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АЛГЕБРАИЧЕСКИЕ УРАВНЕНИЯ И ПОЛИНОМЫ НАД КОЛЬЦОМ *p*-КОМПЛЕКСНЫХ ЧИСЕЛ

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Изучены алгебраические уравнения над кольцом *p*-комплексных чисел. Приведены теоремы о делении с остатком и аналог теоремы Безу для *p*-комплексных полиномов. Для уравнений второй и третьей степени получены условия существования корней, в некоторых случаях даны решения в явном виде. Для полиномов произвольной степени с обратимым старшим коэффициентом доказаны теоремы о разложении на множители с единичным старшим коэффициентом в случаях наличия простых корней, кратных корней и отсутствия корней. Показано, что при отсутствии кратных корней указанное разложение будет единственным, а в случае наличия кратных корней полином допускает бесконечное множество разложений.

Ключевые слова: дуальное число; многочлен; кольцо *p*-комплексных чисел; *p*-комплексный полином; делитель нуля; формула Кардано; разложение на множители.

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ALGEBRAIC EQUATIONS AND POLYNOMIALS OVER THE RING OF *p*-COMPLEX NUMBERS

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In this paper, we study the algebraic equations over the ring of *p*-complex numbers. Remainder division theorems and an analogue of Bezout's theorem for *p*-complex polynomials are represented. For equations of the 2nd and 3rd degrees, conditions for the existence of roots are obtained, in some cases solutions are given in an explicit form. For polynomials of an arbitrary degree with an invertible leading coefficient, theorems on factorisation with a unit leading coefficient are proven in the cases where there are simple roots, multiple roots, and no roots. It is shown that in the absence of multiple roots, this decomposition will be unique, and in the case of the presence of multiple roots, the polynomial admits an infinite number of expansions.

Keywords: dual number; polynomial; ring of *p*-complex numbers; *p*-complex polynomial; zero divisor; Cardano's formula; polynomial factorisation.

Introduction

In the mathematical literature, the theory of *p*-complex (dual) numbers and functions of a *p*-complex variable is not enough explored. In connection with the existing applications in geometry and physics, further research in this direction is topical. Earlier, in papers [1–3], the properties of *h*-complex funtions and polynomials were obtained. In this article, the solution of algebraic equations and the factorisation of polynomials over the ring of *p*-complex numbers are considered. The properties of these numbers and some of the results obtained earlier are given in papers [4–8].

Some general theorems about polynomials over the ring of *p*-complex numbers

Let $\mathbb{C}_p$ be a ring of *p*-complex numbers of the form $z = x + jy$, where $x, y \in \mathbb{R}$, $j^2 = 0$, $j \neq 0$. Number $\operatorname{Re} z = x$ is the real part, and number $\operatorname{Par} z = y$ is the parabolic part of z. The ring $\mathbb{C}_p$ has zero divisors of the form jy, where $y \in \mathbb{R}$, all other elements of this ring are invertible.

We consider an algebraic equation

$$P_n(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0 = 0, \tag{1}$$

where $c_k \in \mathbb{C}_p$ and $n \in \mathbb{N}$. Let $c_k = a_k + jb_k$, z_0 is the root of equation (1). Notice, that $P_n(z) = Q_n(x) + j\left(yQ_n'(x) + T_n(x)\right)$, where $Q_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ and $T_n(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0$ are real polynomials.

The following theorem gives a general idea of the roots of the given equation. The proof of the theorem is given in the source [6, p. 33–39].

Theorem 1. *a) Equation (1) has roots if and only if its real part $Q_n(x)$ has roots.*

b) Equation (1) has two types of roots:

1) isolated one with the form $z_0 = x_0 + j\left(-\dfrac{T_n(x_0)}{Q_n'(x_0)}\right)$, if and only if x_0 is the simple root of $Q_n(x)$, and z_0 is real, if and only if x_0 is the root of $T_n(x)$;

2) non-isolated one with the form $z_0 = x_0 + jy$, where $y \in \mathbb{R}$ is arbitrary, if and only if x_0 is multiple root of the polynomial $Q_n(x)$ and the root of the polynomial $T_n(x)$.

