Southeast Asian Bulletin of Mathematics (2002) 26: 421–432

Southeast Asian
Bulletin of
Mathematics


# Results on Nonlinear Hybrid Contractions Satisfying a Rational Inequality

Mohammad Imdad and Tariq Iqtadar Khan

Department of Mathematics, Aligarh Muslim University, Aligarh (U.P.)-202002, India
E-mail: tariq1@justice.com

AMS Subject Classification (1991): Primary 54H25, Secondary 44H10

**Abstract.** Results on coincidence and fixed points for non-linear hybrid contractions satisfying a rational inequality are proved which unify and generalize several results due to Fisher, Hardy-Rogers and others. Illustrative examples are also furnished.

**Keywords:** weakly commuting mappings, compatible mappings, coincidently idempotent mappings, coincidence points and fixed points

## 1. Introduction

The study of fixed points for multi-valued mappings was initiated and studied by Nadler [17] wherein he proved a multi-valued version of Banach Contraction Principle which is sometime referred as Nadler's contraction principle. Since then various well known metrical fixed point theorems were extended to multi-valued mappings and by now there exists an extensive literature on this subject. For the work of this kind one can be referred to Kaneko [9, 10], Sessa [18], Singh [15] and others.

Recently some non-linear hybrid contractions, i.e., contractive conditions involving single-valued and multi-valued mappings, have been studied by Mukherjee [16], Rhoades et al. [19], Sessa et al. [20, 21] and Imdad-Ahmad [6].

In this paper, using certain weak conditions of commutativity (cf. [2], [7], [8], [22]) we prove results on coincidence points for single-valued and multi-valued mappings satisfying a general rational inequality which unify several well known results due to Fisher [3, 4], Kannan [12, 13], Hardy-Rogers [5] and others.

Our improvement is two fold: Firstly the contraction condition in examination is quite general secondly the number of involved maps are increased from two to six. Apart from these two improvements, we employ weak conditions of commutativity instead of commutativity which accomodates a wider class of mappings.

## 2. Preliminaries

Let $CB(X)$ denote the family of all nonempty closed and bounded subsets of a metric space $(X, d)$. The Hausdorff metric $H$ on $CB(X)$ induced by the metric $d$ is defined as

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}$$

for $A, B \in CB(X)$, where $d(x, A) = \inf_{y\in A} d(x, y)$.

Clearly, $\big(CB(X), H\big)$ is a metric space, and if a metric space $(X, d)$ is complete, then so is $\big(CB(X), H\big)$ (cf. Kuratowski [14]).

**Lemma 2.1** (Nadler [17]). *Let $A, B$ be in $CB(X)$. Then for all $\varepsilon > 0$ and $a \in A$ there exists a point $b \in B$ such that $d(a, b) \leq H(A, B) + \varepsilon$.*

In what follows, $I : X \to X$ and $F : X \to CB(X)$ be single-valued and multivalued mappings, on a metric space $(X, d)$.

**Definition 2.1** ([11]). *The mappings I and F are said to be* weakly commuting *if for all $x \in X$, $IFx \in CB(X)$ and*

$$H(FIx, IFx) \leq d(Ix, Fx),$$

*where H is the Hausdorff metric defined on $CB(X)$.*

**Definition 2.2** ([2, 8]). *A pair of self-mappings $(S, I)$ on X is said to be* coincidently commuting *if both the partners S and I are commuting at the coincidence points of S and I.*

**Definition 2.3** ([11]). *The mappings I and F are said to be* compatible *if and only if $IFx \in CB(X)$ for all $x \in X$ and $H(FIx_n, IFx_n) \to 0$, whenever $\{x_n\}$ is a sequence in X such that $Fx_n \to M \in CB(X)$ and $Ix_n \to t \in M$.*

*Remark 2.1.* If $F$ is restricted to be a single-valued mapping on $X$ in Definitions 2.1 and 2.3, then we deduce the concepts of *weak commutativity* (cf. [22]) and *compatibility* (cf. [7]) for single-valued mappings.

We also need the following:

**Lemma 2.2** ([7]). *Let f and g be mappings from a metric space $(X, d)$ into itself. If f and g are compatible and $fz = gz$ for some $z \in X$, then*

$$fgz = ggz = gfz = ffz.$$

The following lemma (cf. [11]) is adopted for weak commutativity.

