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Some properties for meromorphic functions associated with integral operators

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Abstract

In the present paper we aim at proving some subordinations properties for meromorphic functions analytic in the punctured unit disc $\Delta^* = \{z : 0 < |z| < 1\}$ with a simple pole at the origin. The functions under investigation are associated with two integral operators $\mathcal{P}_\sigma^\gamma$ and $\mathcal{Q}_\sigma^\gamma$ (see Lashin in Comput Math Appl 59:524–531, 2010, https://doi.org/10.1016/j.camwa.2009.06.015). Several other results and numerical examples are also obtained.

Keywords: Meromorphic functions, Differential subordination, Integral operators

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Introduction and Preliminaries

Let Σ denote the class of functions of the form

$$f(z) = 1/z + \sum_{\kappa=1}^{\infty} c_\kappa z^\kappa \tag{1}$$

which are analytic in the punctured unit disc $\Delta^* = \{z : 0 < |z| < 1\}$ with a simple pole at the origin.

Definition 1 For $f(z) \in \Sigma$, given by (1) and $h(z) \in \Sigma$ defined by

$$h(z) = 1/z + \sum_{\kappa=1}^{\infty} h_\kappa z^\kappa,$$

Hadamard product (or convolution) of $f(z)$ and $h(z)$ is given by

$$(f * h)(z) = 1/z + \sum_{\kappa=1}^{\infty} c_\kappa h_\kappa z^\kappa = (h * f)(z).$$

Definition 2 [2] For two functions f and g, analytic in $\Delta = \{z : |z| < 1\}$, we say that the function f is subordinate to g in Δ, written $f \prec g$, if there exists a Schwarz function $\omega(z)$ which is analytic in Δ, satisfying the following conditions:

$$\omega(0)=0 \qquad \text{and} \qquad |\omega(z)|<1; \quad (z\in\Delta),$$

such that

$$f(z)=g(\omega(z)); \qquad (z\in\Delta).$$

In particular, if the function g is univalent in Δ, we have the following equivalence (see also [3, 4]):

$$f(z)\prec g(z) \quad (z\in\Delta) \quad \Longleftrightarrow \quad f(0)=g(0) \quad \text{and} \quad f(\Delta)\subset g(\Delta).$$

In 2010, Lashin [1] defined the following integral operators $\mathcal{P}_\sigma^\gamma$, $\mathcal{Q}_\sigma^\gamma$, $\mathcal{J}_\sigma:\Sigma\to\Sigma$ as follows:

$$\mathcal{P}_\sigma^\gamma f(z)=\frac{\sigma^\gamma}{z^{\sigma+1}\Gamma(\gamma)}\int_0^z s^\sigma\left(log\left(\frac{z}{s}\right)\right)^{\gamma-1}f(s)ds \qquad (\gamma,\sigma>0,\ z\in\Delta^*),$$
$$\mathcal{Q}_\sigma^\gamma f(z)=\frac{\Gamma(\sigma+\gamma)}{z^{\sigma+1}\Gamma(\sigma)\Gamma(\gamma)}\int_0^z s^\sigma\left(1-\frac{s}{z}\right)^{\gamma-1}f(s)ds \qquad (\gamma,\sigma>0,\ z\in\Delta^*),$$

and

$$\mathcal{J}_\sigma f(z)=\frac{\sigma}{z^{\sigma+1}}\int_0^z s^\sigma f(s)ds \qquad (\sigma>0,\ z\in\Delta^*),$$

where $\Gamma(\gamma)$ is the familiar Gamma function.

Using the integral representation of the Gamma functions, it can be shown that

Definition 3 Let $f(z)\in\Sigma$ be given by (1), then

$$\mathcal{P}_\sigma^\gamma f(z)=1/z+\sum_{\kappa=1}^{\infty}\left(\frac{\sigma}{\kappa+\sigma+1}\right)^\gamma c_\kappa z^\kappa \qquad (\gamma,\sigma>0,\ z\in\Delta^*), \tag{2}$$

$$\mathcal{Q}_\sigma^\gamma f(z)=1/z+\frac{\Gamma(\sigma+\gamma)}{\Gamma(\sigma)}\sum_{\kappa=1}^{\infty}\frac{\Gamma(\kappa+\sigma+1)}{\Gamma(\kappa+\sigma+\gamma+1)}c_\kappa z^\kappa \qquad (\gamma,\sigma>0,\ z\in\Delta^*), \tag{3}$$

and

$$\mathcal{J}_\sigma f(z)=1/z+\sum_{\kappa=1}^{\infty}\frac{\sigma}{\kappa+\sigma+1}c_\kappa z^\kappa \qquad (\sigma>0,\ z\in\Delta^*). \tag{4}$$

By (3) and (4), we can easily obtain that

$$z(\mathcal{P}_\sigma^\gamma f(z))'=\sigma\mathcal{P}_\sigma^{\gamma-1}f(z)-(\sigma+1)\mathcal{P}_\sigma^\gamma f(z) \qquad (\sigma>0,\ \gamma>1) \tag{5}$$

and

$$z(\mathcal{Q}_\sigma^\gamma f(z))'=(\sigma+\gamma-1)\mathcal{Q}_\sigma^{\gamma-1}f(z)-(\sigma+\gamma)\mathcal{Q}_\sigma^\gamma f(z) \qquad (\sigma>0,\ \gamma>1). \tag{6}$$

Remark 1

(i) Putting $\gamma = 1$ in the integral operators $\mathcal{P}_\sigma^\gamma$ and $\mathcal{Q}_\sigma^\gamma$, we obtain $\mathcal{Q}_\sigma^1 = \mathcal{P}_\sigma^1 = \mathcal{J}_\sigma$;
(ii) The operator $\mathcal{J}_\sigma$ was defined and studied by Kumar and Shukla [5, Theorem 4.2, with $p = 1$].

