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Research Article

Generalized Distribution and Its Geometric Properties Associated with Univalent Functions

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The purpose of the present paper is to introduce a generalized discrete probability distribution and obtain some results regarding moments, mean, variance, and moment generating function for this distribution. Further, we show that for specific values it reduces to various well-known distributions. Finally, we give a beautiful application of this distribution on certain analytic univalent functions.

1. Introduction

Let the series $\sum_{n=0}^{\infty} a_n$, where $a_n \geq 0$, $\forall n \in N$ is convergent and its sum is denoted by S, that is,

$$S = \sum_{n=0}^{\infty} a_n. \tag{1}$$

Now, we introduce the generalized discrete probability distribution whose probability mass function is

$$p(n) = \frac{a_n}{S}, \quad n = 0, 1, 2, \ldots. \tag{2}$$

Obviously $p(n)$ is a probability mass function because $p(n) \geq 0$ and $\sum_n p_n = 1$.

Now, we introduce the series

$$\phi(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{3}$$

From (1) it is easy to see that the series given by (3) is convergent for $|x| < 1$ and for $x = 1$ it is also convergent.

Definition 1. If X is a discrete random variable which can take the values $x_1, x_2, x_3, \ldots$ with respective probabilities $p_1, p_2, p_3, \ldots$ then expectation of X, denoted by $E(X)$, is defined as

$$E(X) = \sum_{n=1}^{\infty} p_n x_n. \tag{4}$$

Definition 2. The rth moment of a discrete probability distribution about $X = 0$ is defined by

$$\mu_r' = E(X^r). \tag{5}$$

Here μ_1' is known as mean of the distribution and variance of the distribution is given by $\mu_2' - (\mu_1')^2$.

Moments about the Origin

(1)

$$\begin{aligned} \mu_1' &= \sum_{n=0}^{\infty} n p(n) \\ &= \sum_{n=0}^{\infty} n \frac{a_n}{S} \\ &= \frac{1}{S} \sum_{n=1}^{\infty} n a_n \\ &= \frac{\phi'(1)}{S}. \end{aligned} \tag{6}$$

(2)

$$\begin{aligned} \mu_2' &= \sum_{n=0}^{\infty} n^2 p(n) \\ &= \sum_{n=0}^{\infty} n^2 \frac{a_n}{S} \end{aligned}$$

$$= \frac{1}{S}\sum_{n=0}^{\infty}\{n(n-1)+n\}a_n$$
$$= \frac{1}{S}\left[\sum_{n=2}^{\infty}n(n-1)a_n+\sum_{n=1}^{\infty}na_n\right]$$
$$= \frac{1}{S}\left[\phi''(1)+\phi'(1)\right]. \quad (7)$$

(3)

$$\mu_3' = \sum_{n=0}^{\infty}n^3p(n)$$
$$= \sum_{n=0}^{\infty}\{n(n-1)(n-2)+3n(n-1)+n\}\frac{a_n}{S}$$
$$= \frac{1}{S}\left[\sum_{n=3}^{\infty}n(n-1)(n-2)a_n+3\sum_{n=2}^{\infty}n(n-1)a_n + \sum_{n=1}^{\infty}na_n\right] \quad (8)$$
$$= \frac{1}{S}\left[\phi'''(1)+3\phi''(1)+\phi'(1)\right].$$

(4)

$$\mu_4' = \sum_{n=0}^{\infty}n^4p(n)$$
$$= \sum_{n=0}^{\infty}\{n(n-1)(n-2)(n-3)+6n(n-1)(n-2) + 7n(n-1)+n\}\frac{a_n}{S} \quad (9)$$
$$= \frac{1}{S}\left[\phi^{iv}(1)+6\phi'''(1)+7\phi''(1)+\phi'(1)\right].$$

Definition 3. The mean of the distribution is given by

$$\text{Mean} = \mu_1' = \frac{\phi'(1)}{S}. \quad (10)$$

Definition 4. The variance of the distribution is given by

$$\text{Variance} = \mu_2' - (\mu_1')^2 = \frac{1}{S}\left[\phi''(1)+\phi'(1)-\frac{(\phi'(1))^2}{S}\right]. \quad (11)$$

Definition 5. The moment generating function (m.g.f.) of a random variable X is denoted by $M_X(t)$ and defined by

$$M_X(t) = E\left(e^{tX}\right). \quad (12)$$

Theorem 6. *The moment generating function of generalized discrete probability distribution is given by*

$$M_X(t) = \frac{\phi(e^t)}{S}. \quad (13)$$

Proof. One has

$$M_X(t) = E\left(e^{tX}\right) = \sum_{n=0}^{\infty}e^{tn}p(n) = \sum_{n=0}^{\infty}e^{tn}\frac{a_n}{S} = \frac{\phi(e^t)}{S}. \quad (14)$$

□

2. Some Consequences

By specializing the values of a_n, we obtain the following well-known discrete probability distributions.