**Lemma 2.3** ([11]). *Let $I : X \to X$ and $F : X \to CB(X)$ be* weakly commuting. *If $Iz \in Fz$ for some $z \in X$, then $IFz = FIz$.*

## 3. Results

Our results in this paper are amply motivated by a theorem of Fisher [3] which we opt to state before presenting our results.

**Theorem 3.1.** *Let $S$ and $T$ be two self-mappings of a complete metric space $(X, d)$ such that for all $x$, $y$ in $X$, either*

$$\text{(a)} \quad d(Sx, Ty) \le \frac{b[d(x, Ty)]^2 + c[d(y, Sx)]^2}{d(x, Ty) + d(y, Sx)}$$

*if $d(x, Ty) + d(y, Sx) \neq 0$, $b, c \ge 0$ and $b + c < 1$, or*

$$\text{(b)} \quad d(Sx, Ty) = 0 \quad \text{if } d(x, Ty) + d(y, Sx) = 0.$$

*If one of $S$ and $T$ is continuous then $S$ and $T$ have a unique common fixed point.*

We use the following definition to prove our main theorem which merely restricts the full force of idempotence.

**Definition 3.1.** *A pair of self-mappings $(S, I)$ on $X$ is said to be* coincidently idempotent *if both the partners $S$ and $I$ are idempotent at the coincidence points of $S$ and $I$.*

We now prove our main result as follows:

**Theorem 3.2.** *Let $S, T, I$ and $J$ be self-mappings of a complete metric space $(X, d)$ with $SI$ and $TJ$ as $d$-continuous whereas $F, G : X \to CB(X)$ are multi-valued mappings such that*

(i) *$G(X) \subseteq SI(X)$ and $F(X) \subseteq TJ(X)$,*
(ii) *the pairs $(SI, F)$ and $(TJ, G)$ are weakly commuting,*
(iii) *for all $x, y \in X$,*

$$H(Fx, Gy) \le \alpha\left[\frac{\{D(Fx, TJy)\}^2 + \{D(Gy, SIx)\}^2}{D(Fx, TJy) + D(Gy, SIx)}\right] + \beta[D(Fx, SIx) + D(Gy, TJy)] + \gamma d(SIx, TJy). \tag{3.2.1}$$

*If $D(Fx, TJy) + D(Gy, SIx) \neq 0$, $\alpha, \beta, \gamma \ge 0$ with $x \neq y$, $Fx \neq Fy$, $Gx \neq Gy$ and $2\alpha + 2\beta + \gamma < 1$.*

Then the following conclusions hold:

(a) There exists a point $z \in X$ such that $SIz = TJz \in Fz \cap Gz$, i.e., $z$ is a coincidence point of the pairs $(SI, F)$ and $(TJ, G)$.

(b) For each $x \in X$ either (i) $SIx \neq (SI)^2 x \Rightarrow SIx \notin Fx$ (resp. $TJx \neq (TJ)^2 x \Rightarrow TJx \notin Gx$) or (ii) $SIx \in Fx \Rightarrow (SI)^n x \to z$ for some $z \in X$ (resp. $TJx \in Gx \Rightarrow (TJ)^n x \to z$ for some $z \in X$), then $z$ is a common fixed point of the pair $(SI, F)$ (resp. $(TJ, G)$) provided $F$ and $G$ are $H$-continuous.

(c) Moreover, if the pairs of self-mappings $(S, I), (SI, S)$ (resp. $(T, J), (TJ, T)$) are coincidently commuting whereas the pairs $(S, I)$ (resp. $(T, J)$) are coincidently idempotent then $z$ is a common fixed point of $S, I, SI$ and $F$ (resp. $T, J, TJ$ and $G$).