Theorem 2 (on division with a remainder). *Let $f(z)$, $g(z) \in \mathbb{C}_p[z]$, where the leading coefficient of the polynomial $g(z)$ is not a zero divisor. Then there are unique polynomials $q(z)$ and $r(z)$, such that*

$$f(z) = g(z)q(z) + r(z),$$

with the degree $r(z)$ less than $g(z)$ or $r(z) = 0$.

The proof of this theorem completely repeats the analogous one for polynomials over an arbitrary field [9, p. 134–135].

An analogue of Bezout's theorem directly follows from this theorem.

Theorem 3. *The remainder of the division of a polynomial* $P_n(z)$ *by* $(z-a)$, *where* $a \in \mathbb{C}_p$, *equals* $P_n(a)$.

We further assume that the leading coefficient of the polynomial $P_n(z)$ is not a zero divisor and consider some special cases.

Quadratic equations over $\mathbb{C}_p$

At the beginning, we consider

$$z^2 = D, \tag{2}$$

where $D = d_1 + jd_2 \in \mathbb{C}_p$. Let $d_1 > 0$. Then $x^2 + 2jyx = d_1 + jd_2$, from here we find two solutions $z_{1,2} = \pm\sqrt{d_1} \pm \pm j\dfrac{d_2}{2\sqrt{d_1}} = \pm\sqrt{D}$. If $D = 0$, then $x = 0$, and $y \in \mathbb{R}$ is arbitrary, then the solution of the equation is an arbitrary zero divisor $z_0 = jy$. If $d_1 = 0$, and $d_2 \neq 0$, then $x = 0$ and we have $0 = 0 + 2jy0 = jd_2 \neq 0$, thus the equation has no roots, as in the case $d_1 < 0$.

Let us consider the equation

$$P_2(z) = Az^2 + Bz + C = 0, \tag{3}$$

where A, B, $C \in \mathbb{C}_p$ and $\operatorname{Re}A \neq 0$. As in the real case, equation (3) can be transformed to the form

$$(2Az + B)^2 = B^2 - 4AC.$$

By analogy with the real case, we introduce the discriminant $D = B^2 - 4AC$.

If $\operatorname{Re}D > 0$, then we have the first case for equation (2):

$$2Az + B = \pm\sqrt{B^2 - 4AC}.$$

It follows that the roots of equation (3) are calculated by the usual formulas $z_{1,2} = \dfrac{-B \pm \sqrt{D}}{2A}$, and the polynomial $P_2(z)$ is uniquely expressed as the product of two linear factors with leading coefficients equal to one

$$P_2(z) = A(z - z_1)(z - z_2).$$

If $\operatorname{Re}D < 0$ or $\operatorname{Re}D = 0$, $\operatorname{Par}D \neq 0$, equation (3) has no solutions over $\mathbb{C}_p$ and the polynomial $P_2(z)$ can not be expressed as the product of two linear factors with invertible leading coefficients.

Else, if $D = 0$, then equation (3) has an infinite set of solutions of the form $z = \dfrac{-B}{2A} + jy$, where $y \in \mathbb{R}$ is arbitrary. At the same time $z_0 = \dfrac{-B}{2A}$ is the unique multiple root of equation (3) [6, p. 37–38].

Using theorem 1, one can show that for $D = 0$ the general solution of equation (3) can be expressed as $z = x_0 + jy$, where $y \in \mathbb{R}$, x_0 is the multiple root of equation (3):

$$(\operatorname{Re}A)x^2 + (\operatorname{Re}B)x + \operatorname{Re}C = 0.$$

The polynomial $P_2(z)$ for $D = B^2 - 4AC = 0$ admits a continuum of expressions as the product of two linear factors. Let $z_0 = \dfrac{-B}{2A} = x_0 + jy_0$, $z_1 = x_0 + jy_1$, $z_2 = x_0 + jy_2$ be the roots of equation (3). It is easy to verify that the condition $2y_0 = y_1 + y_2$ is necessary and sufficient for

$$Az^2 + Bz + C = A(z - z_0)^2 = A(z - z_1)(z - z_2).$$

Equation (3) is considered in the research [6, p. 37–38], the question of factorisation in this form is considered for the first time.

