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A MAXIMALITY THEOREM FOR CONTINUOUS FIRST ORDER THEORIES

A b s t r a c t. In this paper we prove a Lindström like theorem for the logic consisting of arbitrary Boolean combinations of first order sentences. Specifically we show the logic obtained by taking arbitrary, possibly infinite, Boolean combinations of first order sentences in countable languages is the unique maximal abstract logic which is closed under finitary Boolean operations, has occurrence number ω_1, has the downward Löwenheim-Skolem property to ω and the upward Löwenheim-Skolem property to uncountability, and contains all complete first order theories in countable languages as sentences of the abstract logic. We will also show a similar result holds in the continuous logic framework of [5], i.e. we prove a Lindström like theorem for the abstract continuous logic consisting of Boolean combinations of first order closed conditions. Specifically we show the abstract continuous logic consisting of arbitrary Boolean combinations of closed conditions is the unique maximal abstract continuous logic which is closed under approximate isomorphisms on countable structures, is closed under finitary Boolean operations, has occurrence number ω_1, has the downward Löwenheim-Skolem property to ω, the upward Löwenheim-Skolem property to uncountability and contains all first order theories in countable languages as sentences of the abstract logic.

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1. Introduction

One of the most significant advances in the study of abstract model theory is the collection of characterizations of first order logic by Lindström. These characterizations identify first order logic as the unique maximal logic, on classical structures, satisfying various properties. Specifically, Lindström showed that first order logic is the unique maximal logic with countable occurrence number which has both the upward and downward Löwenheim-Skolem property, and which is closed under finite Boolean operations along with existential quantification (see [11, Thm. 3]).

While the theorem above characterizes first order logic among those logics closed under existential quantification, it fails if we look at first order logic among those logics with just the upward and downward Löwenheim-Skolem property and closed under finitary Boolean operations. In fact, if we add to our logic, as a new sentence, any complete first order theory in a countable language, then our logic will still have both the upward and downward Löwenheim-Skolem properties. It is therefore natural to ask what happens if we replace "closed under existential quantification" with "closed under arbitrary Boolean operations " (in a fixed language)? Let $\bigvee^{\omega} Th^{\omega}(\mathcal{L}_{\omega,\omega})$ be the logic which is the closure of $\mathcal{L}_{\omega,\omega}$ under arbitrary Boolean combinations of sentences (in a fixed countable language). In Theorem 3.9 we show that $\bigvee^{\omega} Th^{\omega}(\mathcal{L}_{\omega,\omega})$ is the unique maximal logic with countable occurrence number which contains first order logic, has the downward Löwenheim-Skolem property to countable structures, the upward Löwenheim-Skolem property to uncountable structures, and which is closed under arbitrary Boolean operations in fixed countable languages.

Since Lindström first provided his characterizations of first order logic several other logics, including $\mathcal{L}_{\infty,\omega}$, have been characterized as the unique maximal logic satisfying certain properties. Recall that a logic is bounded if whenever $<$ is a binary relation there is no sentence which both always interprets $<$ as a well-ordering and which has structures in which $<$ has arbitrarily large domain. As an example of such a characterization of $\mathcal{L}_{\infty,\omega}$, one can show $\mathcal{L}_{\infty,\omega}$ is the unique maximal logic closed under existential quantification, finite Boolean operations, potential isomorphisms and is bounded (see [2, Thm III.3.1] for more details). As $\bigvee^{\omega} Th^{\omega}(\mathcal{L}_{\omega,\omega})$ sits squarely between $\mathcal{L}_{\omega,\omega}$ and $\mathcal{L}_{\infty,\omega}$ Theorem 3.9 gives motivation to the idea that $\bigvee^{\omega} Th^{\omega}(\mathcal{L}_{\omega,\omega})$ is a natural intermediate logic to study.

Over the last decade continuous first order logic has emerged as a powerful analog of classical first order logic for dealing with structures built out of complete metric spaces. In this paper, in Theorem 1.1, we review the notions of continuous first order logic as well as introduce analogs of abstract logics in the continuous case. With these notions in hand we then proceed to prove a continuous version of Theorem 3.9. Let $\bigvee^{\omega} Th^{\omega}(\overline{\mathcal{L}}^{c}_{\omega,\omega})$ be the logic which is the closure of continuous first order logic under arbitrary Boolean combinations of sentences (in a fixed countable language). In Theorem 5.4 we show that $\bigvee^{\omega} Th^{\omega}(\overline{\mathcal{L}}^{c}_{\omega,\omega})$ is the unique maximal logic with countable occurrence number which contains continuous first order logic, has the downward Löwenheim-Skolem property to structures with countable density, the upward Löwenheim-Skolem property to structures with uncountable density, is closed under arbitrary Boolean operations in fixed countable language and which is closed under approximate isomorphisms.

1.1. Continuous First Order Logic

We now give a brief description of the set up. For more details see [5]. By a **continuous language** we mean a classical multi-sorted language where

- for each sort S there is a distinguished relation symbol d_S of type $S \times S$,
- each relation symbol and each function symbol X has associated to it a **modulus of uniform continuity**, m_X, i.e. a non-decreasing map from $(0, 1]$ to $(0, 1]$ whose limit at 0 is 0.

In this paper τ, σ and their variants will always represent continuous languages. A **continuous τ-structure $\mathcal{M}$** consists of the following.

- For each sort S there is a complete bounded metric space, $(S^{\mathcal{M}}, d_S^{\mathcal{M}})$, such that $d_S^{\mathcal{M}}$ is the distance function on the space.
- If S_0, S_1 are sorts and T is the product sort then $(S_0^{\mathcal{M}}, d_{S_0}^{\mathcal{M}}) \times (S_1^{\mathcal{M}}, d_{S_1}^{\mathcal{M}}) = (T^{\mathcal{M}}, d_T^{\mathcal{M}})$, i.e. the complete metric spaces associated to a product sort is the product of the complete metric space associated to each sort.
- For each function symbol f from sort S_0 to sort S_1 there is a function $f^{\mathcal{M}} \colon S_0^{\mathcal{M}} \to S_1^{\mathcal{M}}$.
- For each relation symbol R of sort S there is a function $R^{\mathcal{M}} \colon S^{\mathcal{M}} \to [0, 1]$.
- For each relation symbol R of type S we have $(\forall x, y \in S^{\mathcal{M}}) \, |R^{\mathcal{M}}(x) - R^{\mathcal{M}}(y)| \leq m_R(d_S^{\mathcal{M}}(x, y))$.
- For each function symbol f with domain sort S and codomain T we have $(\forall x, y \in S^{\mathcal{M}}) \, d_T^{\mathcal{M}}(f^{\mathcal{M}}(x), f^{\mathcal{M}}(y)) \leq m_f(d_S^{\mathcal{M}}(x, y))$.

We will treat constant symbols as functions from the unique trivial sort, which necessarily is a 1-point space. We define the **arity** of a tuple to be the product of the sorts to which the elements belong. We define the **type** of a relation as the product of sorts assigned to it and the **type** of a function symbol from sort S to T as $S \to T$.

If $\tau_0 \subseteq \tau_1$ are continuous languages and $\mathcal{M}$ is a continuous τ_1-structure we let $\mathcal{M}|_{\tau_0}$ be the continuous τ_0-structure obtained by ignoring all function symbols, relation symbols and sorts not in τ_0.

The functions $\dot{-}\colon [0,1]\times[0,1]$ and $\frac{\cdot}{2}\colon [0,1]\to[0,1]$ will be important. These functions are defined by $x \dot{-} y = \max\{0, x-y\}$ and $\frac{\cdot}{2}(x) = \frac{x}{2}$. Note that the function $\wedge\colon [0,1]\times[0,1]\to[0,1]$ given by $\wedge(x,y) = x\dot{-}(x\dot{-}y)$ is such that $(\forall x,y\in[0,1])\,\wedge(x,y) = \min\{x,y\}$. Similarly the function $\vee\colon [0,1]\times[0,1]\to[0,1]$ given by $\vee(x,y) = 1\dot{-}(\min(1\dot{-}x, 1\dot{-}y))$ is such that $(\forall x,y\in[0,1])\vee(x,y) = \max\{x,y\}$. We will write $\wedge(x,y)$ as $x\wedge y$ and $\vee(x,y)$ as $x\vee y$.

A useful intuition is to imagine $[0,1]$ is the collection of *truth values* where 0 represents *full truth* and the truth value of a statement is how far from full truth the statement is. With this intuition a relation is a map from an (interpretation of) a sort to $[0,1]$. Also, under this interpretation, $\inf_{x\in X}$ is the analog of the existential quantifier from first order logic and $\sup_{x\in X}$ is the analog of the universal quantifier from first order logic.

In continuous first order logic it is important that each relation in a language has a uniform bound throughout all interpretations of the language. However, it is also rarely important what that specific bound is. This sets up a choice in the presentation of continuous structures. On the one hand we could allow relations to take arbitrary values in $[0,\infty)$ and add to the language a bound to each relation which must hold in any interpretation of the relation. On the other hand we could impose a uniform bound, e.g. 1, on all relations in all structures. While the first choice on its face seems slightly more general, there is in reality very little difference between the two presentations. As such we have chosen the latter approach and required all relations to be uniformly bounded by 1. This loses little in terms of generality as all results in this paper go through immediately for the other presentation of continuous structures, however it will simplify the notation we are using.

We will let ORD represent the class of ordinals.

Definition 1.1. Suppose τ_0, τ_1 are continuous languages. A **renaming** is a bijection $\rho\colon \tau_0\to\tau_1$ which takes sorts to sorts, functions symbols to function symbols, and relation symbols to relation symbols such that

- for any sorts $S_0,\dots,S_n$ in τ_0, $\rho(S_0\times\cdots\times S_n) = \rho(S_0)\times\cdots\times\rho(S_n)$,
- for any sort S in τ_0, $d_{\rho(S)} = \rho(d_S)$,
- if R is a relation symbol of type S then $\rho(R)$ is a relation symbol of type $\rho(S)$,
- if f is a function symbol of type $S\to T$ then $\rho(S)$ is a function symbol of type $\rho(S)\to\rho(T)$,
- if X is a relation symbol or function symbol in τ_0 with modulus of uniform continuity m_X, then the modulus of uniform continuity of $\rho(X)$ is also m_X, i.e. $m_X = m_{\rho(X)}$.

Given such a renaming and a continuous τ_0-structure $\mathcal{M}$ we define the **renamed** τ_1-structure $\rho(\mathcal{M})$ in the obvious way.

We now make precise our definition of continuous first order logic. See [5] for more details.

Definition 1.2. Suppose τ is a continuous language we let $\mathcal{L}^c_{0,0}[\tau]$ be the collection of atomic τ-formulas. We let $\mathcal{L}^c_{\omega,\omega}[\tau]$ be the smallest collection such that

- $\mathcal{L}^c_{0,0}[\tau] \subseteq \mathcal{L}^c_{\omega,\omega}[\tau]$,
- if $\psi \in \mathcal{L}^c_{\omega,\omega}[\tau]$ of type $X \times Y$ then $\inf_{x\in X}\psi, \sup_{x\in X}\psi \in \mathcal{L}^c_{\omega,\omega}[\tau]$ and are of type Y,
- if $\psi_0, \dots, \psi_{n-1} \in \mathcal{L}^c_{\omega,\omega}[\tau]$ are of type X and $\alpha\colon [0,1]^n \to [0,1]$ is a continuous map then $\alpha(\psi_0, \dots, \psi_{n-1}) \in \mathcal{L}_{\omega,\omega}[\tau]$ of type X,
- if $\psi \in \mathcal{L}^c_{\omega,\omega}[\tau]$ is of type X and Y is a sort then there is a formula $\psi(\pi_0)$ of type $X \times Y$ and $\psi(\pi_1)$ of type $Y \times X$.

If $\mathcal{M}$ is a τ structure and $\psi \in \mathcal{L}_{\omega,\omega}[\tau]$ of type X we define $\psi^{\mathcal{M}}\colon X^{\mathcal{M}} \to [0,1]$ by induction on the complexity of the formula in the obvious way.

We call $\mathcal{L}^c_{\omega,\omega}[\tau]$ **continuous first order logic**.

By a **closed condition** we mean an equation of the form $\psi = 0$ where $\psi \in \mathcal{L}^c_{\omega,\omega}[\tau]$ is of type 1 (i.e. has no free variables).

Motivated by Definition 1.2 we will refer to (uniformly) continuous maps from $[0,1]^n$ to $[0,1]$ as **connectives**. Note that the constants $0, 1$ and the functions $\dot{-}$ and $\frac{\cdot}{2}$ are connectives.

Definition 1.3. By a **continuous first order theory** we mean a collection T of closed conditions in a single continuous language τ such that there is some continuous τ-structure with $\mathcal{M} \models \bigwedge T$. We say a such a theory is maximal if there are no larger theories in the same language.

Definition 1.4. We say a continuous τ-structure $\mathcal{M}$ is **discrete** if each relation, including the distance relation, takes values in $\{0,1\}$.

In particular if a continuous structure is discrete then every point is open. Note that being discrete can be defined by a collection of closed conditions.