*Proof.* Assume $\theta = \frac{\alpha+\beta+\gamma}{1-\alpha-\beta}$, let $x_0 \in X$ and $y_1$ be an arbitrary point in $Fx_0$. Since $Fx_0 \subseteq TJ(X)$, there exists a point $x_1$ in $X$ such that $y_1 = TJx_1 \in Fx_0$ and so there exists a point $y_2 \in Gx_1$ such that

$$d(y_1, y_2) \leq H(Fx_0, Gx_1) + \frac{1-\alpha-\beta}{1+\alpha+\beta}\theta,$$

which is always possible in view of Lemma 2.1. Since $Gx_1 \subseteq SI(X)$ there exists a point $x_2 \in X$ such that $y_2 = SIx_2$ and so we can find $y_3 \in Fx_2$ such that

$$d(y_2, y_3) \leq H(Gx_1, Fx_2) + \frac{1-\alpha-\beta}{1+\alpha+\beta}\theta^2,$$

Inductively, one can define a sequence $\{y_n\}$ in $X$ such that

$$y_{2n} = SIx_{2n} \in Gx_{2n-1}, \quad y_{2n+1} = TJx_{2n+1} \in Fx_{2n}$$

Now

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &\leq H(Fx_{2n}, Gx_{2n+1}) + \frac{1-\alpha-\beta}{1+\alpha+\beta}\theta^{2n+1}, \qquad (3.2.2)\\ &\leq \alpha\left[\frac{\{D(Fx_{2n}, TJx_{2n+1})\}^2 + \{D(Gx_{2n+1}, SIx_{2n})\}^2}{D(Fx_{2n}, TJx_{2n+1}) + D(Gx_{2n+1}, SIx_{2n})}\right]\\ &\quad + \beta[D(Fx_{2n}, SIx_{2n}) + D(Gx_{2n+1}, TJx_{2n+1})]\\ &\quad + \gamma d(SIx_{2n}, TJx_{2n+1}) + \frac{1-\alpha-\beta}{1+\alpha+\beta}\theta^{2n+1}, \end{aligned}$$

which on simplifying reduces to

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) \leq\ &\alpha[D(Fx_{2n}, TJx_{2n+1}) + D(Gx_{2n+1}, SIx_{2n})]\\ &+ \beta[D(Fx_{2n}, SIx_{2n}) + D(Gx_{2n+1}, TJx_{2n+1})]\\ &+ \gamma d(SIx_{2n}, TJx_{2n+1}) + \frac{1-\alpha-\beta}{1+\alpha+\beta}\theta^{2n+1}, \end{aligned}$$

so that

$$d(y_{2n+1}, y_{2n+2}) \leq \frac{\alpha+\beta+\gamma}{1-\alpha-\beta} d(y_{2n}, y_{2n+1}) + \frac{\theta^{2n+1}}{1+\alpha+\beta}$$
$$= \theta d(y_{2n}, y_{2n+1}) + \frac{\theta^{2n+1}}{1+\alpha+\beta}. \tag{3.2.3}$$

Also from

$$d(y_{2n}, y_{2n+1}) \leq H(Fx_{2n}, Gx_{2n-1}) + \frac{1-\alpha-\beta}{1+\alpha+\beta}\theta^{2n}. \tag{3.2.4}$$

and using (3.2.1), one can obtain

$$d(y_{2n}, y_{2n+1}) \leq \theta d(y_{2n}, y_{2n-1}) + \frac{\theta^{2n}}{1+\alpha+\beta} \tag{3.2.5}$$

Combining (3.2.3) and (3.2.5), we obtain

$$d(y_{n+1}, y_{n+2}) \leq \theta^2 d(y_n, y_{n-1}) + \frac{2\theta^{n+1}}{1+\alpha+\beta} \leq \cdots \leq \theta^n d(y_1, y_2) + \frac{n\theta^{n+1}}{1+\alpha+\beta}. \tag{3.2.6}$$

Thus a straight forward computation shows that $\{y_n\}$ is a Cauchy sequence in the complete metric space $(X, d)$ and so has a limit point $z$ in $X$. On the otherhand, subsequences $\{SIx_{2n}\}$ and $\{TJx_{2n+1}\}$ of $\{y_n\}$ also converge to $z$.

Now suppose that $SI$ is continuous, then $(SI)^2 x_{2n}$ converges to $SIz$. Using weak commutativity of the pair $(SI, F)$, we have $SI(Fx_{2n}) \in CB(X), x_{2n} \in X$, then it follows that

$$H\big(F(SIx_{2n}), SI(Fx_{2n})\big) \leq D(Fx_{2n}, SIx_{2n}) \leq d(y_{2n+1}, y_{2n}) \to 0 \quad \text{as } n \to \infty.$$

But $D\big(SI(TJx_{2n+1}), F(SIx_{2n})\big) \leq H\big(SI(Fx_{2n}), F(SIx_{2n})\big)$. So in view of the continuity of $SI$, we get

$$D\big(SIz, F(SIx_{2n})\big) \to 0 \quad \text{as } n \to \infty. \tag{3.2.7}$$

Similarly, using weak commutativity of the pair $(TJ, G)$, we get

$$D\big(TJz, G(TJx_{2n+1})\big) \to 0 \quad \text{as } n \to \infty, \tag{3.2.8}$$

which is always possible in view of the continuity of $TJ$.