Cubic equations over $\mathbb{C}_p$

At first, we consider a cubic equation over $\mathbb{R}$

$$x^3 + px + q = 0, \tag{4}$$

where p, $q \in \mathbb{R}$. The solution of the cubic equation in the general case will be expressed through the solution of equation (4). If $p = 0$, equation (4) is equivalent to $x^3 = -q$ and has only one root $x = -\sqrt[3]{q}$. If $p \neq 0$, we represent equation (4) in the form

$$x^3 + px + q = \left(x + \frac{3q}{2p}\right)^2 \left(x - 3\frac{q}{p}\right) + \left(p + \frac{27q^2}{4p^2}\right)\left(x + \frac{q}{p}\right). \tag{5}$$

We introduce the discriminant $D = q^2 + \frac{4}{27}p^3 = \frac{4}{27}p^2\left(\frac{27q^2}{4p^2} + p\right)$ and consider different cases.

Let $D = 0$. Then the formula (5) simplifies to

$$x^3 + px + q = \left(x + \frac{3q}{2p}\right)^2 \left(x - 3\frac{q}{p}\right). \tag{6}$$

Equation (4), in this case, has a simple root $x = 3\frac{q}{p}$ and also has a multiple root $x = -\frac{3q}{2p}$.

Let $f(x) = x^3 + px + q$, then $f'(x) = 3x^2 + p$. For $D = 0$, the root $x = -\frac{3q}{2p}$ coincides with one of the roots of the derivative, which means $p < 0$ and $q^2 = \frac{4}{9}p^2\left(-\frac{p}{3}\right)$. If $q > 0$, then $q = -\frac{2p}{3}\sqrt{-\frac{p}{3}}$ and $-\frac{3q}{2p} = \sqrt{-\frac{p}{3}}$. If $q < 0$, then $q = \frac{2p}{3}\sqrt{-\frac{p}{3}}$ and $-\frac{3q}{2p} = -\sqrt{-\frac{p}{3}}$. The case $q = 0$ for $D = 0$ is not possible.

Now consider the case $D > 0$. Let us find solutions (4) in the form $x = \alpha + \beta$, where $\alpha, \beta \in \mathbb{R}$:

$$(\alpha + \beta)^3 + p(\alpha + \beta) + q = 0 \Leftrightarrow \left(\alpha^3 + \beta^3\right) + (\alpha + \beta)(3\alpha\beta + p) + q = 0. \tag{7}$$

Let us choose α, β so that

$$\begin{cases} \alpha^3 + \beta^3 = -q, \\ \alpha\beta = -\frac{p}{3} \end{cases} \Leftrightarrow \begin{cases} \alpha^3 + \beta^3 = -q, \\ \alpha^3\beta^3 = -\frac{p^3}{27} \end{cases} \Leftrightarrow \begin{cases} \beta^3 = -\alpha^3 - q, \\ \alpha^3\left(-\alpha^3 - q\right) = -\frac{p^3}{27} \end{cases} \Leftrightarrow \begin{cases} \beta^3 = -\alpha^3 - q, \\ \alpha^6 + q\alpha^3 - \frac{p^3}{27} = 0. \end{cases}$$

Solving this system, taking into account the symmetry of formula (7), we obtain the following solution for equation (4):

$$x = \sqrt[3]{\frac{-q + \sqrt{q^2 + \frac{4}{27}p^3}}{2}} + \sqrt[3]{\frac{-q - \sqrt{q^2 + \frac{4}{27}p^3}}{2}}, \tag{8}$$

which is an analogue of the Cardano formula [9, p. 233–241].

