Guaranteeing the homotopy type of a set defined by non-linear inequalities.

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Abstract. This paper provides an effective method to create an abstract simplicial complex homotopy equivalent to a given set $\mathbb{S}$ described by non-linear inequalities (polynomial or not). To our knowledge, no other numerical algorithm is able to deal with this type of problem. The proposed approach divides $\mathbb{S}$ into subsets that have been proven to be contractible using interval arithmetic. The method is close to Čech cohomology and uses the nerve theorem. Some examples illustrate the principle of the approach. This algorithm has been implemented.

Keywords: Triangulation; homotopy equivalence; interval analysis.

1. Introduction

Topological properties of a set $\mathbb{S}$ are significant information and have a lot of applications in different areas like robotics, computer-aided design .... There exist different approaches for computing topological properties of sets.

- In the case of semi-algebraic sets[1], G.E. Collins (G.E. Collins, 1975) proposes a finite partition into semi-algebraic subsets homeomorphic to open boxes. This algorithm is called cylindrical algebraic decomposition. Then, from this cell decomposition, a triangulation homeomorphic to the semi-algebraic set is created. (S. Basu, R. Pollack and M.-F. Roy, 2005) propose a method which computes more efficiently the number of connected components and the first Betti number of semi-algebraic sets.

[1] A semi-algebraic set is a set defined by a quantifier-free Boolean formula with atoms $P < 0$, $P = 0$, $P \in \mathcal{P}$ where $\mathcal{P}$ is a finite collection of polynomials.

- Stander and Hart (John C. Hart and Barton T. Stander, 1997) combine interval analysis (R. E. Moore, 1966) (A. Neumaier, 1990) and Morse theory (John Willard Milnor, 1963) to compute the topology of an implicit surface defined by :

$$\{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = 0, \text{ with } f \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})\}. \tag{1}$$

In our article, sets are defined by a quantifier-free Boolean formula with atoms $f \leq 0$, $f \in \mathcal{F}$ where $\mathcal{F}$ is a finite collection of $\mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$. The algorithm divides a set, denoted by $\mathbb{S}$, with a finite cover $\{\mathbb{S}_i\}_{i \in I}$ such that :

$$\forall J \subset I, \bigcap_{j \in J} \mathbb{S}_j \text{ is contractible or empty.} \tag{2}$$

With this cover, an abstract simplicial complex homotopy equivalent to $\mathbb{S}$ is generated. Homotopy equivalence is important because in algebraic topology most concepts cannot distinguish homotopy equivalent spaces (pathconnectedness, fundamental group, homotopy groups, homology groups, ...). Combining these results with the Kenzo (F. Sergeraert, 1998) program, the homology groups of a given set $\mathbb{S}$ (and the Betti numbers) can be computed.

The paper is organized as follows : Section 2 presents a sufficient condition to prove algorithmically that a set is contractible. Next, from a cover satisfying (2) created by a subdivision process, an algorithm which provides a simplicial complex homotopy equivalent to $\mathbb{S}$ is described in Section 3. To finish, some examples illustrate the principle of the approach. These illustrations have been computed by a new solver we developed in C++ called `HIA` (**H**omotopy via **I**nterval type **A**nalysis (HIA, 2006)).

2. Sufficient condition of contractibility

In this section, we recall definitions about star-shaped sets, homotopy equivalence between maps and topological sets. We describe basic relations between these notions. At the end, a sufficient condition of contractibility is given.

DEFINITION 2.1. *A point v is a* star *for a subset X of an Euclidean set if X contains all the line segments connecting any of its points and v.*

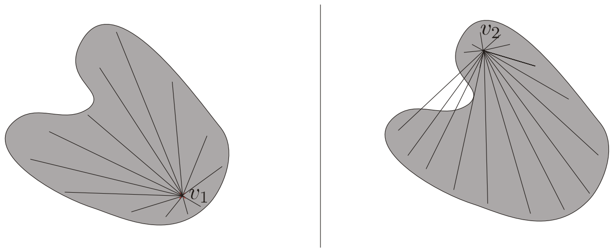

Figure 1. v_1 is a star for this subset of $\mathbb{R}^2$ whereas v_2 is not.

DEFINITION 2.2. *If v is a star for subset X of an Euclidean set, one says that X is* star-shaped *or* v-star-shaped *if one wants to clarify that v is a star.*

PROPOSITION 2.1. *Let X and Y be two v-star-shaped sets, then $X \cap Y$ and $X \cup Y$ are also v-star-shaped.*

DEFINITION 2.3 (Homotopic maps). *Two continuous maps $f, g : X \to Y$ are* homotopic *(or f is* homotopic *to g) if there exists a continuous map $F : X \times [0,1] \to Y$, such that : $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, $\forall x \in X$. The map F is said to be a* homotopy*, and we write $f \simeq g$ for "f is homotopic to g". Figure 2 illustrates this notion.*

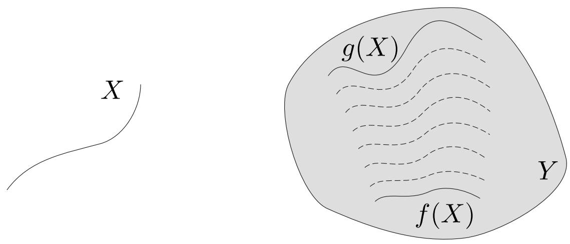

Figure 2. The two continuous maps $f, g : X \to Y$ are homotopic. ($f \simeq g$)

DEFINITION 2.4 (Homotopy equivalence between sets). *Two spaces X and Y are* homotopy-equivalent *(or* of the same homotopy type*) if there exist continuous maps $f : X \to Y$ and $g : Y \to X$, such that : $f \circ g \simeq 1_X$ and $g \circ f \simeq 1_Y$, where 1_X and 1_Y are the identity maps of X and Y respectively. In this case, f is a* homotopy equivalence *and g is the* homotopy inverse *of f. We write $X \simeq Y$ for "X is homotopy-equivalent to Y". Figure 3 illustrates this notion.*

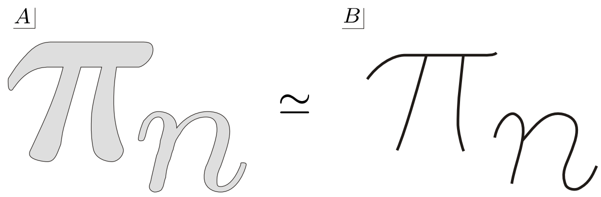

Figure 3. These two sets are homotopy equivalent. ($A \simeq B$)

DEFINITION 2.5. *A topological space X which is homotopy-equivalent to a point, is* contractible*.*

REMARK 2.1. *A star-shaped set is contractible as illustrated in Figure 4.*

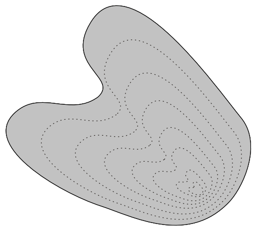

Figure 4. A star shaped set is contractible.