Definition 1.5. We say a continuous τ-structure $\mathcal{M}$ has **density character** κ if there is a dense subset of $\mathcal{M}$ of size κ and no dense subset of size $< \kappa$.

The density character of a continuous τ-structure is, in many cases, the right analog of cardinality. To see why this is the case observe that if $\mathcal{M}$ is a continuous τ-structure and $X \subseteq \mathcal{M}$ is a dense subset of $\mathcal{M}$ then $\mathcal{M}$ can be uniquely recovered from X (by taking the closure). As such, if $\mathcal{M}$ has density character κ, $\mathcal{M}$ can be completely characterized by a set of size κ, even if $\mathcal{M}$ itself has larger cardinality. As a consequence many results of first order logic which deal with cardinality have continuous analogs with respect to density character, even when the obvious analog in terms of cardinality might not be true. A quintessential example of this phenomenon deals with the upwards and downwards Löwenheim-Skolem theorem for continuous first order logic. Any countable continuous first order theory with an infinite model must have a model of any infinite density character. However, there are countable continuous first order theories, such as the theory of $[0,1]$ as a metric space with constants for all rationals, which have infinite models but no models of size less than the continuum.

Definition 1.6. A continuous first order formula $\varphi \in \mathcal{L}^c_{\omega,\omega}[\tau]$ is ***k*-restricted** if it is built from atomic formulas using only connectives in $\{0, 1, \frac{\cdot}{2}, \dot{-}\}$ (where $0, 1$ are the constant functions) and it has only k-many subformulas. We say a continuous first order formula is **restricted** if it is k-restricted for some k. We will denote the collection of restricted formulas in $\mathcal{L}^c_{\omega,\omega}[\tau]$ by S^τ.

The following is a standard fact about restricted formulas (see [5, Thm. 6.3, Prop. 6.6]).

Lemma 1.7. *For every $\varphi \in \mathcal{L}^c_{\omega,\omega}[\tau]$ and every $\epsilon > 0$ there is a restricted formula $\varphi_\epsilon \in \mathcal{L}^c_{\omega,\omega}[\tau]$ such that for all continuous τ-structures $\mathcal{M}$*

$$\sup_{\mathbf{a}\in\mathcal{M}} |\varphi(\mathbf{a}) - \varphi_\epsilon(\mathbf{a})| < \epsilon.$$

Given two formulas $\psi_0, \psi_1 \in \mathcal{L}^c_{\omega,\omega}[\tau]$ we say ψ_0 is **equivalent** to ψ_1 if for all continuous τ-structures $\mathcal{M}$,

$$\sup_{\mathbf{a}\in\mathcal{M}} |\psi_0(\mathbf{a}) - \psi_1(\mathbf{a})| = 0.$$

Therefore Lemma 1.7 tells us that for any formula ψ we can find restricted formulas which are arbitrarily close to being equivalent to ψ.

In particular given two maximal theories T_0 and T_1, if they contain the same closed conditions of the form $\varphi = 0$ where φ is restricted then they are the same. Note, if τ is countable there are only countably many restricted formulas.

Definition 1.8. We define the **quantifier rank** of a restricted formula φ, $\mathrm{qr}(\varphi)$ by induction as follows.

- If φ is an atomic formula, i.e. is built only using relations, functions symbols and the constants $0, 1$, then $\mathrm{qr}(\varphi) = 0$.
- $\mathrm{qr}(\frac{\varphi}{2}) = \mathrm{qr}(\varphi)$.
- $\mathrm{qr}(\varphi \dot{-} \psi) = \sup\{\mathrm{qr}(\varphi), \mathrm{qr}(\psi)\}$.
- $\mathrm{qr}(\sup_x \varphi) = \mathrm{qr}(\inf_x \varphi) = \mathrm{qr}(\varphi) + 1$.

If $\mathrm{qr}(\varphi) = 0$ we say φ is **quantifier-free**.

Note the notion of quantifier rank can easily be extended to non-restricted formulas, however we will not need it here. See [4] for more details on general quantifier rank.

In what follows it will be important to consider finite subsets of restricted formulas which are sufficiently closed under translations. We now make this notion precise.

Definition 1.9. For $\epsilon > 0$ we say that a finite set S of restricted formulas has **magnitude ϵ-translations** if whenever $\varphi \in S$ then there is a formula in S equivalent to $\varphi \dot{-} \epsilon$ and there is a formula in S equivalent to $\inf\{1, \varphi + \epsilon\}$.

The intuition is that S has magnitude ϵ-translations if whenever φ is in S then we can translate everything in S by ϵ (truncating outside of $[0, 1]$ appropriately), and remain in S.

Lemma 1.10. *Suppose S is a collection of restricted formulas. For all $\epsilon > 0$ there is a smallest collection $\hat{S}_\epsilon$ of restricted formulas which contains S, has magnitude ϵ-translations, and is closed under finite Boolean operations. Further, if S is finite so is $\hat{S}_\epsilon$.*

Proof. We define a collection S^n_ϵ by induction on n. First we let $S^0_\epsilon = S$. Now suppose $n \in \omega$ and we have defined S^n_ϵ. We then let $S^{n+1,*}_\epsilon$ be the result of adding, for all $\varphi \in S^n_\epsilon$ the formulas among $\varphi \dot{-} \epsilon, \inf\{r, \varphi + \epsilon\}, 1 \dot{-} \varphi$ which are not equivalent to formulas in S^n_ϵ. We then have if S_ϵ is finite then $|S^{n+1,*}_\epsilon| \leq |S^n_\epsilon| \cdot 4$. Now let S^{n+1}_ϵ be the result of closing $S^{n+1,*}_\epsilon$ under conjunctions (only adding formulas which are not already equivalent to ones in $S^{n+1,*}_\epsilon$). We then have $|S^{n+1}_\epsilon| \leq |\mathfrak{P}(S^{n+1,*}_\epsilon)|$.

It is easy to check that at some finite stage k, which only depends on ϵ, we have $S^k_\epsilon = S^{k+1}_\epsilon$ and this S^k_ϵ is the desired minimal collection $\hat{S}_\epsilon$ containing S with magnitude ϵ-translations and closed under finite Boolean operations. $\square$

Lemma 1.10 tells us that adding magnitude ϵ-translations to a finite collection of restricted formulas results in a finite collection of restricted formulas.

Definition 1.11. Suppose S is a finite collection of restricted formulas. We define $Q_\epsilon(S, 0) = S$ and for $n \in \omega$, $Q_\epsilon(S, n+1)$ is the smallest collection of restricted formulas containing $\{\varphi, \inf_x \varphi, \sup_x \varphi \colon \varphi \in Q_\epsilon(S, n)\}$ and closed under magnitude ϵ-translations and finite Boolean operations.

Note, by Lemma 1.10, for any finite collection of restricted relations S, any n, and any $\epsilon > 0$, $Q_\epsilon(S, n)$ is finite.

1.2. Related Work

Lindström's work (see [11]) gives us a sense in which first order logic is canonical, i.e. his work shows us that there are a few basic properties of first order logic which isolate it among all possible logics. While continuous logic in its current form arose from the work of Ben-Yaacov, Berenstein, Henson, Usvyatsov, and others, equivalent notions have been studied since Chang and Keisler's book [8] in 1966. In particular, one of the first abstract logic results about continuous first order logic was by Iovino in [10]. This result was not done in terms of continuous first order logic but rather in terms of the positive bound formulas and an approximate satisfaction relation logic of Henson (see [9]). Iovino showed that there was no logic which properly extended Henson's logic of positive bound formulas and which satisfies compactness and has the elementary chain property. Like Lindström's theorem did for first order logic, this result gave the first concrete evidence that there was a maximal logic for continuous structures that satisfied many of the important properties of classical first order logic.

Lindström's most well-known result characterizing classical first order logic, and what is often known as Lindström's theorem, says that if a logic is Boolean, has the downward Löwenheim-Skolem property and satisfies compactness, then it must be first order logic. Recently an analog of this theorem was proved for continuous first order logic by Caicedo in [6].

Another of Lindström's results characterizes classical first order logic as maximal among Boolean logics which are closed under the existential quantifier and which have the λ-omitting types property for an uncountable regular λ. In [7], Caicedo and Iovino give an analog of this for continuous logic.

2. Abstract Continuous Logic

Over the years there have been many different types of logics which have been studied on classical structures, e.g. first order logic, higher order logic, infinitary logic, logics with a game quantifier, etc. Because of the breadth of different logics it is important to have the notion of an *abstract logic*.

Intuitively the notion of an abstract logic has two parts. First, for every classical language the logic must give a collection of sentences of that signature. While in most concrete logics sentences are built up from the relation symbols, function symbols and constant symbols by some concrete recursive procedure, in the most general case the collection of sentences only need be a set. Second, there needs to be a *satisfaction* relation which allows us to determine whether or not a structure in a given signature satisfies a sentence in that signature. Once again in most concrete cases where sentences are built from simpler components there is some method for determining whether or not a structure satisfies a sentences by looking at how the structure interacts with simpler components.

However, in the abstract case this is not necessary and we allow any notion of satisfaction between sentences and structures which satisfies four basic properties.

The first property the satisfaction relation must have is that it is preserved under isomorphism of structure. The second property is that it is preserved under renamings (which can be thought of as isomorphisms of languages). The third property is that any sentence in a smaller language is also a sentence in a larger language. And lastly, if we have a sentence in one language and a structure in a larger language, then whether or not the structure satisfies the sentence depends only on the reduct of the structure to the language of the sentence.

2.1. Basic Definitions

In this section we introduce the notion of an abstract logic for continuous structures. Note that these conditions are the analogs of the corresponding conditions in abstract model theory for classical structures. In particular we assume the reader is familiar with [2, Ch. 1-3]. While the main focus of this paper is on continuous logic, we will in Section 3 consider classical abstract logic. Therefore, we will always use *abstract continuous logic* when referring to the abstract logic on continuous structures and *abstract classical logic* when referring to abstract logic on classical structures. It will be useful to let L^{c} denote the class of all continuous languages.

Definition 2.1. An **abstract continuous logic** is a pair $(\mathfrak{L}, \models_{\mathfrak{L}})$ where

- $\mathfrak{L}$ is a function which takes continuous languages as arguments,
- $\models_{\mathfrak{L}}$ is a relation between continuous structures and $\bigcup_{\tau\in\mathrm{L}^{\mathrm{c}}} \mathfrak{L}[\tau]$,

and which satisfy the following where σ, τ and τ^* are arbitrary continuous languages and $\mathcal{M}$ and $\mathcal{N}$ are arbitrary continuous τ-structures.

(i) If $\tau \subseteq \sigma$ then $\mathfrak{L}[\tau] \subseteq \mathfrak{L}[\sigma]$.

(ii) If $\mathcal{M} \models_{\mathfrak{L}} \varphi$ then $\varphi \in \mathfrak{L}[\tau]$.

(iii) (Isomorphism property) If $\mathcal{M} \models_{\mathfrak{L}} \varphi$ and $\mathcal{N} \cong \mathcal{M}$ then $\mathcal{N} \models_{\mathfrak{L}} \varphi$.

(iv) (Reduct Property) If $\varphi \in \mathfrak{L}[\tau^*]$, $\tau^* \subseteq \tau$ then

$$\mathcal{M} \models_{\mathfrak{L}} \varphi \text{ if and only if } \mathcal{M}|_{\tau^*} \models_{\mathfrak{L}} \varphi,$$

where $\mathcal{M}|_{\tau^*}$ is the reduct of $\mathcal{M}$ to τ^*.

(v) (Renaming Property) Let $\rho\colon \tau \to \tau^*$ be a renaming. Then for each $\varphi \in \mathfrak{L}[\tau]$ there is a sentence $\varphi^\rho \in \mathfrak{L}[\tau^*]$ such that for all τ-structures $\mathcal{M}$

$$\mathcal{M} \models_{\mathfrak{L}} \varphi \text{ if and only if } \rho(\mathcal{M}) \models_{\mathfrak{L}} \varphi^\rho.$$

It is worth observing that there is an unfortunate clash of terminology between continuous first order logic and classical abstract logic with regards to the notion of a sentence. This clash comes from the dichotomy that occurs in continuous first order logic between the notion of *internal truth values* vs *external truth values* (a dichotomy which is shared with model theory in a topos). In continuous first order logic the collection of possible truth values is the set $[0, 1]$. One therefore has a notion of an internal formula which is a map from some sort in a structure to the set of truth values. An internal sentence is then simply a map from the terminal object (i.e. the one element sort) into the truth values. However, often when studying a continuous structure one needs to know whether or not a given formula/sentence takes a given value. This gives rise to the notion of an *external sentence*, i.e. one whose truth value is in $\{\top, \bot\}$. In continuous logic these external sentences are called *closed conditions*.

When studying abstract classical logic the objects of study are classes of structures and we are interested in whether a given structure is in the class or not, i.e. a fact with truth value $\{\top, \bot\}$. In abstract classical logic such classes of structures are termed *abstract sentences*. Similarly in abstract continuous logic the objects of study will be classes of models, i.e. external facts which have truth values in $\{\top, \bot\}$. Unlike in continuous first order logic though there need not be a notion of internal formula associated to these external facts. As such we will choose to follow the terminology of abstract classical logic and, for τ a continuous language, refer to $\mathfrak{L}[\tau]$ as the collection of **$\mathfrak{L}$-sentences of language τ**.