Let us show that this solution is unique. If $p > 0$, then $f'(x) = 3x^2 + p > 0$, it follows that f is strictly increasing and equation (4) has a unique solution (8). If $p < 0$, then f has a strict local maximum at $x_1 = -\sqrt{\frac{-p}{3}}$ and a strict local maximum at $x_2 = \sqrt{\frac{-p}{3}}$. For $q > 0$ the value of $f(x_2) = -\frac{p}{3}\sqrt{\frac{-p}{3}} + p\sqrt{\frac{-p}{3}} + q = \frac{2p}{3}\sqrt{\frac{-p}{3}} + q > 0$, and hence equation (4) has a unique solution (8). At $q < 0$ the value of $f(x_1) = \frac{p}{3}\sqrt{\frac{-p}{3}} - p\sqrt{\frac{-p}{3}} + q = -\frac{2p}{3}\sqrt{\frac{-p}{3}} + q < 0$ and equation (4) has a unique solution (8). The case $q = 0$ is impossible with $p < 0$, if $D > 0$.

Let now $D < 0$, then formula (8) is not defined over $\mathbb{R}$. Since $f(x_1) > 0$ and $f(x_2) < 0$, then equation (4) has three simple roots.

Now let us consider the cubic equation over $\mathbb{C}_p$ with p-complex coefficients $p = a + jb$, $q = c + jd$ such that $a, c \neq 0$

$$z^3 + pz + q = 0. \tag{9}$$

This equation is equivalent to the system

$$\begin{cases} x^3 + ax + c = 0, \\ y\left(3x^2 + a\right) + bx + d = 0. \end{cases} \tag{10}$$

Let $D_1 = c^2 + \frac{4}{27}a^3$. If $D_1 = 0$, then by virtue of formula (6) the first equation in system (10) has a simple root $x = \frac{3c}{a}$ and a root $x = -\frac{3c}{2a}$ of multiplicity 2. The following cases are possible:

1) $3cb = 2ad$. In this case, the second equation in system (10) has a root $x = -\frac{3c}{2a}$ and equation (9) has infinitely many solutions of the form $z = -\frac{3c}{2a} + jy$, where $y \in \mathbb{R}$ is arbitrary, and a solution $z = \frac{3c}{a} + j\left(-\frac{3a^2 d}{27c^2 + a^3}\right)$;

2) $3cb \neq 2ad$. Equation (9) has a unique solution $z = \frac{3c}{a} + j\left(-\frac{3abc + a^2 d}{27c^2 + a^3}\right)$.

Else, if $D_1 > 0$, the first equation in system (10) has a unique solution, substituting it into the second, we find the only value y. Thus, equation (9) has a unique solution, which is expressed in radicals using the Cardano formula.

If $D_1 < 0$, then equation (9) has three different p-complex solutions, which can not be expressed using the Cardano formula, since this formula in the case is not defined over $\mathbb{C}_p$.

Let now in formulas (9)–(10) $a = 0$, $c \neq 0$. Then $x = -\sqrt[3]{c}$, therefore $y = \frac{b\sqrt[3]{c} - d}{3\sqrt[3]{c^2}}$. In this case, equation (9) has a single root $z = -\sqrt[3]{c} + j\frac{b\sqrt[3]{c} - d}{3\sqrt[3]{c^2}}$.

In case $a \neq 0$, $c = 0$ the first equation in system (10) has a root $x = 0$, which corresponds $y = \frac{-d}{a}$, and hence equation (9) has a root $z = -j\frac{d}{a}$. If $a > 0$, then the root is unique. If $a < 0$, then the first equation in system (10) has two more roots $x_{1,2} = \pm\sqrt{-a}$, which correspond to $y_{1,2} = \frac{\pm b\sqrt{-a} + d}{2a}$. In this case, equation (9) has two more roots $z_{1,2} = \pm\sqrt{-a} + j\frac{\pm b\sqrt{-a} + d}{2a}$.

If $a = c = 0$, then the first equation in system (10) has a single root $x = 0$, therefore, in the second equation of the system we have the expression $y \cdot 0 + d = 0$. Else, if $d = 0$, then equation (9) has infinitely many solutions of the form $z = jy$, where $y \in \mathbb{R}$ is arbitrary, otherwise the equation has no roots.

Let us consider now the general cubic equation over $\mathbb{C}_p$:

$$P_3(z) = Az^3 + Bz^2 + Cz + D = 0, \tag{11}$$

where A is not a zero divisor.