The next result is a sufficient condition to prove that a set defined by only one inequality is star-shaped. This sufficient condition can be checked using interval analysis (L. Jaulin and M. Kieffer and O. Didrit and E. Walter, 2001).

PROPOSITION 2.2. *Let f be a C^1 function from $\mathbb{R}^n$ to $\mathbb{R}$, D be a convex set and $\mathbb{S} = \{x \in D \subset \mathbb{R}^n \mid f(x) \leq 0\}$. If there exists v in $\mathbb{S}$ such that*

$$\{x \in D \mid f(x) = 0 \text{ and } \nabla f(x) \cdot (x - v) \leq 0\} = \emptyset \tag{3}$$

then $\mathbb{S}$ is star-shaped.

PROOF 2.1. *Proof See (*N. Delanoue, L. Jaulin, B. Cottenceau, 2004*).*

COROLLARY 2.1. *Let $\mathbb{S}$ be a set defined by a quantifier-free Boolean formula with atoms $f \leq 0$, $f \in \{f_i\}_{i \in I}$ where $\{f_i\}_{i \in I}$ is a finite collection of $\mathcal{C}^1(D \subset \mathbb{R}^n, \mathbb{R})$. If D is convex and if there exists $v \in D$ such that*

$$\forall i \in I, \{x \in D \mid f_i(x) = 0 \text{ and } \nabla f_i(x) \cdot (x - v) \leq 0\} = \emptyset \tag{4}$$

then $\mathbb{S}$ is contractible.

PROOF 2.2. *Combine Proposition 2.1, Remark 2.1 and Proposition 2.2.*

3. Discretization

Section 2 shows that proving the contractibility of a set often amounts to checking that a set defined by non-linear inequalities is empty. From a set $\mathbb{S}$, which can be an infinite collection of elements, a single point set $\{v\}$ is created holding the homotopy type. Most of the subsets $\mathbb{S}$ of $\mathbb{R}^n$ are not contractible. The algorithm, presented in this section, produces a set $\mathcal{K}(\mathbb{S})$ (a simplicial complex) which is homotopy equivalent to $\mathbb{S}$. A simplicial complex can be finitely represented.

The section is organized as follows. First, we recall definitions and some properties related to simplicial complexes. In the next subsection, we present the idea of our approach and a proof that the two sets $\mathbb{S}$ and $\mathcal{K}(\mathbb{S})$ are homotopy equivalent. The last subsection presents explicitly our algorithm `Homotopy_via_Interval_Analysis` and examples.

3.1. Simplicial complexes

This subsection is concerned with building up spaces from certain elementary spaces called *simplices*. A simplex is a generalization to n dimensions of a triangle or a tetrahedron. These simplices are fitted together in such a way that two simplices meet (if at all) in a common face.

DEFINITION 3.1. *A* simplicial complex K *is a finite set of simplices, all contained in $\mathbb{R}^n$. Furthermore :*

1. *if σ_n is a simplex of K and τ_p is a face of σ_n, then τ_p is in K.*
2. *if σ_n and σ_p are simplices of K, then either $\sigma_n \cap \sigma_p$ is empty, or it is a common face of σ_n and σ_p.*

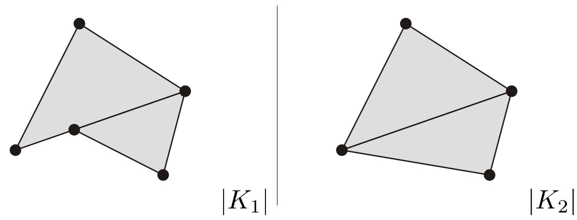

Figure 5. Two subsets of $\mathbb{R}^2$.

For example (Fig. 5, K_2 is a simplicial complex whereas K_1 is not. A simplicial complex K is not a topological space, this is only a set whose elements are simplices. The set of points that lie in at least one of the simplices of K, topologized as a subspace of $\mathbb{R}^n$, is a topological space, called the *polyhedron* of K, written $|K|$. Simplicial complexes are sets of simplices lying in one particular Euclidean space $\mathbb{R}^n$. To free ourselves of this restriction, let us define the notion of *abstract simplicial complex.*

DEFINITION 3.2. *An* abstract simplicial complex $\mathcal{K}$ *is a finite set of elements* $a^0, a^1, \ldots$ *called* abstract vertices*, together with a collection of subsets*
$(a^{i_0}, a^{i_1}, \ldots, a^{i_n}), \ldots$ *called* abstract simplices *, with the property that any subset of a simplex is itself a simplex; i.e.*

$$\sigma \in \mathcal{K}, \sigma' \subset \sigma \Rightarrow \sigma' \in \mathcal{K} \tag{5}$$

The dimension *of an abstract simplicial complex is the maximum of the dimension of its simplices* [2].

EXAMPLE 3.1. *The set*

$$\begin{aligned} \mathcal{K} \;=\; & \{\emptyset, \{a^0\}, \{a^1\}, \{a^2\}, \{a^3\}, \{a^4\}, \{a^0, a^1\}, \ldots \\ & \qquad \ldots \{a^1, a^2\}, \{a^0, a^2\}, \{a^3, a^4\}, \{a^0, a^1, a^2\}\}\,. \end{aligned} \tag{6}$$

is an abstract simplicial complex.
Its dimension is the dimension of $\{a^0, a^1, a^2\}$*, i.e. 2.*

The enumeration of all elements of an abstract simplicial complex $\mathcal{K}$ seems to be useless, since $\{a^0, a^1, a^2\} \in \mathcal{K}$ implies that $\{a^0\}$, $\{a^1\}$, $\{a^2\}$, $\{a^0, a^1\}$, $\{a^1, a^2\}$, $\{a^0, a^2\}$ are also elements of $\mathcal{K}$.