We refer to $\models_{\mathfrak{L}}$ as the **satisfaction relation** of $\mathfrak{L}$ and will omit the subscript when it is clear from context. We will abuse notation and use $\mathfrak{L}$ to refer to the pair $(\mathfrak{L}, \models_{\mathfrak{L}})$ when no confusion can arise. We will also abuse notation and say that $\varphi \in \mathfrak{L}$ if $\varphi \in \bigcup_{\tau \in \mathrm{L}^c} \mathfrak{L}[\tau]$.

For a sentence φ of $\mathfrak{L}$ we let $[\varphi]_{\mathfrak{L}}$ be the (class sized) collection of continuous structures $\mathcal{M}$ such that $\mathcal{M} \models \varphi$.

Definition 2.2. We say an abstract continuous logic $\mathfrak{L}_1$ is **stronger** than an abstract continuous logic $\mathfrak{L}_0$, written $\mathfrak{L}_1 \geq \mathfrak{L}_0$, if for all continuous languages τ, and for all $\varphi_0 \in \mathfrak{L}_0[\tau]$ there is a formula $\varphi_1 \in \mathfrak{L}_1[\tau]$ where for any continuous τ-structure $\mathcal{M}$

$$\mathcal{M} \models \varphi_0 \text{ if and only if } \mathcal{M} \models \varphi_1$$

i.e. for every sentence in $\mathfrak{L}_0$ there is an equivalent sentence in $\mathfrak{L}_1$.

2.2. Properties of Abstract Continuous Logics

Now that we have our notion of abstract continuous logic we consider some properties which we might want an abstract continuous logic to have.

Definition 2.3. We say a logic $\mathfrak{L}$ has **occurrence number** κ if κ is minimal such that for all continuous languages τ and all $\varphi \in \mathfrak{L}[\tau]$ there is a subset τ_0 such that $\varphi \in \mathfrak{L}[\tau_0]$

and $|\tau_0| < \kappa$.

We then define

$$\mathbf{gf}(\mathfrak{L}) := \sup\{|\mathfrak{L}[\tau]| \colon |\tau| < \kappa\}.$$

Note that if $\mathfrak{L}$ has occurrence number κ then each $\mathfrak{L}$-sentence can informally be thought of as having a *general form* in a language of size $< \kappa$, i.e. an equivalent sentence in a language of size $< \kappa$.

Definition 2.4. We say a continuous logic $\mathfrak{L}$ has the **downward Löwenheim-Skolem property to** κ if for all continuous languages τ and all $\varphi \in \mathfrak{L}[\tau]$ for which there is a continuous τ-structure $\mathcal{M}$ of density character $\geq \kappa$ with $\mathcal{M} \models \varphi$, then there is a τ-structure $\mathcal{M}^*$ such that

- $\mathcal{M}^*$ has density character κ, and
- $\mathcal{M}^* \models \varphi$.

There is a similar notion of an upward Löwenheim-Skolem property to κ, however for our purposes we will only need something weaker. We will want to start with a structure with countable density character satisfying a sentence and know that there exists a structure with uncountable density character satisfying the same sentence. For our purposes we will not care about what the uncountable density character is, and in particular will not require that all satisfiable sentences have a model with the same uncountable density. This motivates the next definition.

Definition 2.5. We say a continuous logic $\mathfrak{L}$ has the **upward Löwenheim-Skolem property to uncountability** if whenever τ is a countable continuous language, $\varphi \in \mathfrak{L}[\tau]$, and $\mathcal{M}$ is a τ-structure satisfying φ, then there exists a τ-structure $\mathcal{M}^*$ such that

- $\mathcal{M}^*$ has uncountable density character, and
- $\mathcal{M}^* \models \varphi$.

We now discuss several Boolean operations a logic can be closed under.

Definition 2.6. Suppose $\mathfrak{L}$ is an abstract continuous logic and τ is a continuous language.

- If for every $\varphi \in \mathfrak{L}[\tau]$ there is a formula $\neg\varphi \in \mathfrak{L}[\tau]$ such that for all continuous τ-structures $\mathcal{M}$

  $$\text{not } (\mathcal{M} \models \varphi) \text{ if and only if } \mathcal{M} \models \neg\varphi$$

  then we say $\mathfrak{L}$ is **closed under negation** at τ.

- If for all $(\varphi)_{i\in\gamma} \subseteq \mathfrak{L}[\tau]$ with $\gamma < \kappa$ there is a formula $\bigvee_{i\in\gamma} \varphi_i \in \mathfrak{L}[\tau]$ such that for all continuous τ-structures $\mathcal{M}$

$$\mathcal{M} \models \bigvee_{i\in\gamma} \varphi_i \text{ if and only if } (\exists i \in \gamma)(\mathcal{M} \models \varphi_i)$$

  then we say $\mathfrak{L}$ is **closed under $<\kappa$-disjunctions at τ**. We say a logic is **closed under $<\kappa$-disjunctions** if it is closed under $<\kappa$-disjunctions at τ for all continuous languages τ.

- If for all $(\varphi)_{i\in\gamma} \subseteq \mathfrak{L}[\tau]$ with $\gamma < \kappa$ there is a formula $\bigwedge_{i\in\gamma} \varphi_i \in \mathfrak{L}[\tau]$ such that for all continuous τ-structures $\mathcal{M}$

$$\mathcal{M} \models \bigwedge_{i\in\gamma} \varphi_i \text{ if and only if } (\forall i \in \gamma)(\mathcal{M} \models \varphi_i)$$

  then we say $\mathfrak{L}$ is **closed under $<\kappa$-conjunctions at τ**. We say a logic is **closed under $<\kappa$-conjunctions** if it is closed under $<\kappa$-conjunctions at τ for all continuous languages τ.

We say $\mathfrak{L}$ is **Boolean** if it is closed under negation, $<\omega$-conjunctions and $<\omega$-disjunctions. We say $\mathfrak{L}$ is **completely Boolean** if it has an occurrence number κ and for all continuous languages τ with $|\tau| < \kappa$ we have

- $\mathfrak{L}$ is closed under negations at τ,
- $\mathfrak{L}$ is closed under $< (\mathbf{gf}(\mathfrak{L}))^+$-conjunctions and $< (2^{\mathbf{gf}(\mathfrak{L})})^+$-disjunctions at τ.

A logic is Boolean if its sentences are closed under all finitary Boolean operations. A logic is completely Boolean if and only if every Boolean combination of sentences (in a single language) is itself a sentence (in that language), provided that language has size at most the occurrence number of $\mathfrak{L}$.

In mathematics one rarely distinguishes between isomorphic objects. As such in our definition of an abstract continuous logic we have required the satisfaction relation to be closed under isomorphism. However, when one lifts results from classical logic to continuous logic one often finds that the results don't lift precisely, but only *up to arbitrary accuracy*. This idea motivates the following definition.

Definition 2.7. Suppose $\mathcal{M}$ and $\mathcal{N}$ are continuous τ-structures, $\epsilon \geq 0$ and $S \subseteq \mathcal{L}_{\omega,\omega}[\tau]$ contains all atomic formulas. By an **(ϵ, S)-approximate isomorphism** from $\mathcal{M}$ to $\mathcal{N}$ we mean a relation $F_{\epsilon,S}$ between two sets A and B where:

- A is dense in $\mathcal{M}$ and B is dense in $\mathcal{N}$.

- Whenever $\varphi \in S$, $a_0, \ldots, a_{k-1} \in A$ and $b_0, \ldots, b_{k-1} \in B$ are of the same arity, and $\bigwedge_{i<k} F_{\epsilon,S}(a_i, b_i)$ then

$$|\varphi^{\mathcal{M}}(a_0, \ldots, a_{k-1}) - \varphi^{\mathcal{N}}(b_0, \ldots, b_{k-1})| \leq \epsilon.$$

- For all $a \in A$ there is a $b \in B$ such that $F_{\epsilon,S}(a, b)$ holds.
- For all $b \in B$ there is a $a \in A$ such that $F_{\epsilon,S}(a, b)$ holds.

We say $\mathcal{M}$ and $\mathcal{N}$ are **S-approximately isomorphic** if for every $\epsilon > 0$ there is an (ϵ, S)-approximate isomorphism. We will omit mention of S when S is the collection of all quantifier-free restricted formulas.

The following two lemmas are immediate.

Lemma 2.8. *For τ-structures $\mathcal{M}$ and $\mathcal{N}$ the following are equivalent.*

- *$\mathcal{M}$ and $\mathcal{N}$ are isomorphic.*
- *$\mathcal{M}$ and $\mathcal{N}$ are $(0, \mathcal{L}^c_{0,0})$-approximately isomorphic.*
- *$\mathcal{M}$ and $\mathcal{N}$ are $(0, \mathcal{L}^c_{\omega,\omega})$-approximately isomorphic.*

Lemma 2.9. *The relation of being approximately isomorphic is an equivalence relation.*

Note that this definition is similar in spirit to the notion of approximate isomorphism from [1] except we are not requiring that the ϵ-approximate isomorphisms give rise to a map of metric structures.

Definition 2.10. We say an abstract continuous logic $\mathfrak{L}$ is **closed under S-approximate isomorphisms at** ω if whenever $\varphi \in \mathfrak{L}[\tau]$ and $\mathcal{M}$, $\mathcal{N}$ are τ-structures of density character ω which are S-approximately isomorphic, then $\mathcal{M} \models \varphi$ if and only if $\mathcal{N} \models \varphi$.

In classical logic the analog of $\mathcal{L}^c_{0,0}$-approximate isomorphisms is the notion of *potential isomorphism*. In particular an important fact which is used in the proof of the classical version of our main theorem, i.e. Lemma 3.6, is that two countable models which are potentially isomorphic are isomorphic and hence satisfy the same sentences. For the continuous analog though we need to explicitly assume that our logic is closed under the analog of potential isomorphism. However, for our purposes we will only need to assume that our logics are closed under $\mathcal{L}^c_{\omega,\omega}$-approximate isomorphisms and not necessarily under $\mathcal{L}^c_{0,0}$-approximate isomorphisms.

2.3. Examples of Abstract Continuous Logic

We now discuss several important examples of abstract continuous logics.

Definition 2.11. We let $\overline{\mathcal{L}}^c_{\omega,\omega}$ denote **continuous first order logic** treated as an abstract continuous logic, i.e. the abstract continuous logic where the sentences are the closed conditions of continuous first order logic.

We now define an abstract continuous logic built up from classical first order logic.

Definition 2.12. Suppose τ is a continuous language. Let τ_c be the classical language with the following properties.

- The sorts of τ_c consist of the sorts of τ along with a new distinguished sort R.
- The set of function symbols of τ_c is the union of (1) the set of function symbols of τ, (2) the set of function symbols $\{X_c\colon X \text{ is a relation symbol of } \tau\}$, (3) a function symbol $\Delta_c\colon R \to R$ for all moduli of uniform continuity Δ and (4) a function symbol $d_R\colon R \times R \to R$, and a function symbol $+\colon R \times R \to R$.
- The set of constants of τ_c is the union of the set of constants in τ along with constants $\{c_q\}_{q\in\mathbb{Q}\cap[0,1]}$ of sort R.
- There is a single relation symbol $\leq$ of arity $R \times R$.
- For every relation symbol X of τ the function symbol X_c has the same domain sorts as the relation symbol of τ and has codomain sort R.

If $\mathcal{M}$ is a continuous τ-structure we let $\mathcal{M}_c$ be the corresponding τ_c-structure where $(R, \{c_q\}_{q\in\mathbb{Q}}, \leq)$ is interpreted in $\mathcal{M}$ by $([0,1], \{q\}_{q\in\mathbb{Q}\cap[0,1]}, \leq^{[0,1]})$.

Given a continuous τ-structure $\mathcal{M}$, we can think of $\mathcal{M}_c$ as the *same* object, but interpreted as a classical τ_c-structure.

Definition 2.13. Let $\mathcal{L}^*_{\omega,\omega}$ be the abstract continuous logic which takes a continuous language τ and returns the collection of classical first order sentences in τ_c and where, for $\varphi \in \mathcal{L}^*_{\omega,\omega}(\tau)$ and $\mathcal{M}$ a continuous τ-structure,

$$\mathcal{M} \models \varphi \text{ if and only if } \mathcal{M}_c \models \varphi.$$

Note it is easy to see that $\mathcal{L}^*_{\omega,\omega}$ has occurrence number ω and is Boolean. It is worth noting that while some properties of classical first order logic generalize to $\mathcal{L}^*_{\omega,\omega}$, in general $\mathcal{L}^c_{\omega,\omega}$ and $\mathcal{L}^*_{\omega,\omega}$ can behave very differently.