Note that equation (11) can be reduced to form (9) by the change of variable $z = t - \frac{B}{A}$.

We will assume that the coefficients p, q of the equation obtained as a result of such a change are not zero divisors, then we use the results obtained above to obtain the following conclusions regarding the factorisation of equation (11).

If z_1, z_2, z_3 are different roots of equation (11), then $Az^3 + Bz^2 + Cz + D = A(z - z_1)(z - z_2)(z - z_3)$. If z_0 is the unique root $Az^3 + Bz^2 + Cz + D = (z - z_0)(Az^2 + Ez + R)$. Note that such expressions are unique due to theorems 2 and 3. If z_0 is the root of multiplicity 2, then, similarly to the case of a quadratic equation, the polynomial $P_3(z)$ admits a continuum of expressions of the form

$$Az^3 + Bz^2 + Cz + D = A(z - z_0)^2(z - z_1) = A(z - z_1)(z - z_2)(z - z_3),$$

if and only if $2y_0 = y_2 + y_3$, where $y_0 = \mathrm{Par}\, z_0$, $y_2 = \mathrm{Par}\, z_2$, $y_3 = \mathrm{Par}\, z_3$.

If z_0 is the root of multiplicity 3, then the polynomial $P_3(z)$ admits a continuum of expressions of the form

$$P_3(z) = A(z - z_0)^3 = A(z - z_1)(z - z_2)(z - z_3),$$

if and only if $3y_0 = y_1 + y_2 + y_3$, where $y_0 = \mathrm{Par}\, z_0$, $y_1 = \mathrm{Par}\, z_1$, $y_2 = \mathrm{Par}\, z_2$, $y_3 = \mathrm{Par}\, z_3$.

Factorisation of polynomials with an invertible leading coefficient over $\mathbb{C}_p$

Consider the polynomial

$$P_n(z) = z^n + c_{n-1}z^{n-1} + \ldots + c_1 z + c_0, \tag{12}$$

where $c_k \in \mathbb{C}_p$ and $n \in \mathbb{N}$. Let $c_k = a_k + jb_k$, $z = x + jy$, where $a_k, b_k, x, y \in \mathbb{R}$. Note that the real part $Q_n(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ of the polynomial (of an odd degree) $P_n(z)$ always can be expressed in the form $Q_n(x) = (x^2 + p_1x + q_1)\cdots(x^2 + p_mx + q_m)$, where $n = 2m$.

Theorem 4. *Let the polynomial of an even degree (12) have no roots. If in the factorisation of its real part* $Q_n(x) = (x^2 + p_1x + q_1)\cdots(x^2 + p_mx + q_m)$ *all square trinomials are distinct, it can be expressed in a unique way as*

$$P_n(z) = (z^2 + \alpha_1 z + \beta_1)\cdots(z^2 + \alpha_m z + \beta_m), \tag{13}$$

where $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m \in \mathbb{C}_p$.

Proof. We express $P_n(z)$ in the form $P_n(z) = P_{n-2}(z)(z^2 + \alpha_1 z + \beta_1)$, where $P_{n-2} = z^{n-2} + l_{n-3}z^{n-3} + \ldots + l_1z + l_0$. We use the method of mathematical induction on m. The base of the induction is the case $m = 1$, then $n = 2$ and $P_2(z) = (z^2 + \alpha_1 z + \beta_1)$, and hence the statement of the theorem becomes trivial. Let us assume that expression (13) is unique for $(m-1)$. Let us show that there will also be a unique expression

$$P_n(z) = (z^{n-2} + l_{n-3}z^{n-3} + \ldots + l_1z + l_0)(z^2 + \alpha_1 z + \beta_1).$$