[2] The dimension of an abstract simplex is the number of vertices minus 1.

NOTATION 3.1. *With $\mathcal{V}$ a finite collection of elements (abstract vertices) $\mathcal{V} = \{a^0, a^1, \ldots, a^n\}$ and $2^{\mathcal{V}}$ the power set of $\mathcal{V}$, a simplicial complex is a subset of $2^{\mathcal{V}}$ satisfying (5). Letting $\{\sigma_1, \ldots, \sigma_m\}$ be included in $2^{\mathcal{V}}$, (not necessarily an abstract simplicial complex), we denote by $\sigma_1 + \ldots + \sigma_m$ the following abstract simplicial complex*[3] *:*

$$\sigma_1 + \ldots + \sigma_m := \bigcup_{i=1}^{i=m} 2^{\sigma_i}$$

Thanks to this notation, the abstract simplicial complex $\mathcal{K}$ defined in example 3.1 is written : $\mathcal{K} = \{a^0, a^1, a^2\} + \{a^3, a^4\}$, and if no ambiguity can arise, we will write $\mathcal{K} = a^0a^1a^2 + a^3a^4$.

DEFINITION 3.3. *A realization of an abstract simplicial complex $\mathcal{K}$ is a simplicial complex K having $\mathcal{K}$ as an abstraction.*

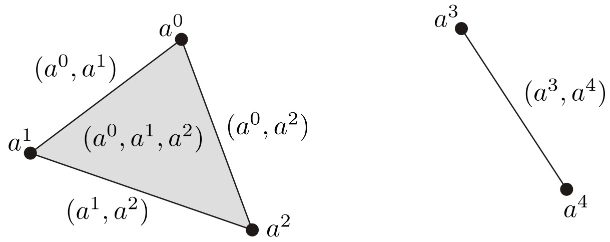

Figure 6. A realization of $\mathcal{K}$.

REMARK 3.1. *If K_1 and K_2 are two realizations of an abstract simplicial complex $\mathcal{K}$, then $|K_1|$ and $|K_2|$ are homeomorphic (*C.R.F. Maunder, 1970*). Hence, topological properties of an abstract simplicial complex $\mathcal{K}$ are well defined.*

DEFINITION 3.4. *Let $\mathcal{K}$ be an abstract simplicial complex, and $\{x\}$ a vertex. We denote by $\mathcal{C}(x, \mathcal{K})$ the following set :*

$$\mathcal{C}(x, \mathcal{K}) = \mathcal{K} \cup \bigcup_{s \in \mathcal{K}} \{\{x\} \cup s\}.$$

$\mathcal{C}(x, \mathcal{K})$ is an abstract simplicial complex called the cone *of $\mathcal{K}$ from x. With notation 3.1, a cone can be interpreted as a product noted by $*$; with $K = \sigma_1 + \ldots + \sigma_m$ one has :*

$$\mathcal{C}(x, \mathcal{K}) = x * (\sigma_1 + \ldots + \sigma_m) = x\sigma_1 + \ldots + x\sigma_m.$$

[3] The reader can check that $\sigma_1 + \ldots + \sigma_m$ is the smallest, with inclusion defined on $2^{2^{\mathcal{V}}}$ as order relation, abstract simplicial complex that contains σ_1, $\ldots$, σ_m, as simplices.

EXAMPLE 3.2. *For $\mathcal{K}$ defined in example 3.1, $\mathcal{C}(x,\mathcal{K})$ is equal to*

$$\begin{aligned} \mathcal{C}(x,\mathcal{K}) \;=\; & \mathcal{K} \cup \{\{x\},\{x,a^0\},\{x,a^1\},\{x,a^2\},\{x,a^3\},\{x,a^4\},\dots \\ & \dots \{x,a^0,a^1\},\{x,a^1,a^2\},\{x,a^2,a^3\},\{x,a^4,a^5\},\{x,a^0,a^1,a^2\}\}\,. \end{aligned} \tag{7}$$

With notation 3.1 :

$$\begin{aligned} \mathcal{C}(x,\mathcal{K}) &= x * \{a^0a^1a^2 + a^3a^4\} \\ &= xa^0a^1a^2 + xa^3a^4 \end{aligned}$$

Figure 7 shows a realization of $\mathcal{C}(x,\mathcal{K})$.

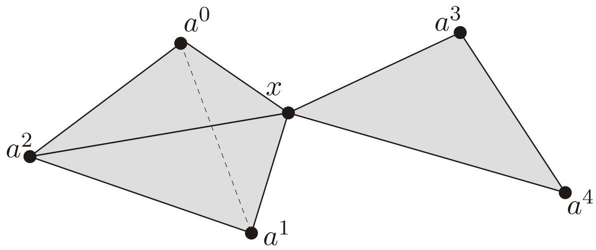

Figure 7. A realization of $\mathcal{C}(x,\mathcal{K})$.

PROPOSITION 3.1. *Let $\mathcal{K}$ be an abstract simplicial complex, and $\mathcal{C}(x,\mathcal{K})$ a cone of $\mathcal{K}$, then $\mathcal{C}(x,\mathcal{K}) \simeq \{x\}$.*

PROOF 3.1. *Proof Let K be a realization of $\mathcal{K}$, by construction $|K|$ is $|\{x\}|$-star-shaped.*

3.2. Homotopy type is preserved

To guarantee a simplicial complex has the same homotopy type as set $\mathbb{S}$, the main idea of our approach is to create an abstract simplicial complex from a finite cover $\{\mathbb{S}_i\}_{i\in I}$ of $\mathbb{S}$. This cover has to be such that each $\mathbb{S}_i$ is *contractible* and compact, moreover, the intersection of elements of any sub collection of $\{\mathbb{S}_i\}_{i\in I}$ has to be contractible (or empty). To simplify our presentation, let us introduce the notation 3.2.

NOTATION 3.2. *With I an index set and J a subset of I, let us denote by $\mathbb{S}_J$ the set $\bigcap_{j\in J}\mathbb{S}_j$. With this notation : $\mathbb{S}_3\cap\mathbb{S}_4\cap\mathbb{S}_9$ is denoted by $\mathbb{S}_{\{3,4,9\}}$.*

DEFINITION 3.5. *Let $\mathbb{S}$ be a topological subset of $\mathbb{R}^n$, $\{\mathbb{S}_i\}_{i\in I}$ is a* compact contractible cover *of $\mathbb{S}$ if :*

- *I is finite.*
- *$\forall i \in I, \mathbb{S}_i$ is compact.*
- *$\forall J \subset I, \mathbb{S}_J$ is contractible or empty.*

Figure 8 is an example of compact contractible cover.