Lemma 2.14. *Suppose*

- κ *is an infinite cardinal,*
- τ *is a continuous language of size* $\leq \kappa$,
- $\{\varphi_i\}_{i\in\kappa}$ *is a collection of sentences in* $\mathcal{L}^*_{\omega,\omega}(\tau)$,
- $\mathcal{M}$ *is a continuous* τ*-structure such that for all* $i \in \kappa$, $\mathcal{M} \models \varphi_i$, *and*
- $X \subseteq \mathcal{M}$.

Then there is a continuous τ*-structure* $\mathcal{M}^*$ *such that*

- $X \subseteq \mathcal{M}^* \subseteq \mathcal{M}$,
- *for all* $i \in \kappa$, $\mathcal{M}^* \models \varphi_i$, *and*
- $\mathcal{M}^*$ *has cardinality (and hence density character) at most* $(|X| + \kappa)^\omega$.

Proof. Let $(V, \in)$ be the background model of ZFC. We now define for each $i \in \omega_1+1$ a subset $\mathcal{M}_i \subseteq \mathcal{M}$ of size $\leq (|X| + \kappa)^\omega$.

Let $\mathcal{M}_0 := X$. Next, if we have defined $\mathcal{M}_i$ for all $i \leq \omega \cdot \alpha$, then let $\mathcal{M}_{\omega\cdot\alpha} := \bigcup_{i\in\omega\cdot\alpha} \mathcal{M}_i$.

Finally suppose we have defined $\mathcal{M}_i$ and for all $j < i$ we have defined a set V_j of size at most $(|X| + \kappa)^\omega$. By the downward Löwenheim-Skolem theorem for first order logic there is subset $V_i \subseteq V$ such that

- $(V_i, \in)$ is an elementary substructure of $(V, \in)$,
- $|V_i| \leq (|X| + \kappa)^\omega$, and
- $\bigcup_{j<i} V_i \cup \{\mathcal{M}\} \cup \mathcal{TC}(\tau) \cup \{x \in \mathcal{M}_j\}^\omega \in V_i$,

where $\mathcal{TC}(\tau)$ is the transitive closure of τ as a set. Note that $V_i \models (\mathcal{M} \models \bigwedge_{i\in\kappa} \varphi_i)$. Let $\mathcal{M}_i := \{m \in \mathcal{M} \colon m \in V_i\}$.

Suppose $(c_i)_{i\in\omega}$ is a Cauchy sequence in $\mathcal{M}_{\omega_1}$ converging to c. Then there must be some $i < \omega_1$ such that $\{c_i\}_{i\in\omega} \subseteq \mathcal{M}_i$. Therefore $c \in \mathcal{M}_{\omega_1}$ and, as the Cauchy sequence was arbitrary, $\mathcal{M}_{\omega_1}$ is Cauchy complete. But $\mathcal{M}_{\omega_1} \models \bigwedge_{i\in\kappa} \varphi_i$ and $|\mathcal{M}_{\omega_1}| \leq \omega_1 \cdot (|X| + \kappa)^\omega = (|X| + \kappa)^\omega$. Hence $\mathcal{M}_{\omega_1}$ is our desired $\mathcal{M}^*$. $\square$

Note the above lemma shows that $\mathcal{L}^*_{\omega,\omega}$ has the downward Löwenheim-Skolem property to ω^ω for theories. It is an interesting question whether or not $\mathcal{L}^*_{\omega,\omega}$ has the downward Löwenheim-Skolem property to ω.

Lemma 2.15. *$\mathcal{L}^*_{\omega,\omega}$ does not have the upward Löwenheim-Skolem property to uncountability.*

Proof. Let τ_0 be the continuous language with a single sort S, constant symbols $\{c_0, c_1\}$, no function symbols, and with d_S the only relation symbol. Now consider the sentence T which is the conjunction of the following

- $d_S(c_0, c_1) = 1$
- $(\forall x, y : S)\ d_S(c_0, x) = d_S(c_0, y) \to x = y$
- $(\forall \epsilon : R)\ \epsilon \leq 1 \to (\exists x)\ d_S(c_0, x) = \epsilon$
- $(\forall x : S)\ d_S(c_0, x) \leq 1$
- Let $\eta_{\leq}(x, y) := d_S(c_0, x) + d_S(x, y) = d_S(c_0, y)$. Then $\eta_{\leq}(x, y)$ is a linear ordering.

It is then easily checked that any continuous model of T must be isomorphic to $([0, 1], |\cdot - \cdot|)$ as a complete metric space. $\square$

Note that classical first order logic does have an upward Löwenheim-Skolem theorem and our result here is making fundamental use of the fact that our structures are continuous and non-classical (and in particular that any two elements whose distance is *arbitrarily* small, i.e. less than $\frac{1}{n}$ for all $n \in \mathbb{N}$, must have distance 0 and hence be the same).

Similarly to classical first order logic, the logic $\overline{\mathcal{L}}^c_{\omega,\omega}$ has occurrence number ω, has both the downward Löwenheim-Skolem property to ω (see [5, Prop. 7.3]) and upward Löwenheim-Skolem property to uncountability (which follows from the compactness of $\mathcal{L}^c_{\omega,\omega}$ via a standard argument). However, even though $\mathcal{L}^c_{\omega,\omega}$ has $< \omega$-conjunctions and $< \omega$-disjunctions it is not Boolean as it is not closed under negations.

2.4. Building Logics

When dealing with abstract logics there is often a tension between wanting the logic to have nice properties, e.g. downward Löwenheim-Skolem property, being completely Boolean, etc. and wanting to maximize the expressive power of the logic. In this section we will give several ways of extending abstract continuous logics and consider the corresponding extensions of abstract continuous first order logic.

Given an abstract continuous logic there are some obvious ways to build a (potentially) bigger logic from it. One of the simplest is to add to the logic as sentences the collection of the maximal theories of the logic.

Definition 2.16. We define a **satisfiable theory** in τ (for $\mathfrak{L}$) to be a collection $T := \{\varphi_i\}_{i \in I} \subseteq \mathfrak{L}[\tau]$ such that there is some $\mathcal{M}$ where $(\forall i \in I)\ \mathcal{M} \models \varphi_i$. In this case we say $\mathcal{M} \models T$.

We say a satisfiable theory T is **maximal** if there is no strictly larger satisfiable theory in the same continuous language containing it.

In this paper we will only consider satisfiable theories and so we will drop the adjective "satisfiable".

It is worth noting that it need not be the case that the union of an increasing sequence of theories T_i is itself a theory. Also worth noting is that while every maximal $\mathfrak{L}[\tau]$ theory is of the form

$$Th_{\mathfrak{L}}(\mathcal{M}) := \{\varphi \in \mathfrak{L}[\tau] \colon \mathcal{M} \models \varphi\},$$

if $\mathfrak{L}$ is not closed under negation it is possible for there to be a τ-structure $\mathcal{M}$ such that $Th_{\mathfrak{L}}(\mathcal{M})$ is not maximal. However, at least for first order continuous logic, this can't happen.

Lemma 2.17. *If $\mathcal{M}$ is a continuous τ-structure then $Th_{\overline{\mathcal{L}}^c_{\omega,\omega}}(\mathcal{M})$ is a maximal τ-theory.*

Proof. Suppose toward contradiction that $Th_{\overline{\mathcal{L}}^c_{\omega,\omega}}(\mathcal{M})$ is not maximal, i.e. there is a continuous τ-structure $\mathcal{N}$ and a closed condition $\psi \in \overline{\mathcal{L}}^c_{\omega,\omega}[\tau] \setminus Th_{\overline{\mathcal{L}}^c_{\omega,\omega}}(\mathcal{M})$ such that $\mathcal{N} \models Th_{\overline{\mathcal{L}}^c_{\omega,\omega}}(\mathcal{M}) \cup \{\psi\}$. Let ψ^* be the continuous τ-formula with no free variables such that ψ is $\psi^* = 0$. Now as $\neg(\mathcal{M} \models \psi)$ there must be some $r \in (0, 1]$ such that $\mathcal{M} \models \psi^* = r$. But then $\mathcal{M} \models \max\{\psi^* - r, r - \psi^*, 0\} = 0$. So $\max\{\psi^* - r, r - \psi^*, 0\} = 0 \in Th_{\overline{\mathcal{L}}^c_{\omega,\omega}}(\mathcal{M})$ and hence $\mathcal{N} \models \max\{\psi^* - r, r - \psi^*, 0\} = 0$. But this contradicts our assumption that $\mathcal{N} \models \psi^* = 0$. ☐

Definition 2.18. For $\kappa \in \mathrm{ORD} \cup \{\mathrm{ORD}\}$ let $Th^\kappa(\mathfrak{L})$ be the abstract continuous logic where

- If $|\tau| \leq \kappa$ then $Th^\kappa(\mathfrak{L})[\tau] := \{T \colon T \text{ is a maximal } \mathfrak{L}[\tau]\text{-theory}\}$.
- If $|\tau| > \kappa$ then $Th^\kappa(\mathfrak{L})[\tau] := \bigcup\{Th^\kappa(\mathfrak{L})[\tau_0] \colon \tau_0 \subseteq \tau, |\tau_0| \leq \kappa\}$.

It is straightforward to check for $\kappa, \gamma \in \mathrm{ORD} \cup \{\mathrm{ORD}\}$ that $Th^\kappa(Th^\gamma(\mathfrak{L}))$ is equivalent to $Th^{\max\{\kappa,\gamma\}}(\mathfrak{L})$. The following is also immediate from the definition.

Lemma 2.19. *For any abstract continuous logic $\mathfrak{L}$, $Th^\kappa(\mathfrak{L})$ has occurrence number $\leq \kappa^+$.*

Specializing to the case of $\overline{\mathcal{L}}^c_{\omega,\omega}$ we have the following result.

Lemma 2.20. *$Th^\kappa(\overline{\mathcal{L}}^c_{\omega,\omega})$ has the downward Löwenheim-Skolem property to κ, and the upward Löwenheim-Skolem property to uncountability.*

Proof. The downward Löwenheim-Skolem property follows immediately from [5, Prop. 7.3]. The upward Löwenheim-Skolem property follows from the compactness theorem for continuous first order logic (see [5, Thm. 5.8]) in the standard way. □

Classically when we move from sentences of first order logic to theories of first order logic we move from a Boolean logic to a non-Boolean logic (as theories are not in general closed under negation). The fact that theories are not closed under negation can cause difficulties and there are results which hold for the collection of first order sentences which do not hold for the collection of first order theories because of it.

In continuous first order logic however the move from sentences to theories is often not as significant, as even the first order sentences of continuous first order logic are not closed under negation.

Definition 2.21. Let $\bigvee^{\kappa} \mathfrak{L}$ be the smallest logic containing $\mathfrak{L}$ such that for any language τ with $|\tau| \leq \kappa$ and any collection $(\psi_i)_{i\in I} \subseteq \mathfrak{L}[\tau]$ there is a τ-sentence $\bigvee_{i\in I} \psi_i \in \bigvee^{\kappa} \mathfrak{L}[\tau]$ such that for any continuous τ-structure $\mathcal{M}$

$$\mathcal{M} \models \bigvee_{i\in I} \psi_i \text{ if and only if } (\exists i \in I)\ \mathcal{M} \models \psi_i.$$

Note that the following two lemmas are immediate from the definitions.

Lemma 2.22. *If $\mathfrak{L}$ has the downward Löwenheim-Skolem property to λ or the upward Löwenheim-Skolem property to uncountability then so does $\bigvee^{\kappa}(\mathfrak{L})$.*

Lemma 2.23. *If $\mathfrak{L}$ has occurrence number κ^+ so does $\bigvee^{\kappa} \mathfrak{L}$.*

Lemma 2.24. *$\bigvee^{\omega} Th^{\omega}(\overline{\mathcal{L}}^{c}_{\omega,\omega})$ is closed under $\mathcal{L}^{c}_{\omega,\omega}$-approximate isomorphisms.*

Proof. Suppose τ is countable and $\mathcal{M}$ and $\mathcal{N}$ are τ-structures which are $\mathcal{L}^{c}_{\omega,\omega}$-approximately isomorphic. It suffices to show that $Th_{\overline{\mathcal{L}}^{c}_{\omega,\omega}}(\mathcal{M}) = Th_{\overline{\mathcal{L}}^{c}_{\omega,\omega}}(\mathcal{N})$. But for any formula $\varphi \in \overline{\mathcal{L}}^{c}_{\omega,\omega}[\tau]$ with no free variables and any ϵ we must have $|\varphi^{\mathcal{M}} - \varphi^{\mathcal{N}}| \leq \epsilon$ as there is an $(\epsilon, \mathcal{L}^{c}_{\omega,\omega})$-approximate isomorphism from $\mathcal{M}$ to $\mathcal{N}$. But this implies $\varphi^{\mathcal{M}} = \varphi^{\mathcal{N}}$ and so $Th_{\overline{\mathcal{L}}^{c}_{\omega,\omega}}(\mathcal{M}) = Th_{\overline{\mathcal{L}}^{c}_{\omega,\omega}}(\mathcal{N})$. □

Putting this together we have the following

Proposition 2.25. *The logic $\bigvee^{\omega} Th^{\omega}(\overline{\mathcal{L}}^{c}_{\omega,\omega})$ has the following properties.*

(a) It has occurrence number ω_1.