(1) Let $a_n = \rho B(n,\rho+1)$, where $\rho > 0$, $n \in N = \{1,2,3,\ldots\}$; then it reduces to Yule–Simon Distribution [1].

(2) Let $a_n = p^n/n$, where $0 < p < 1$, $n \in N = \{1,2,3,\ldots\}$; then it reduces to Logarithmic Distribution [1].

(3) Let $a_n = m^n/n!$ and then it reduces to Poisson distribution [1, 2].

(4) Let $a_n = {}^kC_n(p/(1-p))^n$, $n = 0,1,2,\ldots,k$; then it reduces to Binomial Distribution [1, 2].

(5) Let $a_n = {}^kC_nB(n+\alpha, n-k+\beta)$, $n = 0,1,2,\ldots,k$; then it reduces to Beta-Binomial Distribution [1].

(6) Let $a_n = (1-p)^n$, $n = 0,1,2,\ldots$; then it reduces to Geometric Distribution [1].

(7) Let $a_n = 1/n^s$, where $s \in (1,\infty)$ and $n \in N$; then it reduces to Zeta Distribution [1].

(8) Let

$$a_n = \begin{cases} p, & \text{if } n = 1 \\ 1-p, & \text{if } n = 0 \end{cases} \quad (15)$$

and then it reduces to Bernoulli Distribution [1, 2].

3. Applications on Certain Classes of Univalent Functions

Let $\mathscr{A}$ denote the class of functions f of the following form:

$$f(z) = z + \sum_{n=2}^{\infty}A_nz^n, \quad (16)$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in C$ and $|z| < 1\}$. As usual, by $\mathscr{S}$ we shall represent the class of all functions in $\mathscr{A}$ which are univalent in $\mathbb{U}$ and further, we denote $\mathscr{T}$ be the subclass of $\mathscr{S}$ consisting of functions of the following form:

$$f(z) = z - \sum_{n=2}^{\infty}|A_n|z^n. \quad (17)$$

In 1988, Altintas and Owa [3] introduced the class $T(\lambda,\alpha)$, (α ($0 \le \alpha < 1$), λ ($0 \le \lambda < 1$)), being the subclass of $\mathcal{T}$ consisting of functions which satisfy the following condition:

$$\Re\left\{\frac{zf'(z)}{\lambda zf'(z)+(1-\lambda)f(z)}\right\} > \alpha, \quad (z \in \mathbb{U}). \tag{18}$$

Also, they introduce $C(\lambda,\alpha)$, (α ($0 \le \alpha < 1$), λ ($0 \le \lambda < 1$)), being the subclass of $\mathcal{T}$ consisting of functions which satisfy the following condition:

$$\Re\left\{\frac{f'(z)+zf''(z)}{f'(z)+\lambda zf''(z)}\right\} > \alpha, \quad (z \in \mathbb{U}). \tag{19}$$

By using (18) and (19) we have

$$f(z) \in C(\lambda,\alpha) \Longleftrightarrow zf'(z) \in T(\lambda,\alpha). \tag{20}$$

It is easy to see that for $\lambda = 0$ the classes $T(\lambda,\alpha)$ and $C(\lambda,\alpha)$ reduce to the classes of starlike functions of order α ($0 \le \alpha < 1$), $T^*(\alpha)$ and the convex functions of order α ($0 \le \alpha < 1$), $C(\alpha)$, respectively, studied by Silverman [4].

Mostafa [5] and Porwal and Dixit [6] obtain certain conditions for hypergeometric functions and generalized Bessel functions, respectively, for these classes.

