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A double construction of quadratic anticenter-symmetric Jacobi-Jordan algebras

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Abstract. This work addresses some relevant characteristics and properties of anticenter-symmetric Jacobi-Jordan algebras such as bimodules, matched pairs. Besides, the Jacobi-Jordan admissible algebra is defined; a special emphasis is given to a double construction of quadratic anticenter-symmetric algebras. We then follow this theory with the main properties and related algebraic structures of an anticenter-symmetric JJ algebra, namely the anti-Zinbiel algebras. Finally, we discuss the double construction of some classes of the two dimensional anticenter-symmetric JJ algebras.

1. Introduction

Jacobi-Jordan algebras (JJ algebras for short) were introduced in [4] in 2014 as vector spaces $\mathcal{A}$ over a field $\mathbb{K}$, equipped with a bilinear map $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfying the Jacobi identity and the commutativity $x \cdot y = y \cdot x$, for all $x, y \in \mathcal{A}$. This class of algebras appear under different names in the litterature reflecting, perhaps, the fact that it was considered from different viewpoints by different communities, sometimes not aware of each other's results. In [11, 13, 10] and other Jordan litterature, these algebras are called *Jordan algebras of nil index* 3. In [9] they are called *Lie-Jordan algebras* (superalgebras are also considered there). In [5] and [14] they were called *mock-Lie algebras*.

In [12] Wörz-Busekros relates these type of algebras with Bernstein algebras. One crucial remark is that JJ algebras are examples of the more popular and well-referenced Jordan algebras [1, 8] introduced in order to achieve an axiomatization for the algebra of observables in quantum mechanics. In [4] the authors achieved the classification of these algebras up

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to dimension 6 over an algebraically closed field of characteristic different from 2 and 3. As it was explained in the paper and in [1], JJ algebras are objects fundamentally different from both associative and Lie algebras though their definition differs from the latter only modulo a sign. In [1] it's proven that there exists a rich and very interesting theory behind the JJ algebras which deserves to be developed further mainly for three important reasons (see [1] for more details). They mainly discuss the general extension (GE) problem (which is a kind of generalization of the classical Holder extension problem) for JJ algebras. Interestingly, they authors prove that a finite dimensional JJ algebras is Frobenius if and only if there exists an invariant non degenerate bilinear form (Proposition 1.8).

On the other hand Frobenius algebra is an associative algebra equipped with a non-degenerate invariant bilinear form. This type of algebras plays an important role in different areas of mathematics and physics, such as statistical models over two-dimensional graphs [3] and topological quantum field theory [7]. In [2], C. Bai described associative analogs of Drinfeld's double constructions for Frobenius algebras and for associative algebras equipped with non-degenerate Connes cocycles.

Using the essential fact that JJ algebras are Frobenius below susmentioned condition, our purpose is to define an anticenter-symmetric JJ algebra, that is a JJ admissible algebra endowed with a symmetric antiassociator. We further proceed the double construction of quadratic anti-center symmetric JJ algebra which is a Frobenius analog for the anti-center symmetric JJ algebra.

Particularly, we study anticenter-symmetric Jacobi-Jordan algebras. We explicitly show that an algebra of this type equiped with an anticommutator is a JJ algebra. We specially study their bimodules, matched pairs and the double construction of quadratic anticenter-symmetric Jacobi-Jordan algebras. Furthermore, an illustration of the theory is given based on some classes of two dimensional anticenter-symmetric Jacobi-Jordan algebras with a special emphasis on the double construction.

The paper is organized as follow. In Section 2, after recalling some basics concepts and necessary properties on antiassociative and JJ algebras, we discuss on the matched pairs of JJ algebras, we define the notion of anticenter-symmetric Jacobi-Jordan algebra and give the its main properties. Section 3 discusses bimodules and matched pairs of anticenter-symmetric Jacobi-Jordan algebras. Section 4 is devoted to double construction of quadratic anticenter-symmetric Jacobi-Jordan algebras. In Section

5, we study the main properties and related algebraic structures of a special class of anticenter-symmetric JJ algebra namely anti-Zinbiel algebras. A detailed survey is done in Section 6 on the 2-dimensional anticenter-symmetric Jacobi-Jordan algebras. In Section 7, we end with some concluding remarks.

2. Anticenter-symmetric Jacobi-Jordan algebras

2.1. Preliminaries

In this section, we will give some basic facts on antiassociative algebras and JJ algebras. Throughout this work, we consider $\mathcal{A}$ as a finite dimensional vector space over the field $\mathbb{K}$ of characteristic different from $2, 3$ together with a bilinear product $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $(x, y) \mapsto x \cdot y$.

Definition 2.1. Let "$\cdot$" be a bilinear product in a vector space $\mathcal{A}$. Suppose that it satisfies the following law:

$$(x \cdot y) \cdot z = -x \cdot (y \cdot z). \tag{1}$$

Definition 2.2. An algebra $(\mathcal{A}, [,])$ over $\mathbb{K}$ is called *JJ* if it is commutative:

$$[x, y] = [y, x], \tag{2}$$

and satisfies the Jacobi identity:

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \tag{3}$$

for any x,y, $z \in \mathcal{A}$.

Theorem 2.3. (cf. [14]) *Given an antiassociative algebra $(\mathcal{A}, \cdot)$, the new algebra $\mathcal{A}^\dagger$ with multiplication give by the "anticommutator"*

$$[x, y] = \frac{1}{2}\left(x \cdot y + y \cdot x\right),$$

is a JJ algebra.

Since JJ algebras are commutative, the left and right actions of an algebra coincide, so we can speak about just modules.

Definition 2.4. (cf. [14]) A vector space V is a module over a JJ algebra $\mathcal{A}$, if there is a linear map (a representation) $\rho : \mathcal{A} \to End(V)$ such that

$$\rho([x, y])(v) = -\rho(x)(\rho(y)v) - \rho(y)(\rho(x)v) \tag{4}$$

for any $x, y \in \mathcal{A}$ and $v \in V$.

Proposition 2.5. (cf. [6]) *Let $(\mathcal{A}, [,])$ be a JJ algebra and (V, ρ) be a representation of $\mathcal{A}$. The direct summand $\mathcal{A} \oplus V$ with a bracket defined by*

$$[(x+u),(y+w)]_{\mathcal{A}\oplus V} := [x,y] + \rho(x)(w) + \rho(y)(u) \quad \forall x,y \in \mathcal{A},\ \forall u,w \in V \tag{5}$$

is a JJ algebra.

Definition 2.6. (cf. [6]) Let $(\mathcal{A}, [,])$ be a JJ algebra. Two representations (V, ρ) and (V', ρ') of $\mathcal{A}$ are said to be isomorphic if there exists a linear map $\phi\ : V \to V'$ such that

$$\forall x \in \mathcal{A}, \quad \rho'(x) \circ \phi = \phi \circ \rho(x).$$

Theorem 2.7. (cf. [6]) *Let $\mathcal{A}$ and $\mathcal{B}$ be fixed two JJ algebras and let $\mu : \mathcal{B} \to \mathfrak{gl}(\mathcal{A})$ and $\rho : \mathcal{A} \to \mathfrak{gl}(\mathcal{B})$ be two JJ algebra representations. Then, $(\mathcal{A}, \mathcal{B}, \rho, \mu)$ is a matched pair of the JJ algebras if and only if μ and ρ satisfy:* $\forall x, y \in \mathcal{A},\ a, b \in \mathcal{B}$,

$$\rho(x)\,[a,b] + [\rho(x)a, b] + [a, \rho(x)b] + \rho(\mu(a)x)b + \rho(\mu(b)x)a = 0, \tag{6}$$

$$\mu(a)\,[x,y] + [\mu(a)x, y] + [x, \mu(a)y] + \mu(\rho(x)a)y + \mu(\rho(y)a)x = 0. \tag{7}$$

$(\mathcal{A} \oplus \mathcal{B}, *)$ defines a JJ algebra with respect to the product $*$ satisfying:

$$(x+a)*(y+b) = [x,y] + \mu(a)y + \mu(b)x + [a,b] + \rho(x)b + \rho(y)a. \tag{8}$$

2.2. Anticenter-symmetric Jacobi-Jordan algebras

Definition 2.8. A *JJ admissible algebra* is a nonassociative algebra which an anticommutator is a JJ algebra.

Example 2.9. Any antiassociative algebra is a JJ admissible algebra.

Definition 2.10. $(\mathcal{A}, \cdot)$, (or simply $\mathcal{A}$), is said to be an *anticenter-symmetric JJ algebra* if $\forall\, x, y, z \in \mathcal{A}$, the antiassociator of the bilinear product $\cdot$ defined by $(x,y,z)_{-1} := (x \cdot y) \cdot z + x \cdot (y \cdot z)$, is symmetric in x and z, i.e.

$$(x,y,z)_{-1} = -(z,y,x)_{-1}. \tag{9}$$

As matter of notation simplification, we will denote $x \cdot y$ by xy if not any confusion.

Proposition 2.11. *Any antiassociative algebra is an anticenter-symmetric JJ algebra, and*

$$\mu \circ (\mu \otimes \mathrm{id}) + \mu \circ (\mathrm{id} \otimes \mu) + (\mu \circ \tau) \circ (\mathrm{id} \otimes (\mu \circ \tau)) \\ + (\mu \circ \tau) \circ ((\mu \circ \tau) \otimes \mathrm{id}) = 0, \quad (10)$$

where μ is the multiplication operator and τ is such that $\tau(x \otimes y) = y \otimes x$.

Proof. Let $(\mathcal{A}, \cdot)$ be an antiassociative algebra. We have $\forall x, y, z \in \mathcal{A}$

$$(x \cdot y) \cdot z = -x \cdot (y \cdot z) \Leftrightarrow (x, y, z)_{-1} = 0 = -(z, y, x)_{-1}.$$

Thus $(x, y, z)_{-1} + (z, y, x)_{-1} = 0$, which implies that

$$(xy)z + x(yz) + (zy)x + z(yx) = 0.$$

Supposing μ be the bilinear product on $\mathcal{A}$, then the previous relation can be written as

$$\mu(\mu(x \otimes y) \otimes z) + \mu(x \otimes \mu(y \otimes z)) \\ + (\mu \circ \tau)((\mu \circ \tau)(x \otimes y) \otimes z) + (\mu \circ \tau)(x \otimes (\mu \circ \tau) y \otimes z) = 0. \quad (11)$$

Thus

$$\mu \circ (\mu \otimes \mathrm{id}) + \mu \circ (\mathrm{id} \otimes \mu) + (\mu \circ \tau) \circ (\mathrm{id} \otimes (\mu \circ \tau)) + (\mu \circ \tau) \circ ((\mu \circ \tau) \otimes \mathrm{id})$$

is equal to 0. □

Considering the representations of the left L and right R multiplication operations:

$$\begin{array}{rcl} L : \mathcal{A} & \longrightarrow & \mathfrak{gl}(\mathcal{A}) \\ x & \longmapsto & L_x : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ y & \longmapsto & x \cdot y, \end{array} \end{array} \quad (12)$$

$$\begin{array}{rcl} R : \mathcal{A} & \longrightarrow & \mathfrak{gl}(\mathcal{A}) \\ x & \longmapsto & R_x : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ y & \longmapsto & y \cdot x, \end{array} \end{array} \quad (13)$$

we infer the adjoint representation $\mathrm{ad} := L + R$ of the sub-adjacent JJ algebra of an anticenter-symmetric JJ algebra $\mathcal{A}$ as follows:

$$\begin{array}{rcl} \mathrm{ad} : \mathcal{A} & \longrightarrow & \mathfrak{gl}(\mathcal{A}) \\ x & \longmapsto & \mathrm{ad}_x : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ y & \longmapsto & [x, y], \end{array} \end{array} \quad (14)$$

such that $\forall\, x, y \in \mathcal{A}, \mathrm{ad}_x(y) := (L_x + R_x)(y)$.

Proposition 2.12. *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric JJ algebra. For any $x, y \in \mathcal{A}$, the following relations are satisfied:*

- *The anticommutator associated to the bilinear product " $\cdot$ " given by $[x, y] = x \cdot y + y \cdot x$ defines a JJ algebra structure on $\mathcal{A}$,*
- $[L_x, R_y] = -[R_x, L_y]$
- $L_{xy} + L_x L_y = -R_{yx} - R_x R_y$,
- *$ad = L + R$ a linear representation of the sub-adjacent JJ algebra of $(\mathcal{A}, \cdot)$ and, $[ad_x, ad_y] = ad_{[x,y]}$.*

  The linear map R is the right multiplication operator associated to the bilinear product " $\cdot$ " on $\mathcal{A}$.