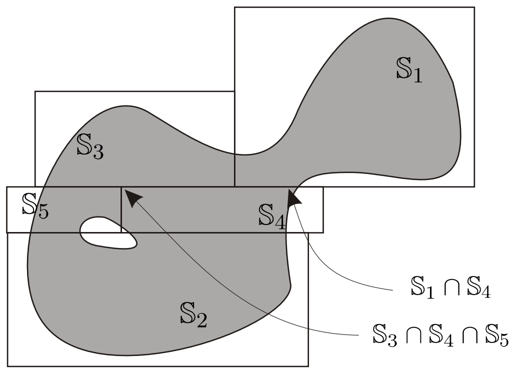

Figure 8. A compact contractible cover $\{\mathbb{S}_i\}_{i\in I}$ of $\mathbb{S}$.

DEFINITION 3.6. *Let $\{\mathbb{S}_i\}_{i\in I}$ be a compact contractible cover of a set $\mathbb{S}$. We denote by $\mathcal{J}$ those index sets J such that $\mathbb{S}_J$ is non-empty. An abstract simplicial complex $\mathcal{K}(\mathbb{S})$ is said to be* adapted *to $\{\mathbb{S}_i\}_{i\in I}$ if it is the smallest simplicial complex satisfying the following conditions :*

- *$\forall J \in \mathcal{J}$, an abstract vertex (a^J) is in $\mathcal{K}(\mathbb{S})$.*
- *$\forall J \in \mathcal{J}$, an abstract simplicial complex $\mathcal{K}_J$ defined by*

$$\mathcal{K}_J = a^J * \left(\sum_{J' \in \mathcal{J} | \mathbb{S}_{J'} \subset \mathbb{S}_J} \mathcal{K}_{J'} \right).$$

  is an abstract simplicial subcomplex of $\mathcal{K}(\mathbb{S})$. $\mathcal{K}(\mathbb{S})$ is usually called the nerve of the cover $\{\mathbb{S}_i\}_{i\in I}$.

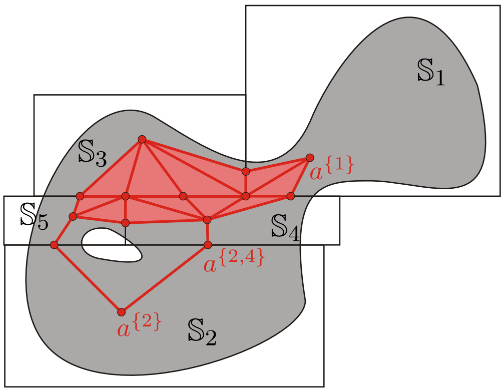

Figure 9. An abstract simplicial complex adapted to $\{\mathbb{S}_i\}_{i\in I}$.

THEOREM 3.1. *If $\{\mathbb{S}_i\}_{i\in I}$ is a compact contractible cover of $\mathbb{S} \subset \mathbb{R}^n$ and $\mathcal{K}(\mathbb{S})$ an abstract simplicial complex adapted to $\{\mathbb{S}_i\}_{i\in I}$ then $\mathcal{K}(\mathbb{S})$ and $\mathbb{S}$ are of the same homotopy type.*

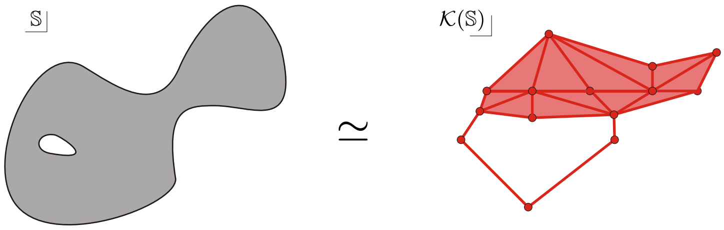

Figure 10. $\mathbb{S}$ and $\mathcal{K}(\mathbb{S})$ are homotopy equivalent.

PROOF 3.2. *$\mathcal{K}(\mathbb{S})$ is the nerve of the cover $\{\mathbb{S}_i\}_{i\in I}$. From the nerve theorem (*H. Edelsbrunner, N. R. Shha, 1994*), $\mathcal{K}(\mathbb{S})$ and $\mathbb{S}$ are of the same homotopy type.*

3.3. A new algorithm : Homotopy type via Interval Analysis

This subsection presents a new algorithm, called `HIA` (**H**omotopy via **I**nterval **A**nalysis). This algorithm is often able to create a compact contractible cover of a set defined by non-linear inequalities. In a second pass, it creates an abstract simplicial complex adapted to this cover. The cover $\{\mathbb{S}_i\}_{i\in I}$ is created thanks to a tiling. Therefore, before introducing the main step of this algorithm, definitions and notations are introduced.

DEFINITION 3.7. *A box is a Cartesian product of compact real intervals.*

Figure 3.3 shows a box of $\mathbb{R}^2$ and one of $\mathbb{R}^3$.

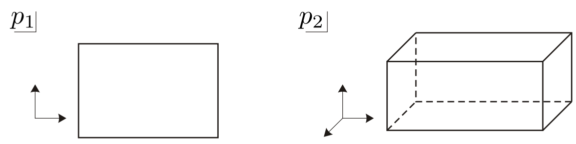

Figure 11. p_1 and p_2 are respectively examples of boxes of $\mathbb{R}^2$ and $\mathbb{R}^3$.

DEFINITION 3.8. *A tiling $\mathcal{P}$ is a finite collection of boxes $\{p_i\}_{i\in I}$ satisfying :*

$$i \neq j \Rightarrow m(p_i \cap p_j) = 0,$$

where m is the classical Lebesgue Measure.

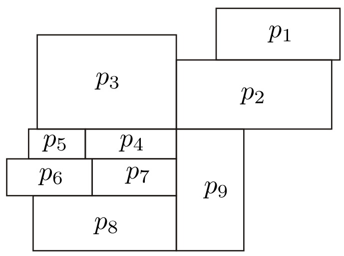

Figure 12. Example of a tiling $\mathcal{P}$ with 9 boxes. $\mathcal{P} = \{p_1, \ldots, p_9\}$.