(b) It has the upward Löwenheim-Skolem property to uncountability and the downward Löwenheim-Skolem property to ω.

(c) It is the minimal completely Boolean logic containing $\overline{\mathcal{L}}^{c}_{\omega,\omega}$.

(d) It is closed under $\mathcal{L}^c_{\omega,\omega}$-approximate isomorphisms.

Proof. Condition (a) follows from Lemma 2.19 and Lemma 2.23. Condition (b) follows from Lemma 2.20 and Lemma 2.22. Condition (d) follows from Lemma 2.24.

It is immediate that $\bigvee^\omega Th^\omega(\overline{\mathcal{L}}^c_{\omega,\omega})$ is contained in any logic that is completely Boolean and contains $\overline{\mathcal{L}}^c_{\omega,\omega}$. However, because $Th_{\overline{\mathcal{L}}^c_{\omega,\omega}}(\mathcal{M})$ is a maximal theory for each $\mathcal{M}$ we know that for any sentence $\varphi \in \overline{\mathcal{L}}^c_{\omega,\omega}[\tau]$ we have

$$[\varphi]_{\overline{\mathcal{L}}^c_{\omega,\omega}} = \bigcup\{[Th_{\overline{\mathcal{L}}^c_{\omega,\omega}}(\mathcal{M})]_{Th^\omega(\overline{\mathcal{L}}^c_{\omega,\omega})} \colon \mathcal{M} \models \varphi\}.$$

Therefore every element of $\overline{\mathcal{L}}^c_{\omega,\omega}[\tau]$ is a disjunction of maximal theories and hence in $\bigvee^\omega Th^\omega(\overline{\mathcal{L}}^c_{\omega,\omega})$. Further, for every countable τ we have $\{[\eta] \in \bigvee^\omega Th^\omega(\overline{\mathcal{L}}^c_{\omega,\omega})[\tau]\}$ forms a complete Boolean algebra whose atoms are $\{[T] \colon T \in Th^\omega(\overline{\mathcal{L}}^c_{\omega,\omega})[\tau]\}$. Hence $\bigvee^\omega Th^\omega(\overline{\mathcal{L}}^c_{\omega,\omega})$ is completely Boolean and (c) holds.

□

3. The Classical Case

Before we move on to the proof of our main theorem it will be useful to first consider the analogous result for classical structures, as the proof in the continuous case will be essentially identical (modulo the different notion of back and forth system). For the rest of this section (and this section only) we will assume all structures and languages are classical (i.e. not continuous). In particular $\mathcal{L}_{\omega,\omega}$ will denote ordinary first order logic.

The following result is due to Lindström (see [11, Thm. 3]) and gives a characterization of first order logic in terms of the downward Löwenheim-Skolem property to ω and the upward Löwenheim-Skolem property to uncountability.

Theorem 3.1 (Lindström). *Suppose $\mathfrak{L}$ is an abstract classical logic such that*

(a) $\mathcal{L}_{\omega,\omega} \leq \mathfrak{L}$.

(b) $\mathfrak{L}$ has the downward Löwenheim-Skolem property to ω as well as the upward Löwenheim-Skolem property to uncountability, where these are the obvious classical analogs.

(c) $\mathfrak{L}$ has occurrence number ω_1.

(d) $\mathfrak{L}$ is Boolean.

(e) $\mathfrak{L}$ is closed under existential quantification.

Then $\mathfrak{L} = \mathcal{L}_{\omega,\omega}$.

Now notice that if T is a maximal first order theory in a countable language then $\mathcal{L}_{\omega,\omega} \cup \{T\}$ also satisfies conditions (a), (b) and (c). Further, note that both the downward Löwenheim-Skolem property to ω and the upward Löwenheim-Skolem property to uncountability are closed under arbitrary Boolean operations on first order theories. Therefore, if we let A_T be the smallest Boolean logic containing $\mathcal{L}_{\omega,\omega} \cup \{T\}$ we have that A_T satisfies (a), (b), (c) and (d). In particular, if we remove condition (e), i.e. the requirement that our logic be closed under existential quantification, then $\mathcal{L}_{\omega,\omega}$ is no longer maximal. We will show that in the situation where we have a logic which contains all maximal first order theories in countable languages as sentences, if the logic also satisfies (a), (b), (c) and (d) then every sentence is equivalent to a disjunction of countable complete first order theories.

Definition 3.2. Let $Th^\omega(\mathcal{L}_{\omega,\omega})$ be the logic such that

- If τ is countable then $Th^\omega(\mathcal{L}_{\omega,\omega})[\tau] = \{Th_{\mathcal{L}_{\omega,\omega}}(\mathcal{M})\colon \mathcal{M} \text{ is a } \tau\text{-structure}\}$ where $Th_{\mathcal{L}_{\omega,\omega}}(\mathcal{M})$ is the complete theory of $\mathcal{M}$.
- If τ is uncountable then $Th^\omega(\mathcal{L}_{\omega,\omega})[\tau] = \bigcup_{\substack{\tau_0 \subseteq \tau \\ |\tau_0| \le \omega}} Th^\omega(\mathcal{L}_{\omega,\omega})[\tau_0]$.

Definition 3.3. Let $\bigvee^\omega Th^\omega(\mathcal{L}_{\omega,\omega})$ be the smallest logic such that

- $\bigvee^\omega Th^\omega(\mathcal{L}_{\omega,\omega})$ contains $\mathcal{L}_{\omega,\omega}$.
- If τ is countable then $\bigvee^\omega Th^\omega(\mathcal{L}_{\omega,\omega})[\tau]$ contains $Th^\omega(\mathcal{L}_{\omega,\omega})[\tau]$ and is closed under arbitrary disjunctions.
- If τ is uncountable then $\bigvee^\omega Th^\omega(\mathcal{L}_{\omega,\omega})[\tau] = \bigcup_{\substack{\tau_0 \subseteq \tau \\ |\tau_0| \le \omega}} \bigvee^\omega Th^\omega(\mathcal{L}_{\omega,\omega})[\tau_0]$.

So $\bigvee^\omega Th^\omega(\mathcal{L}_{\omega,\omega})$ is the classical analog of $\bigvee^\omega Th^\omega(\overline{\mathcal{L}}^c_{\omega,\omega})$. We now recall some standard definitions which will be important that we will need for the classical analog of our main theorem.

Definition 3.4. Suppose $\mathcal{M}, \mathcal{N}$ are τ-structures. A **partial isomorphism** is an isomorphism $p\colon A \to B$ where A is a finite substructure of $\mathcal{M}$ and B is a finite substructure of $\mathcal{N}$.

By a **back and forth system** of length n between $\mathcal{M}$ and $\mathcal{N}$ we mean a sequence $\langle I_i \rangle_{i \le n}$ such that

- for all $i < n$, $\{\emptyset\} \subseteq I_{i+1} \subseteq I_i$,
- $\bigcup_{i \le n} I_i$ consists of partial isomorphisms from $\mathcal{M}$ to $\mathcal{N}$, and
- for $i < n$, if $p \in I_{i+1}$ then

– for all $a \in \mathcal{M}$ there is a $b \in \mathcal{N}$ such that $p \cup \{(a, b)\} \in I_i$, and
– for all $b \in \mathcal{N}$ there is a $a \in \mathcal{M}$ such that $p \cup \{(a, b)\} \in I_i$.

We say a set I is a **potential isomorphism** between $\mathcal{M}$ and $\mathcal{N}$ if

- I is non-empty and consists of partial isomorphisms from $\mathcal{M}$ to $\mathcal{N}$,
- for all $p \in I$ and $a \in \mathcal{M}$ there is a $b \in \mathcal{N}$ such that $p \cup \{(a, b)\} \in I$, and
- for all $p \in I$ and $b \in \mathcal{N}$ there is a $a \in \mathcal{M}$ such that $p \cup \{(a, b)\} \in I$.

For more on the notion of back and forth systems and potential isomorphisms see [3, Ch. VII.5-7]. Although they use the term *partially isomorphic* for potential isomorphism. The following lemmas are standard.

Lemma 3.5. *The following are equivalent for τ-structures $\mathcal{M}$ and $\mathcal{N}$.*

- *$\mathcal{M}$ and $\mathcal{N}$ satisfy the same first order theory.*
- *For all finite $\tau_0 \subseteq \tau$ and all $n \in \omega$ there is a back and forth system of length n between $\mathcal{M}|_{\tau_0}$ and $\mathcal{N}|_{\tau_0}$.*

Lemma 3.6. *If $\mathcal{M}$ and $\mathcal{N}$ are countable τ-structures and there is a potential isomorphism from $\mathcal{M}$ to $\mathcal{N}$ then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.*

We now use Lemma 3.5 and Lemma 3.6 to prove the following classical analog of Theorem 5.1.

Theorem 3.7. *Suppose $\mathfrak{L}$ is an abstract classical logic such that*

(a) $Th^\omega(\mathcal{L}_{\omega,\omega}) \leq \mathfrak{L}$,

(b) *$\mathfrak{L}$ has the downward Löwenheim-Skolem property to ω as well as the upward Löwenheim-Skolem property to uncountability,*

(c) *$\mathfrak{L}$ has occurrence number ω_1, and*

(d) *$\mathfrak{L}$ is Boolean.*

Then $\mathfrak{L} \leq \bigvee^\omega Th^\omega(\mathcal{L}_{\omega,\omega})$.

Proof. Suppose to get a contradiction that $\mathfrak{L} \not\leq \bigvee^\omega Th^\omega(\mathcal{L}_{\omega,\omega})$. Then there must be a sentence $\varphi \in \mathfrak{L}[\tau]$ which is not equivalent to any sentence in $\bigvee^\omega Th^\omega(\mathcal{L}_{\omega,\omega})$. Further, as $\mathfrak{L}$ has occurrence number ω_1 we can assume without loss of generality that τ is countable. Let $\langle \tau_i \rangle_{i\in\omega}$ be an increasing enumeration of the finite sublanguages of τ such that $\bigcup_{i\in\omega} \tau_i = \tau$.

Claim 3.8. *There are τ-structures $\mathcal{M}$ and $\mathcal{N}$ such that $Th_{\mathcal{L}_{\omega,\omega}}(\mathcal{M}) = Th_{\mathcal{L}_{\omega,\omega}}(\mathcal{N})$, $\mathcal{M} \models \varphi$ and $\mathcal{N} \models \neg\varphi$.*

Proof. Suppose not. Then for every complete τ-theory T we have either all $\mathcal{M}$ that satisfy T also satisfy φ or all $\mathcal{M}$ which satisfy T do not satisfy φ. Let T_φ be the collection of theories such that all models of a theory in T_φ satisfy φ. Then $\{\mathcal{M} : \mathcal{M} \models \varphi\} = \{\mathcal{M} : \mathcal{M} \models \bigvee T_\varphi\}$ and so φ is equivalent to $\bigvee T_\varphi \in \bigvee^\omega Th^\omega(\mathcal{L}_{\omega,\omega})[\tau]$ contradicting our assumption on φ. $\square$

Let $\mathcal{M}$ and $\mathcal{N}$ be as in Claim 3.8 with $T = Th_{\mathcal{L}_{\omega,\omega}}(\mathcal{M}) = Th_{\mathcal{L}_{\omega,\omega}}(\mathcal{N})$. As $\mathfrak{L}$ is Boolean and $T \in \mathfrak{L}[\tau]$ we have $T \wedge \{\varphi\} \in \mathfrak{L}[\tau]$ and $T \wedge \{\neg\varphi\} \in \mathfrak{L}[\tau]$ and so by applying the downward Löwenheim-Skolem property to ω we can assume without loss of generality that $\mathcal{M}$ and $\mathcal{N}$ are countable.

Now let τ^* be the language which consists of the following:

(a) Two copies of τ, denoted τ^+, τ^- (with isomorphisms of languages $i_+ : \tau \to \tau^+$ and $i_- : \tau \to \tau^-$ witnessing they are copies of τ).

(b) A new sort N, with a relation S of arity $N \times N$, which is intended to represent the natural numbers with S the successor relation.

(c) For every sort X in τ^* a function $f_X : N \to X$.

(d) For every finite sequence of sorts X in τ, a new relation I_X of arity $X^+ \times X^- \times N \times N$, which is intended to represent a collection of partial isomorphisms whose domain / codomain have arity X.

Let K^* be the τ^*-structure where

(A) The reduct of K^* to τ^+ is isomorphic to $i_+(\mathcal{M})$.

(B) The reduct of K^* to τ^- is isomorphic to $i_-(\mathcal{N})$.

(C) (N, S) in K^* is isomorphic to $(\mathbb{N}, Suc)$.

(D) f_X in K^* is a surjection for every sort X.

(E) $I_X(\mathbf{a}, \mathbf{b}, m, n)$ holds in K^* if and only if there is a back and forth system $\langle I^n_i \rangle_{i \leq m}$ between $\mathcal{M}|_{\tau_n}$ and $\mathcal{N}|_{\tau_n}$ with $(\mathbf{a}, \mathbf{b}) \in I^n_m$.

Let $T^* := Th_{\mathcal{L}_{\omega,\omega}}(K^*)$.