Proof. Consider the anticenter-symmetric JJ algebra $(\mathcal{A}, \cdot)$. For $x, y, z \in \mathcal{A}$ we have

$$\begin{aligned}
&[x,[y,z]] + [y,[z,x]] + [z,[y,x]] \\
&= [x, yz + zy] + [y, zx + xz] + [z, xy + yx] \\
&= x(yz) + x(zy) + (yz)x + (zy)x \\
&\quad + y(zx) + y(xz) + (zx)y + (xz)y \\
&\quad + z(xy) + z(yx) + (xy)z + (yx)z \\
&= \{x(yz) + (yz)x + y(zx) + (zx)y + z(xy) + (xy)z\} \\
&\quad + \{(zy)x + x(zy) + y(xz) + (xz)y + z(yx) + (yx)z\} \\
&= \{(x,y,z)_{-1} + (y,z,x)_{-1} + (z,x,y)_{-1}\} \\
&\quad + \{(z,y,x)_{-1} + (x,z,y)_{-1} + (y,x,z)_{-1}\} \\
&= \{(y,x,z)_{-1} + (z,x,y)_{-1}\} + \{(z,y,x)_{-1} + (x,y,z)_{-1}\} \\
&\quad + \{(x,z,y)_{-1} + (y,z,x)_{-1}\} \\
&= 0.
\end{aligned}$$

On other hand, we have for all $x, y, z \in \mathcal{A}$

$$\begin{aligned}
(x,y,z)_{-1} = -(z,y,x)_{-1} &\Leftrightarrow (xy)z + x(yz) = -(zy)x - z(yx) \\
&\Leftrightarrow (xy)z + z(yx) = -(zy)x - x(yz) \\
&\Leftrightarrow (L_{xy} + R_{yx})(z) = -(R_x R_y + L_x L_y)(z).
\end{aligned}$$

Therefore, the relation

$$L_{xy} + R_{yx} = -R_x R_y - L_x L_y$$

holds $\forall x, y \in \mathcal{A}$.

We also have for all $x, y, z \in \mathcal{A}$

$$\begin{aligned}
[L_x, R_y](z) &= L_x\left(R_y(z)\right) + R_y\left(L_x(z)\right) \\
&= x(zy) + (xz)y = (x, z, y)_{-1} = -(z, x, y)_{-1} \\
&= -\left((zx)y + z(xy)\right) = -\left(R_y\left(R_x(z)\right) + R_{xy}(z)\right) \\
&= -\left(R_y R_x + R_{xy}\right)(z) \\
&= (z, y, x)_{-1} = (zy)x + z(yx) = R_x R_y(z) + R_{xy}(z) \\
&= -(y, z, x) - (yz)x - y(zx) \\
&= -R_x L_y(z) - L_y R_x(z) \\
&= -[L_y, R_x](z).
\end{aligned}$$

Therefore $[L_x, R_y] = -[L_y, R_x]$.

We have for all $x, y \in \mathcal{A}$,

$$\begin{aligned}
[ad_x, ad_y] &= [L_x, L_y] + [L_x, R_y] + [R_x, L_y] + [R_x, R_y] \\
&= [L_x, L_y] + [R_x, R_y] \\
&= (L_x L_y + R_x R_y) + (L_y L_x + R_y R_x) \\
&= (L + R)_{xy} + (L + R)_{yx} \\
&= (L + R)_{xy+yx} \\
&= (L + R)_{[x,y]} \\
&= ad_{[x,y]}
\end{aligned}$$

which completes the proof. □

3. Bimodules of anticenter-symmetric JJ algebras

Definition 3.1. Let $\mathcal{A}$ be an anticenter-symmetric JJ algebra, V be a vector space. Suppose $l, r : \mathcal{A} \to \mathfrak{gl}(V)$ be two linear maps satisfying: for all $x, y \in \mathcal{A}$,

$$[l_x, r_y] = -[l_y, r_x] \tag{15}$$

$$l_{xy} + l_x l_y = -r_{yx} - r_x r_y. \tag{16}$$

Then, (l, r, V) (or simply (l, r)) is called a *bimodule of the anticenter-symmetric JJ algebra* $\mathcal{A}$.

Proposition 3.2. *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric JJ algebra and V be a vector space over $\mathbb{K}$. Consider two linear maps, $l, r : \mathcal{A} \to \mathfrak{gl}(V)$. Then, (l, r, V) is a bimodule of $\mathcal{A}$ if and only if, the semi-direct sum $\mathcal{A} \oplus V$ of vector spaces is turned into an anticenter-symmetric JJ algebra by defining the multiplication in $\mathcal{A} \oplus V$ by $\forall x_1, x_2 \in \mathcal{A},\ v_1, v_2 \in V$,*

$$(x_1 + v_1) * (x_2 + v_2) = x_1 \cdot x_2 + (l_{x_1} v_2 + r_{x_2} v_1),$$

We denote it by $\mathcal{A} \ltimes^{-1}_{l,r} V$ or simply $\mathcal{A} \ltimes^{-1} V$.

Proof. It is obvious that the semi-direct sum of two vector spaces is also a vector space. Now suppose that (l, r, V) is a bimodule of $\mathcal{A}$ and show that $(\mathcal{A} \oplus V, *)$ is an anticenter-symmetric JJ algebra. Since $*$ is a bilinear product, for all $x_1, x_2, x_3 \in \mathcal{A}$ and for all $v_1, v_2, v_3 \in V$, we have:

$$\begin{aligned}
((x_1 + v_1), (x_2 + v_2), (x_3 + v_3))_{-1} &= \{(x_1 + v_1) * (x_2 + v_2)\} * (x_3 + v_3) \\
&\quad + (x_1 + v_1) * \{(x_2 + v_2) * (x_3 + v_3)\} \\
&= (x_1 x_2) x_3 + l_{x_1 x_2} v_3 + r_{x_3}(l_{x_1} v_2 + r_{x_2} v_1) \\
&\quad + x_1(x_2 x_3) + l_{x_1}(l_{x_2} v_3) + l_{x_1}(r_{x_3} v_2) + r_{x_2 x_3} v_1 \\
&= (x_1 x_2) x_3 + l_{x_1 x_2} v_3 + r_{x_3}(l_{x_3} v_2) + r_{x_3}(r_{x_2} v_1) \\
&\quad + x_1(x_2 x_3) + l_{x_1}(l_{x_2} v_3) + l_{x_1}(r_{x_3} v_2) + r_{x_2 x_3} v_1 .
\end{aligned}$$

$$\begin{aligned}
((x_1 + v_1), (x_2 + v_2), (x_3 + v_3))_{-1} &= (x_1, x_2, x_3)_{-1} + (l_{x_1 x_2} + l_{x_1} l_{x_2}) v_3 \\
&\quad + (r_{x_3} l_{x_1} + l_{x_1} r_{x_3}) v_2 + (r_{x_3} r_{x_2} + r_{x_2 x_3}) v_1 .
\end{aligned}$$

$$\begin{aligned}
((x_3 + v_3), (x_2 + v_2), (x_1 + v_1))_{-1} &= (x_3, x_2, x_1)_{-1} + (l_{x_3 x_2} + l_{x_3} l_{x_2}) v_1 \\
&\quad + (r_{x_1} l_{x_3} + l_{x_3} r_{x_1}) v_2 + (r_{x_1} r_{x_2} + r_{x_2 x_1}) v_3 .
\end{aligned}$$

Therefore, form the definition of anticenter-symmetric JJ algebra we have

$$\begin{aligned}
(x_1 + v_1, x_2 + v_2, x_3 + v_3)_{-1} &= -(x_3 + v_3, x_2 + v_2, x_1 + v_1)_{-1} \\
&\Leftrightarrow \begin{cases} l_{x_1 x_2} + l_{x_1} l_{x_2} = -r_{x_1} r_{x_2} - r_{x_2 x_1} \\ r_{x_3} l_{x_1} + l_{x_1} r_{x_3} = -r_{x_1} l_{x_3} - l_{x_1} r_{x_3} \\ l_{x_1 x_3} + l_{x_3} l_{x_2} = -r_{x_3} r_{x_2} - r_{x_2 x_3} \end{cases} \\
&\Leftrightarrow (\mathcal{A} \oplus V, *)
\end{aligned}$$

is an anticenter symmetric JJ algebra. □

Proposition 3.3. *Let $\mathcal{A}$ be an anticenter-symmetric JJ algebra and V be a vector space over $\mathbb{K}$. Consider two linear maps, $l, r : \mathcal{A} \to \mathfrak{gl}(V)$, such that (l, r, V) is a bimodule of $\mathcal{A}$. Then, the map $l + r : \mathcal{A} \longrightarrow \mathfrak{gl}(V)$, $x \longmapsto l_x + r_x$, is a linear representation of the sub-adjacent JJ algebra of $\mathcal{A}$.*

Proof. Let (l, r, V) be a bimodule of the center-symmetric algebra $\mathcal{A}$. Then, $\forall x, y \in \mathcal{A} \quad [l_x, r_y] = -[l_y, r_x]; l_{xy} + l_x l_y = -r_x r_y - r_{yx}$. Besides, it is a matter of straightforward computation to show that $l + r$ is a linear map on $\mathcal{A}$. Then, we have:

$$\begin{aligned}
[(l+r)(x), (l+r)(y)] &= [l_x + r_x, l_y + r_y] \\
&= [l_x, l_y] + [l_x, r_y] + [r_x, l_y] + [r_x, r_y] \\
&= [l_x, l_y] + [r_x, r_y] \\
&= l_x l_y + l_y l_x + r_x r_y + r_y r_x \\
&= \{l_x l_y + r_x r_y\} + \{l_y l_x + r_y r_x\} \\
&= \{l_{xy} + r_{yx}\} + \{l_{yx} + r_{xy}\} \\
&= (l+r)_{xy} + (l+r)_{yx} = (l+r)_{[x,y]}.
\end{aligned}$$

Therefore, (l, r, V) is a bimodule of $\mathcal{A}$ implies that $l + r$ is a representation of the linear representation of the sub-adjacent JJ algebra of $\mathcal{A}$. □

Example 3.4. According to the Proposition 2.12, one can deduce that $(L, R, \mathcal{A})$ is a bimodule of the anticenter-symmetric JJ algebra $\mathcal{A}$, where L and R are the left and right multiplication operator representations, respectively.

Theorem 3.5. *Let $(\mathcal{A}, \cdot)$ and $(\mathcal{B}, \circ)$ be two anticenter-symmetric JJ algebras. Suppose that $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B})$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A})$ are bimodules of $\mathcal{A}$ and $\mathcal{B}$, respectively, obeying the relations:*

$$\begin{aligned}
&r_{\mathcal{A}}(x)(a \circ b) + r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + a \circ (r_{\mathcal{A}}(x)b) \\
&+ l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a + (l_{\mathcal{A}}(x)b) \circ a + l_{\mathcal{A}}(x)(b \circ a) = 0,
\end{aligned} \tag{17}$$

$$\begin{aligned}
&r_{\mathcal{B}}(a)(x \cdot y) + r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x + x \cdot (r_{\mathcal{B}}(a)y) \\
&+ l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x + (l_{\mathcal{B}}(a)y) \cdot x + l_{\mathcal{B}}(a)(y \cdot x) = 0,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&a \circ (l_{\mathcal{A}}(x)b) + (r_{\mathcal{A}}(x)b) \circ a + (r_{\mathcal{A}}(x)a) \circ b + l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b \\
&+ r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a + l_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + b \circ (l_{\mathcal{A}}(x)a) + r_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b = 0,
\end{aligned} \tag{19}$$

$$\begin{aligned}
&x \cdot (l_{\mathcal{B}}(a)y) + (r_{\mathcal{B}}(a)y) \cdot x + (r_{\mathcal{B}}(a)x) \cdot y + l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y \\
&+ r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x + l_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x + y \cdot (l_{\mathcal{B}}(a)x) + r_{B}(r_{\mathcal{A}}(x)a)y = 0,
\end{aligned} \tag{20}$$

for all $x, y \in A$ and $a, b \in \mathcal{B}$. Then, there is an anticenter-symmetric JJ algebra structure on $\mathcal{A} \oplus \mathcal{B}$ given by:

$$(x+a)*(y+b) = (x \cdot y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) + (a \circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a). \tag{21}$$

We denote this anticenter-symmetric JJ algebra by $\mathcal{A} \bowtie^{-1,l_\mathcal{A},r_\mathcal{A}}_{l_\mathcal{B},r_\mathcal{B}} \mathcal{B}$, or simply by $\mathcal{A} \bowtie^{-1} \mathcal{B}$. Then $(\mathcal{A},\mathcal{B},l_\mathcal{A},r_\mathcal{A},l_\mathcal{B},r_\mathcal{B})$ satisfying the above conditions is called the *matched pair of the anticenter-symmetric JJ algebras* $\mathcal{A}$ and $\mathcal{B}$.