In order to prove our main results, we recall the following lemmas:

Lemma 1 [6, 7] *If* $g(z)$ *is a convex function in* Δ *with* $g(0) = 1$, $q(z) = 1 + q_1(z) + q_2(z) + \ldots$ *is analytic in* Δ *and* $\delta \in \mathbb{C}, \mathrm{Re}(\delta) > 0$, *then*

$$q(z) + \frac{zq'(z)}{\delta} \prec g(z)$$

implies

$$q(z) \prec \delta z^{-\delta} \int_0^z s^{\delta-1} g(s)ds = \hbar(z) \prec g(z)$$

and $\hbar(z)$ *is the best dominant.*

Definition 4 Let a, b and c be complex and real numbers with $c \neq 0, -1, -2, \ldots$. The Gaussian hypergeometric function is defined by the power series

$$\begin{aligned} {}_2F_1(a,b;c;z) &= 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + \ldots \\ &= \sum_{\kappa=0}^{\infty} \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa (1)_\kappa} z^\kappa, \end{aligned}$$

where $(a)_\kappa = \Gamma(a+\kappa)/\Gamma(a)$ is the Pochhammer symbol.

Lemma 2 [8, 9] *For* a, b, c *real numbers other than* $0, -1, -2, \ldots$ *and* $c > b > 0$, *we have*

$$\int_0^1 s^{b-1}(1-s)^{c-b-1}(1-sz)^{-a}ds = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a,b;c;z) \tag{7}$$

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\, {}_2F_1(a, c-b; c; z/(z-1)) \tag{8}$$

$${}_2F_1(1,1;2;z) = -(1/z)ln(1-z) \tag{9}$$

$$\begin{aligned} &c(c-1)(z-1)\, {}_2F_1(a,b;c-1;z) \\ &+ c[c-1-(2c-a-b-1)z]\, {}_2F_1(a,b;c;z) \\ &+ (c-a)(c-b)z\, {}_2F_1(a,b;c+1;z) = 0. \end{aligned} \tag{10}$$

Lemma 3 *For any real number* $\alpha \neq 0$, *we have*

$${}_2F_1(1,1;2;\alpha z/(\alpha z+1)) = \frac{(1+\alpha z)ln(1+\alpha z)}{\alpha z}, \tag{11}$$

$$ {}_2F_1(1,1;3;\alpha z/(\alpha z+1)) = \frac{2(1+\alpha z)}{\alpha z}\left[1-\frac{ln(1+\alpha z)}{\alpha z}\right], \tag{12} $$

$$ {}_2F_1(1,1;4;\alpha z/(\alpha z+1)) = \frac{3(1+\alpha z)}{2(\alpha z)^3}[2ln(1+\alpha z)-\alpha z(2-\alpha z)], \tag{13} $$

$$ {}_2F_1(1,1;5;\alpha z/(\alpha z+1)) = \frac{2(1+\alpha z)}{(\alpha z)^3}\left[\frac{2(\alpha z)^2-3\alpha z+6}{3}-\frac{2ln(1+\alpha z)}{\alpha z}\right]. \tag{14} $$

Proof To prove the relation (11), substituting for $z = \alpha z/(\alpha z+1)$ in (9), we obtain

$$ \begin{aligned} {}_2F_1(1,1;2;\alpha z/(\alpha z+1)) &= -\left(\frac{1+\alpha z}{\alpha z}\right)\ln\left(1-\frac{\alpha z}{\alpha z+1}\right) \\ &= -\left(\frac{1+\alpha z}{\alpha z}\right)\ln\left(\frac{1}{\alpha z+1}\right) \\ &= \frac{(1+\alpha z)ln(1+\alpha z)}{\alpha z}. \end{aligned} $$

Now, we prove the relation (12), by replacing $a = b = 1$ and $c = 2$ in (10), we obtain

$$ 2(z-1){}_2F_1(1,1;1;z) + 2(1-z)\,{}_2F_1(1,1;2;z) + z\,{}_2F_1(1,1;3,z) = 0 $$

then

$$ \begin{aligned} {}_2F_1(1,1;3,z) &= \frac{1}{z}[-2(z-1){}_2F_1(1,1;1;z) - 2(1-z)\,{}_2F_1(1,1;2;z)] \\ &= \frac{2}{z}\left[1+\frac{1-z}{z}ln(1-z)\right]. \end{aligned} $$

By replacing $z = \alpha z/(\alpha z+1)$ in the above equation, we obtain

$$ \begin{aligned} {}_2F_1(1,1;3,\alpha z/(\alpha z+1)) &= \frac{2(1+\alpha z)}{\alpha z}\left[1+\frac{1}{\alpha z}ln\left(1-\frac{\alpha z}{1+\alpha z}\right)\right] \\ &= \frac{2(1+\alpha z)}{\alpha z}\left[1-\frac{ln(1+\alpha z)}{\alpha z}\right]. \end{aligned} $$

Similarly, we can obtain the relations (13) and (14). □

The purpose of this paper is to prove some subordinations properties for meromorphic functions analytic in the punctured unit disc $\Delta^* = \{z : 0 < |z| < 1\}$. The functions under investigation are associated with two integral operators $\mathcal{P}_\sigma^\gamma$ and $\mathcal{Q}_\sigma^\gamma$. Several other results and numerical examples are also obtained.