Then by Lemma 3.5 T^* implies the following:

(1) The restriction to τ^+ satisfies $i_+(T)$ and the restriction to τ^- satisfies $i_-(T)$.

(2) Each f_X is surjective.

(3) S is the graph of an injective function on N such that there is a unique element which is not in its image (which we think of as 0).

(4) If $I_X(\mathbf{a}, \mathbf{b}, m, n)$ holds then

- if m is a k-fold successor and $n^* \leq n$ then $\mathbf{a}$ and $\mathbf{b}$ satisfy all the same quantifier-free formulas in τ_{n^*}, i.e. the map p with $p(\mathbf{a}) = \mathbf{b}$ is a partial isomorphism from the $i_+(\tau_{n^*})$-structure to the $i_-(\tau_{n^*})$-structure, and
- if m is the successor of m^* and Y is a sort in τ then for all n
  - $(\forall y^+ : Y^+)(\exists y^- : Y^-)\ I_{X\times Y}(\mathbf{a}y^+, \mathbf{b}y^-, m^*, n)$, and
  - $(\forall y^- : Y^-)(\exists y^+ : Y^+)\ I_{X\times Y}(\mathbf{a}y^+, \mathbf{b}y^-, m^*, n)$.

(5) $(\forall m, n : N)(\forall \mathbf{a} : X^+)(\exists \mathbf{b} : X^-)\ I_X(\mathbf{a}, \mathbf{b}, m, n)$.

(6) $(\forall m, n : N)(\forall \mathbf{b} : X^-)(\exists \mathbf{a} : X^+)\ I_X(\mathbf{a}, \mathbf{b}, m, n)$.

In other words T^* says that for every m and n there is a back and forth system of length m between the τ_n^+-reduct and the τ_n^--reduct along with surjections from N to every sort. Note that $T^* \in \bigvee^\omega Th^\omega(\mathcal{L}_{\omega,\omega})[\tau^*]$ and so, as $\mathfrak{L}$ is Boolean, we have $T^* \wedge i_+(\varphi) \wedge i_-(\neg\varphi) \in \mathfrak{L}[\tau^*]$ and $K^* \models T^* \wedge i_+(\varphi) \wedge i_-(\neg\varphi)$. Note we are using the fact that $\mathcal{M}$ and $\mathcal{N}$ are countable to get the surjection from N in K^* to each sort of K^*.

Now by the upward Löwenheim-Skolem property to uncountability there is a model K^{**} of T^* which is uncountable. As K^{**} is uncountable, in K^{**} the sort N must be uncountable. Therefore, there must be some a in K^{**} of arity N which is a k-fold successor for every $k \in \omega$, i.e. a non-standard natural number. Let $\tau^{**} := \tau^* \cup \{c\}$ where c is a constant of arity N. Let T^{**} be the τ^{**}-theory which contains T^* and says c is a k-fold successor (for every k). Note as being a k-fold successor can be represented by a single formula of $\mathcal{L}_{\omega,\omega}[\tau^{**}]$, T^{**} is a countable theory in $\mathfrak{L}[\tau^{**}]$. Further, interpreting c by a in K^{**} shows that T^{**} has a model.

But if T^{**} has a model it must have a countable model K° as $\mathfrak{L}$ has the downward Löwenheim-Skolem property to ω. Let Q be the smallest subset of N in K° which contains c and is such that whenever $K^\circ \models S(x, y)$ then $\{x, y\} \cap Q \neq \emptyset$ implies $\{x, y\} \subseteq Q$ (i.e. Q is the smallest $(\mathbb{Z}, Suc)$ copy containing c). Let

$$I := \{p\colon (\exists q \in Q)\ K^\circ \models I_X(\mathrm{dom}(p), \mathrm{range}(p), q, q), \text{ where } X \text{ is the arity of } \mathrm{dom}(p)\}.$$

Note that if $q \in Q$, $p \in I$ with the arity of $\mathrm{dom}(p)$ equal to X, Y is any sort of τ, and $K^\circ \models I_X(\mathrm{dom}(p), \mathrm{range}(p), q, q)$ then by (4) we have

- $K^\circ \models (\forall a\colon Y^-)(\exists b\colon Y^+)\, I_{X\times Y}(\mathbf{a}a, \mathbf{b}b, q-1, q-1)$
- $K^\circ \models (\forall b\colon Y^+)(\exists a\colon Y^-)\, I_{X\times Y}(\mathbf{a}a, \mathbf{b}b, q-1, q-1)$

Therefore I is a potential isomorphism between $\mathcal{M}^\circ := i_+^{-1}(K^\circ|_{\tau^+})$ and $\mathcal{N}^\circ := i_-^{-1}(K^\circ|_{\tau^-})$. But as K° is countable so are $\mathcal{M}^\circ$ and $\mathcal{N}^\circ$. Therefore by Lemma 3.6 we must have $\mathcal{M}^\circ$ and $\mathcal{N}^\circ$ are isomorphic, contradicting the fact that $\mathcal{M}^\circ \models \varphi$ and $\mathcal{N}^\circ \models \neg\varphi$.

□

The following theorem is an immediate consequence of Theorem 3.7.

Theorem 3.9. $\bigvee^\omega Th^\omega(\mathcal{L}_{\omega,\omega})$ *is the unique maximal logic which*

(a) is stronger than $\mathcal{L}_{\omega,\omega}$*,*

(b) has the downward Löwenheim-Skolem property to ω *as well as the upward Löwenheim-Skolem property to uncountability,*

(c) has occurrence number ω_1*,*

(d) is completely Boolean.

4. Metric Scott Analysis

Classically for any two τ-structures $\mathcal{M}$ and $\mathcal{N}$ satisfying the same theory, any finite $\tau_0 \subseteq \tau$, and for any $n \in \mathbb{N}$ there is a maximal back and forth system $(I_i)_{i\leq n}$ from $\mathcal{M}|_{\tau_0}$ to $\mathcal{N}|_{\tau_0}$ where $(\mathbf{a}, \mathbf{b}) \in I_n$ if and only if $\mathbf{a}$ and $\mathbf{b}$ satisfy the same formulas of $\mathcal{L}_{\omega,\omega}$ of quantifier rank n. This association allows one to study in fine-grained detail the quantifier depth of formulas on which two tuples agree. This technique surrounding back and forth systems is often called the *Scott analysis* of a pair of structures, the study of Ehrenfeucht–Fraïssé games.

When we move to the continuous setting the analog of this back and forth system will be a sequence of functions which we now describe. Note these functions are very similar to the $r_n^{A,B}$ functions of [4], where a continuous analog of the Scott analysis is developed, except we allow ourselves to start with an arbitrary collection of restricted quantifier-free formulas in the definition.

Definition 4.1. Suppose S is a finite collection of restricted quantifier-free τ-formulas and $\mathcal{M}, \mathcal{N}$ are τ-structures. For $n \in \mathbb{N}$ we define $r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}, \mathbf{b})$ for $\mathbf{a} \in \mathcal{M}$ and $\mathbf{b} \in \mathcal{N}$ (of the same arity) by induction on n as follows.

$$r_{0,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}, \mathbf{b}) = \sup_{\varphi\in S} |\varphi^{\mathcal{M}}(\mathbf{a}) - \varphi^{\mathcal{N}}(\mathbf{b})|$$

where the supremum is over formulas in S with arities compatible with $\mathbf{a}$ and $\mathbf{b}$.

We define

$$r_{n+1,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}, \mathbf{b}) = \sup_{c\in\mathcal{M}, d\in\mathcal{N}} \inf_{c'\in\mathcal{M}, d'\in\mathcal{N}} r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}c, \mathbf{b}d') \vee r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}c', \mathbf{b}d).$$

Note as there are only finitely many formulas in S that $r_{0,S}^{\mathcal{M},\mathcal{N}}$ is definable by a formula in $\mathcal{L}_{\omega,\omega}^c$. Therefore for all n, $r_{n,S}^{\mathcal{M},\mathcal{N}}$ is definable by a formula in $\mathcal{L}_{\omega,\omega}^c$ as well.

Lemma 4.2. *Suppose S is a finite collection of restricted τ-formulas.*

(a) If $S \subseteq S^*$ where S^* is a finite collection of restricted formulas then for all $\mathbf{a} \in \mathcal{M}, \mathbf{b} \in \mathcal{N}$ and $n \in \omega$,

$$r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}, \mathbf{b}) \leq r_{n,S^*}^{\mathcal{M},\mathcal{N}}(\mathbf{a}, \mathbf{b}).$$

(b) If $n \leq n^*$ then for all $\mathbf{a} \in \mathcal{M}, \mathbf{b} \in \mathcal{N}$ and all collections S of restricted formulas,

$$r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}, \mathbf{b}) \leq r_{n^*,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}, \mathbf{b}).$$

(c) Suppose τ^* consists of two copies of τ, denoted τ^+, τ^- (with isomorphisms of languages $i_+ \colon \tau \to \tau^+$ and $i_- \colon \tau \to \tau^-$ witnessing they are copies of τ). Then there is a formula $r_{n,S}^* \in \mathcal{L}_{\omega,\omega}^c[\tau^*]$ such that for any τ^*-structure K and any $\mathbf{a} \in K|_{\tau^+}$ and $\mathbf{b} \in K|_{\tau^-}$

$$(r_{n,S}^*)^K(\mathbf{a}, \mathbf{b}) = r_{n,S}^{\mathcal{M}^\circ,\mathcal{N}^\circ}(\mathbf{a}, \mathbf{b})$$

where $\mathcal{M}^\circ = i_+^{-1}(K|_{\tau^+})$ and $\mathcal{N}^\circ = i_-^{-1}(K|_{\tau^-})$.

Proof. Condition (a) is immediate from the definition of $r_{n+1,S}^{\mathcal{M},\mathcal{N}}$. (c) follows immediately from the fact that S is finite and so there is a formula describing $r_{0,S}^{\mathcal{M},\mathcal{N}}$.

We now show condition (b). First note that if $\mathbf{a} \in \mathcal{M}$, $a_0 \in \mathbf{a}$, and $\mathbf{b}, d \in \mathcal{N}$ then $r_{0,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}a_0, \mathbf{b}d) \geq r_{0,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}, \mathbf{b})$. Therefore, by a straightforward induction we have for all $n \in \omega$,

$$r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}a_0, \mathbf{b}d) \geq r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}, \mathbf{b}).$$

Also note that if $a_0 \in \mathbf{a}$ and $b_0 \in \mathbf{b}$ then $r_{0,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}a_0, \mathbf{b}b_0) = r_{0,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}, \mathbf{b})$ and so, by induction,

$$r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}a_0, \mathbf{b}b_0) = r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}, \mathbf{b}).$$

Putting these two equations together, and using the symmetry between $\mathcal{M}$ and $\mathcal{N}$ in the definition of $r_{n,S}^{\mathcal{M},\mathcal{N}}$ we have whenever $a_0 \in \mathbf{a}$ and $b_0 \in \mathbf{b}$ that

$$\inf_{c' \in \mathcal{M}, d' \in \mathcal{N}} r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}a_0, \mathbf{b}d') \vee r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}c', \mathbf{b}b_0) = r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}, \mathbf{b}).$$

But this then implies

$$\sup_{c \in \mathcal{M}, d \in \mathcal{N}} \inf_{c' \in \mathcal{M}, d' \in \mathcal{N}} r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}c, \mathbf{b}d') \vee r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}c', \mathbf{b}d) \geq r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}, \mathbf{b}),$$

and we are done. $\square$

Suppose S^ω is the collection of all quantifier-free restricted τ-formulas and S^ω_n is then the collection of all formulas of quantifier rank at most n. We have the following lemma which is [4, Thm. 3.5].

Lemma 4.3. *For $\mathbf{a} \in \mathcal{M}$ and $\mathbf{b} \in \mathcal{N}$ and $n \in \omega$,*

$$r^{\mathcal{M},\mathcal{N}}_{n,S^\omega}(\mathbf{a},\mathbf{b}) = \sup_{\varphi \in S^\omega_n} |\varphi^{\mathcal{M}}(\mathbf{a}) - \varphi^{\mathcal{N}}(\mathbf{b})|.$$

In particular we have the following consequence of Lemma 4.3.

Lemma 4.4. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are continuous τ-structures which are approximately isomorphic. Then $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}^c_{\omega,\omega}[\tau]$-approximately isomorphic.*

Proof. Suppose $\epsilon > 0$, $A_\epsilon \subseteq \mathcal{M}$ and $B \subseteq \mathcal{N}$ are dense and F_{ϵ,S^ω} satisfies the conditions of Definition 2.7. We therefore have whenever $F_{\epsilon,S^\omega}(\mathbf{a},\mathbf{b})$ holds and φ is quantifier-free that $|\varphi^{\mathcal{M}}(\mathbf{a}) - \varphi^{\mathcal{N}}(\mathbf{b})| \leq \epsilon$. In particular this implies that $r^{\mathcal{M},\mathcal{N}}_{0,S^\omega}(\mathbf{a},\mathbf{b}) \leq \epsilon$.