Proof. For $x, y \in \mathcal{A}$ and $a, b \in \mathcal{B}$ we have

$$(x + a) * (y + b) = (xy + l_\mathcal{B}(a)y + r_\mathcal{B}(b)x) + (a \circ b + l_\mathcal{A}(x)b + r_\mathcal{A}(y)a).$$

By considering this bilinear product (by definition), let us compute its associator and find their relations with $(x,y,z)_{-1}, (x,y,c)_{-1}, (x,b,z)_{-1}, (x,b,c)_{-1}, (a,y,z)_{-1}, (a,y,c)_{-1}, (a,b,c)_{-1}$. The first part of the associator reads:

$$\begin{aligned}
&\{(x + a) * (y + b)\} * (z + c)\\
&= \{(xy + l_\mathcal{B}(a)y + r_\mathcal{B}(b)x) + (a \circ b + l_\mathcal{A}(y) + r_\mathcal{A}(y)a)\} * (z + c)\\
&= (xy)z + (l_\mathcal{B}(a)y) \cdot z + (r_\mathcal{B}(b)x) \cdot z l_\mathcal{B}(a \circ b)z + l_\mathcal{B}(l_\mathcal{A}(x)b)z\\
&\quad + l_\mathcal{B}(r_\mathcal{A}(y)a)z + r_\mathcal{B}(c)(x \cdot y) + r_\mathcal{B}(c)(l_\mathcal{B}(a)y) + r_\mathcal{B}(c)(r_\mathcal{B}(b)x)\\
&\quad + (a \circ b) \circ c + (l_\mathcal{A}(x)b) \circ c + (r_\mathcal{A}(y)a) \circ c + l_\mathcal{A}(x \cdot y)c\\
&\quad + l_\mathcal{A}(l_\mathcal{B}(a)y)c + l_\mathcal{A}(r_\mathcal{B}(b)x)c + r_\mathcal{A}(z)(a \circ b) + r_\mathcal{A}(z)(l_\mathcal{A}(x)b)\\
&\quad + r_\mathcal{A}(z)(r_\mathcal{A}(y)a),
\end{aligned}$$

while its second part:

$$\begin{aligned}
&(x + a) * \{(y + b) * (z + c)\}\\
&= (x + a) * \{yz + l_\mathcal{B}(b)z + r_\mathcal{B}(c)y + b \circ c + l_\mathcal{A}(y)c + r_\mathcal{A}(z)b\}\\
&= x(yz) + x \cdot (l_\mathcal{B}(b)z) + x \cdot (r_\mathcal{B}(c)y) + l_\mathcal{B}(a)(y \cdot z)\\
&\quad + l_\mathcal{B}(a)(l_\mathcal{B}(b)z) + l_\mathcal{B}(a)(r_\mathcal{B}(c)y) + r_\mathcal{B}(b \circ c)x + r_\mathcal{B}(l_\mathcal{A}(y)c)x\\
&\quad + r_\mathcal{B}(r_\mathcal{A}(z)b)x + a \circ (b \circ c) a \circ (l_\mathcal{A}(y)c) + a \circ (r_\mathcal{A}(z)b)\\
&\quad + l_\mathcal{A}(x)(b \circ c) + l_\mathcal{A}(x)(l_\mathcal{A}(y)c) + l_\mathcal{A}(x)(r_\mathcal{A}(z)b) + r_\mathcal{A}(y \cdot z)a\\
&\quad + r_\mathcal{A}(l_\mathcal{B}(b)z)a + r_\mathcal{A}(r_\mathcal{B}(c)y)a.
\end{aligned}$$

and the associator takes the form:

$$\begin{aligned}
&(x + a, y + b, z + c)_{-1}\\
&= (x, y, z)_{-1} + (a, b, c)_{-1} + \{r_\mathcal{B}(c)(x \cdot y) + l_\mathcal{A}(x \cdot y)c + x \cdot (r_\mathcal{B}(c)y)\\
&\quad + l_\mathcal{A}(x)(l_\mathcal{A}(y)c) + r_\mathcal{B}(l_\mathcal{A}(y)c)x\} + \{r_\mathcal{B}(c)(r_\mathcal{B}(b)x) + l_\mathcal{A}(r_\mathcal{B}(b)x)c\\
&\quad + r_\mathcal{B}(b \circ c)x + (l_\mathcal{A}(x)b) \circ c + l_\mathcal{A}(x)(b \circ c)\} + \{(r_\mathcal{B}(b)x) \cdot z l_\mathcal{B}(l_\mathcal{A}(x)b)z\\
&\quad + r_\mathcal{A}(z)(l_\mathcal{A}(x)b) + x \cdot (l_\mathcal{B}(b)z) + r_\mathcal{B}(r_\mathcal{A}(z)b)x + l_\mathcal{A}(x)(r_\mathcal{A}(z)b)\}\\
&\quad + \{(l_\mathcal{B}(a)y) \cdot z + l_\mathcal{B}(r_\mathcal{A}(y)a)z + r_\mathcal{A}(z)(r_\mathcal{A}(y)a) + l_\mathcal{B}(a)(y \cdot z)\\
&\quad + r_\mathcal{A}(y \cdot z)a\} + \{r_\mathcal{B}(c)(l_\mathcal{B}(a)y) + (r_\mathcal{A}(y)a) \circ c + l_\mathcal{A}(l_\mathcal{B}(a)y)c\\
&\quad + l_\mathcal{B}(a)(r_\mathcal{B}(c)y) + a \circ (l_\mathcal{A}(y)c) + r_\mathcal{A}(r_\mathcal{B}(c)y)a\} + \{l_\mathcal{B}(a \circ b)z\\
&\quad + r_\mathcal{A}(z)(a \circ b) + l_\mathcal{B}(a)(l_\mathcal{B}(b)z) + a \circ (r_\mathcal{A}(z)b) + r_\mathcal{A}(l_\mathcal{B}(b)z)a\}.
\end{aligned}$$

Further, we have

$$\begin{aligned}
(x,y,c)_{-1} &= r_{\mathcal{B}}(c)(x\cdot y)+x\cdot(r_{\mathcal{B}}(c)y)+r_{\mathcal{B}}(l_{\mathcal{A}}(y)c)x+l_{\mathcal{A}}(x\cdot y)c\\
&\quad +l_{\mathcal{A}}(x)(l_{\mathcal{A}}(y)c),\\
(x,b,c)_{-1} &= r_{\mathcal{B}}(c)(r_{\mathcal{B}}(b)x)+l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)c+(l_{\mathcal{A}}(x)b)\circ c+r_{\mathcal{B}}(b\circ c)x\\
&\quad +l_{\mathcal{A}}(x)(b\circ c),\\
(x,b,z)_{-1} &= (r_{\mathcal{B}}(b)x)\cdot z+l_{\mathcal{B}}(l_{\mathcal{A}}(x)b)z+r_{\mathcal{A}}(z)(l_{\mathcal{A}}(x)b)+x\cdot(l_{\mathcal{B}}(b)z)\\
&\quad +r_{\mathcal{B}}(r_{\mathcal{A}}(z)b)x+l_{\mathcal{A}}(x)(r_{\mathcal{A}}(z)b),\\
(a,y,z)_{-1} &= (l_{\mathcal{B}}(a)y)\cdot z+l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)z+r_{\mathcal{A}}(z)(r_{\mathcal{A}}(y)a)+l_{\mathcal{B}}(a)(y\cdot z)\\
&\quad +r_{\mathcal{A}}(y\cdot z)a,\\
(a,y,c)_{-1} &= r_{\mathcal{B}}(c)(l_{\mathcal{B}}(a)y)+(r_{\mathcal{A}}(y)a)\circ c+l_{\mathcal{A}}(l_{\mathcal{B}}(a)y)c+l_{\mathcal{B}}(a)(r_{\mathcal{B}}(c)y)\\
&\quad +a\circ(l_{\mathcal{A}}(y)c)+r_{\mathcal{A}}(r_{\mathcal{B}}(c)y)a,\\
(a,b,z)_{-1} &= l_{\mathcal{B}}(a\circ b)z+r_{\mathcal{A}}(z)(a\circ b)+l_{\mathcal{B}}(a)(l_{\mathcal{B}}(b)z)+r_{\mathcal{A}}(l_{\mathcal{B}}(b)z)a\\
&\quad +r_{\mathcal{A}}(l_{\mathcal{B}}(b)z)a+a\circ(r_{\mathcal{A}}(z)b),
\end{aligned}$$

which can also be re-expressed as:

$$\begin{aligned}
(x{+}a,y{+}b,z{+}c)_{-1} &= (x,y,z)_{-1}+(x,y,c)_{-1}+(x,b,z)_{-1}+(x,b,c)_{-1}\\
&\quad +(a,y,z)_{-1}+(a,y,c)_{-1}+(a,b,z)_{-1}+(a,b,c)_{-1}. \qquad (22)
\end{aligned}$$

Similarly,

$$\begin{aligned}
(z{+}c,y{+}c,x{+}a)_{-1} &= (z,y,x)_{-1}+(z,y,a)_{-1}+(z,b,x)_{-1}+(z,b,a)_{-1}\\
&\quad +(c,y,x)_{-1}+(c,b,a)_{-1}+(c,y,a)_{-1}+(c,b,x)_{-1}. \qquad (23)
\end{aligned}$$

Using the fact that $(l_{\mathcal{A}},r_{\mathcal{A}})$ is a bimodule of $\mathcal{A}$ and $(l_{\mathcal{B}},r_{\mathcal{B}})$ is a bimodule of $\mathcal{B}$, one arrives at the following result:

$$(x+a,y+b,z+c)_{-1} = -(z+c,y+b,x+a)_{-1}$$

$$\Leftrightarrow \begin{cases}
(x,y,z)_{-1} = -(z,y,x)_{-1}\\
(x,y,c)_{-1} = -(c,y,x)_{-1}\\
(x,b,z)_{-1} = -(z,b,x)_{-1}\\
(x,b,c)_{-1} = -(c,b,x)_{-1}\\
(a,y,z)_{-1} = -(z,y,a)_{-1}\\
(a,y,c)_{-1} = -(c,y,a)_{-1}\\
(a,b,z)_{-1} = -(z,b,a)_{-1}\\
(a,b,c)_{-1} = -(c,b,a)_{-1}
\end{cases}
\Leftrightarrow \begin{cases}
(l_{\mathcal{A}},r_{\mathcal{A}},\mathcal{B}) \ , \ (l_{\mathcal{B}},r_{\mathcal{B}},\mathcal{A})\\
(x,y,c)_{-1} = -(c,y,x)_{-1} & (18)\\
(x,b,z)_{-1} = -(z,b,x)_{-1} & (20)\\
(x,b,c)_{-1} = -(c,b,x)_{-1} & (17)\\
(a,y,c)_{-1} = -(c,y,a)_{-1}. & (19)
\end{cases}$$

This last relation ends the proof. □

Corollary 3.6. *Let $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ be a matched pair of anticenter-symmetric JJ algebras. Then, $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{B}), l_{\mathcal{A}} + r_{\mathcal{A}}, l_{\mathcal{B}} + r_{\mathcal{B}})$ is a matched pair of sub-adjacent JJ algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$.*