Main results

Unless otherwise mentioned, we assume throughout this paper that $\gamma > 2$, $\sigma > 0$ and $-1 \le B < A \le 1$.

Theorem 1 *Let*

$$\frac{\mathcal{P}_\sigma^{\gamma-1}f(z)}{\mathcal{P}_\sigma^{\gamma}f(z)}\left(1+\frac{\mathcal{P}_\sigma^{\gamma-2}f(z)}{\mathcal{P}_\sigma^{\gamma-1}f(z)}-\frac{\mathcal{P}_\sigma^{\gamma-1}f(z)}{\mathcal{P}_\sigma^{\gamma}f(z)}\right)\prec\frac{1+Az}{1+Bz}\qquad(\sigma>0,\ \gamma>2)\tag{15}$$

then we have

$$\frac{\mathcal{P}_\sigma^{\gamma-1}f(z)}{\mathcal{P}_\sigma^{\gamma}f(z)}\prec\hbar(z)\prec\frac{1+Az}{1+Bz}\qquad(z\in\Delta)$$

where

$$\hbar(z)=(1+Bz)^{-1}\left[{}_2F_1\left(1,1;1+\sigma;Bz/(1+Bz)\right)+\frac{\sigma Az}{\sigma+1}\,{}_2F_1\left(1,1;2+\sigma;Bz/(1+Bz)\right)\right]\tag{16}$$

and $\hbar(z)$ *is the best dominant. Furthermore,*

$$\operatorname{Re}\left(\frac{\mathcal{P}_\sigma^{\gamma-1}f(z)}{\mathcal{P}_\sigma^{\gamma}f(z)}\right)>\eta$$

where

$$\eta=(1-B)^{-1}\left[{}_2F_1\left(1,1;1+\sigma;B/(B-1)\right)-\frac{\sigma A}{\sigma+1}\,{}_2F_1\left(1,1;2+\sigma;B/(B-1)\right)\right].\tag{17}$$

Proof Suppose that

$$q(z)=\frac{\mathcal{P}_\sigma^{\gamma-1}f(z)}{\mathcal{P}_\sigma^{\gamma}f(z)},\tag{18}$$

then $q(z)=1+a_1z+a_2z^2+...$ is analytic in Δ with $q(0)=1$. Using the logarithmic differentiation of the both sides of (18) with respect to z and with the aid of the identity (5), we get

$$\left(\frac{1}{\sigma}\right)\frac{zq'(z)}{q(z)}=\frac{\mathcal{P}_\sigma^{\gamma-2}f(z)}{\mathcal{P}_\sigma^{\gamma-1}f(z)}-\frac{\mathcal{P}_\sigma^{\gamma-1}f(z)}{\mathcal{P}_\sigma^{\gamma}f(z)}\tag{19}$$

By using (18) and (19), we obtain

$$\frac{\mathcal{P}_\sigma^{\gamma-1}f(z)}{\mathcal{P}_\sigma^{\gamma}f(z)}\left(1+\frac{\mathcal{P}_\sigma^{\gamma-2}f(z)}{\mathcal{P}_\sigma^{\gamma-1}f(z)}-\frac{\mathcal{P}_\sigma^{\gamma-1}f(z)}{\mathcal{P}_\sigma^{\gamma}f(z)}\right)=q(z)+\left(\frac{1}{\sigma}\right)zq'(z).$$

Thus, by using Lemma 1, for $\delta=\sigma$, we have

$$\frac{\mathcal{P}_\sigma^{\gamma-1}f(z)}{\mathcal{P}_\sigma^{\gamma}f(z)}\prec\sigma z^{-\sigma}\int_0^z s^{\sigma-1}\frac{1+As}{1+Bs}ds=\hbar(z).$$

Using the identities (7) and (8), we can written $\hbar(z)$ as following:

$$\begin{aligned}\hbar(z) =&\sigma\int_0^1 t^{\sigma-1}\frac{1+Atz}{1+Btz}dt\\ =&\sigma\left[\int_0^1 t^{\sigma-1}(1+Btz)^{-1}dt + Az\int_0^1 t^{\sigma}(1+Btz)^{-1}dt\right]\\ =&(1+Bz)^{-1}\left[{}_2F_1\left(1,1;1+\sigma,Bz/(1+Bz)\right)+\frac{\sigma Az}{\sigma+1}{}_2F_1\left(1,1;2+\sigma,Bz/(1+Bz)\right)\right].\end{aligned}$$

This completes the proof of (16) of Theorem 1. Now, to prove the assertion (17) of Theorem 1, it suffices to show that

$$\inf_{|z|<1}\{\hbar(z)\}=\hbar(-1). \tag{20}$$

Indeed, for $|z|\le r<1$,

$$\mathrm{Re}\left(\frac{1+Az}{1+Bz}\right)\ge\frac{1-Ar}{1-Br}.$$

Upon setting

$$K(t,z)=\frac{1+Atz}{1+Btz} \text{ and } d\nu(t)=\sigma t^{\sigma-1}dt \qquad (0\le t\le 1,\ z\in\Delta),$$

which is a positive measure on [0, 1], we get

$$\hbar(z)=\int_0^1 K(t,z)d\nu(t),$$

so that

$$\mathrm{Re}(\hbar(z))\ge\int_0^1\left(\frac{1-Atr}{1-Btr}\right)d\nu(t)=\hbar(-r) \qquad (|z|\le r<1).$$