Using (3.2.1), we have

$$
\begin{aligned}
&D\big(F(SIx_{2n}), TJz\big)\\
&\quad\leq H\big(F(SIx_{2n}), G(TJx_{2n+1})\big) + D\big(G(TJx_{2n+1}), TJz\big),\\
&\quad\leq \alpha\left[\frac{\big\{D\big(F(SIx_{2n}), TJ(TJx_{2n+1})\big)\big\}^2 + \big\{D\big(G(TJx_{2n+1}), SI(SIx_{2n})\big)\big\}^2}{D\big(F(SIx_{2n}), TJ(TJx_{2n+1})\big) + D\big(G(TJx_{2n+1}), SI(SIx_{2n})\big)}\right]\\
&\qquad + \beta\big[D\big(F(SIx_{2n}), SI(SIx_{2n})\big) + D\big(G(TJx_{2n+1}), TJ(TJx_{2n+1})\big)\big]\\
&\qquad + \gamma d\big(SI(SIx_{2n}), TJ(TJx_{2n+1})\big) + D\big(G(TJx_{2n+1}), TJz\big),
\end{aligned}
$$

which on using (3.2.7), (3.2.8) and letting $n \to \infty$, reduces to

$$d(SIz, TJz) \leq (2\alpha + \gamma)d(SIz, TJz),$$

a contradiction, giving thereby $SIz = TJz$.

Further from (3.2.1), we get

$$
\begin{aligned}
D(SIz, Fz) &\leq D\big(SIz, G(TJx_{2n+1})\big) + H\big(G(TJx_{2n+1}), Fz\big),\\
&\leq D\big(SIz, G(TJx_{2n+1})\big)\\
&\quad + \alpha\left[\frac{\big\{D\big(Fz, TJ(TJx_{2n+1})\big)\big\}^2 + \big\{D\big(G(TJx_{2n+1}), SIz\big)\big\}^2}{D\big(Fz, TJ(TJx_{2n+1})\big) + D\big(G(TJx_{2n+1}), SIz\big)}\right]\\
&\quad + \beta\big[D(Fz, SIz) + D\big(G(TJx_{2n+1}), TJ(TJx_{2n+1})\big)\big]\\
&\quad + \gamma d\big(SIz, TJ(TJx_{2n+1})\big),
\end{aligned}
$$

which on using (3.2.7), (3.2.8), $SIz = TJz$ and letting $n \to \infty$, reduces to

$$D(SIz, Fz) \leq (\alpha + \beta)D(SIz, Fz),$$

a contradiction giving thereby $SIz \in Fz$.

Again using (3.2.1), we have

$$
\begin{aligned}
D(TJz, Gz) &\leq D\big(TJz, F(SIx_{2n})\big) + H\big(F(SIx_{2n}, Gz)\big),\\
&\leq D\big(TJz, F(SIx_{2n})\big)\\
&\quad + \alpha\left[\frac{\big\{D\big(F(SIx_{2n}), TJz\big)\big\}^2 + \big\{D\big(Gz, SI(SIx_{2n})\big)\big\}^2}{D\big(F(SIx_{2n}), TJz\big) + D\big(Gz, SI(SIx_{2n})\big)}\right]\\
&\quad + \beta\big[D\big(F(SIx_{2n}), SI(SIx_{2n})\big) + D(Gz, TJz)\big] + \gamma d\big(SI(SIx_{2n}), TJz\big),
\end{aligned}
$$

which on using (3.2.7), (3.2.8), $SIz = TJz$ and letting $n \to \infty$, reduces to

$$D(TJz, Gz) \leq (\alpha + \beta)D(TJz, Gz),$$

a contradiction yielding thereby $TJz \in Gz$. Thus we have shown that $SIz = TJz \in Fz \cap Gz$.