Proof. By using the Proposition 3.3 and the bimodules $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B})$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A})$, we have: $\mathrm{ad}_{\mathcal{A}} := l_{\mathcal{A}} + r_{\mathcal{A}}$ and $\mathrm{ad}_{\mathcal{B}} := l_{\mathcal{B}} + r_{\mathcal{B}}$ are the linear representations of the sub-adjacent JJ algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$ of the anticenter-symmetric JJ algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Then, the statement that $\mathcal{G}(\mathcal{A}) \bowtie_{\mathrm{ad}_{\mathcal{B}}}^{-1,\mathrm{ad}_{\mathcal{A}}} \mathcal{G}(\mathcal{B})$ is a matched pair of JJ algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$ follows from Theorem 3.5. By analogous step giving:

$$\begin{aligned} &\mathrm{ad}_{\mathcal{A}}(x)\,[a,b] + [\mathrm{ad}_{\mathcal{A}}(x)a, b] + [a, \mathrm{ad}_{\mathcal{A}}(x)b] \\ &+\mathrm{ad}_{\mathcal{A}}(\mathrm{ad}_{\mathcal{B}}(a)x)b + \mathrm{ad}_{\mathcal{A}}(\mathrm{ad}_{\mathcal{B}}(b)x)a = 0 \end{aligned} \tag{24}$$

$$\begin{aligned} &\mathrm{ad}_{\mathcal{B}}(a)\,[x,y] + [\mathrm{ad}_{\mathcal{B}}(a)x, y] + [x, \mathrm{ad}_{\mathcal{B}}(a)y] \\ &+\mathrm{ad}_{\mathcal{B}}(\mathrm{ad}_{\mathcal{A}}(x)a)y + \mathrm{ad}_{\mathcal{B}}(\mathrm{ad}_{\mathcal{A}}(y)a)x = 0 \end{aligned} \tag{25}$$

Thus, we first have:

$$\begin{aligned} &\mathrm{ad}_{\mathcal{A}}(x)\,[a,b] + [\mathrm{ad}_{\mathcal{A}}(x)a, b] + [a, \mathrm{ad}_{\mathcal{A}}(x)b] + \mathrm{ad}_{\mathcal{A}}(\mathrm{ad}_{\mathcal{B}}(a)x)b + \mathrm{ad}_{\mathcal{A}}(\mathrm{ad}_{\mathcal{B}}(b)x)a \\ &= (l_{\mathcal{A}} + r_{\mathcal{A}})(x)\,[a,b] + [(l_{\mathcal{A}} + r_{\mathcal{A}})(x)a, b] + [a, (l_{\mathcal{A}} + r_{\mathcal{A}})(x)b] \\ &\quad +(l_{\mathcal{A}} + r_{\mathcal{A}})((l_{\mathcal{B}} + r_{\mathcal{B}})(a)x)b + (l_{\mathcal{A}} + r_{\mathcal{A}})((l_{\mathcal{B}} + r_{\mathcal{B}})(b)x)a \\ &= l_{\mathcal{A}}(x)[a,b] + r_{\mathcal{A}}(x)[a,b] + [l_{\mathcal{A}}(x)a, b] + [r_{\mathcal{A}}(x)a, b] + [a, l_{\mathcal{A}}(x)b] \\ &\quad + [a, r_{\mathcal{A}}(x)b] + l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b + l_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b + r_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b + r_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b \\ &\quad +l_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a + r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a \\ &= l_{\mathcal{A}}(x)(a \circ b) + l_{\mathcal{A}}(x)(b \circ a) + r_{\mathcal{A}}(x)(a \circ b) + r_{\mathcal{A}}(x)(b \circ a) + (l_{\mathcal{A}}(x)a) \circ b \\ &\quad +b \circ (l_{\mathcal{A}}(x)a) + (r_{\mathcal{A}}(x)a) \circ b + b \circ (r_{\mathcal{A}}(x)a) + a \circ (l_{\mathcal{A}}(x)b) + (l_{\mathcal{A}}(x)b) \circ a \\ &\quad +a \circ (r_{\mathcal{A}}(x)b) + (r_{\mathcal{A}}(x)b) \circ a + l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b + l_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b + r_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b \\ &\quad +r_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b + l_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a + r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a \\ &= \{r_{\mathcal{A}}(x)(a \circ b) + r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + a \circ (r_{\mathcal{A}}(x)b) + l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a + (l_{\mathcal{A}}(x)b) \circ a \\ &\quad +l_{\mathcal{A}}(x)(b \circ a)\} + \{a \circ (l_{\mathcal{A}}(x)b) + (r_{\mathcal{A}}(x)b) \circ a + (r_{\mathcal{A}}(x)a) \circ b + l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b \\ &\quad +r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a + l_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + b \circ (l_{\mathcal{A}}(x)a) + r_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b\} + \{r_{\mathcal{A}}(x)(b \circ a) \\ &\quad +r_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b + b \circ (r_{\mathcal{A}}(x)a) + l_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b + (l_{\mathcal{A}}(x)a) \circ b + l_{\mathcal{A}}(x)(a \circ b)\} = 0. \end{aligned}$$

Secondly:

$$\begin{aligned} &\mathrm{ad}_{\mathcal{B}}(a)\,[x,y] + [\mathrm{ad}_{\mathcal{B}}(a)x, y] + [x, \mathrm{ad}_{\mathcal{B}}(a)y] + \mathrm{ad}_{\mathcal{B}}(\mathrm{ad}_{\mathcal{A}}(x)a)y + \mathrm{ad}_{\mathcal{B}}(\mathrm{ad}_{\mathcal{A}}(y)a)x \\ &= (l_{\mathcal{B}} + r_{\mathcal{B}})(a)\,[x,y] + [(l_{\mathcal{B}} + r_{\mathcal{B}})(a)x, y] + [x, (l_{\mathcal{B}} + r_{\mathcal{B}})(a)y] \\ &\quad +(l_{\mathcal{B}} + r_{\mathcal{B}})((l_{\mathcal{A}} + r_{\mathcal{A}})(x)a)y + (l_{\mathcal{B}} + r_{\mathcal{B}})((l_{\mathcal{A}} + r_{\mathcal{A}})(y)a)x \\ &= l_{\mathcal{B}}(a)[x,y] + r_{\mathcal{B}}(a)[x,y] + [l_{\mathcal{B}}(a)x, y] + [r_{\mathcal{B}}(a)x, y] + [x, l_{\mathcal{B}}(a)y] \\ &\quad + [x, r_{\mathcal{B}}(a)y] + l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y + l_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y + r_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y + r_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y \\ &\quad +l_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x + l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x + r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x + r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x \\ &= l_{\mathcal{B}}(a)(x \cdot y) + l_{\mathcal{B}}(a)(y \cdot x) + r_{\mathcal{B}}(a)(x \cdot y) + r_{\mathcal{B}}(a)(y \cdot x) + (l_{\mathcal{B}}(a)x) \cdot y \end{aligned}$$

$$
\begin{aligned}
&+y\cdot(l_{\mathcal{B}}(a)x)+(r_{\mathcal{B}}(a)x)\cdot y+y\cdot(r_{\mathcal{B}}(a)x)+x\cdot(l_{\mathcal{B}}(a)y)+(l_{\mathcal{B}}(a)y)\cdot x\\
&+x\cdot(r_{\mathcal{B}}(a)y)+(r_{\mathcal{B}}(a)y)\cdot x+l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y+l_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y+r_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y\\
&+r_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y+l_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x+l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x+r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x+r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x\\
=&\ \{r_{\mathcal{B}}(a)(x\cdot y)+r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x+x\cdot(r_{\mathcal{B}}(a)y)+l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x+(l_{\mathcal{B}}(a)y)\cdot x\\
&+l_{\mathcal{B}}(a)(y\cdot x)\}+\{x\cdot(l_{\mathcal{B}}(a)y)+(r_{\mathcal{B}}(a)y)\cdot x+(r_{\mathcal{B}}(a)x)\cdot y+l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y\\
&+r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x+l_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x+y\cdot(l_{\mathcal{B}}(a)x)+r_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y\}+\{r_{\mathcal{B}}(a)(y\cdot x)\\
&+r_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y+y\cdot(r_{\mathcal{B}}(a)x)+l_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y+(l_{\mathcal{B}}(a)x)\cdot y+l_{\mathcal{B}}(a)(x\cdot y)\}=0.
\end{aligned}
$$

Thus, we obtain

$$\mathrm{ad}_{\mathcal{A}}(x)[a,b]+[\mathrm{ad}_{\mathcal{A}}(x)a,b]+[a,\mathrm{ad}_{\mathcal{A}}(x)b]+\mathrm{ad}_{\mathcal{A}}(\mathrm{ad}_{\mathcal{B}}(a)x)b+\mathrm{ad}_{\mathcal{A}}(\mathrm{ad}_{\mathcal{B}}(b)x)a=0,$$

$$\mathrm{ad}_{\mathcal{B}}(a)[x,y]+[\mathrm{ad}_{\mathcal{B}}(a)x,y]+[x,\mathrm{ad}_{\mathcal{B}}(a)y]+\mathrm{ad}_{\mathcal{B}}(\mathrm{ad}_{\mathcal{A}}(x)a)y+\mathrm{ad}_{\mathcal{B}}(\mathrm{ad}_{\mathcal{A}}(y)a)x=0.$$

Hence, $(\mathcal{G}(\mathcal{A}),\mathcal{G}(\mathcal{B}),\mathrm{ad}_{\mathcal{A}},\mathrm{ad}_{\mathcal{B}})$ is a matched pair of sub-adjacent JJ algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$. □

Definition 3.7. Let (l,r,V) be a bimodule of an anticenter-symmetric JJ algebra $\mathcal{A}$, where V is a finite dimensional vector space. The dual maps l^*,r^* of the linear maps l,r, are defined, respectively, as: $l^*,r^*:\mathcal{A}\to\mathfrak{gl}(V^*)$ such that: for all $x\in\mathcal{A},\ u^*\in V^*,\ v\in V$,

$$
\begin{array}{rcl}
l^*:\mathcal{A} & \longrightarrow & \mathfrak{gl}(V^*)\\
x & \longmapsto & l^*_x: \begin{array}{rcl} V^* & \longrightarrow & V^*\\ u^* & \longmapsto & l^*_xu^*: \begin{array}{rcl} V & \longrightarrow & \mathbb{K}\\ v & \longmapsto & \langle l^*_xu^*,v\rangle:=\langle u^*,l_xv\rangle\,, \end{array}\end{array}
\end{array}
\tag{26}
$$

$$
\begin{array}{rcl}
r^*:\mathcal{A} & \longrightarrow & \mathfrak{gl}(V^*)\\
x & \longmapsto & r^*_x: \begin{array}{rcl} V^* & \longrightarrow & V^*\\ u^* & \longmapsto & r^*_xu^*: \begin{array}{rcl} V & \longrightarrow & \mathbb{K}\\ v & \longmapsto & \langle r^*_xu^*,v\rangle:=\langle u^*,r_xv\rangle\,. \end{array}\end{array}
\end{array}
\tag{27}
$$

Proposition 3.8. *Let $(\mathcal{A},\cdot)$ be an anticenter-symmetric JJ algebra and $l,r:\mathcal{A}\to\mathfrak{gl}(V)$ be two linear maps, where V is a finite dimensional vector space. The following conditions are equivalent:*

(1) *(l,r,V) is a bimodule of $\mathcal{A}$.*

(2) *(r^*,l^*,V^*) is a bimodule of $\mathcal{A}$.*

Proof. (1) $\Rightarrow$ (2). Suppose that (l, r, V) is a bimodule of $(\mathcal{A}, \cdot)$ and show that (r^*, l^*, V^*) is also a bimodule of $(\mathcal{A}, \cdot)$. We have:

$$\begin{aligned}\left\langle (r^*_{xy} + r^*_x r^*_y)u^*, v\right\rangle &= \left\langle r^*_{xy}u^*, v\right\rangle + \left\langle (r^*_x r^*_y)u^*, v\right\rangle \\ &= \langle r_{xy}(v), u^*\rangle + \langle r_y(r_x(v)), u^*\rangle \\ &= \langle (r_{xy} + r_y r_x)(v), u^*\rangle = \langle -(l_{yx} + l_y l_x)(v), u^*\rangle \\ &= -\langle l_{yx}(v), u^*\rangle - \langle (l_y l_x)(v), u^*\rangle \\ &= -\left\langle l^*_{yx}u^*, v\right\rangle - \left\langle (l^*_x l^*_y)u^*, v\right\rangle \\ &= \left\langle -(l^*_{yx} + l^*_x l^*_y)u^*, v\right\rangle .\end{aligned}$$

Therefore,

$$l^*_{yx} + l^*_x l^*_y = -r^*_{xy} - r^*_x r^*_y, \ \forall\, x, y\ \mathcal{A} \tag{28}$$

$$\begin{aligned}\left\langle [l^*_x, r^*_y]u^*, v\right\rangle &= \left\langle l^*_x(r^*_y)u^*, v\right\rangle + \left\langle r^*_y(l^*_x)u^*, v\right\rangle = \left\langle l_x(v), r^*_y u^*\right\rangle + \langle r_y v, l^*_x u^*\rangle \\ &= \langle r_y(l_x(v)), u^*\rangle + \langle l_x(r_y(v)), u^*\rangle = \langle [r_y, l_x]v, u^*\rangle \\ &= \langle -[r_x, l_y]v, u^*\rangle = \langle -(r_x(l_y) + l_y(r_x))v, u^*\rangle \\ &= \left\langle -(l^*_y r^*_x + r^*_x l^*_y)u^*, v\right\rangle = \left\langle -[l^*_y, r^*_x]u^*, v\right\rangle .\end{aligned}$$

Thus,

$$[l^*_x, r^*_y] = -[l^*_y, r^*_x], \ \forall\, x, y\ \in \mathcal{A}. \tag{29}$$

By considering the relations (28) and (29), we conclude that (r^*, l^*, V) is a bimodule of $(\mathcal{A}, \cdot)$.