Letting $r\to 1^-$ in the above inequality, we obtain the assertion (20). The result in (17) is the best possible as the function $\hbar(z)$ is the best dominant of (16). The proof of Theorem 1 is completed. □

Letting $\sigma=1$ in Theorem 1 and using the identities (11) and (12), we have

Corollary 1 *Let*

$$\frac{\mathcal{P}^{\gamma-1}f(z)}{\mathcal{P}^{\gamma}f(z)}\left(1+\frac{\mathcal{P}^{\gamma-2}f(z)}{\mathcal{P}^{\gamma-1}f(z)}-\frac{\mathcal{P}^{\gamma-1}f(z)}{\mathcal{P}^{\gamma}f(z)}\right)\prec\frac{1+Az}{1+Bz} \qquad (\gamma>2)$$

then we have

$$\frac{\mathcal{P}^{\gamma-1}f(z)}{\mathcal{P}^{\gamma}f(z)}\prec\hbar(z)\prec\frac{1+Az}{1+Bz} \qquad (z\in\Delta)$$

where

$$\hbar(z) = \begin{cases} \left(1 - \frac{A}{B}\right)\frac{\ln(1+Bz)}{Bz} + \frac{A}{B} & B \neq 0 \\ 1 + \frac{A}{2}z & B = 0, \end{cases} \tag{21}$$

and $\hbar(z)$ *is the best dominant. Furthermore,*

$$\mathrm{Re}\left(\frac{\mathcal{P}^{\gamma-1}f(z)}{\mathcal{P}^{\gamma}f(z)}\right) > \eta$$

where

$$\eta = \begin{cases} \left(\frac{A}{B} - 1\right)\frac{\ln(1-B)}{B} + \frac{A}{B} & B \neq 0 \\ 1 - \frac{A}{2} & B = 0. \end{cases} \tag{22}$$

Letting $B \neq 0$ in Corollary 1, we have

Corollary 2 *If*

$$\frac{\mathcal{P}^{\gamma-1}f(z)}{\mathcal{P}^{\gamma}f(z)}\left(1 + \frac{\mathcal{P}^{\gamma-2}f(z)}{\mathcal{P}^{\gamma-1}f(z)} - \frac{\mathcal{P}^{\gamma-1}f(z)}{\mathcal{P}^{\gamma}f(z)}\right) \prec \frac{1 + \frac{B\ln(1-B)}{B+\ln(1-B)}z}{1 + Bz} \qquad (\gamma > 2)$$

then

$$\mathrm{Re}\left(\frac{\mathcal{P}^{\gamma-1}f(z)}{\mathcal{P}^{\gamma}f(z)}\right) > 0 \qquad (z \in \Delta).$$

Letting $B = -1$ in Corollary 2, we obtain the following special case

Example 1 If

$$\mathrm{Re}\left(\frac{\mathcal{P}^{\gamma-1}f(z)}{\mathcal{P}^{\gamma}f(z)}\left(1 + \frac{\mathcal{P}^{\gamma-2}f(z)}{\mathcal{P}^{\gamma-1}f(z)} - \frac{\mathcal{P}^{\gamma-1}f(z)}{\mathcal{P}^{\gamma}f(z)}\right)\right) > \frac{2\ln 2 - 1}{2\ln 2 - 2} \approx -0.61 \qquad (\gamma > 2)$$

then

$$\mathrm{Re}\left(\frac{\mathcal{P}^{\gamma-1}f(z)}{\mathcal{P}^{\gamma}f(z)}\right) > 0 \qquad (z \in \Delta).$$

Letting $A = 1 - 2\lambda$, $(0 \leq \lambda < 1)$ and $B = -1$ in Corollary 1, we have

Corollary 3 *If*

$$\mathrm{Re}\left(\frac{\mathcal{P}^{\gamma-1}f(z)}{\mathcal{P}^{\gamma}f(z)}\left(1 + \frac{\mathcal{P}^{\gamma-2}f(z)}{\mathcal{P}^{\gamma-1}f(z)} - \frac{\mathcal{P}^{\gamma-1}f(z)}{\mathcal{P}^{\gamma}f(z)}\right)\right) > \lambda \qquad (\gamma > 2)$$

then

$$\operatorname{Re}\left(\frac{\mathcal{P}^{\gamma-1}f(z)}{\mathcal{P}^{\gamma}f(z)}\right) > (2\lambda-1)+2(1-\lambda)\ln 2.$$

Letting $\sigma = 2$ in Theorem 1 and using the identities (12) and (13), we have

Corollary 4 *Let*

$$\frac{\mathcal{P}_2^{\gamma-1}f(z)}{\mathcal{P}_2^{\gamma}f(z)}\left(1+\frac{\mathcal{P}_2^{\gamma-2}f(z)}{\mathcal{P}_2^{\gamma-1}f(z)}-\frac{\mathcal{P}_2^{\gamma-1}f(z)}{\mathcal{P}_2^{\gamma}f(z)}\right) \prec \frac{1+Az}{1+Bz} \qquad (\gamma > 2)$$

then we have

$$\frac{\mathcal{P}_2^{\gamma-1}f(z)}{\mathcal{P}_2^{\gamma}f(z)} \prec \hbar(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \Delta)$$

where

$$\hbar(z) = \begin{cases} \left(\frac{A}{B}-1\right)\left(\frac{2[\ln(1+Bz)-Bz]}{B^2z^2}\right)+\frac{A}{B} & B \neq 0 \\ \frac{2}{3}Az+1 & B=0, \end{cases} \tag{23}$$

and $\hbar(z)$ *is the best dominant. Furthermore,*

$$\operatorname{Re}\left(\frac{\mathcal{P}_2^{\gamma-1}f(z)}{\mathcal{P}_2^{\gamma}f(z)}\right) > \eta$$

where

$$\eta = \begin{cases} \left(\frac{A}{B}-1\right)\frac{2[\ln(1-B)+B]}{B^2}+\frac{A}{B} & B \neq 0 \\ 1-\frac{2}{3}A & B=0, \end{cases} \tag{24}$$