Now assume that $n \in \omega$ and for all $\mathbf{a} \in A$, $\mathbf{b} \in B$, $F_{\epsilon,S^\omega}(\mathbf{a},\mathbf{b})$ implies $r^{\mathcal{M},\mathcal{N}}_{n,S^\omega}(\mathbf{a},\mathbf{b}) \leq \epsilon$. Then, as A and B are dense in $\mathcal{M}$ and $\mathcal{N}$ we have for all $\mathbf{a} \in A$, $\mathbf{b} \in B$ with $F_{\epsilon,S^\omega}(\mathbf{a},\mathbf{b})$,

$$\begin{aligned} r^{\mathcal{M},\mathcal{N}}_{n+1,S^\omega} &= \sup_{c \in \mathcal{M}, d \in \mathcal{N}} \inf_{c' \in \mathcal{M}, d' \in \mathcal{N}} r^{\mathcal{M},\mathcal{N}}_{n,S^\omega}(\mathbf{a}c, \mathbf{b}d') \vee r^{\mathcal{M},\mathcal{N}}_{n,S}(\mathbf{a}c', \mathbf{b}d) \\ &= \sup_{c \in A, d \in B} \inf_{c' \in A, d' \in B} r^{\mathcal{M},\mathcal{N}}_{n,S^\omega}(\mathbf{a}c, \mathbf{b}d') \vee r^{\mathcal{M},\mathcal{N}}_{n,S}(\mathbf{a}c', \mathbf{b}d) \leq \epsilon. \end{aligned}$$

Therefore, by Lemma 4.3 we have that $|\varphi^{\mathcal{M}}(\mathbf{a}) - \varphi^{\mathcal{N}}(\mathbf{b})| \leq \epsilon$ for all formulas of S^ω_{n+1}. So, byLemma 1.7 and by induction $(F_{\epsilon,S^\omega})_{\epsilon>0}$ witnesses that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}^c_{\omega,\omega}$-approximately isomorphic. □

The following corollary is then immediate.

Corollary 4.5. $\overline{\mathcal{L}}^c_{\omega,\omega}$ *is closed under approximate isomorphisms at ω.*

We would like an analog of Lemma 4.3 to hold where S^ω is replaced by an arbitrary finite collection of restricted formulas. Unfortunately generalizing the proof of Lemma 4.3 runs into a fundamental issue when S is not closed under all translations of formulas by a fixed constant (i.e. when we don't necessarily have $\varphi + c \in S$ whenever $\varphi \in S$). However, if we are closed under sufficiently small translations we can get a bound on the difference between $r^{\mathcal{M},\mathcal{N}}_{n,S}(\mathbf{a},\mathbf{b})$ and $\sup_{\varphi \in S_n} |\varphi^{\mathcal{M}}(\mathbf{a}) - \varphi^{\mathcal{N}}(\mathbf{b})|$.

Proposition 4.6. *Suppose S is a finite collection of restricted quantifier-free τ-formulas closed under subformulas and finite Boolean operations and which has magnitude ϵ-translations. Let $\mathcal{M}, \mathcal{N}$ be τ-structures. Then for $\mathbf{a} \in \mathcal{M}$ and $\mathbf{b} \in \mathcal{N}$ we have the following.*

$$\left| r^{\mathcal{M},\mathcal{N}}_{n,S}(\mathbf{a},\mathbf{b}) - \sup_{\varphi \in Q_\epsilon(S,n)} |\varphi^{\mathcal{M}}(\mathbf{a}) - \varphi^{\mathcal{N}}(\mathbf{b})| \right| \leq n \cdot \epsilon$$

where $Q_\epsilon(S,n)$ is as in Definition 1.11.

Proof. If $n=0$ this follows from the definition of $r_{0,S}^{\mathcal{M},\mathcal{N}}$. Assume as the inductive hypothesis that the lemma is true for n. Fix $t \in [0,1]$ and assume $r_{n+1,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a},\mathbf{b}) > t$ to show $\sup_{\varphi\in Q_\epsilon(S,n)} |\varphi^{\mathcal{M}}(\mathbf{a}) - \varphi^{\mathcal{N}}(\mathbf{b})| > t-(n+1)\cdot\epsilon$.

By the definition of $r_{n+1,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a},\mathbf{b})$ either there is a $c\in\mathcal{M}$ such that for all $d\in\mathcal{N}$, $r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}c,\mathbf{b}d) > t$ or there is a $d\in\mathcal{N}$ such that for all $c\in\mathcal{M}$, $r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}c,\mathbf{b}d) > t$. We assume the former with the case of the latter being identical.

By the inductive hypothesis for each $d\in\mathcal{N}$ there is a $\varphi_d \in Q_\epsilon(S,n)$ such that $|\varphi_d^{\mathcal{M}}(\mathbf{a}c) - \varphi_d^{\mathcal{N}}(\mathbf{b}d)| > t-n\cdot\epsilon$. But, because $Q_\epsilon(S,n)$ has magnitude ϵ-translations, we can choose φ_d such that $\varphi_d^{\mathcal{M}}(\mathbf{a}c) > t-n\cdot\epsilon \geq \epsilon \geq \varphi_d^{\mathcal{N}}(\mathbf{b},d) \geq 0$.

Note that there are at most finitely many non-equivalent φ_d as each $\varphi_d \in Q_\epsilon(S,n)$ and $Q_\epsilon(S,n)$ is finite. Hence if we let $\varphi^* := \bigwedge_{d\in\mathcal{N}} \varphi_d$ then $\varphi^* \in Q_\epsilon(S,n)$. Now let $\psi(\boldsymbol{x}) := \sup_y \varphi^*(\boldsymbol{x},y)$. Therefore $\psi^{\mathcal{M}}(\mathbf{a}) > t-n\cdot\epsilon \geq \epsilon \geq \psi^{\mathcal{N}}(\mathbf{b})$. But as $\psi \in (S,n+1)$ we have $\sup_{\varphi\in(S,n+1)} |\varphi^{\mathcal{M}}(\mathbf{a}) - \varphi^{\mathcal{N}}(\mathbf{b})| > t-(n+1)\cdot\epsilon$.

Now assume $\sup_{\varphi\in Q_\epsilon(S,n+1)} |\varphi^{\mathcal{M}}(\mathbf{a}) - \varphi^{\mathcal{N}}(\mathbf{b})| > t$. Then there must be some formula $\varphi \in Q_\epsilon(S,n+1)$ such that $|\varphi^{\mathcal{M}}(\mathbf{a}) - \varphi^{\mathcal{N}}(\mathbf{b})| > t$. If φ is of the form $\frac{\psi}{2}$ then we replace φ by ψ. If φ is of the form $\psi_0 \dot{-} \psi_1$ then $|\psi_i^{\mathcal{M}}(\mathbf{a}) - \psi_i^{\mathcal{N}}(\mathbf{b})| > t$ for some $i\in\{0,1\}$. We then replace φ by ψ_i. By repeatedly doing this we can find a φ such that either $\varphi = \inf_x \psi$ or $\varphi = \sup_x \psi$ (for some $\psi \in Q_\epsilon(S,n)$).

We will now consider the case when $\varphi = \sup_x \psi$, the case when $\varphi = \inf_x \psi$ being similar. As $Q_\epsilon(S,n)$ has magnitude ϵ-translations, by replacing ψ with an appropriate translation, we can assume that the following inequalities hold, $\varphi^{\mathcal{M}}(\mathbf{a}) > t \geq \epsilon \geq \varphi^{\mathcal{N}}(\mathbf{b}) \geq 0$. So there is a $c\in\mathcal{M}$ such that $\psi^{\mathcal{M}}(\mathbf{a}c) > t$ and for all $d\in\mathcal{N}$, $\psi^{\mathcal{N}}(\mathbf{b},d) \leq \epsilon$. We therefore have by the inductive hypothesis that for all $d\in\mathcal{N}$, $r_{n,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a}c,\mathbf{b}d) > t-n\cdot\epsilon-\epsilon$ and so $r_{n+1,S}^{\mathcal{M},\mathcal{N}}(\mathbf{a},\mathbf{b}) > t-(n+1)\cdot\epsilon$.

□

5. The Continuous Case

We now prove our main theorem, which is the continuous analog of Theorem 3.7.

Theorem 5.1. *Suppose $\mathfrak{L}$ is an abstract continuous logic such that*

(a) $Th^\omega(\overline{\mathcal{L}}^c_{\omega,\omega}) \leq \mathfrak{L}$.

(b) $\mathfrak{L}$ has the downward Löwenheim-Skolem property to ω as well as the upward Löwenheim-Skolem property to uncountabiliy.

(c) $\mathfrak{L}$ has occurrence number ω_1.

(d) $\mathfrak{L}$ is Boolean.

(e) $\mathfrak{L}$ *is closed under approximate isomorphisms at* ω.

Then $\mathfrak{L} \leq \bigvee^{\omega} Th^{\omega}(\overline{\mathcal{L}}^{c}_{\omega,\omega})$.

Proof. Suppose the contrary, namely that $\mathfrak{L} \not\leq \bigvee^{\omega} Th^{\omega}(\overline{\mathcal{L}}^{c}_{\omega,\omega})$. Then there must be a sentence $\varphi \in \mathfrak{L}[\tau]$ which is not equivalent to any sentence in $\bigvee^{\omega} Th^{\omega}(\overline{\mathcal{L}}^{c}_{\omega,\omega})$. Further, as $\mathfrak{L}$ has occurrence number ω_1 we can assume without loss of generality that τ is countable.

Claim 5.2. *There are* τ*-structures* $\mathcal{M}$ *and* $\mathcal{N}$ *such that* $Th_{\mathcal{L}^{c}_{\omega,\omega}}(\mathcal{M}) = Th_{\mathcal{L}^{c}_{\omega,\omega}}(\mathcal{N})$, $\mathcal{M} \models \varphi$ *and* $\mathcal{N} \models \neg\varphi$.

Proof. Suppose not. Then for every complete τ-theory T we have that either all $\mathcal{M}$ which satisfy T also satisfy φ or all $\mathcal{M}$ which satisfy T do not satisfy φ. Let T_φ be the collection of theories such that all models of a theory in T_φ satisfy φ. Then $\{\mathcal{M} : \mathcal{M} \models \varphi\} = \{\mathcal{M} : \mathcal{M} \models \bigvee T_\varphi\}$ and so φ is equivalent to $\bigvee T_\varphi$. But $\bigvee T_\varphi \in \bigvee^{\omega} Th^{\omega}(\overline{\mathcal{L}}^{c}_{\omega,\omega})[\tau]$, contradicting our assumption on φ. $\square$

Let $\mathcal{M}$ and $\mathcal{N}$ be as in Claim 5.2 with $T = Th_{\mathcal{L}^{c}_{\omega,\omega}}(\mathcal{M}) = Th_{\mathcal{L}^{c}_{\omega,\omega}}(\mathcal{N})$. As $\mathfrak{L}$ is Boolean and $T \in \mathfrak{L}[\tau]$ we have $T \wedge \{\varphi\} \in \mathfrak{L}[\tau]$ and $T \wedge \{\neg\varphi\} \in \mathfrak{L}[\tau]$, and so by applying the downward Löwenheim-Skolem property to ω we can assume without loss of generality that $\mathcal{M}$ and $\mathcal{N}$ have countable density character.

Now let $\langle S_i \rangle_{i \in \omega}$ be an increasing sequence of finite sets of restricted τ-formulas such that

- All restricted τ-formulas are in $\bigcup_{i \in \omega} S_i$.
- For all $k \in \omega$, S_k has magnitude $\frac{1}{2^k}$-translations.

Now let τ^* be the language which consists of the following.

- Two copies of τ, denoted τ^+, τ^- (with isomorphisms of languages $i_+ : \tau \to \tau^+$ and $i_- : \tau \to \tau^-$ witnessing they are copies of τ).
- A new sort N, with a relation S which is intended to represent the natural numbers with S the successor relation along with a relation $\leq$ of arity $N \times N$ which is intended to represent a linear ordering.
- For every sort X in τ^* a function $f_X : N \to X$.
- For every finite sequence of sorts X in τ new relations I_X of arity $X^+ \times X^- \times N \times N$.

Let K^* be the τ^* structure such that

- $K^*|_{\tau^+} = i_+(\mathcal{M})$ and $K^*|_{\tau^-} = i_-(\mathcal{N})$,
- $(N, S) = (\mathbb{N}, Suc)$,

- $I_X(\mathbf{a}, \mathbf{b}, m, n) = r_{m,S_n}(\mathbf{a}, \mathbf{b})$,
- f_X is any map onto a dense subset of X in K^*. Note because the distances of X in K^* are bounded, any such map is uniformly continuous.

Let $T^* := Th_{\overline{\mathcal{L}}^c_{\omega,\omega}}(K^*)$. Note T^* satisfies all of the following.