(2) $\Rightarrow$ (1). The converse, (i.e. by supposing that (r^*, l^*, V) is a bimodule of $(\mathcal{A}, \cdot)$ then (l, r, V) is also a bimodule of $(\mathcal{A}, \cdot)$), can be proved by direct calculations by using similar relations as for the first part of the proof. □

Theorem 3.9. *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric JJ algebra. Suppose that there exists an anticenter-symmetric JJ algebra structure " $\circ$ " on its dual space $\mathcal{A}^*$. Then, $(\mathcal{A}, \mathcal{A}^*, R^*_\cdot, L^*_\cdot, R^*_\circ, L^*_\circ)$ is a matched pair of anticenter-symmetric JJ algebras $\mathcal{A}$ and $\mathcal{A}^*$ if and only if $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{A}^*), -\mathrm{ad}^*_\cdot, -\mathrm{ad}^*_\circ)$ is a matched pair of JJ algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{A}^*)$.*

Proof. By considering the Theorem 3.5, setting $l_{\mathcal{A}} = R^*_\cdot$, $r_{\mathcal{A}} = L^*_\cdot$, $l_{\mathcal{B}} = R^*_\circ$, $r_{\mathcal{B}} = L^*_\circ$, and exploiting the Definition 2.7 with $\mathcal{G} = \mathcal{G}(\mathcal{A})$, $\mathcal{H} = \mathcal{G}(\mathcal{A}^*)$, $\rho = R^*_\cdot + L^*_\cdot$, $\mu = R^*_\circ + L^*_\circ$, and the relations (24) and (25), we have

• The equation (6) is equivalent to both the equations (17) and (19), i.e.:

$$
\begin{aligned}
&(R^*_. + L^*_.)(x)[a,b] + [(R^*_. + L^*_.)(x)a, b] + [a, (R^*_. + L^*_.)(x)b] \\
&+(R^*_. + L^*_.)((R^*_\circ + L^*_\circ)(a)x)b + (R^*_. + L^*_.)((R^*_\circ + L^*_\circ(b)x)a \\
=& R^*_.(x)[a,b] + L^*_.(x)[a,b] + [R^*_.(x)a, b] + [L^*_.(x)a, b] + [a, R^*_.(x)b] \\
&+[a, L^*_.(x)b] + R^*_.(R^*_\circ(a)x)b + R^*_.(L^*_\circ(a)x)b + R_.(R^*_\circ(b)x)a \\
&+R^*_.(L^*_\circ(b)x)a + L^*_.(L^*_\circ(a)x)b + L^*_.(R_\circ(a)x)b + L^*_.(L^*_\circ(b)x)a + L^*_.(R^*_\circ(b)x)a \\
=& R^*_.(a\circ b) + R^*_.(b\circ a) + L^*_.(x)[a,b] + (R^*_.(x)a)\circ b + b\circ(R^*_.(x)a) \\
&+(L^*_.(x)a)\circ b + b\circ(L^*_.(x)a) + a\circ(R^*_.(x)b) + (R^*_.(x)b)\circ a + a\circ(L^*_.(x)b) \\
&+(L^*_.(x)b)\circ a + R^*_.(R^*_\circ(a)x)b + R^*_.(L^*_\circ(a)x)b + R_.(R^*_\circ(b)x)a + R^*_.(L^*_\circ(b)x)a \\
&+L^*_.(L^*_\circ(a)x)b + L^*_.(R_\circ(a)x)b + L^*_.(L^*_\circ(b)x)a + L^*_.(R^*_\circ(b)x)a \\
=& \{R^*_.(a\circ b) + R^*_.(R^*_\circ(a)x)b + R^*_.(L^*_\circ(a)x)b + (R^*_.(x)a)\circ b \\
&+(L^*_.(x)a)\circ b + L^*_.(L^*_\circ(b)x)a + a\circ(R^*_.(x)b)\} + \{R^*_.(b\circ a) \\
&+R^*_.(R^*_\circ(b)x)a + R^*_.(L^*_\circ(b)x)a + (R^*_.(x)b)\circ a + (L^*_.(x)b)\circ a \\
&+b\circ(R^*_.(x)a) + L^*_.(L^*_\circ(a)x)b\} + \{L^*_.(x)[a,b] + b\circ(L^*_.(x)a) \\
&+a\circ(L^*_.(x)b) + L^*_.(R_\circ(a)x)b + L^*_.(R_\circ(b)x)a\} = 0.
\end{aligned}
$$

The two first relations in brace on the last equality give zero (see (18)) and the last one brace also yields zero (see (17)).

• The equation (7) is equivalent to both the equations (19) and (20), i.e.:

$$
\begin{aligned}
&(R^*_\circ + L^*_\circ)(a)[x,y] + [(R^*_\circ + L^*_\circ)(a)x, y] + [x, (R^*_\circ + L^*_\circ)(a)y] \\
&+(R^*_\circ + L^*_\circ)((R^*_. + L^*_.)(x)a)y + (R^*_\circ + L^*_\circ)((R^*_. + L^*_.(y)a)x \\
=& R^*_.(a)[x,y] + L^*_.(a)[x,y] + [R^*_.(a)x, y] + [L^*_.(a)x, y] + [x, R^*_.(a)y] \\
&+[x, L^*_.(a)y] + R^*_.(R^*_\circ(x)a)y + R^*_.(L^*_\circ(x)a)y + R_.(R^*_\circ(y)a)x \\
&+R^*_.(L^*_\circ(y)a)x + L^*_.(L^*_\circ(x)a)y + L^*_.(R_\circ(x)a)y + L^*_.(L^*_\circ(y)a)x \\
&+L^*_.(R^*_\circ(y)a)x \\
=& R^*_.(x\circ y) + R^*_.(y\circ x) + L^*_.(a)[x,y] + (R^*_.(a)x)\circ y + y\circ(R^*_.(a)x) \\
&+(L^*_.(a)x)\circ y + y\circ(L^*_.(a)x) + x\circ(R^*_.(a)y) + (R^*_.(a)y)\circ x \\
&+x\circ(L^*_.(a)y) + (L^*_.(a)y)\circ x + R^*_.(R^*_\circ(x)a)y + R^*_.(L^*_\circ(x)a)y \\
&+R_.(R^*_\circ(y)a)x + R^*_.(L^*_\circ(y)a)x + L^*_.(L^*_\circ(x)a)y + L^*_.(R_\circ(x)a)y \\
&+L^*_.(L^*_\circ(y)a)x + L^*_.(R^*_\circ(y)a)x \\
=& \{R^*_.(x\circ y) + R^*_.(R^*_\circ(x)a)y + R^*_.(L^*_\circ(x)a)y + (R^*_.(a)x)\circ y \\
&+(L^*_.(a)x)\circ y + L^*_.(L^*_\circ(y)a)x + x\circ(R^*_.(a)y)\} + \{R^*_.(y\circ x) \\
&+R^*_.(R^*_\circ(y)a)x + R^*_.(L^*_\circ(y)a)x + (R^*_.(a)y)\circ x + (L^*_.(a)y)\circ x \\
&+L^*_.(L^*_\circ(x)a)y\} + \{L^*_.(a)[x,y] + y\circ(L^*_.(a)x) + x\circ(L^*_.(a)y) \\
&+y\circ(R^*_.(a)x) + L^*_.(R_\circ(x)a)y + L^*_.(R_\circ(y)a)x = 0.
\end{aligned}
$$

The two first relations in brace on the last equality gives zero (see (19)) and the last one brace also leads to zero (see (20)). □

4. Double construction

In this section, we define and establish the double construction of quadratic anticenter-symmetric JJ algebras.

Definition 4.1. We call $(\mathcal{A}, \mathfrak{B}_{\mathcal{A}})$ the *double construction* of anticenter-symmetric JJ algebras associated to $\mathcal{A}_1$ and $\mathcal{A}_1^*$ if it satisfies the conditions

(1) $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_1^*$ as the direct sum of vector spaces;

(2) $\mathcal{A}_1$ and $\mathcal{A}_1^*$ are anticenter-symmetric JJ subalgebras of $\mathcal{A}$;

(3) $\mathfrak{B}_{\mathcal{A}}$ is the natural non-degerenate invariant symmetric bilinear form on $\mathcal{A}_1 \oplus \mathcal{A}_1^*$ given by

$$\mathfrak{B}_{\mathcal{A}}(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle a^*, y \rangle \tag{30}$$

for all $x, y \in \mathcal{A}_1, a^*, b^* \in \mathcal{A}_1^*$ where $\langle , \rangle$ is the natural pair between the vector space $\mathcal{A}_1$ and its dual space $\mathcal{A}_1^*$.

Theorem 4.2. *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric JJ algebra. Suppose that there is an anticenter-symmetric JJ algebra structure " $\circ$ " on its dual space $\mathcal{A}^*$. Then, there is a double construction of a quadratic anticenter-symmetric JJ algebra associated to $(\mathcal{A}, \cdot)$ and $(\mathcal{A}^*, \circ)$ if and only if $(\mathcal{A}, \mathcal{A}^*, R_\cdot^*, L_\cdot^*; R_\circ^*, L_\circ^*)$ is a matched pair of anticenter-symmetric JJ algebras.*

Proof. By considering that $(\mathcal{A}, \mathcal{A}^*, R_\cdot^*, L_\cdot^*; R_\circ^*, L_\circ^*)$ is a matched pair of anticenter-symmetric JJ algebras, it follows that the bilinear product $*$ defined in the Theorem 3.5 is anticenter-symmetric JJ algebra on the direct sum of underlying vectors spaces, $\mathcal{A} \oplus \mathcal{A}^*$. We have $\forall x, y, z \in \mathcal{A}; a, b, c \in \mathcal{A}^*$.

$$\begin{aligned}
&\mathfrak{B}_{\mathcal{A}}((x + a) * (y + b), z + c) \\
&= \langle xy + R_\circ^*(a)y + L_\circ^*(b)x, c \rangle + \langle z, a \circ b + R_\cdot^*(x)b + L_\cdot^*(y)a \rangle \\
&= \langle xy, c \rangle + \langle R_\circ^*(a)y, c \rangle + \langle L_\circ^*(b)x, c \rangle + \langle z, a \circ b \rangle + \langle z, R_\cdot^*(x)b \rangle \\
&\quad + \langle z, L_\cdot^*(y)a \rangle = \langle xy, c \rangle + \langle y, R_a(c) \rangle + \langle x, L_b(c) \rangle + \langle z, a \circ b \rangle \\
&\quad + \langle R_x(z), b \rangle + \langle L_y(z), a \rangle = \langle xy, c \rangle + \langle y, c \circ a \rangle \\
&\quad + \langle x, b \circ c \rangle + \langle z, a \circ b \rangle + \langle zx, b \rangle + \langle yz, a \rangle .
\end{aligned}$$

$$
\begin{aligned}
&\mathfrak{B}_{\mathcal{A}}\left((x+a),(y+b)*(z+c)\right)\\
&=\langle x, b\circ c+R^*_{\cdot}(y)c+L^*_{\cdot}(z)b\rangle+<yz+R^*_{\circ}(b)z\\
&\quad+L^*_{\circ}(c)y,a>+\langle x,b\circ c\rangle+\langle x,R^*_{\cdot}(y)c\rangle+\langle x,L^*_{\cdot}(z)b\rangle\\
&\quad+\langle yz,a\rangle+\langle R^*_{\circ}(b)z,a\rangle+\langle L^*_{\circ}(c)y,a\rangle\\
&=\langle x,b\circ c\rangle+\langle R_y(x),c\rangle+\langle L_z(x),b\rangle\\
&\quad+\langle yz,a\rangle+\langle z,R_b(a)\rangle+\langle y,L_c(a)\rangle\\
&=\langle x,b\circ c\rangle+\langle xy,c\rangle+\langle zx,b\rangle+\langle yz,a\rangle\\
&\quad+\langle z,a\circ b\rangle+\langle y,c\circ a\rangle.
\end{aligned}
$$