For proving (b), assume that $SIx \neq (SI)^2x$ which implies that $SIx \notin Fx$, we deduce that $SIx = (SI)^2x \in SI(Fx) = F(SIx)$, which is always possible in view of Lemma 2.3. Assuming that $SIx \in Fx$ implies that $(SI)^n x \to z$ for some $z$ in $X$, then it is straight forward to note that $SIz = z$ by continuity of $SI$. We assert that $(SI)^n x \in F(SI)^{n-1}x$ for each $n$. To see this, let $(SI)^2x = SI(SIx) \in SI(Fx) = F(SIx)$. Also $(SI)^3x = SI\big((SI)^2x\big) \in SI\big(F(SI)x\big) = F\big((SI)^2x\big)$. Repeating this argument, one inductively obtains $(SI)^n x \in F\big((SI)^{n-1}x\big)$ which together with the continuity of $F$ gives

$$d(z, Fz) \leq d\big(z, (SI)^n x\big) + d\big((SI)^n x, Fz\big) \leq d\big(z, (SI)^n x\big) + H\big(F(SI)^{n-1}x, Fz\big) \to 0,$$

i.e., $z \in Fz$ as $Fz$ is closed. Hence $z$ is a common fixed point of the pair $(SI, F)$ (resp. $(TJ, G)$).

For proving (c), let us write

$$Sz = S(SIz) = S(ISz) = SI(Sz) = IS(Sz) = I(S^2z) = I(Sz) = SIz = z,$$
$$Iz = I(SIz) = IS(Iz) = SI(Iz) = S(I^2z) = S(Iz) = SIz = z,$$

which show that $z$ is a common fixed point of $S, I, SI$ and $F$. Similarly it can be shown that $z$ is also a common fixed point of $T, J, TJ$ and $G$.

**Corollary 3.1.** *Theorem* 3.2 *remains true if contraction condition* (3.2.1) *is replaced by any one of the following: for all* $x, y$ *in* $X$ (*with* $D(Fx, TJy) + D(Gy, SIx) \neq 0$).

$$\text{(A)}\quad H(Fx, Gy) \leq \alpha\left[\frac{\{D(Fx, TJy)\}^2 + \{D(Gy, SIx)\}^2}{D(Fx, TJy) + D(Gy, SIx)}\right] + \beta[D(Fx, SIx) + D(Gy, TJy)]$$

*with* $2\alpha + 2\beta < 1$, *or*

$$\text{(B)}\quad H(Fx, Gy) \leq \alpha\left[\frac{\{D(Fx, TJy)\}^2 + \{D(Gy, SIx)\}^2}{D(Fx, TJy) + D(Gy, SIx)}\right] + \gamma d(SIx, TJy)]$$

*with* $2\alpha + \gamma < 1$, *or*

$$\text{(C)}\quad H(Fx, Gy) \leq \alpha\left[\frac{\{D(Fx, TJy)\}^2 + \{D(Gy, SIx)\}^2}{D(Fx, TJy) + D(Gy, SIx)}\right] \quad \textit{with } \alpha > 0, \alpha < \frac{1}{2}, \quad \textit{or}$$

$$\text{(D)}\quad H(Fx, Gy) \leq \alpha[D(Fx, TJy) + D(Gy, SIx)] + \beta[D(Fx, SIx) + D(Gy, TJy)] + \gamma d(SIx, TJy)$$

*with* $2\alpha + 2\beta + \gamma < 1$, *or*

$$\text{(E)}\quad H(Fx, Gy) \leq \alpha[D(Fx, TJy) + D(Gy, SIx)] \quad \textit{with } \alpha < \frac{1}{2}, \quad \textit{or}$$

$$\text{(F)} \quad H(Fx, Gy) \le \beta[D(Fx, SIx) + D(Gy, TJy)] \quad \textit{with } \beta < \frac{1}{2}, \quad \textit{or}$$

$$\text{(G)} \quad H(Fx, Gy) \le \gamma d(SIx, TJy) \quad \textit{with } \gamma < 1.$$

*Proof.* Corollaries corresponding to contractions (A), (B), and (C) can be deduced directly from Theorem 3.2 by choosing $\gamma = 0$, $\beta = 0$, $\beta = \gamma = 0$, respectively. The corollary corresponding to contraction condition (D) also follows from Theorem 3.2 by noting that

$$\begin{aligned} \frac{\{D(Fx, TJy)\}^2 + \{D(Gy, SIx)\}^2}{D(Fx, TJy) + D(Gy, SIx)} &\le \frac{[D(Fx, TJy) + D(Gy, SIx)]^2}{D(Fx, TJy) + D(Gy, SIx)} \\ &= D(Fx, TJy) + D(Gy, SIx). \end{aligned}$$

Finally, one may note that the contraction conditions (E), (F) and (G) are special cases to the contraction condition (D).