NOTATION 3.3. *Let $\mathbb{S}$ be a part of $\mathbb{R}^n$ and $\{p_i\}_{i\in I}$ a tiling such that $\mathbb{S} \subset \bigcup_{i\in I} p_i$. Let us denote by $\{\mathbb{S}_i\}_{i\in I}$ the cover of $\mathbb{S}$ where $\mathbb{S}_i = \mathbb{S} \cap p_i, i \in I$. So the bisection of $\mathbb{S}_i$ is defined via the bisection of the box p_i.*

The main idea of this algorithm is a subdivision process. From a cover $\{\mathbb{S}_i\}_{i\in I}$ created thanks to a tiling $\{p_i\}_{i\in I}$ (see notation 3.3), a sub-algorithm checks if :

$$\forall J \subset I, \bigcap_{j\in J} \mathbb{S}_j \text{ is contractible or empty. } (\mathbb{S}_j := \mathbb{S} \cap p_j)$$

If it is not satisfied, then each $\mathbb{S}_i$ responsible for this failure is bisected. The following algorithm uses :

- the tiling $\mathcal{P}_*$ is such that : $\forall \{p_j\}_{j \in J} \subset \mathcal{P}_*, \ \bigcap_{j \in J} \mathbb{S}_j$ is contractible or empty.

- the tiling $\mathcal{P}_\Delta$, nothing is known about its boxes.

At the end of the algorithm, using $\mathcal{P}_* = \{p_i\}_{i \in I}$, the sub-algorithm `Adapted_Triangulation` creates an abstract simplicial complex $\mathcal{K}(\mathbb{S})$ adapted to the cover $\{\mathbb{S}_i := \mathbb{S} \cap p_i\}_{i \in I}$.

Alg. 1 `HIA` - Homotopy type via Interval Analysis

Require: $\mathbb{S}$ a subset of $\mathbb{R}^n$, X_0 a box of $\mathbb{R}^n$ which contains $\mathbb{S}$.
Ensure: A triangulation $\mathcal{K}(\mathbb{S})$ which is homotopy equivalent to $\mathbb{S}$.
1: Initialization : $\mathcal{P}_* := \emptyset$, $\mathcal{P}_\Delta := \{X_0\}$
2: **while** $\mathcal{P}_\Delta \neq \emptyset$ **do**
3: Pop from the last element of $\mathcal{P}_\Delta$ into the box p
4: **if**

$$\forall \{p_j\}_{j \in J} \subset \mathcal{P}_* \cup \{p\}, \quad \bigcap_{j \in J} \mathbb{S}_j \text{ is proven contractible or empty}$$

then
5: Push $\{p\}$ into $\mathcal{P}_*$;
6: **else**
7: *Bisect*(p); then Push back the two resulting boxes into $\mathcal{P}_\Delta$;
8: **end if**
9: **end while**
10: $\mathcal{K}(\mathbb{S}) \leftarrow$ `Adapted_Triangulation` from the cover : $\{\mathbb{S}_i\}_{i \in I}$, (where $\mathbb{S}_i := \mathbb{S} \cap p_i, p_i \in \mathcal{P}_*$).

REMARK 3.2. *The condition at step 4 is checked using results presented in Section 2.*

`Adapted_Triangulation` is a new algorithm (Alg. 2) which creates an adapted triangulation $\mathcal{K}(\mathbb{S})$ from a cover $\{\mathbb{S}_i\}_{i \in I}$ of $\mathbb{S}$. The idea is to add a cone to $\mathcal{K}(\mathbb{S})$ for each J in $\mathcal{J}$, and to do this, in such away, that any two cones can be attached by a cone previously created.

Alg. 2 `Adapted_Triangulation`

Require: A set $\mathbb{S}$ and a cover $\{\mathbb{S}_i\}_{i\in I}$ of $\mathbb{S}$.
Ensure: An adapted triangulation $\mathcal{K}(\mathbb{S})$ of $\{\mathbb{S}_i\}_{i\in I}$.
  $\{$*Initialization :*$\}$
1: $\mathcal{K}(\mathbb{S}) \leftarrow \emptyset$, $\mathcal{J} \leftarrow \emptyset$.
  $\{$*Remove useless indices, i.e. Create* $\mathcal{J}$ *:*$\}$
2: **for all** $J \subset I$ **do : if** $\mathbb{S}_J$ is contractible **then** $\mathcal{J} \leftarrow \mathcal{J} \cup \{J\}$ **end for**
  $\{$*Add cones :*$\}$
3: $\mathcal{K}(\mathbb{S}) \leftarrow \sum_{i\in I}$`Cone`$(\{i\})$

Above, `Cone`(J) with $J \in \mathcal{J}$ is defined recursively by :

$$\texttt{Cone}(J) = a^J * \left(\sum_{J' \in \mathcal{J} | \mathbb{S}_{J'} \subset \mathbb{S}_J} \texttt{Cone}(J') \right)$$

with convention that `Cone`$(\emptyset) = \emptyset$ and $a^J * \emptyset = a^J$.

Illustration: To illustrate Alg. 2, let us use the cover given in Fig. 13. In this case, the set : $\{J|J \subset I\}$ has 2^5 elements since $\#I = 5$. But only some of those are such that $\bigcap_{j\in J} \mathbb{S}_j$ is contractible. After step 2, one has :

$$\begin{aligned} \mathcal{J} \ = \ & \{\{1\}, \{1,4\}, \{1,3\}, \{1,3,4\}, \{2\}, \{2,4\}, \{2,5\}, \{3\}, \{3,4\}, \{3,5\}, \\ & \{3,4,5\}, \{4\}, \{4,5\}, \{5\}\} \end{aligned}$$

To explain what is done at step 3, let us see how `Cone`$(\{1\})$ is computed, for example. To do it, the collection of $J' \in \mathcal{J}$ such that $\mathbb{S}_{J'} \subset \mathbb{S}_{\{1\}}$ has to be known. This collection is composed of $\mathbb{S}_{\{1\}}, \mathbb{S}_{\{1,4\}}, \mathbb{S}_{\{1,3\}}$ and $\mathbb{S}_{\{1,3,4\}}$. More generally, elements of $\{\mathbb{S}_J, J \in \mathcal{J}\}$ can be partially ordered by inclusion. Fig. 13 shows how these elements are ordered where $\mathbb{S}_{J'} \subset \mathbb{S}_J$ is represented by $J' \leftarrow J$.