Therefore, the following relation

$$\mathfrak{B}_{\mathcal{A}}((x+a)*(y+b),(z+c))=\mathfrak{B}_{\mathcal{A}}\left((x+a),(y+b)*(z+c)\right) \tag{31}$$

holds, which expresses the invariance of the standard bilinear form on $\mathcal{A}\oplus\mathcal{A}^*$. Therefore, $(A\oplus\mathcal{A}^*,B)$ is the standard double construction of the anticenter-symmetric JJ algebras $\mathcal{A}$ and $\mathcal{A}^*$. □

Proposition 4.3. *Let $(\mathcal{A},\cdot)$ be an anticenter-symmetric JJ algebra and $(\mathcal{A}^*,\circ)$ be an anticenter-symmetric JJ algebra structure on its dual space $\mathcal{A}^*$. Then the following conditions are equivalent:*

(1) *$(\mathcal{A}\oplus\mathcal{A}^*,\mathfrak{B}_{\mathcal{A}})$ is the standard double construction of considered anticenter-symmetric JJ algebras;*

(2) *$(\mathcal{G}(\mathcal{A}),\mathcal{G}(\mathcal{A}^*),-\mathrm{ad}^*_{\cdot},-\mathrm{ad}^*_{\circ})$ is a matched pair of sub-adjacent JJ algebras;*

(3) *$(\mathcal{A},\mathcal{A}^*,R^*_{\cdot},L^*_{\cdot},R^*_{\circ},L^*_{\circ})$ is a matched pair of anticenter-symmetric JJ algebras.*

Proof. From Theorem 3.9, (2) $\Leftrightarrow$ (3), while from Theorem 4.2 shows that (1) $\Leftrightarrow$ (3). Then (1) $\Leftrightarrow$ (2). □

5. Anti-Zinbiel algebra

In this section we provide a particular class of an anticenter symmetric JJ algebra namely an anti-Zinbiel algebra. We derive some important results such as Proposition 5.5, Theorem 5.6, Theorem 5.8, Proposition 5.13 and Theorem 5.15.

Definition 5.1. Consider a vector space $\mathcal{A}$ equipped with a bilinear product $* : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

• The couple $(\mathcal{A}, *)$ is called a *left anti-Zinbiel algebra* if for all $x, y, z \in \mathcal{A}$,

$$(x * y) * z = -x * (y * z) + x * (z * y), \tag{32}$$

which is equivalent to

$$(x, y, z)_{-1} = x * (z * y). \tag{33}$$

• The couple $(\mathcal{A}, *)$ is called a *right anti-Zinbiel algebra* if for all $x, y, z \in \mathcal{A}$,

$$x * (y * z) = -(x * y) * z + (y * x) * z, \tag{34}$$

which is equivalent to

$$(x, y, z)_{-1} = (y * x) * z. \tag{35}$$

Remark 5.2. By setting $\mu = * : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, id the identity map on a vector space $\mathcal{A}$, and τ the exchange map on $\mathcal{A} \otimes \mathcal{A}$, then the equation (32) is equivalent to

$$\mu \circ (\mu \otimes \mathrm{id}) = -\mu \circ (\mathrm{id} \otimes \mu) + \mu \circ (\mathrm{id} \otimes (\mu \circ \tau)), \tag{36}$$

and the equation (34) is equivalent to

$$\mu \circ (\mathrm{id} \otimes \mu) = -\mu \circ (\mu \otimes \mathrm{id}) + \mu \circ ((\mu \circ \tau) \otimes \mathrm{id}). \tag{37}$$

Proposition 5.3. *Consider a vector space $\mathcal{A}$ equipped with a bilinear product $* := \mu$. Suppose τ be the exchange map defined on $\mathcal{A} \otimes \mathcal{A}$ and* id *the identity map on $\mathcal{A}$.*

(1) *If $(\mathcal{A}, *)$ is a left anti-Zinbiel algebra, then the following relation holds*

$$-x * (y * z) = y * (x * z), \forall x, y, z \in \mathcal{A}, \tag{38}$$

which is equivalent to

$$-\mu \circ (\mathrm{id} \otimes \mu) = \mu \circ (\mathrm{id} \otimes \mu) \circ (\tau \otimes \mathrm{id}), \tag{39}$$

(2) *If $(\mathcal{A}, *)$ is a right anti-Zinbiel algebra, then the following relation holds $\forall x, y, z \in \mathcal{A}$,*

$$-(x * y) * z = (x * z) * y, \tag{40}$$

which is equivalent to

$$-\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau). \tag{41}$$

Proposition 5.4. *The opposite algebra of the left (right) anti-Zinbiel algebra is a right (left) anti-Zinbiel algebra under the same underlying vector space.*

In the following of this note, without any further clarification, both left and right anti-Zinbiel algebra are called an anti-Zinbiel algebra.

Proposition 5.5. *Let $(\mathcal{A}, *)$ be an anti-Zinbiel algebra. Then, $(\mathcal{A}, \{,\}_*)$ is an anticommutative antiassociative algebra, where $\{x, y\}_* = x*y - y*x$ for all $x, y \in \mathcal{A}$.*

Proof. Consider the anti-Zinbiel algebra $(\mathcal{A}, *)$. By definition, the product given by, for all $x, y \in \mathcal{A}$, $\{x, y\}_* = x*y - y*x$ is bilinear and anticommutative. Then it remains to prove that

$$\forall x, y, z \in \mathcal{A},\ \ \{x, \{y, z\}_*\}_* + \{\{x, y\}_*, z\}_* = 0.$$

We have for all $x, y, z \in \mathcal{A}$.

$$\begin{aligned}
\{x, \{y, z\}_*\}_* &= x*(y*z - z*y) - (y*z - z*y)*x \\
&= x*(y*z) - x*(z*y) - (y*z)*x + (z*y)*x \\
&= -(x*y)*z + (y*x)*z + (x*z)*y - (z*x)*y \\
&\quad -(y*z)*x + (z*y)*x \\
&= (-(z*x)*y + (x*z)*y) + ((z*y)*x - (y*z)*x) \\
&\quad -(x*y)*z + (y*x)*z \\
&= z*(x*y) - z*(y*x) - (x*y)*z + (y*x)*z \\
&= z*(x*y - y*x) - (x*y - y*x)*z \\
&= -\{\{x, y\}_*, z\}_*.
\end{aligned}$$

Therefore the bilinear product $\{,\}_*$ gives an antiassociative algebra structure on $\mathcal{A}$ and hence the algebra $(\mathcal{A}, \{,\}_*)$ is an anticommutative antiassociative algebra. □

Theorem 5.6. *Let $(\mathcal{A}, *)$ be an anti-Zinbiel algebra. Then, $(\mathcal{A}, [,]_*)$ is a JJ algebra, where for all $x, y \in \mathcal{A}$, $[x, y]_* = x*y + y*x$.*

Proof. Consider an anti-Zinbiel algebra $(\mathcal{A}, * = \mu)$. By definition, the product $[x, y]_* = x*y + y*x$ is bilinear product is bilinear and skew symmetric. In addition, we have, for all $x, y, z \in \mathcal{A}$,

$$\begin{aligned}
[x, [y, z]_*]_* + [y, [z, x]_*]_* + [z, [x, y]_*]_* &= [x, y*z + z*y]_* + [y, z*x + x*z]_* \\
&\quad + [z, x*y + x*y]_*
\end{aligned}$$

$$
\begin{aligned}
&= [x, y * z]_* + [y, z * x]_* + [z, x * y]_* \\
&\quad + [x, z * y]_* + [y, x * z]_* + [z, y * x]_* \\
&= x * (y * z) + (y * z) * x + y * (z * x) \\
&\quad + (z * x) * y + z * (x * y) + (x * y) * z \\
&\quad x * (z * y) + (z * y) * x + y * (x * z) \\
&\quad + (x * z) * y + z * (y * x) + (y * x) * z \\
&= \{x * (y * z) + y * (x * z)\} + \{y * (z * x) \\
&\quad + z * (y * x)\} + \{z * (x * y) + x * (z * y)\} \\
&\quad + \{(y * x) * z + (y * z) * x\} + \{(z * y) * x \\
&\quad + (z * x) * y\} + \{(x * z) * y + (x * y) * z\}.
\end{aligned}
$$

By using the relations (38), (39), (41), (40), and the Proposition 5.4, in the right hand side of the last equation, we get that for all $x, y, z \in \mathcal{A}$,

$$[x, [y, z]_*]_* + [y, [z, x]_*]_* + [z, [x, y]_*]_* = 0,$$

i.e. the Jacobi identity and then $(\mathcal{A}, [,]_*)$ is a JJ algebra. □

In the following of this note, we denote the subadjacent anticommutative antiassociative algebra of an anti-Zinbiel algebra $(\mathcal{A}, *)$ by $\mathcal{G}(\mathcal{A})_{ass}$ and that of underlying Lie algebra of $(\mathcal{A}, *)$ by $\mathcal{G}(\mathcal{A})$.

Remark 5.7. According to the Theorem 5.6, anti-Zinbiel algebras are JJ admissible algebras, i.e. their anticommutators define JJ algebras.

Theorem 5.8. *Any anti-Zinbiel algebra is an anticenter-symmetric JJ algebra.*

Proof. Consider an anti-Zinbiel algebra $(\mathcal{A}, \circ)$. In view of the equality (33), we have for all $x, y, z \in \mathcal{A}$, $(x, y, z)_{-1,\circ} = x \circ (z \circ y)$. Using the relation (38), $x \circ (z \circ y) = -z \circ (x \circ y)$ we obtain, for all $x, y, z \in \mathcal{A}$, $(x, y, z)_{-1,\circ} = -z \circ (x \circ y) = (z, y, x)_{-1,\circ}$ which is exactly the relation (9). Therefore, $(\mathcal{A}, \circ)$ is an anticenter-symmetric JJ algebra. □

Proposition 5.9. *Let $(\mathcal{A}, \mu)$ be an anti-Zinbiel algebra, $\tau : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\tau(x \otimes y) = y \otimes x$ be the exchange operator defined on $\mathcal{A}$. Then*

$$\mu \circ (\mathrm{id} \otimes (\mu \circ \tau)) = -\mu \circ (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) + (\mu \circ \tau) \circ (\mathrm{id} \otimes (\mu \circ \tau)) \circ (\tau \otimes \mathrm{id}), \quad (42)$$

$$(\mu \circ \tau) \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) - (\mu \circ \tau) \circ (\mathrm{id} \otimes (\mu \circ \tau)) \circ (\tau \otimes \mathrm{id}). \quad (43)$$

Proof. Consider an anti-Zinbiel algebra $(\mathcal{A}, \mu)$, i.e. $\mathcal{A}$ is a vector space and μ satisfies the relation

$$\mu \circ (\mathrm{id} \otimes \mu) = \mu \circ (\mu \otimes \mathrm{id}) + \mu \circ ((\mu \circ \tau) \otimes \mathrm{id}).$$

We have for all $x, y, z \in \mathcal{A}$,

$$\begin{aligned}\mu\circ(\mathrm{id} \otimes(\mu\circ\tau))(x, y, z) &= \mu(x, z * y) = x * (z * y) = -(x * z) * y + (z * x) * y \\ &= \mu(x * z, y) + \mu \circ \tau(y, z * x) \\ &= -\mu \circ (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes\tau)(x, y, z) \\ &\quad +(\mu \circ \tau) \circ (\mathrm{id} \otimes(\mu \circ \tau)) \circ (\tau \otimes \mathrm{id})(x, y, z).\end{aligned}$$

Therefore holds the equality

$$\begin{aligned}\mu \circ (\mathrm{id} \otimes(\mu \circ \tau)) &= -\mu \circ (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes\tau) \\ &\quad +(\mu \circ \tau) \circ (\mathrm{id} \otimes(\mu \circ \tau)) \circ (\mathrm{id} \otimes\tau) \circ (\tau \otimes \mathrm{id}),\end{aligned}$$

which is the relation (42).