*Remark 3.1.* The foregoing corollary presents generalized hybrid fixed point theorems corresponding to the results contained in Fisher [3, 4], Kannan [12, 13] and Hardy-Rogers [5].

**Theorem 3.3.** *Let $S, T, I, J, F$ and $G$ be the same as defined in Theorem* 3.2 *satisfying* (i), (iii) *and condition* (ii) *is replaced by*

(ii)′ *the pairs $(SI, F)$ and $(TJ, G)$ are compatible.*

*Then the conclusions* (a), (b) *and* (c) (*of Theorem* 3.2) *remain true*.

*Proof.* Proceeding as in Theorem 3.2, one can show that $\{y_n\}$ is a Cauchy sequence which converges to a point $z$ in $X$. Further, from (3.2.2) and (3.2.3), we recall that

$$H(Fx_{2n}, Gx_{2n+1}) \le \theta d(y_{2n}, y_{2n+1}) + \frac{\alpha + \beta}{1 + \alpha + \beta}\theta^{2n+1} \quad (n = 0, 1, 2, \ldots)$$

which yields that the sequence

$$\{Fx_0, Gx_1, Fx_2, \ldots, Gx_{2n-1}, Fx_{2n}, Gx_{2n+1}, \ldots\}$$

is a Cauchy sequence in the complete metric space $(CB(X), H)$ and hence converges to some $M \in CB(X)$. Consequently, the subsequences $\{Fx_{2n}\}$ and $\{Gx_{2n+1}\}$ also converge to $M$.

Now

$$\begin{aligned} D(z, M) &\le d(z, TJx_{2n+1}) + D(TJx_{2n+1}, M), \\ &\le d(z, TJx_{2n+1}) + H(Fx_{2n}, M). \end{aligned}$$

On letting $n \to \infty$, we get $z \in M$ as $M$ is closed. Further the compatibility of $F$ and $SI$ implies that

$$H\big(F(SIx_{2n}), SI(Fx_{2n})\big) \to 0 \quad \text{as } n \to \infty.$$

But $D\big(SI(TJx_{2n+1}), F(SIx_{2n})\big) \le H\big(F(SIx_{2n}), SI(Fx_{2n})\big)$. So in view of the continuity of $SI$, we get $D\big(SIz, F(SIx_{2n})\big) \to 0$ as $n \to \infty$.

Similarly, as the pair $(TJ, G)$ is compatible with $TJ$ continuous, we get

$$D\big(TJz, G(TJx_{2n+1})\big) \to 0 \quad \text{as } n \to \infty.$$

Now arguing as in Theorem 3.2, one can prove (a), (b) and (c).

Next, using Theorem 3.3, we give a common fixed point theorem for single-valued mappings in metric spaces. For this let $F$ and $G$ denotes the single-valued mappings from a metric sapce $(X, d)$ into itself in Theorem 3.3, then we have the following:

**Theorem 3.4.** *Let $S, T, I, J, F$ and $G$ be continuous self-mappings of a metric space $(X, d)$ such that the pairs $(SI, F)$ and $(TJ, G)$ are compatible. If $F(X) \subseteq TJ(X)$, $G(X) \subseteq SI(X)$ and for all $x, y \in X$, either*

$$d(Fx, Gy) \le \alpha\left[\frac{\{d(Fx, TJy)\}^2 + \{d(Gy, SIx)\}^2}{d(Fx, TJy) + d(Gy, SIx)}\right] + \beta[d(Fx, SIx) + d(Gy, TJy)] + \gamma d(SIx, TJy), \tag{3.4.1}$$