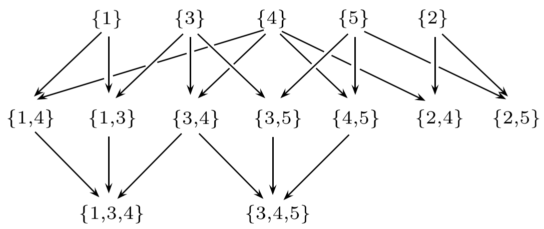

Figure 13. Partial order relation on $\{\mathbb{S}_J, J \in \mathcal{J}\}$

To make clearer our explanation, let us rename $a^{\{1\}}$ by a, $a^{\{3\}}$ by b, $a^{\{4\}}$ by c, etc. Fig. 14 shows how the a^J are renamed.

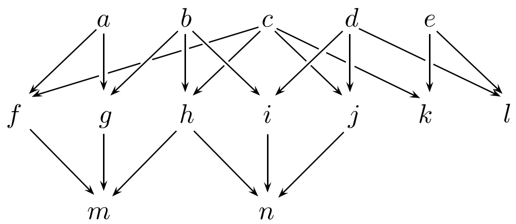

Figure 14. Partial order relation on $\{\mathbb{S}_J, J \in \mathcal{J}\}$ with renamed elements.

After step 3, one obtains this abstract simplicial complex :

$$\begin{aligned}\mathcal{K}(\mathbb{S}) &= a*(f*m+g*m)+b*(g*m+h*(m+n)+i*n)+\\ &\quad c*(f*m+h*(m+n)+j*n+k)+d*(i*n+j*n+l)+e*(k+l)\end{aligned}$$

$$\begin{aligned}\mathcal{K}(\mathbb{S}) &= afm+agm+bgm+bhm+bhm+bin+\\ &\quad cfm+chm+chn+cjn+ck+din+djn+dl+ek+el\end{aligned}$$

Geometrical illustration is given at Fig. 21 in which a realization of the abstract simplicial complex $\mathcal{K}$ is provided.

EXAMPLE 3.3. *Fig. 15 and Fig. 16 are respectively examples of realizations of simplicial complexes generated by* `HIA`[4] *for the sets :*

$$\mathbb{S}_1 = \left\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2+xy-2 \leq 0 and - x^2 - y^2 - xy + 1 \leq 0\right\}$$

[4] `HIA` algorithm has been implemented in C++ / OpenGL and can be downloaded at
`http://www.istia.univ-angers.fr/~delanoue/`

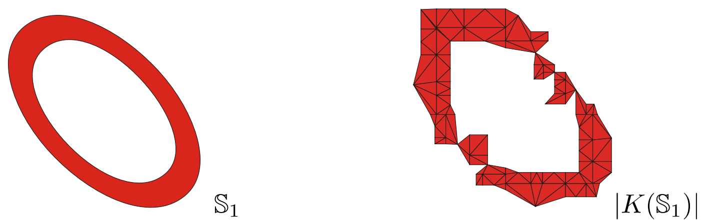

Figure 15. *Example of a set and a realization of the simplicial complex generated by* `HIA`.

and

$$\mathbb{S}_2 = \Big\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - 6 \leq 0 \textit{ and } 0.2\cos(x - y) - \sin(yx) - 0.6 \leq 0\Big\}$$

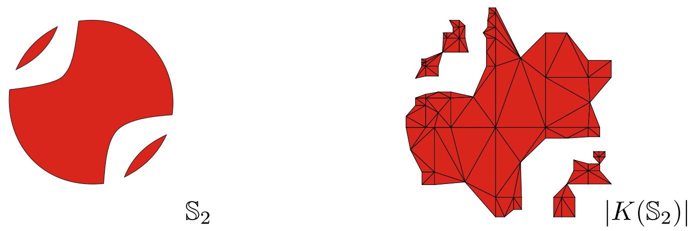

Figure 16. *Example of a set and a realization of the simplicial complex generated by* `HIA`.

EXAMPLE 3.4. *This example illustrates why topological properties of sets can be really useful in robotics. To our knowledge, the set studied here is not a semi-algebraic set. Let us consider a rope, with length L suspended between two points A and B, which are endpoints of two arms (see Figure 17).*

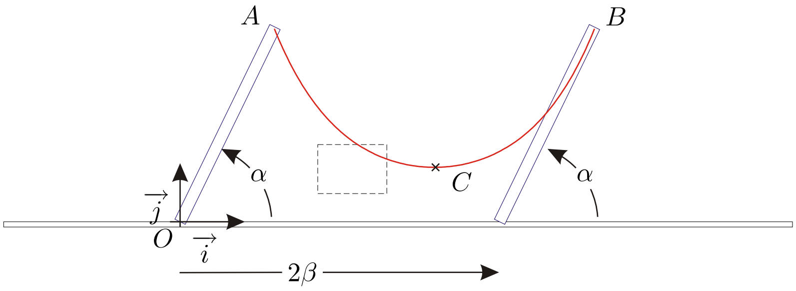

Figure 17. A rope suspended between two points A and B.

The independent parameters needed to specify an object configuration are the real numbers α and β. The robot has to satisfy some constraints. The rope must not touch the floor, and the lower point of the rope, denoted by C, must be outside the area delimited by the dotted lines. One would like to study the topological properties of the feasible configuration set $\mathbb{S}$.

$$\mathbb{S} = \{(\alpha, \beta) \in [0, \pi] \times [\beta^-, \beta^+] \text{ such that } y_C \geq 0 \text{ and } C \notin \boxed{}\}$$

To apply our algorithm, first, it is needed that the set $\mathbb{S}$ could be described by a quantifier-free Boolean formula with atoms $f \leq 0$, $f \in \mathcal{F}$ where $\mathcal{F}$ is a finite collection of $\mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$. Secondly, to prove that a subset $\mathbb{S}_J$ of $\mathbb{S}$ is contractible, one also needs to be able to evaluate Df on a bounded box for all $f \in \mathcal{F}$. The next paragraphs show how these two problems can be solved.