Similarly, for all $x, y, z \in \mathcal{A}$,

$$\begin{aligned}(\mu \circ \tau) \circ (\mu \otimes \mathrm{id})(x, y, z) &= (\mu \circ \tau)(x * y, z) = z * (x * y) \\ &= -(z * x) * y + (x * z) * y \\ &= -(\mu \circ \tau)(y, z * x) + \mu(x * z, y) \\ &= -(\mu \circ \tau)(y, (\mu \circ \tau)(x, z)) + \mu(x * z, y) \\ &= -(\mu \circ \tau)(\mathrm{id} \otimes(\mu \circ \tau))(\tau \otimes \mathrm{id})(x, y, z) \\ &\quad +\mu \circ (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes\mu)(x, y, z) \\ &= -((\mu \circ \tau)(\mathrm{id} \otimes(\mu \circ \tau))(\tau \otimes \mathrm{id}) \\ &\quad +\mu \circ (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes\mu))(x, y, z),\end{aligned}$$

then the following relation is satisfied

$$(\mu \circ \tau) \circ (\mu \otimes \mathrm{id}) = -(\mu \circ \tau)(\mathrm{id} \otimes(\mu \circ \tau))(\tau \otimes \mathrm{id}) + \mu \circ (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes\mu),$$

which is exactly the equation (43). □

Proposition 5.10. *Let $(\mathcal{A}, \cdot)$ be an anti-Zinbiel algebra. Suppose R and L the right and left multiplication representation, respectively. The following relations hold, $\forall x, y \in \mathcal{A}$*

$$L_x L_y = -L_{x\cdot y} + L_{y\cdot x}, \tag{44}$$

$$L_x R_y = -R_y L_x + R_y R_x = -R_{x\cdot y}. \tag{45}$$

5.1. Bimodule and matched pair of Zinbiel algebras

Definition 5.11. A *bimodule of an anti-Zinbiel algebra* is a triple $(l, r, \mathcal{A})$, where $\mathcal{A}$ is a vector space equipped with an anti-Zinbiel algebra structure "$\cdot$", V a vector space and $l, r : \mathcal{A} \to \mathfrak{gl}(V)$ two linear maps satisfying the following relations, for all $x, y \in \mathcal{A}$,

$$l_x l_y = -l_{x\cdot y} + l_{y\cdot x} \tag{46}$$

$$l_x r_y = -r_{x\cdot y} = r_y r_x - r_y l_x. \tag{47}$$

Example 5.12. Let $(\mathcal{A}, \cdot)$ be an anti-Zinbiel algebra. Suppose L and R be respectively the representations of the left and right multiplication operators. Then $(L, R, \mathcal{A})$ is a bimodule of the anti-Zinbiel algebra $(\mathcal{A}, \cdot)$.

Proposition 5.13. *Let $(\mathcal{A}, \cdot)$ be an anti-Zinbiel algebra. Consider a vector space V over the field $\mathbb{K}$ and two linear maps given by $l, r : \mathcal{A} \to \mathfrak{gl}(V)$. Then the triple (l, r, V) is a bimodule of $\mathcal{A}$ if and only if there is an anti-Zinbiel algebra structure on the vector space $\mathcal{A}\oplus V$ given by, for all $x, y \in \mathcal{A}$, $u, v \in V$*

$$(x + u) * (y + v) = x \cdot y + (l_x v + r_y u). \tag{48}$$

Proof. Suppose that (l, r, V) is a bimodule of the anti-Zinbiel algebra $(\mathcal{A}, \cdot)$ and consider the bilinear product " $*$ " defined on the vector space $\mathcal{A} \oplus V$ by: $\forall x, y \in \mathcal{A}, u, v \in V$, $(x + u) * (y + v) = x \cdot y + l_x v + r_y u$. We have for all $x, y \in \mathcal{A}$, and for all $u, v \in V$,

$$\{(x + u) * (y + v)\} * (z + w) = (x \cdot y) \cdot z + l_{x\cdot y} w + r_z l_x v + r_z r_y u, \tag{49}$$

$$\{(y + v) * (x + u)\} * (z + w) = (y \cdot x) \cdot z + l_{y\cdot x} w + r_z l_y u + r_z r_x v, \tag{50}$$

$$(x + u) * \{(y + v) * (z + w)\} = x \cdot (y \cdot z) + l_x l_y w + l_x r_z v + r_{y\cdot z} u. \tag{51}$$

Since (l, r, V) is a bimodule, then

$$\begin{aligned}\{(x + u) * (y + v)\} * (z + w) + \{(y + v) * (x + u)\} * (z + w)\\ = (x + u) * \{(y + v) * (z + w)\},\end{aligned}$$

is equivalent to: $(\mathcal{A}, \cdot)$ is an anti-Zinbiel algebra and the relations (46) (47) are satisfied.

Conversely, if $(\mathcal{A}, \cdot)$ is an anti-Zinbiel algebra and the bilinear maps l, r satisfying the relations (46) and (47), there is an anti-Zinbiel algebra structure on $\mathcal{A} \oplus V$ given by, $\forall x, y \in \mathcal{A}$, $u, v \in V$, $(x + u) * (y + v) = x \cdot y + l_x v + r_y u$. □

Proposition 5.14. *Let (l, r, V) be a bimodule of an anti-Zinbiel algebra $(\mathcal{A}, \cdot)$, where V is a vector space and $l, r : \mathcal{A} \to \mathfrak{gl}(V)$ are two linear maps. Then the following conditions are satisfied, for all $x, y \in A$,*

$$l_{xy} + r_y l_x = 0, \tag{52}$$

$$l_x l_y + l_y l_x = 0. \tag{53}$$

Proof. In view of the definition of the bimodule of an anti-Zinbiel algebra, we deduce that the linear maps l, r satisfy the relations (46) and (47). According to Proposition 5.13, it follows that the bilinear product given in the relation (48) satisfies the relation (38) (or (40)) that implies the relations (52) and (53). □

Theorem 5.15. *Let $(\mathcal{A}, \cdot)$ and $(\mathcal{B}, \circ)$ be two anti-Zinbiel algebras. Suppose that $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B})$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A})$ are bimodules of $\mathcal{A}$ and $\mathcal{B}$, respectively, obeying the relations:*

$$r_B(a)(y \cdot x - x \cdot y) - x \cdot (r_B(a)y) - r_B(l_A(y)a)x = 0, \tag{54}$$

$$r_A(x)(b \circ a - a \circ b) - a \circ (r_A(x)b) - r_A(l_B(b)x)a = 0, \tag{55}$$

$$l_B(a)(x \cdot y) \;=\; l_B((l_A - r_A)(x)a)y - ((l_B - r_B)(a)x) \cdot y, \tag{56}$$

$$l_A(x)(a \circ b) \;=\; l_A((l_B - r_B)(a)x)b - ((l_A - r_A)(x)a) \circ b, \tag{57}$$

for all $x, y \in A$ and $a, b \in \mathcal{B}$. Then, there is an anti-Zinbiel algebra structure on $\mathcal{A} \oplus \mathcal{B}$ given by:

$$(x+a) * (y+b) = (x \cdot y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) + (a \circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a). \tag{58}$$

We denote the anti-Zinbiel algebra given on $\mathcal{A} \oplus \mathcal{B}$ by $\mathcal{A} \bowtie^{l_{\mathcal{A}}, r_{\mathcal{A}}}_{l_{\mathcal{B}}, r_{\mathcal{B}}} \mathcal{B}$, or simply by $\mathcal{A} \bowtie \mathcal{B}$. Then $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ satisfying the above conditions is called the *matched pair of the anti-Zinbiel algebras* $\mathcal{A}$ and $\mathcal{B}$.

Proof. Suppose $x, y, z \in \mathcal{A}$ and $a, b, c \in \mathcal{B}$. By using the bilinear product $*$ defined on $\mathcal{A} \oplus \mathcal{B}$ as $(x+a) * (y+b) = \{x \cdot y + l_{\mathcal{B}}(x)y + r_{\mathcal{B}}(b)x\} + \{a \circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a\}$, such that $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B})$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A})$ define a bimodule of $\mathcal{A}$ and $\mathcal{B}$ respectively, we easily get:

$$\begin{aligned}&\{(x+a) * (y+b)\} * (z+c) = (x \cdot y) \cdot z + ((l_{\mathcal{B}}(a)y) \cdot z + l_{\mathcal{B}}(r_{\mathcal{A}}(y)az))\\ &+l_{\mathcal{B}}(a \circ b)z + r_{\mathcal{B}}(c)(x \cdot y) + ((r_{\mathcal{B}}(b)x) \cdot z + l_{\mathcal{B}}(l_{\mathcal{A}}(x)b)z) + r_{\mathcal{B}}(c)(l_{\mathcal{B}}(a)y)\\ &+r_{\mathcal{B}}(c)(r_{\mathcal{B}}(b)x) + (a \circ b) \circ c + ((l_{\mathcal{A}}(x)b) \circ c + l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)c) + l_{\mathcal{A}}(x \cdot y)c)\\ &+r_{\mathcal{A}}(z)(a \circ b + r_{\mathcal{A}}(z)(l_{\mathcal{A}}(x)b) + ((r_{\mathcal{A}}(y)a) \circ c + l_{\mathcal{A}}(l_{\mathcal{B}}(a)y)c)\\ &+r_{\mathcal{A}}(z)(r_{\mathcal{A}}(y)a),\end{aligned} \tag{59}$$

$$\begin{aligned}&\{(y+b) * (x+a)\} * (z+c) = (y \cdot x) \cdot z + ((l_{\mathcal{B}}(b)x) \cdot z + l_{\mathcal{B}}(r_{\mathcal{A}}(x)bz))\\ &+l_{\mathcal{B}}(b \circ a)z + r_{\mathcal{B}}(c)(y \cdot x) + ((r_{\mathcal{B}}(a)y) \cdot z + l_{\mathcal{B}}(l_{\mathcal{A}}(y)a)z) + r_{\mathcal{B}}(c)(l_{\mathcal{B}}(b)x)\\ &+r_{\mathcal{B}}(c)(r_{\mathcal{B}}(a)y) + (b \circ a) \circ c + ((l_{\mathcal{A}}(y)a) \circ c + l_{\mathcal{A}}(r_{\mathcal{B}}(a)y)c) + l_{\mathcal{A}}(y \cdot x)c\\ &+r_{\mathcal{A}}(z)(b \circ a) + ((r_{\mathcal{A}}(x)b) \circ c + l_{\mathcal{A}}(l_{\mathcal{B}}(b)x)c) + r_{\mathcal{A}}(z)(l_{\mathcal{A}}(y)a) + r_{\mathcal{A}}(z)(r_{\mathcal{A}}(x)b),\end{aligned} \tag{60}$$

$$\begin{aligned}&(x+a) * \{(y+b) * (z+c)\} = x \cdot (y \cdot z) + (x \cdot (l_{\mathcal{B}}(b)z) + r_{\mathcal{B}}(r_{\mathcal{A}}(z)b)x)\\ &+l_{\mathcal{B}}(a)(y \cdot z) + l_{\mathcal{B}}(a)(l_{\mathcal{B}}(b)z) + l_{\mathcal{B}}(a)(r_{\mathcal{B}}(c)y) + r_{\mathcal{B}}(b \circ c)x + (x \cdot (r_{\mathcal{B}}(c)y)\\ &+r_{\mathcal{B}}(l_{\mathcal{A}}(y)c)x) + a \circ (b \circ c) + (a \circ (l_{\mathcal{A}}(y)c) + r_{\mathcal{A}}(r_{\mathcal{B}}(c)y)a) + (a \circ (r_{\mathcal{A}}(z)b)\\ &+r_{\mathcal{A}}(l_{\mathcal{B}}(b)z)a) + l_{\mathcal{A}}(x)(b \circ c) + l_{\mathcal{A}}(x)(l_{\mathcal{A}}(y)c) + l_{\mathcal{A}}(x)(r_{\mathcal{A}}(z)b) + r_{\mathcal{A}}(y \cdot z)a.\end{aligned} \tag{61}$$

The relation (60)–(59)=(61) is equivalent to the following results: $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B})$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A})$ are bimodule of the Zinbiel algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, $l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}$ and $r_{\mathcal{B}}$ satisfying the equations (54), (55), (56) and (57). □

6. 2-dimensional anticenter-symmetric JJ algebras

Let $\mathcal{A}$ be an anticenter-symmetric JJ algebra such that there is an anticenter-symmetric JJ structure " $\circ$ " on its dual space $\mathcal{A}^*$ spanned by $\{e_1, e_2\}$ and $\{e_1^*, e_2^*\}$ respectively. Formula (9) leads to the following relations:

$$(e_i \cdot e_j) \cdot e_k + e_i \cdot (e_j \cdot e_k) = (e_k \cdot e_j) \cdot e_i + e_k \cdot (e_j \cdot e_i), \quad \text{where } i, j, k = 1, 2.$$

Let $e_1 \cdot e_1 = a_1 e_1 + a_2 e_2$, $e_1 \cdot e_2 = b_1 e_1 + b_2 e_2$, $e_2 \cdot e_1 = c_1 e_1 + c_2 e_2$, $e_2 \cdot e_2 = d_1 e_1 + d_2 e_2$, where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{C}$.