Now, we introduce a power series whose coefficients are probabilities of the generalized distribution:

$$K_\phi(z) = z + \sum_{n=2}^{\infty}\frac{a_{n-1}}{S}z^n. \tag{21}$$

Further, we define the following function:

$$TK_\phi(z) = z - \sum_{n=2}^{\infty}\frac{a_{n-1}}{S}z^n. \tag{22}$$

The convolution (or Hadamard product) of two series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series:

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \tag{23}$$

Next, we introduce the convolution operator $TK_\phi(f,z)$ for functions f of the form (17) as follows:

$$TK_\phi(f,z) = TK_\phi(z) * f(z) = z - \sum_{n=2}^{\infty}\left|A_n\right|\frac{a_{n-1}}{S}z^n. \tag{24}$$

Recently, Porwal [7] introduced a Poisson distribution series whose coefficients are probabilities of Poisson distribution and established a correlation between Statistics and Geometric Function Theory which opened up a new direction of research. After the appearance of this paper some researchers (e.g., Ahmad et al. [8], Murugusundaramoorthy [9], and Porwal and Kumar [10]) obtained some new and interesting results by using Hypergeometric Distribution, Poisson Distribution, and Confluent Hypergeometric Distribution. In the present paper motivated with the above-mentioned work, we obtain necessary and sufficient conditions for $TK_\phi(z)$ and $TK_\phi(f,z)$ in the classes $T(\lambda,\alpha)$ and $C(\lambda,\alpha)$.

To prove our main theorem, we need the following lemma.

Lemma 7 (see [11]). *If $f \in R^\tau(A,B)$ is of the form (16) then*

$$\left|A_n\right| \le \frac{(A-B)|\tau|}{n}, \quad (n \in N \setminus \{1\}). \tag{25}$$

The bounds given in (25) are sharp.

Lemma 8 (see [3]). *A function $f(z)$ defined by (17) is in class $T(\lambda,\alpha)$, if and only if*

$$\sum_{n=2}^{\infty}[n-\lambda\alpha n-\alpha+\lambda\alpha]\left|a_n\right| \le 1-\alpha. \tag{26}$$

Lemma 9 (see [3]). *A function $f(z)$ defined by (17) is in class $C(\lambda,\alpha)$, if and only if*

$$\sum_{n=2}^{\infty}n[n-\lambda\alpha n-\alpha+\lambda\alpha]\left|a_n\right| \le 1-\alpha. \tag{27}$$

Theorem 10. *If $TK_\phi(z)$ that is of form (22) is in class $T(\lambda,\alpha)$, if and only if*

$$\frac{1}{S}\left[(1-\alpha\lambda)\phi'(1)+(1-\alpha)\left[\phi(1)-\phi(0)\right]\right] \le 1-\alpha. \tag{28}$$

Proof. Since

$$TK_\phi(z) = z - \sum_{n=2}^{\infty}\frac{a_{n-1}}{S}z^n, \tag{29}$$

according to Lemma 8, we have to show that

$$\sum_{n=2}^{\infty}[n(1-\lambda\alpha)-\alpha(1-\lambda)]\frac{a_{n-1}}{S} \le 1-\alpha. \tag{30}$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1-\alpha\lambda)-\alpha(1-\lambda)]\frac{a_{n-1}}{S}\\
&\quad = \frac{1}{S}\left[(1-\alpha\lambda)\sum_{n=2}^{\infty}(n-1)a_{n-1}+(1-\alpha)\sum_{n=2}^{\infty}a_{n-1}\right]\\
&\quad = \frac{1}{S}\left[(1-\alpha\lambda)\sum_{n=1}^{\infty}na_n+(1-\alpha)\sum_{n=1}^{\infty}a_n\right]\\
&\quad = \frac{1}{S}\left[(1-\alpha\lambda)\phi'(1)+(1-\alpha)\left[\phi(1)-\phi(0)\right]\right]\\
&\quad \le 1-\alpha.
\end{aligned} \tag{31}$$

This completes the proof of Theorem 10. □

Theorem 11. *If $TK_\phi(z)$ that is of form (22) is in class $C(\lambda,\alpha)$, if and only if*

$$\begin{aligned}
&\frac{1}{S}\left[(1-\alpha\lambda)\phi''(1)+(3-2\alpha\lambda-\alpha)\phi'(1)\right.\\
&\qquad\left.+(1-\alpha)\left[\phi(1)-\phi(0)\right]\right] \le 1-\alpha.
\end{aligned} \tag{32}$$

Proof. Since

$$TK_{\phi}(z) = z - \sum_{n=2}^{\infty} \frac{a_{n-1}}{S} z^n, \tag{33}$$

according to Lemma 9, we have to prove that

$$\sum_{n=2}^{\infty} n\left[n(1-\lambda\alpha) - \alpha(1-\lambda)\right] \frac{a_{n-1}}{S} \le 1-\alpha. \tag{34}$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\left[n(1-\alpha\lambda) - \alpha(1-\lambda)\right] \frac{a_{n-1}}{S} \\
&\quad = \frac{1}{S}\left[(1-\alpha\lambda)\sum_{n=2}^{\infty}(n-1)(n-2)a_{n-1}\right. \\
&\quad \left. + (3-2\alpha\lambda-\alpha)\sum_{n=2}^{\infty}(n-1)a_{n-1} + (1-\alpha)\sum_{n=2}^{\infty}a_{n-1}\right] \\
&\quad = \frac{1}{S}\left[(1-\alpha\lambda)\phi''(1) + (3-2\alpha\lambda-\alpha)\phi'(1)\right. \\
&\quad \left. + (1-\alpha)\left[\phi(1)-\phi(0)\right]\right] \le 1-\alpha.
\end{aligned} \tag{35}$$