- The restriction to τ^+ satisfies $i_+(T)$ and the restriction to τ^- satisfies $i_-(T)$.
- (N, d_N) is a discrete metric space.
- For each sort X, $\sup_{x \in X} \inf_{n \in N} d_X(x, f_X(n)) = 0$, i.e. the image of N under f_X is dense.
- S is the graph of an injective function on N such that there is a unique element which is not in its image (which we think of as 0). Note this can be expressed in continuous logic as (N, d_N) is discrete.
- If $n \in N$ is an n_0-fold successor and $m \in N$ is an m_0-fold successor then
$$I_X(\mathbf{a}, \mathbf{b}, m, n) \geq r^*_{m_0,S_{n_0}}(\mathbf{a}, \mathbf{b})$$
where $r^*_{m_0,S_{n_0}}(\boldsymbol{x}, \boldsymbol{y})$ is as the formula defining $r_{m_0,S_{n_0}}(\boldsymbol{x}, \boldsymbol{y})$. This follows from Lemma 4.2 (a) and (b). Note that because $r^*_{m_0,S_{n_0}}(\boldsymbol{x}, \boldsymbol{y})$ is a formula of $\mathcal{L}_{\omega,\omega}(\mathcal{L}^c_{\omega,\omega})$ then this inequality is expressible by a sentence of $\overline{\mathcal{L}}^c_{\omega,\omega}$.
- If $m \in N$ is the successor of $m^* \in N$, $n^* \leq n \in N$, and Y is a sort of τ then
$$I_X(\mathbf{a},\mathbf{b}, m, n) \geq \sup_{c \in Y^+, d \in Y^-} \inf_{c' \in Y^+, d' \in Y^-} I_{X \times Y}(\mathbf{a}c, \mathbf{b}d', m^*, n^*) \vee I_{X \times Y}(\mathbf{a}c', \mathbf{b}d, m^*, n^*).$$
- If $!$ is the empty product of sorts, n is a n_0-fold successor then $I_!(\emptyset, \emptyset, n, n) \leq \frac{n_0}{2^{n_0}}$. This follows from Proposition 4.6 and the fact that $Th_{\overline{\mathcal{L}}^c_{\omega,\omega}}(\mathcal{M}) = Th_{\overline{\mathcal{L}}^c_{\omega,\omega}}(\mathcal{N})$.

In other words T^* says that for $m, n \in \mathbb{N}$, $I_X(\mathbf{a}, \mathbf{b}, m, n)$ acts like $r_{m,S_n}(\mathbf{a}, \mathbf{b})$ (and in particular bounds it).

Note $T^* \in Th^\omega(\overline{\mathcal{L}}^c_{\omega,\omega})[\tau^*]$ by Lemma 2.17. So, as $\mathfrak{L}$ is Boolean, we have $T^* \wedge i_+(\varphi) \wedge i_-(\neg\varphi) \in \mathfrak{L}[\tau^*]$.

Therefore, because $\mathfrak{L}$ has the upward Löwenheim-Skolem property to uncountability there must be an uncountable model K^+ of $T^* \wedge i_+(\varphi) \wedge i_-(\neg\varphi)$. But then in K^+ we must have some sort with uncountable density and so, because the image of N under f_X is dense in each sort, we must have that N is uncountable. Hence there must be some a in K^+ of sort N which is a k-fold successor for every $k \in \omega$. Let $\tau^{**} := \tau^* \cup \{c\}$ where c is a constant of sort N. Let K^{**} be the τ^{**}-structure whose reduct to τ^* is K^+ and where $c^{K^{**}} = a$. Let T^{**} be the τ^{**}-theory of K^{**}. Note T^{**} says c is a k-fold successor (for every k) as being a k-fold successor can be represented by a single formula of $\mathcal{L}^c_{\omega,\omega}[\tau^{**}]$.

But as T^{**} has a model it must have a model K° with countable density character as $T^{**} \in Th^\omega(\overline{\mathcal{L}}^c_{\omega,\omega})[\tau]$ and $Th^\omega(\overline{\mathcal{L}}^c_{\omega,\omega})$ has the downward Löwenheim-Skolem property to ω. Let $\mathcal{M}^\circ := i_+^{-1}(K^\circ|_{\tau^+})$ and $\mathcal{N}^\circ := i_-^{-1}(K^\circ|_{\tau^+})$.

Claim 5.3. *$\mathcal{M}^\circ$ and $\mathcal{N}^\circ$ are approximately isomorphic.*

Proof. For notational convenience, for $k \in \omega$ let $c^\circ - k$ denote the unique element of arity N in K° whose k-fold successor is c.

Suppose $\epsilon > 0$. Let $\{a_i\}_{i\in\omega}$ be a dense subset of $\mathcal{M}^\circ$ and $\{b_i\}_{i\in\omega}$ a dense subset of $\mathcal{N}^\circ$. We now define two sequences $\langle e_j\rangle_{j\in\omega}$ of elements of $\mathcal{M}^\circ$ and $\langle d_j\rangle_{j\in\omega}$ of elements of $\mathcal{N}^\circ$ by induction. For notational convenience for $k \in \omega$ we will let $\mathbf{e}_k = (e_i)_{i<k}$ and $\mathbf{d}_k = (d_i)_{i<k}$.

Our inductive hypothesis will be that

$$I_X(\mathbf{e}_n, \mathbf{d}_n, c^\circ - n, c^\circ - n) \leq \sum_{0\leq j<n} 2^{-j-3}\epsilon < \frac{\epsilon}{4}.$$

<u>Stage 0:</u> T^{**} implies that $I_\emptyset(\emptyset, \emptyset, c^\circ, c^\circ) \leq \frac{k}{2^k}$ for all k such that c° is a k-successor. But as c° is non-standard we have $I_\emptyset(\emptyset, \emptyset, c^\circ, c^\circ) = 0$ and so the inductive hypothesis holds.

<u>Stage $2n$:</u> Let e_{2n} be the least element of $\langle a_i\rangle_{i\in\omega}$ which does not occur in $\mathbf{e}_{2n}$. Suppose e_{2n} is of arity Y_{2n}. We know that

$$\begin{aligned} I_X(\mathbf{e}_{2n}, &\mathbf{d}_{2n}, c^\circ - 2n, c^\circ - 2n) \geq \\ &\sup_{e\in Y^+_{2n}, d\in Y^-_{2n}} \inf_{e'\in Y^+_{2n}, d'\in Y^-_{2n}} \big[I_{X\times Y_{2n}}(\mathbf{e}_{2n}e, \mathbf{d}_{2n}d', c^\circ - (2n+1), c^\circ - (2n+1)) \\ &\qquad\qquad \vee \quad I_{X\times Y_{2n}}(\mathbf{e}_{2n}e', \mathbf{d}_{2n}d, c^\circ - (2n+1), c^\circ - (2n+1))\big]. \end{aligned}$$

Therefore we can find a d_{2n} such that

$$I_{X\times Y_{2n}}(\mathbf{e}_{2n}e_{2n}, \mathbf{d}_{2n}d_{2n}, c^\circ - (2n+1), c^\circ - (2n+1)) \dot{-} I_X(\mathbf{e}_{2n}, \mathbf{d}_{2n}, c^\circ - 2n, c^\circ - 2n) < 2^{-2n-1}\epsilon.$$

The inductive hypothesis therefore holds for this stage.

<u>Stage $2n+1$:</u> Let d_{2n+1} be the least element of $\langle b_i\rangle_{i\in\omega}$ which does not occur in $\mathbf{d}_{2n+1}$.

Suppose d_{2n+1} is of arity Y_{2n+1}. We know that

$$\begin{aligned} I_X(\mathbf{e}_{2n+1}, \mathbf{d}_{2n+1}), c^\circ - (2n+1), c^\circ - (2n+1)) \geq \\ \sup_{e \in Y^+_{2n+1}, d \in Y^-_{2n+1}} \inf_{e' \in Y^+_{2n+1}, d' \in Y^-_{2n+1}} \big[I_{X \times Y_{2n+1}}(\mathbf{e}_{2n+1}e, \mathbf{d}_{2n+1}d', c^\circ - (2n+2), c^\circ - (2n+2)) \\ \vee \quad I_{X \times Y_{2n+1}}(\mathbf{e}_{2n+1}e', \mathbf{d}_{2n+1}d, c^\circ - (2n+2), c^\circ - (2n+2)) \big]. \end{aligned}$$

Therefore we can find a e_{2n+2} such that

$$\begin{aligned} I_{X \times Y_{2n+1}}(\mathbf{e}_{2n+1}e_{2n+1}, \mathbf{d}_{2n+1}d_{2n+1}, c^\circ - (2n+2), c^\circ - (2n+2)) \\ \dot{-}\, I_X(\mathbf{e}_{2n+1}, \mathbf{d}_{2n+1}, c^\circ - (2n+1), c^\circ - (2n+1)) < 2^{-2n-1}\epsilon. \end{aligned}$$

The inductive hypothesis then holds for this stage as well.

Let $E = \{e_i\}_{i \in \omega}$ and $D := \{d_i\}_{i \in \omega}$. Note E is dense in $\mathcal{M}^\circ$ and D is dense in $\mathcal{N}^\circ$. Let $F_\epsilon(e, d)$ hold if and only if $e = e_i$ and $d = d_i$ for some $i \in \omega$. Let $\psi \in \mathcal{L}^c_{\omega,\omega}[\tau]$ and let ψ^* be a restricted formula such that for all continuous τ-structures $\mathcal{M}^*$, $\sup_{\mathbf{a} \in \mathcal{M}^*} |\psi(\mathbf{a}) - \psi^*(\mathbf{a})| \leq \frac{\epsilon}{4}$. Note that such a ψ exists by Lemma 1.7. Let ℓ be the smallest number such that $\psi^* \in Q_{2^{-\ell}}(S_\ell, \ell)$.

Note for any sequence $\boldsymbol{x} = (x_0, \ldots, x_{n-1}) \in E$ and $\boldsymbol{y} = (y_0, \ldots, y_{n-1}) \in D$ such that $\bigwedge_{i \leq n} F_\epsilon(x_i, y_i)$ there is some $k \in \omega$ such that

- $k > \ell$,
- $\frac{k}{2^k} < \frac{\epsilon}{4}$,
- $\{x_i\}_{i<n} \subseteq \{e_i\}_{i<k}$, and
- $\{y_i\}_{i<n} \subseteq \{d_i\}_{i<k}$.

In particular as $I_X(\mathbf{e}_k, \mathbf{d}_k, c^\circ - k, c^\circ - k) \leq \frac{\epsilon}{4}$ holds by the above argument, where X is the arity of $\mathbf{e}_k$, we have $r_{k,S_k}(\boldsymbol{x}, \boldsymbol{y}) \leq r_{k,S_k}(\mathbf{e}_k, \mathbf{d}_k) \leq \frac{\epsilon}{4}$.

Therefore, by Proposition 4.6 and the fact that $\psi^* \in Q_{2^{-\ell}}(S_\ell, \ell) \subseteq Q_{2^{-k}}(S_k, k)$, the latter of which has $\frac{1}{2^k}$ translations, we have

$$\begin{aligned} |\psi^{\mathcal{M}}(\boldsymbol{x}) - \psi^{\mathcal{N}}(\boldsymbol{y})| &\leq |(\psi^*)^{\mathcal{M}}(\boldsymbol{x}) - (\psi^*)^{\mathcal{N}}(\boldsymbol{y})| + 2 \cdot \frac{\epsilon}{4} \\ &\leq \sup_{\varphi \in Q_{2^{-k}}(S_k, k)} |\varphi^{\mathcal{M}}(\mathbf{e}_k) - \varphi^{\mathcal{N}}(\mathbf{d}_k)| + 2 \cdot \frac{\epsilon}{4} \\ &\leq r_{k,S_k}(\mathbf{e}_k, \mathbf{d}_k) + \frac{k}{2^k} + 2 \cdot \frac{\epsilon}{4} \\ &\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + 2 \cdot \frac{\epsilon}{4} = \epsilon \end{aligned}$$

Therefore F is an $(\epsilon, \mathcal{L}^c_{\omega,\omega})$-approximate isomorphism between $\mathcal{M}^\circ$ and $\mathcal{N}^\circ$. So, because ϵ was arbitrary we have that $\mathcal{M}^\circ$ and $\mathcal{N}^\circ$ are $\mathcal{L}^c_{\omega,\omega}$-approximately isomorphic and hence approximately isomorphic.

□

But by construction $\mathcal{M}^\circ \models \varphi$ and $\mathcal{N}^\circ \models \neg\varphi$ and by assumption $\mathfrak{L}$ is closed under approximate isomorphisms at ω, getting us our contradiction.

□

In particular the following is immediate.

Theorem 5.4. $\bigvee^\omega Th^\omega(\overline{\mathcal{L}}^c_{\omega,\omega})$ *is the unique maximal logic which*

(a) is stronger than $\overline{\mathcal{L}}^c_{\omega,\omega}$,

(b) has the downward Löwenheim-Skolem property to ω *as well as the upward Löwenheim-Skolem property to uncountability,*

(c) has occurrence number ω_1,

(d) is completely Boolean, and

(e) is closed under approximate isomorphisms at ω.

6. Open Questions

We now list some open questions.

- Does Theorem 5.1 hold if we remove the condition that our logic is closed under approximate isomorphisms?
- Does $\mathcal{L}^*_{\omega,\omega}$ have the downward Löwenheim-Skolem property to ω?

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