Let us expand brackets in expression (13), taking into account $z = x + jy$, $\alpha_1 = p_1 + jp_2$, $\beta_1 = q_1 + jq_2$, $l_k = h_k + js_k$:

$$\begin{gathered}P_n(z) = x^n + (p_1 + h_{n-3})x^{n-1} + (q_1 + p_1h_{n-3} + h_{n-4})x^{n-2} + \ldots + (q_1h_k + p_1h_{k-1} + h_{k-2})x^k + \ldots + \\ + (p_1h_0 + h_1q)_1 x + q_1h_0 + j\Big(y\big(nx^{n-1} + (n-1)(p_1 + h_{n-3})(p_1 + d_1)x^{n-2} + \ldots + (p_1h_0 + h_1q_1)\big) + \\ + (p_2 + s_{n-3})x^{n-1} + (q_2 + h_{n-3}p_2 + p_1s_{n-3} + s_{n-4})x^{n-2} + \ldots + \\ + (h_kq_2 + q_1s_k + h_{k-1}p_2 + p_1s_{k-1} + s_{k-2})x^k + \ldots + \\ + (h_1q_2 + q_1s_1 + h_0p_2 + p_1s_0)x + (h_0q_2 + q_1s_0)\Big) = Q_n(x) + j\big(Q_n'(x)y + T_n(x)\big),\end{gathered}$$

where $T_n(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_1x + b_0$.

The coefficients in the factorisation of $Q_n(x)$ are known. The unknowns are the coefficients p_2, d_2, s_k, where $k \in \{1, \ldots, n-2\}$. Given that

$$\begin{gathered}(p_2 + s_{n-3})x^{n-1} + (q_2 + h_{n-3}p_2 + p_1s_{n-3} + s_{n-4})x^{n-2} + \ldots + \\ + (h_kq_2 + q_1s_k + h_{k-1}p_2 + p_1s_{k-1} + s_{k-2})x^k + \ldots + \\ + (h_1q_2 + q_1s_1 + h_0p_2 + p_1s_0)x + (h_0q_2 + q_1s_0) = T_n(x),\end{gathered}$$

we have a system with respect to these unknowns

$$\begin{cases} p_2 + s_{n-3} = b_{n-1}, \\ h_{n-3}p_2 + q_2 + p_1s_{n-3} + s_{n-4} = b_{n-2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ h_{k-1}p_2 + h_kq_2 + p_1s_{k-1} + q_1s_k + s_{k-2} = b_k, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ h_0p_2 + h_1q_2 + p_1s_0 + q_1s_1 = b_1, \\ h_0q_2 + q_1s_0 = b_0. \end{cases} \tag{14}$$

This system has a unique solution if and only if the determinant M is non-zero:

$$M = \begin{vmatrix} 1 & 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ h_{n-3} & 1 & p_1 & 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ h_{n-4} & h_{n-3} & q_1 & p_1 & \dots & 0 & 0 & \dots & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ h_{k-1} & h_k & 0 & 0 & \dots & q_1 & p_1 & \dots & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ h_2 & h_3 & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 \\ h_1 & h_2 & 0 & 0 & \dots & 0 & 0 & \dots & p_1 & 1 \\ h_0 & h_1 & 0 & 0 & \dots & 0 & 0 & \dots & q_1 & p_1 \\ 0 & h_0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & q_1 \end{vmatrix}.$$

Note that this determinant is the resultant of the polynomials $\left(x^{n-2} + h_{n-3}x^{n-3} + \ldots + h_1 x + h_0\right)$ and $\left(x^2 + p_1 x + q_1\right)$ (the concept of resultant and its properties can be found in more detail in the source [9, p. 334–345]). Thus, M is the product of pairwise differences between their roots in the algebraic closure of the field $\mathbb{R}$, that is $\mathbb{C}$. Due to the conditions of the theorem, these polynomials have no common roots, then $M \neq 0$. Thus, system (14) has a unique solution, and hence $P_n(z)$ is expressed in the form $\left(z^{n-2} + l_{n-3}z^{n-3} + \ldots + l_1 z + l_0\right)\left(z^2 + \alpha_1 z + \beta_1\right)$ in a unique way. It follows from this, by virtue of the inductive assumption, that the polynomial $P_n(z)$ admits a unique expression in the form (13). The theorem is proved.