Letting $B \neq 0$ in Corollary 4, we have

Corollary 5 *If*

$$\frac{\mathcal{P}_2^{\gamma-1}f(z)}{\mathcal{P}_2^{\gamma}f(z)}\left(1+\frac{\mathcal{P}_2^{\gamma-2}f(z)}{\mathcal{P}_2^{\gamma-1}f(z)}-\frac{\mathcal{P}_2^{\gamma-1}f(z)}{\mathcal{P}_2^{\gamma}f(z)}\right) \prec \frac{1+\frac{B\ln(1-B)}{B+\ln(1-B)}z}{1+Bz} \qquad (\gamma > 2)$$

then

$$\operatorname{Re}\left(\frac{\mathcal{P}_2^{\gamma-1}f(z)}{\mathcal{P}_2^{\gamma}f(z)}\right) > 0 \qquad (z \in \Delta).$$

Letting $B=-1$ in Corollary 5, we obtain the following special case

Example 2

If

$$\mathrm{Re}\left(\frac{\mathcal{P}_2^{\gamma-1}f(z)}{\mathcal{P}_2^{\gamma}f(z)}\left(1+\frac{\mathcal{P}_2^{\gamma-2}f(z)}{\mathcal{P}_2^{\gamma-1}f(z)}-\frac{\mathcal{P}_2^{\gamma-1}f(z)}{\mathcal{P}_2^{\gamma}f(z)}\right)\right)>\frac{4\ln 2-3}{4\ln 2-2}\approx -0.29 \qquad (\gamma>2)$$

then

$$\mathrm{Re}\left(\frac{\mathcal{P}_2^{\gamma-1}f(z)}{\mathcal{P}_2^{\gamma}f(z)}\right)>0 \qquad (z\in\Delta).$$

Letting $A=1-2\lambda$, $(0\le\lambda<1)$ and $B=-1$ in Corollary 4, we have

Corollary 6 *If*

$$\mathrm{Re}\left(\frac{\mathcal{P}_2^{\gamma-1}f(z)}{\mathcal{P}_2^{\gamma}f(z)}\left(1+\frac{\mathcal{P}_2^{\gamma-2}f(z)}{\mathcal{P}_2^{\gamma-1}f(z)}-\frac{\mathcal{P}_2^{\gamma-1}f(z)}{\mathcal{P}_2^{\gamma}f(z)}\right)\right)>\lambda \qquad (\gamma>2)$$

then

$$\mathrm{Re}\left(\frac{\mathcal{P}_2^{\gamma-1}f(z)}{\mathcal{P}_2^{\gamma}f(z)}\right)>(2\lambda-1)-4(1-\lambda)(\ln 2-1).$$

Letting $\sigma=3$ in Theorem 1 and using the identities (13) and (14), we have

Corollary 7 *Let*

$$\frac{\mathcal{P}_3^{\gamma-1}f(z)}{\mathcal{P}_3^{\gamma}f(z)}\left(1+\frac{\mathcal{P}_3^{\gamma-2}f(z)}{\mathcal{P}_3^{\gamma-1}f(z)}-\frac{\mathcal{P}_3^{\gamma-1}f(z)}{\mathcal{P}_3^{\gamma}f(z)}\right)\prec\frac{1+Az}{1+Bz} \qquad (\gamma>2)$$

then we have

$$\frac{\mathcal{P}_3^{\gamma-1}f(z)}{\mathcal{P}_3^{\gamma}f(z)}\prec\hbar(z)\prec\frac{1+Az}{1+Bz} \qquad (z\in\Delta)$$

where

$$\hbar(z)=\begin{cases}\left(1-\frac{A}{B}\right)\frac{3}{B^3z^3}\left[\ln(1+Bz)-Bz+\frac{B^2z^2}{2}\right]+\frac{A}{B} & B\neq 0\\ 1+\frac{3}{4}Az & B=0,\end{cases} \tag{25}$$

and $\hbar(z)$ *is the best dominant. Furthermore,*

$$\mathrm{Re}\left(\frac{\mathcal{P}_3^{\gamma-1}f(z)}{\mathcal{P}_3^{\gamma}f(z)}\right)>\eta$$

where

$$\eta = \begin{cases} \left(\frac{A}{B} - 1\right)\frac{3}{B^3}\left[\ln(1-B) + B + \frac{B^2}{2}\right] + \frac{A}{B} & B \neq 0 \\ 1 - \frac{3}{4}A & B = 0. \end{cases} \tag{26}$$

Letting $B \neq 0$ in Corollary 7, we have

Corollary 8 *If*

$$\frac{\mathcal{P}_3^{\gamma-1}f(z)}{\mathcal{P}_3^{\gamma}f(z)}\left(1 + \frac{\mathcal{P}_3^{\gamma-2}f(z)}{\mathcal{P}_3^{\gamma-1}f(z)} - \frac{\mathcal{P}_3^{\gamma-1}f(z)}{\mathcal{P}_3^{\gamma}f(z)}\right) \prec \frac{1 + \frac{B\ln(1-B)}{B+\ln(1-B)}z}{1+Bz} \qquad (\gamma > 2)$$

then

$$\mathrm{Re}\left(\frac{\mathcal{P}_3^{\gamma-1}f(z)}{\mathcal{P}_3^{\gamma}f(z)}\right) > 0 \qquad (z \in \Delta).$$