*if $d(Fx, TJy) + d(Gy, SIx) \neq 0$, $\alpha, \beta, \gamma \ge 0$, $2\alpha + 2\beta + \gamma < 1$, or*

$$d(Fx, Gy) = 0 \quad \text{if } d(Fx, TJy) + d(Gy, SIx) = 0. \tag{3.4.2}$$

*Then $SI, TJ, F$ and $G$ have a unique common fixed point $z$ in $X$. Moreover, $z$ is a unique common fixed point of the pairs $(SI, F)$ and $(TJ, G)$.*

*Further, if the pairs $(S, I), (IS, I), (S, F), (F, I), (T, J), (JT, J), (T, G)$ and $(G, J)$ commute at the points of coincidence, then $z$ remains a unique common fixed point of $S, I, T, J, F$ and $G$ separately.*

*Proof.* The existence of the point $w$ with $SIw = Fw$ and $TJw = Gw$ for contraction condition (3.4.1) follows from Theorem 3.2. Hence we need to prove the same for condition (3.4.2). For this $d(Fw, TJw) + d(Gw, SIw) = 0$ implies that $d(Fw, Gw) = 0$, which gets us

$$Fw = SIw = TJw = Gw.$$

Since the pair $(SI, F)$ is compatible and $SIw = Fw$, therefore by Lemma 2.2, we have

$$SI(Fw) = FFw = F(SIw) = SI(SIw), \tag{3.4.3}$$

which implies that $d(FFw, TJw) + d\big(Gw, SI(Fw)\big) = 0$, which due to (3.4.2) yields $d(FFw, Gw) = 0$, giving thereby $FFw = Gw$, and we obtain

$$F(SIw) = FFw = Gw = SIw, \tag{3.4.4}$$

Therefore $SIw = z$ is a fixed point of $F$. Further, (3.4.3) and (3.4.4) implies that

$$Fz = SIz = z.$$

Similarly, we can show that

$$Gz = TJz = z.$$

Using (3.4.2), as $d(Fz, TJz) + d(Gz, SIz) = 0$, it follows that $d(Fz, Gz) = 0$ and so $Fz = Gz$. Therefore, the point $z$ is a common fixed point of $SI, TJ, F$ and $G$.

The rest of the proof is straight forward, henced it is ommited. This evidently completes the proof.

## 4. Related Examples

Our first example is furnished to demonstrate the validity of the hypotheses and degree of generality of Theorem 3.2. (resp. Theorem 3.1)

*Example 4.1.* Consider $X = [0,1]$ with usual metric. Define self-mappings $Fx = x/12$, $Tx = x/2$, $Jx = x/4$, $Gx = x/16$, $Sx = x/5$, $Ix = 5x/6$ so that $TJx = x/8$ and $SIx = x/6$. Clearly $G(X) = [0, 1/16] \subset [0, 1/6] = SI(X)$ and $F(X) = [0, 1/12] \subset [0, 1/8] = TJ(X)$. Also the pairs of mappings $(SI, F)$ and $(TJ, G)$ are commuting hence weakly commuting or compatible or weakly compatible.

Now for any $x$, $y$ in $X$, one can have

$$\begin{aligned} H(Fx, Gy) &= d(Fx, Gy) \\ &= \left|\frac{x}{12} - \frac{y}{16}\right| = \frac{1}{2}\left|\frac{x}{6} - \frac{y}{8}\right| = \frac{1}{2} d(SIx, TJy) \\ &\leq \alpha\left[\frac{[d(Fx, TJy)]^2 + [d(Gy, SIx)]^2}{d(Fx, TJy) + d(Gy, SIx)}\right] + \beta[d(Fx, SIx) + d(Gy, TJy)] \\ &\quad + 1/2 d(SIx, TJy), \end{aligned}$$

which verifies the contraction condition (3.2.1) with $\gamma = 1/2$ and $2\alpha + 2\beta < 1/2$. Clearly '0' is the unique common fixed point of $F, G, S, T, I$ and $J$.

However, our unification is genuine because for $x = 0,\ y = 1$ the contraction condition (3.2.1) with $\alpha = \gamma = 0$ implies $1/16 \leq \beta/16$ or $\beta \geq 1$ which is a contradiction. Also for $x = 1,\ y = 0$ the contraction condition (3.2.1) with $\beta = \gamma = 0$ implies $1/12 \leq 5\alpha/36$ or $2\alpha \geq 6/5$ which is again a contradiction.

We conclude by observing that the conditions $x \neq y,\ Fx \neq Fy,\ Gx \neq Gy$ are necessary in Theorem 3.2. To substantiate this, we consider the following example.