From the Hamilton principle, it is possible to show that Cartesian coordinates of C are given by :

$$\begin{aligned} x_C &= g_1(\alpha, \beta) = 2\cos\alpha + \beta \\ y_C &= g_2(\alpha, \beta) = 2\sin(\alpha) - a\cosh\left(\tfrac{\beta}{a}\right) + a \end{aligned}$$

where a is a positive real solution of equation (8) for a fixed real positive number β :

$$a \sinh\left(\frac{\beta}{a}\right) - L = 0 \tag{8}$$

Let $z = \frac{\beta}{a}$, therefore equation (8) is equivalent to equation (9) where the unknown variable is z.

$$f(\beta, z) = \sinh(z) - \frac{L}{\beta} z = 0 \tag{9}$$

We have

$$Df(\beta, z) = \left(\frac{Lz}{\beta^2}, \cosh(z) - \frac{L}{\beta}\right) \tag{10}$$

With $\beta = \beta_0$*, Figure 18 presents the function :*

$$(\mathbb{R}^{+*} \ni z \mapsto f(\beta_0, z) \in \mathbb{R}).$$

z	0		z_0		z^*		∞
$D_2 f(\beta_0, z)$		$-$	0		$+$		
$f(\beta_0, .)$	0	$\searrow$	$f(\beta_0, z_0) < 0$	$\nearrow$	0	$\nearrow$	∞

Figure 18. Variations of the function $z \mapsto f(\beta_0, z)$ *where* $z_0 = \arccos(\frac{L}{\beta})$*.*

From the Weierstrass intermediate value theorem, and since $D_2 f(\beta_0, z) > 0$ *if* $z \in]z_0; +\infty[$*, there exists a unique* z^* *in* $]z_0; +\infty[$ *such that* $f(\beta_0, z^*) = 0$*. Therefore, the function denoted by* ϕ*, which associates with a positive real number* β_0 *a positive real number* z^* *such that* $f(\beta_0, z^*) = 0$*, is well defined. Cartesian coordinates of* C *are given by :*

$$\begin{array}{rcl} x_C & = & g_1(\alpha, \beta) = 2\cos\alpha + \beta \\ y_C & = & g_2(\alpha, \beta) = 2\sin(\alpha) - \frac{\beta}{\phi(\beta)} \cosh\phi(\beta) + \frac{\beta}{\phi(\beta)} \end{array} \tag{11}$$

We can deduce that the feasible configuration set can be described by a quantifier-free Boolean formula with atoms $f \leq 0$*,* $f \in \mathcal{F}$ *where* $\mathcal{F}$ *is a finite collection of* $\mathbb{R}^{\mathbb{R}}$ *:*

$$\mathbb{S} = \{(\alpha, \beta) \in [0, \pi] \times [\beta_-, \beta_+] \text{ such that } c_1 \wedge (c_2 \vee c_3 \vee c_4 \vee c_5)\} \tag{12}$$

where

$$\begin{array}{lclcrcl} c_1 & \Leftrightarrow & (f_1(\alpha, \beta) & = & -g_2(\alpha, \beta) & \leq & 0) \\ c_2 & \Leftrightarrow & (f_2(\alpha, \beta) & = & g_1(\alpha, \beta) - l & \leq & 0) \\ c_3 & \Leftrightarrow & (f_3(\alpha, \beta) & = & -g_1(\alpha, \beta) + r & \leq & 0) \\ c_4 & \Leftrightarrow & (f_4(\alpha, \beta) & = & g_2(\alpha, \beta) - b & \leq & 0) \\ c_5 & \Leftrightarrow & (f_5(\alpha, \beta) & = & -g_2(\alpha, \beta) + t & \leq & 0) \end{array}$$

and $\square =]l, r[\times]b, t[$*.*

It only remains to show that for all i *in* $\{1, \ldots, 5\}$*,* f_i *is a* $\mathcal{C}^1$ *function, and how* Df_i *can be computed. Functions* f_2 *and* f_3 *are clearly* *differentiable since g_1 is. One can obtain Df_2 and Df_3 from Dg_1 :*

$$\begin{aligned} D_1 g_1(\alpha,\beta) &= -2\sin(\alpha) \\ D_2 g_1(\alpha,\beta) &= 1 \end{aligned}$$

Functions f_1, f_4 and f_5 are differentiable if g_2 is. And g_2 is differentiable if ϕ is a differentiable function. Let us show that ϕ is differentiable. Since $D_2 f(\beta, z^*) > 0$, $D_2 f(\beta, z^*)$ is an isomorphism of $\mathbb{R}$, from the implicit function theorem, one has that ϕ is a $\mathcal{C}^1$ function and moreover :

$$\phi'(\beta) = -D_2^{-1} f(\beta, \phi(\beta)) \circ D_1 f(\beta, \phi(\beta))$$

$$\phi'(\beta) = -\frac{\frac{L\phi(\beta)}{\beta^2}}{\cosh(\phi(\beta)) - \frac{L}{\beta}} = \frac{-L\phi(\beta)}{\beta^2 \cosh(\phi(\beta)) - \beta L}$$

One obtains :

$$\begin{aligned} D_1 g_2(\alpha,\beta) &= 2\cos(\alpha) \\ D_2 g_2(\alpha,\beta) &= \frac{L}{\beta\cosh(\phi(\beta)) - L}\left(\frac{1-\cosh\phi(\beta)}{\phi(\beta)} + \sinh\phi(\beta)\right) + \frac{1-\cosh\phi(\beta)}{\phi(\beta)} \end{aligned}$$

Numerical application [5] : *Figure 3.4 presents the feasible configuration set $\mathbb{S}$ with ⬚ $=]0.8, 1.2[\times]0.5, 0.7[$, $L = 4$ and $(\alpha, \beta) \in [0, \pi] \times [1/2, 7/4]$. It also provides a realization of the simplicial complex generated by `HIA`. The simplicial complex $\mathcal{K}(\mathbb{S})$ can be collapsed, and one can affirm that $\mathbb{S}$ is homotopy equivalent to a circle.*

[5] A program (Rope.exe) associated with this example can be found at http://www.istia.univ-angers.fr/ delanoue/

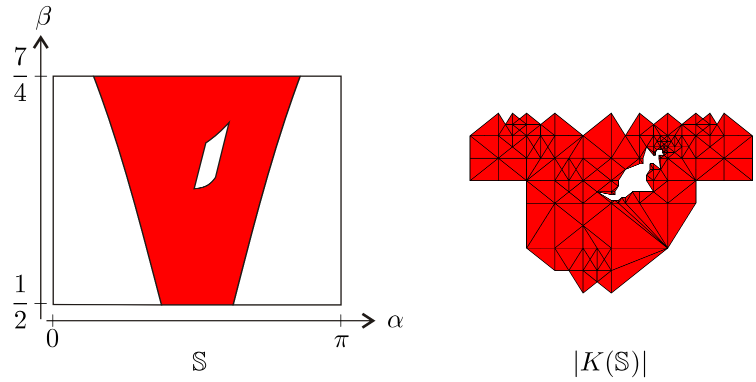

Figure 19. The feasible configuration set $\mathbb{S}$ *and a triangulation generated by* `HIA`.