Thus the proof of Theorem 11 is established. □

Theorem 12. *If $f \in R^{\tau}(A,B)$ is of form (17) and the operator $TK_{\phi}(f,z)$ defined by (24) is in the class $C(\lambda,\alpha)$, if and only if*

$$\begin{aligned}
&\frac{(A-B)|\tau|}{S} \\
&\quad \cdot \sum_{n=2}^{\infty}\left[(1-\alpha\lambda)\phi'(1) + (1-\alpha)(\phi(1)-\phi(0))\right] \le 1 \\
&\quad - \alpha.
\end{aligned} \tag{36}$$

Proof. By Lemma 9, it suffices to prove that

$$P_1 = \sum_{n=2}^{\infty} n\left[n - \lambda\alpha n - \alpha + \lambda\alpha\right]|a_n| \le 1-\alpha. \tag{37}$$

Since $f \in R^{\tau}(A,B)$ then by using Lemma 7 we have

$$|a_n| \le \frac{(A-B)|\tau|}{n}. \tag{38}$$

Hence

$$\begin{aligned}
P_1 &\le \frac{(A-B)|\tau|}{S}\sum_{n=2}^{\infty}\left[n(1-\alpha\lambda) - \alpha(1-\lambda)\right]a_{n-1} \\
&= \frac{(A-B)|\tau|}{S}\sum_{n=2}^{\infty}\left[(n-1)(1-\alpha\lambda) + (1-\alpha)a_{n-1}\right] \\
&= \frac{(A-B)|\tau|}{S} \\
&\quad \cdot \sum_{n=2}^{\infty}\left[(1-\alpha\lambda)\phi'(1) + (1-\alpha)(\phi(1)-\phi(0))\right] \le 1 \\
&\quad - \alpha.
\end{aligned} \tag{39}$$

Thus the proof of Theorem 12 is established. □

4. An Integral Operator

In this section, we introduce an integral operator $TG_{\phi}(z)$ as follows:

$$TG_{\phi}(z) = \int_0^z \frac{TK_{\phi(t)}}{t}dt, \tag{40}$$

and we obtain a necessary and sufficient condition for $TG_{\phi}(z)$ belonging to class $C(\lambda,\alpha)$

Theorem 13. *If $TK_{\phi}(z)$ is defined by (22), then $TG_{\phi}(z)$ defined by (40) is in class $C(\lambda,\alpha)$, if and only if (28) satisfies.*

Proof. Since

$$TG_{\phi}(z) = z - \sum_{n=2}^{\infty}\frac{a_{n-1}}{nS}z^n \tag{41}$$

by Lemma 9, we have to prove that

$$\sum_{n=2}^{\infty} n\left[n(1-\lambda\alpha) - \alpha(1-\lambda)\right]\frac{a_{n-1}}{nS} \le 1-\alpha. \tag{42}$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\left[n(1-\lambda\alpha) - \alpha(1-\lambda)\right]\frac{a_{n-1}}{nS} \\
&\quad = \sum_{n=2}^{\infty}\left[n(1-\lambda\alpha) - \alpha(1-\lambda)\right]\frac{a_{n-1}}{S} \\
&\quad = \frac{1}{S}\left[(1-\alpha\lambda)\sum_{n=2}^{\infty}(n-1)a_{n-1} + (1-\alpha)\sum_{n=2}^{\infty}a_{n-1}\right] \\
&\quad = \frac{1}{S}\left[(1-\alpha\lambda)\sum_{n=1}^{\infty}na_n + (1-\alpha)\sum_{n=1}^{\infty}a_n\right] \\
&\quad = \frac{1}{S}\left[(1-\alpha\lambda)\phi'(1) + (1-\alpha)\left[\phi(1)-\phi(0)\right]\right] \\
&\quad \le 1-\alpha.
\end{aligned} \tag{43}$$

This completes the proof of Theorem 13. □

Conflicts of Interest

There are no conflicts of interest regarding the publication of this manuscript.

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