Theorem 5. *Let the real part of $P_n(z)$ have the following irreducible factorisation:*

$$Q_n(x) = (x - x_1)\cdots(x - x_s)\left(x^2 + p_1 x + q_1\right)\cdots\left(x^2 + p_k x + q_k\right), \tag{15}$$

and all factors be different, then $P_n(z)$ can be expressed in a unique way as

$$P_n(z) = (z - z_1)\cdots(z - z_s)\left(z^2 + \alpha_1 z + \beta_1\right)\cdots\left(z^2 + \alpha_k z + \beta_k\right), \tag{16}$$

where $z_1, \ldots, z_s, \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in \mathbb{C}_p$.

P r o o f. From formula (15) it follows that $P_n(z)$ has s roots, and all of them are simple. So, by virtue of theorems 2 and 3, $P_n(z)$ can be uniquely expressed as

$$P_n(z) = (z - z_1)\cdots(z - z_s)\left(z^{n-s} + l_{n-s-1}z^{n-s-1} + \ldots + l_1 z + l_0\right), \tag{17}$$

where the last factor has no roots. By virtue of the conditions imposed on the factors in formula (15), theorem 4 can be applied to the last factor in formula (17), then

$$\left(z^{n-s} + l_{n-s-1}z^{n-s-1} + \ldots + l_1 z + l_0\right) = \left(z^2 + \alpha_1 z + \beta_1\right)\cdots\left(z^2 + \alpha_k z + \beta_k\right),$$

where $2k = n - s$. This implies the statement of the theorem and expression (16).

Remark. It is easy to see that when the square trinomials coincide, the expansion $P(z) = \left(z^2 + \alpha z + \beta\right)^2$ can be rewritten in another form, for example, $P(z) = \left(z^2 + \alpha z + \beta + j\right)\left(z^2 + \alpha z + \beta - j\right)$.

Now let the polynomial (12) has a unique root z_0 of multiplicity n. The case $n = 2$ has been already discussed, now let us consider the general case.

Theorem 6. *The polynomial $P_n(z) = (z - z_0)^n$ admits factorisation*

$$P_n(z) = (z - z_1)\cdots(z - z_n),$$

if and only if $ny_0 = y_1 + y_2 + \ldots + y_n$, where $y_k = \mathrm{Par}\, z_k$, $k \in \{0, 1, \ldots, n\}$.

P r o o f. It is obvious that $\mathrm{Re}\, z_1 = \mathrm{Re}\, z_2 = \ldots = \mathrm{Re}\, z_n = \mathrm{Re}\, z_0$, otherwise there would be a contradiction with the fact that the root z_0 is multiple. Then

$$(z - z_1)\cdots(z - z_n) = \left(x - x_0 + j(y - y_1)\right)\cdots\left(x - x_0 + j(y - y_n)\right) =$$

$$= (x - x_0)^n + j(x - x_0)^{n-1}(ny - y_1 - y_2 - \ldots - y_n) =$$

$$= (x - x_0)^n + j(x - x_0)^{n-1}(ny - ny_0) + j(x - x_0)^{n-1}(ny - y_1 - y_2 - \ldots - y_n) =$$
$$= (z - z_0)^n + j(x - x_0)^{n-1}(ny - y_1 - y_2 - \ldots - y_n),$$

since $(z - z_0)^n = (x - x_0)^n + jn(x - x_0)^{n-1}(y - y_0)$, due to the properties of $\mathbb{C}_p$. This implies the statement of the theorem.

Summing up all of the above, we get that a polynomial with a leading coefficient equal to one has a unique expression as the product of linear and square factors with a leading coefficient equal to one if and only if all its roots are simple, and all square trinomials in the expansion of its real part are different. If the polynomial has a multiple root or there are two coinciding square trinomials in the expansion of its real part, then the polynomial has a continuum of factorisations, and the corresponding equation has a continuum of solutions.

Conclusions

In this paper, the issue of solvability of algebraic equations in the ring of *p*-complex numbers is studied. Equations of the second and third degree are considered separately, in particular, an analogue of the Cardano formula is obtained. For polynomials of an arbitrary degree with an invertible leading coefficient, theorems on factoring with a unit leading coefficient over the ring of *p*-complex numbers are proven.

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