Letting $B = -1$ in Corollary 8, we obtain the following special case

Example 3

If

$$\mathrm{Re}\left(\frac{\mathcal{P}_3^{\gamma-1}f(z)}{\mathcal{P}_3^{\gamma}f(z)}\left(1 + \frac{\mathcal{P}_3^{\gamma-2}f(z)}{\mathcal{P}_3^{\gamma-1}f(z)} - \frac{\mathcal{P}_3^{\gamma-1}f(z)}{\mathcal{P}_3^{\gamma}f(z)}\right)\right) > \frac{12\ln 2 - 19}{12\ln 2 - 20} \approx 0.91 \qquad (\gamma > 2)$$

then

$$\mathrm{Re}\left(\frac{\mathcal{P}_3^{\gamma-1}f(z)}{\mathcal{P}_3^{\gamma}f(z)}\right) > 0 \qquad (z \in \Delta).$$

Letting $A = 1 - 2\lambda$, $(0 \leq \lambda < 1)$ and $B = -1$ in Corollary 7, we have

Corollary 9 *If*

$$\mathrm{Re}\left(\frac{\mathcal{P}_3^{\gamma-1}f(z)}{\mathcal{P}_3^{\gamma}f(z)}\left(1 + \frac{\mathcal{P}_3^{\gamma-2}f(z)}{\mathcal{P}_3^{\gamma-1}f(z)} - \frac{\mathcal{P}_3^{\gamma-1}f(z)}{\mathcal{P}_3^{\gamma}f(z)}\right)\right) > \lambda \qquad (\gamma > 2)$$

then

$$\mathrm{Re}\left(\frac{\mathcal{P}_3^{\gamma-1}f(z)}{\mathcal{P}_3^{\gamma}f(z)}\right) > (2\lambda - 1) + 3(1-\lambda)(2\ln 2 - 3).$$

Theorem 2 *Let*

$$\frac{\mathcal{Q}_\sigma^{\gamma-1}f(z)}{\mathcal{Q}_\sigma^{\gamma}f(z)}\left(\frac{\mathcal{Q}_\sigma^{\gamma-2}f(z)}{\mathcal{Q}_\sigma^{\gamma-1}f(z)}-\frac{\sigma+\gamma-1}{\sigma+\gamma-2}\frac{\mathcal{Q}_\sigma^{\gamma-1}f(z)}{\mathcal{Q}_\sigma^{\gamma}f(z)}+\frac{\sigma+\gamma-1}{\sigma+\gamma-2}\right)\prec\frac{1+Az}{1+Bz}\qquad(\sigma>0,\ \gamma>2)$$

then we have

$$\frac{\mathcal{Q}_\sigma^{\gamma-1}f(z)}{\mathcal{Q}_\sigma^{\gamma}f(z)}\prec\hbar(z)\prec\frac{1+Az}{1+Bz}\qquad(z\in\Delta)$$

where

$$\begin{aligned}\hbar(z)=&(1+Bz)^{-1}\left[{}_2F_1\left(1,1;\sigma+\gamma-1;Bz/(1+Bz)\right)\right.\\&\left.+\frac{(\sigma+\gamma-2)Az}{\sigma+\gamma-1}\,{}_2F_1\left(1,1;\sigma+\gamma;Bz/(1+Bz)\right)\right]\end{aligned}\tag{27}$$

and $\hbar(z)$ *is the best dominant. Furthermore,*

$$\mathrm{Re}\left(\frac{\mathcal{Q}_\sigma^{\gamma-1}f(z)}{\mathcal{Q}_\sigma^{\gamma}f(z)}\right)>\eta$$

where

$$\eta=(1-B)^{-1}\left[{}_2F_1\left(1,1;\sigma+\gamma-1;B/(B-1)\right)-\frac{(\sigma+\gamma-2)A}{\sigma+\gamma-1}\,{}_2F_1\left(1,1;\sigma+\gamma;B/(B-1)\right)\right].\tag{28}$$

Proof Suppose that

$$q(z)=\frac{\mathcal{Q}_\sigma^{\gamma-1}f(z)}{\mathcal{Q}_\sigma^{\gamma}f(z)},\tag{29}$$

then $q(z)=1+a_1z+a_2z^2+...$ is analytic in Δ with $q(0)=1$. Using the logarithmic differentiation of the both sides of (29) with respect to z, and with the aid of the identities (6), we get

$$\left(\frac{1}{\sigma+\gamma-2}\right)\frac{zq'(z)}{q(z)}=\frac{\mathcal{Q}_\sigma^{\gamma-2}f(z)}{\mathcal{Q}_\sigma^{\gamma-1}f(z)}-\frac{\sigma+\gamma-1}{\sigma+\gamma-2}\frac{\mathcal{Q}_\sigma^{\gamma-1}f(z)}{\mathcal{Q}_\sigma^{\gamma}f(z)}+\frac{1}{\sigma+\gamma-2}\tag{30}$$