*Example 4.2.* Consider $X = [0, 1]$ with usual metric. Define $Sx = 1 - x$, $Ix = 2x$, $Tx = 1 - 2x$, $Jx = x/2$, $Fx = Gx = \{0, 1\}$ so that $SIx = 1 - 2x$ and $TJx = 1 - x$ for all $x \in X$.

It is straight forward to note that all the conditions of Theorem 3.2 (a) are satisfied except $x \neq y,\ Fx \neq Fy,\ Gx \neq Gy$. One can note that $TJ(1/2) = 1/2 \notin F(1/2) \cap G(1/2)$ and $SI(1/3) = 1/3 \notin F(1/3) \cap G(1/3)$ which show that $F, G, SI$ and $TJ$ have no coincidence or fixed points.

## References


1. Cho, Y.J., Fisher, B. and Genga, G.S.: Coincidence theorems for non-linear hybrid contractions, *Internat. J. Math. and Math. Sci.* **20**(2), 249–256 (1997).
2. Dhage, B.C.: On common fixed point of pairs of coincidently commuting mappings in *D*-metric spaces, *Indian J. Pure and Appl. Math.* **30**(4), 395–409 (1999).
3. Fisher, B.: Mappings satisfying rational inequality, *Nanta. Math.* **12**, 195–199 (1979).
4. Fisher, B.: Mappings with a common fixed point, *Math. Sem. Notes* **7**, 81–83 (1979).
5. Hardy, G.E. and Rogers, T.D.: A generalization of a fixed point theorem of Reich, *Canad. Math. Bull.* **16**, 201–206 (1973).
6. Imdad, M. and Ahmad, A.: On a common fixed point of mappings with weak conditions of commutativity, *Publ. Math. Debrecen* **44/1–2**, 105–114 (1994).
7. Jungck, G.: Compatible mappings and common fixed points, *Internat. J. Math. and Math. Sci.* **9**(4), 771–779 (1986).
8. Jungck, G. and Rhoades, B.E.: Fixed points for set-valued functions without continuity, *Indian J. Pure and Appl. Math.* **29**(3), 227–238 (1998).
9. Jungck, G. and Rhoades, B.E.: A comparison of contractive conditions for multi-valued mappings, *Kobe J. Math.* **3**, 37–45 (1986).
10. Kaneko, H.: A common fixed point of weakly commuting multi-valued mappings, *Math. Japon* **33**(5), 741–744 (1988).
11. Kaneko, H. and Sessa, S.: Fixed point theorem for compatible multi-valued and single-valued mappings, *Internat. J. math. and Math. Sci.* **12**(2), 257–262 (1989).
12. Kannan, R.: Some results on fixed points, *Bull. Calcutta Math. Soc.* **60**, 71–76 (1968).
13. Kannan, R.: Some results in fixed points II, *Amer. Math. Monthly* **76**, 405–408 (1969).
14. Kuratowsaki, Topology, *Vol. 1, Academic Press*, 1966.
15. Meade, B.A. and Singh, S.P.: On common fixed point theorems, *Bull. Austral. Math. Soc.* **16**, 49–53 (1977).
16. Mukherjee, R.N.: On fixed point of single-valued and set-valued mappings, *J. Indian Acad. Math.* **4**, 101–103 (1982).
17. Nadler, S.B.: Multi-valued contraction mappings, *Pacific J. Math.* **20**(2), 475–488 (1969).
18. Rhoades, B.E. and Sessa, S.: Common fixed point theorems for three mappings under a weak commutativity condition, *Indian J. Pure. and Appl. Math.* **17**, 47–57 (1986).

19. Rhoades, B.E., Singh, S.L. and Kulshrestha, C.: Coincidence theorems for some multi-valued mappings, *Internat. J. Math. Sci.* **7**, 429–434 (1984).
20. Sessa, S., Mukherjee, R.N. and Som, T.: A common fixed point theorem for weakly commuting mappings, *Math. Japon.* **31**, 235–245 (1986).
21. Sessa, S., Khan, M.S. and Imdad, M.: A common fixed point theorem with a weak commutativity condition, *Glas. Mat. Ser. III* **21**, 225–235 (1986).
22. Sessa, S.: On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math.* **32**(46), 149–153 (1982).