4. Discussion

The proposed method is efficient (e.g. time computing is less than 1 sec for example 3.3) but does not *always* terminate because for some sets, it is not possible to decompose these ones with a *finite collection of star-shaped sets*. The circle $S^1 = \{(x, y) \in \mathbb{R}^2, x^2+y^2-1 = 0\}$ is an example that the `HIA` algorithm cannot handle. In general, the HIA algorithm is not able to deal with manifolds. To expand our algorithm to manifolds, a algorithm could enlarge it without changing the homotopy type. For example, the circle S^1 could be enlarged into $S^1_\epsilon = \{(x, y) \in \mathbb{R}^2, (x^2 + y^2 - 1)^2 \leq \epsilon\}$ as showed in Fig. 4.

Figure 20. The circle S^1 and its enlargement S^1_ϵ with $\epsilon < 1$.

The presence of singularities has not been discussed. If there is a critical point x_1 satisfying $f(x_1) = 0$, `HIA` algorithm won't terminate because the sufficient condition of Proposition 2.2 can not be checked using interval analysis. But on the other hand, thanks to Proposition

2.2, all critical points do not have to be found to guarantee the homotopy type of $M^0 = f^{-1}(]-\infty; 0])$(John Willard Milnor, 1963).

5. Conclusion

This paper addresses to the topological study of solutions $\mathbb{S}$ of systems. It provides a new algorithm that creates a simplicial complex homotopy equivalent to $\mathbb{S}$. This algorithm could be combined with constraint propagation to be faster. It could be interesting to compare the efficiency of this algorithm with the cylindrical algebraic decomposition algorithm of Collins in the case of semi-algebraic sets.

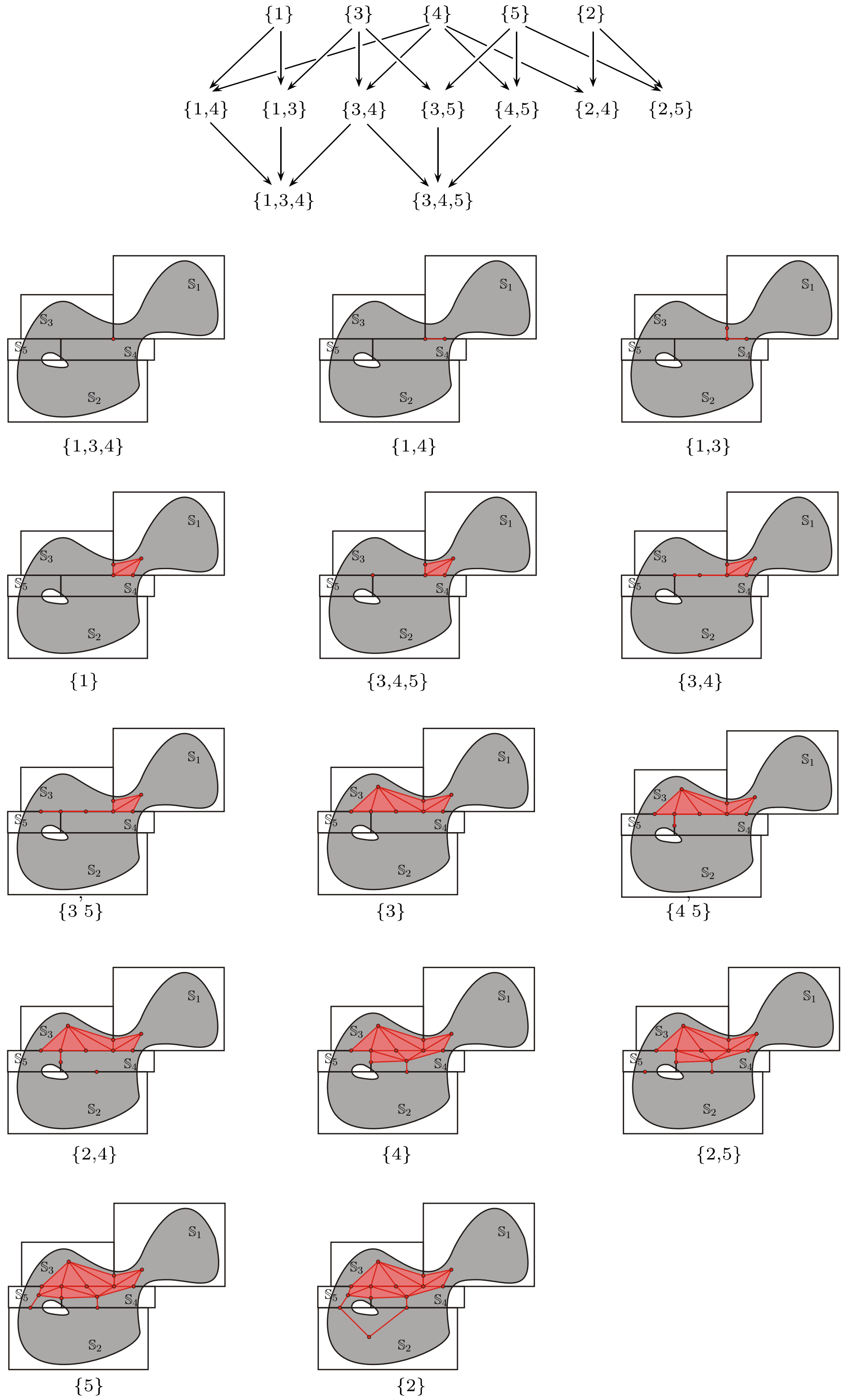

Figure 21. `Adapted_Triangulation` algorithm step by step.

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