By using (29) and (30), we obtain

$$\frac{\mathcal{Q}_\sigma^{\gamma-1}f(z)}{\mathcal{Q}_\sigma^{\gamma}f(z)}\left(\frac{\mathcal{Q}_\sigma^{\gamma-2}f(z)}{\mathcal{Q}_\sigma^{\gamma-1}f(z)}-\frac{\sigma+\gamma-1}{\sigma+\gamma-2}\frac{\mathcal{Q}_\sigma^{\gamma-1}f(z)}{\mathcal{Q}_\sigma^{\gamma}f(z)}+\frac{\sigma+\gamma-1}{\sigma+\gamma-2}\right)=q(z)+\left(\frac{1}{\sigma+\gamma-2}\right)zq'(z).$$

Thus, by using Lemma 1, for $\delta=\sigma+\gamma-2$, the estimates (27) and (28) can be proved on the same lines as that of (16) and (17). This completes the proof of Theorem 2. □

Letting $\gamma = 3 - \sigma$ in Theorem 2 and using the identities (11) and (12), we have

Corollary 10 *Let*

$$\frac{\mathcal{Q}_\sigma^{2-\sigma} f(z)}{\mathcal{Q}_\sigma^{3-\sigma} f(z)} \left(2 + \frac{\mathcal{Q}_\sigma^{1-\sigma} f(z)}{\mathcal{Q}_\sigma^{2-\sigma} f(z)} - 2 \frac{\mathcal{Q}_\sigma^{2-\sigma} f(z)}{\mathcal{Q}_\sigma^{3-\sigma} f(z)} \right) \prec \frac{1 + Az}{1 + Bz} \qquad (0 < \sigma < 1)$$

then we have

$$\frac{\mathcal{Q}_\sigma^{2-\sigma} f(z)}{\mathcal{Q}_\sigma^{3-\sigma} f(z)} \prec \hbar(z) \prec \frac{1 + Az}{1 + Bz} \qquad (z \in \Delta)$$

where $\hbar(z)$ *given as* (21) *and* $\hbar(z)$ *is the best dominant. Furthermore,*

$$\operatorname{Re}\left(\frac{\mathcal{Q}_\sigma^{2-\sigma} f(z)}{\mathcal{Q}_\sigma^{3-\sigma} f(z)} \right) > \eta$$

where η *given as* (22).

Letting $B \neq 0$ in Corollary 10, we have

Corollary 11 *If*

$$\frac{\mathcal{Q}_\sigma^{2-\sigma} f(z)}{\mathcal{Q}_\sigma^{3-\sigma} f(z)} \left(2 + \frac{\mathcal{Q}_\sigma^{1-\sigma} f(z)}{\mathcal{Q}_\sigma^{2-\sigma} f(z)} - 2 \frac{\mathcal{Q}_\sigma^{2-\sigma} f(z)}{\mathcal{Q}_\sigma^{3-\sigma} f(z)} \right) \prec \frac{1 + \frac{B \ln(1-B)}{B + \ln(1-B)} z}{1 + Bz} \qquad (0 < \sigma < 1)$$

then

$$\operatorname{Re}\left(\frac{\mathcal{Q}_\sigma^{2-\sigma} f(z)}{\mathcal{Q}_\sigma^{3-\sigma} f(z)} \right) > 0 \qquad (z \in \Delta).$$

Letting $B = -1$ in Corollary 11, we obtain the following special case

Example 4

If

$$\operatorname{Re}\left(\frac{\mathcal{Q}_\sigma^{2-\sigma} f(z)}{\mathcal{Q}_\sigma^{3-\sigma} f(z)} \left(2 + \frac{\mathcal{Q}_\sigma^{1-\sigma} f(z)}{\mathcal{Q}_\sigma^{2-\sigma} f(z)} - 2 \frac{\mathcal{Q}_\sigma^{2-\sigma} f(z)}{\mathcal{Q}_\sigma^{3-\sigma} f(z)} \right) \right) > \frac{2 \ln 2 - 1}{2 \ln 2 - 2} \approx -0.61 \qquad (0 < \sigma < 1)$$

then

$$\operatorname{Re}\left(\frac{\mathcal{Q}_\sigma^{2-\sigma} f(z)}{\mathcal{Q}_\sigma^{3-\sigma} f(z)} \right) > 0 \qquad (z \in \Delta).$$

Letting $A = 1 - 2\lambda$, $(0 \leq \lambda < 1)$ and $B = -1$ in Corollary 10, we have

Corollary 12 *If*

$$\mathrm{Re}\left(\frac{\mathcal{Q}_{\sigma}^{2-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}\left(2+\frac{\mathcal{Q}_{\sigma}^{1-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{2-\sigma}f(z)}-2\frac{\mathcal{Q}_{\sigma}^{2-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}\right)\right)>\lambda \qquad (0<\sigma<1)$$

then

$$\mathrm{Re}\left(\frac{\mathcal{Q}_{\sigma}^{2-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}\right)>(2\lambda-1)+2(1-\lambda)\ln 2.$$

Letting $\gamma=4-\sigma$ in Theorem 2 and using the identities (12) and (13), we have

Corollary 13 *Let*

$$\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}\left(\frac{3}{2}+\frac{\mathcal{Q}_{\sigma}^{2-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}-\frac{3}{2}\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}\right)\prec\frac{1+Az}{1+Bz} \qquad (0<\sigma<2)$$

then we have

$$\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}\prec\hbar(z)\prec\frac{1+Az}{1+Bz} \qquad (z\in\Delta)$$

where $\hbar(z)$ *given as* (23) *and* $\hbar(z)$ *is the best dominant. Furthermore,*

$$\mathrm{Re}\left(\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}\right)>\eta$$

where η *given as* (24).