Proposition 6.1. *$e_i \cdot e_j = 0$ and $e_1 \cdot e_1 = e_2$ are 2-dimensional non-isomorphic anticenter-symmetric JJ classes.*

Proof. Let $\mathcal{A}$ be a 2-dimensional anticenter-symmetric JJ algebra with basis $\{e_1, e_2\}$. Suppose $x, y, z \in \mathcal{A}$ such that $x = x_1e_1 + x_2e_2$, $y = y_1e_1 + y_2e_2$ and $z = z_1e_1 + z_2e_2$ with $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{C}$. We have eight relations given by

$$(e_i, e_j, e_k) = -(e_k, e_j, e_i), \quad i, j, k = 1, 2.$$

Setting $e_1e_1 = a_1e_1 + a_2e_2$, $e_1e_2 = b_1e_1 + b_2e_2$, $e_2e_1 = c_1e_1 + c_2e_2$ and $e_2e_2 = d_1e_1 + d_2e_2$, in the previous relations one have the following eight relations equivalent to a system of 16 equations:

$$\begin{cases} 2a_1^2 + a_2c_1 + a_2b_1 = 0, \\ 2a_1a_2 + a_2c_2 + a_2b_2 = 0,, \\ 2a_1b_1 + 2a_1c_1 + 2a_2d_1 + b_2b_1 + c_2c_1 = 0, \\ a_2b_2 + 2a_2d_2 + b_1a_2 + b_2^2 + c_1a_2 + a_1c_2 + c_2^2 = 0, \\ b_1a_1 + c_1a_1 + b_2c_1 + c_2b_1 = 0, \\ b_1a_2 + c_1a_2 + 2c_2b_2 = 0, \\ b_1^2 + 2d_1a_1 + c_1^2 + +c_2d_1 + d_2c_1 + b_2d_1 + d_2b_1 = 0 \\ 2b_2d_2 + b_1b_2 + 2d_1a_2 + 2c_2d_2 + c_1c_2 = 0, \\ 2c_1a_1 + 2a_1b_1 + 2a_2d_1 + c_2c_1 + b_2b_1 = 0, \\ 2a_2d_2 + c_2^2 + b_2^2 + c_1a_2 + a_1c_2 + a_1b_2 + b_1a_2 = 0, \\ 2c_1b_1 + c_2d_1 + b_2d_1 = 0, \\ c_1b_2 + c_2d_2 + b_1c_2 + b_2d_2 = 0, \\ 2a_1d_1 + d_2c_1 + c_2d_1 + c_1^2 + b_1^2 + b_2d_1 + d_2b_1 = 0, \\ 2d_1a_2 + 2d_2d_1 + c_1c_2 + 2b_2d_2 + b_1b_2 = 0, \\ d_1b_1 + 2d_2d_1 + d_1c_1 = 0, \\ d_1b_2 + 2d_2^2 + d_1c_2 = 0. \end{cases}$$

For

$a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ b_2 = 0 \Rightarrow c_2 = 0,\ d_2 = 0,\ c_1 = 0,\ b_1 = 0,$
class: $e_2e_2 = d_1e_1$.

For

$a_1 =,\ a_2 \neq 0,\ b_1 = 0 \Rightarrow c_1 = 0,\ d_1 = 0,\ b_2 = 0,\ d_2 = 0,\ c_2 = 0,$
class: $e_1e_1 = a_2e_2$.

Therefore, by isomorphism we get $e_1e_1 = e_2$ and $e_ie_j = 0$. □

In the following we discuss the double constructions.

CASE I. $e_1 \cdot e_1 = e_2$. The considered product on the dual space is $e_2^* \circ e_2^* = e_1^*$.

Using relation (21) when $l_{\mathcal{A}} = R^*$, $r_{\mathcal{A}} = L^*$, $l_{\mathcal{B}} = l_{\mathcal{A}^*} = R^*_\circ$, $r_{\mathcal{B}} = r_{\mathcal{A}^*} = L^*_\circ$, we obtain the double construction $(\mathcal{A} \oplus \mathcal{A}^*, *, \mathcal{B})$ associated to $(\mathcal{A}, \cdot)$ and $(\mathcal{A}^*, \circ)$ given explicitly by the following relations:

$$
\begin{aligned}
(e_1+e_1^*)*(e_1+e_1^*) &= (e_1\cdot e_1+R_\circ^*(e_1^*)e_1+L_\circ^*(e_1^*)e_1)+(e_1^*\circ e_1^*+R_\cdot^*(e_1)e_1^*+L_\cdot^*(e_1)e_1^*),\\
&= e_2,\\
(e_1+e_1^*)*(e_1+e_2^*) &= (e_1\cdot e_1+R_\circ^*(e_1^*)e_1+L_\circ^*(e_2^*)e_1)+(e_1^*\circ e_2^*+R_\cdot^*(e_1)e_2^*+L_\cdot^*(e_1)e_1^*),\\
&= 2e_2+e_1^*,\\
(e_1+e_1^*)*(e_2+e_1^*) &= (e_1\cdot e_2+R_\circ^*(e_1^*)e_2+L_\circ^*(e_1^*)e_1)+(e_1^*\circ e_1^*+R_\cdot^*(e_1)e_1^*+L_\cdot^*(e_2)e_1^*),\\
&= 0,\\
(e_1+e_1^*)*(e_2+e_2^*) &= (e_1\cdot e_2+R_\circ^*(e_1^*)e_2+L_\circ^*(e_2^*)e_1)+(e_1^*\circ e_2^*+R_\cdot^*(e_1)e_2^*+L_\cdot^*(e_2)e_1^*),\\
&= e_2+e_1^*,\\
(e_1+e_2^*)*(e_1+e_1^*) &= (e_1\cdot e_1+R_\circ^*(e_2^*)e_1+L_\circ^*(e_1^*)e_1)+(e_2^*\circ e_1^*+R_\cdot^*(e_1)e_1^*+L_\cdot^*(e_1)e_1^*),\\
&= 2e_2,\\
(e_1+e_2^*)*(e_1+e_2^*) &= (e_1\cdot e_1+R_\circ^*(e_2^*)e_1+L_\circ^*(e_2^*)e_1)+(e_2^*\circ e_2^*+R_\cdot^*(e_1)e_2^*+L_\cdot^*(e_1)e_2^*),\\
&= 3e_2+3e_1^*,\\
(e_2+e_1^*)*(e_1+e_1^*) &= (e_2\cdot e_1+R_\circ^*(e_1^*)e_1+L_\circ^*(e_2^*)e_1)+(e_1^*\circ e_1^*+R_\cdot^*(e_2)e_1^*+L_\cdot^*(e_1)e_1^*),\\
&= e_2,\\
(e_2+e_1^*)*(e_1+e_2^*) &= (e_2\cdot e_1+R_\circ^*(e_1^*)e_1+L_\circ^*(e_2^*)e_2)+(e_1^*\circ e_2^*+R_\cdot^*(e_2)e_2^*+L_\cdot^*(e_1)e_1^*),\\
&= 0,\\
(e_2+e_2^*)*(e_1+e_1^*) &= (e_2\cdot e_1+R_\circ^*(e_2^*)e_1+L_\circ^*(e_1^*)e_2)+(e_2^*\circ e_1^*+R_\cdot^*(e_2)e_1^*+L_\cdot^*(e_1)e_2^*),\\
&= e_2+e_1^*,\\
(e_2+e_2^*)*(e_1+e_2^*) &= (e_2\cdot e_1+R_\circ^*(e_2^*)e_1+L_\circ^*(e_2^*)e_2)+(e_2^*\circ e_2^*+R_\cdot^*(e_2)e_2^*+L_\cdot^*(e_1)e_2^*),\\
&= e_2+2e_1^*,\\
(e_2+e_2^*)*(e_2+e_1^*) &= (e_2\cdot e_2+R_\circ^*(e_2^*)e_2+L_\circ^*(e_1^*)e_2)+(e_2^*\circ e_1^*+R_\cdot^*(e_2)e_1^*+L_\cdot^*(e_2)e_2^*),\\
&= 0,\\
(e_2+e_2^*)*(e_2+e_2^*) &= (e_2\cdot e_2+R_\circ^*(e_2^*)e_2+L_\circ^*(e_2^*)e_2)+(e_2^*\circ e_2^*+R_\cdot^*(e_2)e_2^*+L_\cdot^*(e_2)e_2^*),\\
&= e_1^*,\\
(e_1+e_2^*)*(e_2+e_1^*) &= (e_1\cdot e_2+R_\circ^*(e_2^*)e_2+L_\circ^*(e_1^*)e_1)+(e_2^*\circ e_1^*+R_\cdot^*(e_1)e_1^*+L_\cdot^*(e_2)e_2^*),\\
&= 0,\\
(e_1+e_2^*)*(e_2+e_2^*) &= (e_1\cdot e_2+R_\circ^*(e_2^*)e_2+L_\circ^*(e_2^*)e_1)+(e_2^*\circ e_2^*+R_\cdot^*(e_1)e_2^*+L_\cdot^*(e_2)e_2^*),\\
&= e_2+2e_1^*,\\
(e_2+e_1^*)*(e_2+e_1^*) &= (e_2\cdot e_2+R_\circ^*(e_1^*)e_2+L_\circ^*(e_1^*)e_2)+(e_1^*\circ e_1^*+R_\cdot^*(e_2)e_1^*+L_\cdot^*(e_2)e_1^*),\\
&= 0,\\
(e_2+e_1^*)*(e_2+e_2^*) &= (e_2\cdot e_2+R_\circ^*(e_1^*)e_2+L_\circ^*(e_2^*)e_2)+(e_1^*\circ e_2^*+R_\cdot^*(e_2)e_2^*+L_\cdot^*(e_2)e_1^*),\\
&= 0.
\end{aligned}
$$

CASE II. $e_2 \cdot e_2 = e_1$. The product on the dual space is given by the following relations: $e_i^* \circ e_j^* = 0$. The double construction $(\mathcal{A} \oplus \mathcal{A}^*, *, \mathcal{B})$ associated to $(\mathcal{A}, \cdot)$ and $(\mathcal{A}^*, \circ)$ is given explicitly by the following relations:

$$\begin{aligned}
&(e_1+e_1^*)*(e_1+e_1^*) = 0,\\
&(e_1+e_1^*)*(e_1+e_2^*) = 0,\\
&(e_1+e_1^*)*(e_2+e_1^*) = e_2^*,\\
&(e_1+e_1^*)*(e_2+e_2^*) = e_2^*,\\
&(e_1+e_2^*)*(e_1+e_1^*) = 0,\\
&(e_1+e_2^*)*(e_1+e_2^*) = 0,\\
&(e_2+e_1^*)*(e_1+e_1^*) = e_2^*,\\
&(e_2+e_1^*)*(e_1+e_2^*) = 0,\\
&(e_2+e_2^*)*(e_1+e_1^*) = e_2^*,\\
&(e_2+e_2^*)*(e_1+e_2^*) = 0,\\
&(e_2+e_2^*)*(e_2+e_1^*) = e_1+e_2^*,\\
&(e_2+e_2^*)*(e_2+e_2^*) = e_1,\\
&(e_1+e_2^*)*(e_2+e_1^*) = 0,\\
&(e_1+e_2^*)*(e_2+e_2^*) = 0,\\
&(e_2+e_1^*)*(e_2+e_1^*) = e_1+2e_2^*,\\
&(e_2+e_1^*)*(e_2+e_2^*) = e_1+e_2^*.
\end{aligned}$$

7. Concluding remarks

In this work, we defined anticenter-symmetric JJ algebras and JJ admissible algebras and, discussed their bimodule and matched pair. We established the double construction of quadratic anticenter-symmetric JJ algebras. We gave main properties and some algebraic structures of a special class of anticenter-symmetric JJ algebra: anti-Zinbiel algebras. Finally, we described some double contructions of quadratic anticenter-symmetric JJ algebras in dimension two.

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