\hat{g}(x)_i = a$ is determined as follows: If $x_i = 2$, then $a = 2$. If $x_i \in \{0, 1\}$, and the maximal segment of 2s to the right is 2^j, then let $j = i' + mkn$ where $0 \leq i' < kn$ and $m \in \{0, 1\}$. If this cannot be done since the 2^*-segment is too long, then set $a = x_i$.

Suppose then that we find i' and m. Then if $i' = i$, set $a = 1 - x_i$ (we are inside a segment of type i). If $i' \neq i$ and $m = 0$, set $a = x_i$ (we are properly inside a segment of type i'). If $i' \neq i$, $m = 1$ and $\{i, i'\} \in E$ set $a = x_i$ (we are at the right boundary of a segment of type i', but G_i commutes with $G_{i'}$).

If $i' \neq i$, $m = 1$ and $\{i, i'\} \notin E$, then consider the configuration to the right of the 2^j-word. It starts with an element of $\{0, 1\}$, so suppose it is in $\{0, 1\}2^{j'}$ for some maximal j' (again if the run is very long, set $a = x_i$). Let $j' = i'' + m'kn$.

If $i'' \neq i$, then set $a = x_i$. If $i'' = i$ then check whether $\hat{g}$ will flip this bit. Set $a = 1 - x_i$ if it will, otherwise set $a = x_i$.

Analogously to the proof of Lemma 10, it is easy to see that the actions are strongly faithful when Γ has no edges. ○

Remark 4. In particular, letting Γ be the complement of the disjoint union of n copies of the k-clique in the above theorem, we get

$$\mathbb{Z}^n, F_k \leq F_k^n \leq (\mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2)^n \leq (\mathbb{Z}_2, \mathbb{Z}_2, \ldots, \mathbb{Z}_2)^\Gamma \in \mathrm{Aut}(\{0,1,2\}^\mathbb{Z}).$$

This action is not strongly faithful in general, but as we noted it is when applied to an edgeless graph. As in Lemma 11 it is easy to see that direct products of strongly faithful actions are strongly faithful. From these observations, we get that $\mathbb{Z}^n$ and F_k admit strongly faithful actions in $\mathcal{G}'_\infty$. ○

Theorem 6. *If A is finite abelian and $G \in \mathcal{G}'_\infty$ is Aithful, then $A \wr G \in \mathcal{G}'_\infty$.*

Proof. Let $N = \{0, 1, \ldots, n-1\}$ and suppose G acts Aithfully on $N^\mathbb{N}$. Let $\theta : N^\mathbb{N} \to \mathrm{Hom}(A, A^d)$ be the continuous function from the definition of Aithfulness. For our embedding we pick the alphabet $\Sigma = A^d \sqcup N$. If $g \in G$, $\hat{g}$ acts on maximal finite segments over N as in the above proof (so the word $wa \in N^*$ with $a \in N$ represents wa^∞, and we conjugate the natural action through this identification). Symbols in A^d are not modified by $\hat{g}$.

If $a \in A$, $\hat{a}$ acts trivially on symbols in N. If $x_i \in A^d$, then if also $x_{i+1} \in A^d$ we set $\hat{a}(x)_i = x_i$, while if $x_{i+1} \in N$ we interpret a prefix of $x_{[i+1,\infty)}$ as a configuration in $y \in N^\mathbb{N}$ the same way $\hat{g}$ does, i.e. if $x_{[i+1,\infty)}$ is a one-way infinite word over N, then directly use this configuration, and otherwise take the maximal finite segment over N and interpret the last symbol as being repeated infinitely. Now set $\hat{a}(x)_i = x_i + \theta(y)(a)$. The proof that this gives an embedding of the wreath product is similar to Theorem 3. □

The Aithfulness certificate of G might use a large power of A, so it is difficult to include alphabet sizes in the statement. The proof, combined with ideas from the proof of Theorem 4, shows that $\max_i p_i^{e_i} + 1$ symbols suffice in addition to N if $A = \prod_i \mathbb{Z}_{p_i}^{e_i}$, because A^d acts faithfully by disjoint cycles on a finite set of cycles of cardinalities in $p_i^{e_i}$, and that we can code the type of the cycle using the alphabet itself.

Theorem. *If A is finite abelian and $n \in \mathbb{N}$, then $A \wr F_n, A \wr \mathbb{Z}^n \in \mathcal{G}'_\infty$.*

Proof. This follows from the previous theorem and Remark 4. □

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