Letting $B\neq 0$ in Corollary 13, we have

Corollary 14 *If*

$$\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}\left(\frac{3}{2}+\frac{\mathcal{Q}_{\sigma}^{2-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}-\frac{3}{2}\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}\right)\prec\frac{1+\frac{B\ln(1-B)}{B+\ln(1-B)}z}{1+Bz} \qquad (0<\sigma<2)$$

then

$$\mathrm{Re}\left(\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}\right)>0 \qquad (z\in\Delta).$$

Letting $B=-1$ in Corollary 15, we obtain the following special case

Example 5

If

$$\mathrm{Re}\left(\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}\left(\frac{3}{2}+\frac{\mathcal{Q}_{\sigma}^{2-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}-\frac{3}{2}\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}\right)\right)>\frac{4\ln 2-3}{4\ln 2-2}\approx-0.29 \qquad (0<\sigma<2)$$

then

$$\mathrm{Re}\left(\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}\right) > 0 \qquad (z \in \Delta).$$

Letting $A = 1 - 2\lambda$, $(0 \leq \lambda < 1)$ and $B = -1$ in Corollary 13, we have

Corollary 15 *If*

$$\mathrm{Re}\left(\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}\left(\frac{3}{2}+\frac{\mathcal{Q}_{\sigma}^{2-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}-\frac{3}{2}\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}\right)\right) > \lambda \qquad (0 < \sigma < 2)$$

then

$$\mathrm{Re}\left(\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}\right) > (2\lambda - 1) - 4(1-\lambda)(\ln 2 - 1).$$

Letting $\gamma = 5 - \sigma$ in Theorem 2 and using the identities (13) and (14), we have

Corollary 16 *Let*

$$\frac{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{5-\sigma}f(z)}\left(\frac{4}{3}+\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}-\frac{4}{3}\frac{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{5-\sigma}f(z)}\right) \prec \frac{1+Az}{1+Bz} \qquad (0 < \sigma < 3)$$

then we have

$$\frac{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{5-\sigma}f(z)} \prec \hbar(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \Delta)$$

where $\hbar(z)$ *given as* (25) *and* $\hbar(z)$ *is the best dominant. Furthermore,*

$$\mathrm{Re}\left(\frac{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{5-\sigma}f(z)}\right) > \eta$$

where given as (26).

Letting $B \neq 0$ in Corollary 16, we have

Corollary 17 *If*

$$\frac{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{5-\sigma}f(z)}\left(\frac{4}{3}+\frac{\mathcal{Q}_{\sigma}^{3-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}-\frac{4}{3}\frac{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{5-\sigma}f(z)}\right) \prec \frac{1+\frac{B\ln(1-B)}{B+\ln(1-B)}z}{1+Bz} \qquad (0 < \sigma < 3)$$

then

$$\mathrm{Re}\left(\frac{\mathcal{Q}_{\sigma}^{4-\sigma}f(z)}{\mathcal{Q}_{\sigma}^{5-\sigma}f(z)}\right) > 0 \qquad (z \in \Delta).$$

Letting $B = -1$ in Corollary 17, we obtain the following special case

Example 6

If

$$\mathrm{Re}\left(\frac{\mathcal{Q}_\sigma^{4-\sigma}f(z)}{\mathcal{Q}_\sigma^{5-\sigma}f(z)}\left(\frac{4}{3}+\frac{\mathcal{Q}_\sigma^{3-\sigma}f(z)}{\mathcal{Q}_\sigma^{4-\sigma}f(z)}-\frac{4}{3}\frac{\mathcal{Q}_\sigma^{4-\sigma}f(z)}{\mathcal{Q}_\sigma^{5-\sigma}f(z)}\right)\right)>\frac{12\ln 2-19}{12\ln 2-20}\approx 0.91 \qquad (0<\sigma<3)$$

then

$$\mathrm{Re}\left(\frac{\mathcal{Q}_\sigma^{4-\sigma}f(z)}{\mathcal{Q}_\sigma^{5-\sigma}f(z)}\right)>0 \qquad (z\in\Delta).$$

Letting $A=1-2\lambda$, $(0\le\lambda<1)$ and $B=-1$ in Corollary 16, we have

Corollary 18 *If*

$$\mathrm{Re}\left(\frac{\mathcal{Q}_\sigma^{4-\sigma}f(z)}{\mathcal{Q}_\sigma^{5-\sigma}f(z)}\left(\frac{4}{3}+\frac{\mathcal{Q}_\sigma^{3-\sigma}f(z)}{\mathcal{Q}_\sigma^{4-\sigma}f(z)}-\frac{4}{3}\frac{\mathcal{Q}_\sigma^{4-\sigma}f(z)}{\mathcal{Q}_\sigma^{5-\sigma}f(z)}\right)\right)>\lambda \qquad (0<\sigma<3)$$

then

$$\mathrm{Re}\left(\frac{\mathcal{Q}_\sigma^{4-\sigma}f(z)}{\mathcal{Q}_\sigma^{5-\sigma}f(z)}\right)>(2\lambda-1)+3(1-\lambda)(2\ln 2-3).$$

Conclusion

The purpose of the current work is to demonstrate various subordination characteristics for analytical meromorphic functions in $\Delta^*=\{z:0<|z|<1\}$. Lashin [1] defined and studied the integral operators $\mathcal{P}_\sigma^\gamma$ and $\mathcal{Q}_\sigma^\gamma$ on the meromorphic functions. We use these operators to study and demonstrate some subordination characteristics for meromorphic functions. In addition, we obtain some particular cases and numerical examples of our main results.

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