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Improvements of Approximating Functions Method for Solving Problems with a Dielectric Layer with Media of a High Degree of Nonlinearity

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Abstract

The article introduces analytical modifications of the approximating functions method, a special case of finite element method (FEM), which entail an increase in its computational performance for solving electrodynamics problems in one space dimension and time domain (1D+T) inside inhomogeneity with nonlinear media placed in the linear one, using the Volterra integral equation method, which is an integral equations equivalent to Maxwell's equations. The purpose of this study is not only to make the analytical improvements, but also to adapt them for fast and convenient programming and fast computations. The proposed method was validated on the problems of interaction of electromagnetic waves incident on a layer with second-order and third-order nonlinear medium inserted in linear media.

Keywords: Nonlinear dielectric layer, Approximating functions method, Volterra integral equation method, Analytical-numerical method, Computational efficiency

1. INTRODUCTION

The approximating functions method [1-5] is a particular case of the finite element method (FEM) [6-7] for solving Volterra integral equations of the second kind that describes field evolution inside inhomogeneity placed in the media [8,9]. To solve such equations, the problem definition region is partitioned by mesh of square cells and the approximation of the desired function by second-order Lagrange polynomials of a special form. Calculation of the equation that defines the problem at all points of the mesh reduces the solution of the problem to finding a solution to a system of nonlinear algebraic equations, which is solved by Newton's method [10].

The original integral equation that defines the problem is determined by the Volterra integral equation method [8,9], which is an integral equations equivalent to Maxwell's equations applied to the electromagnetic problem. Key features of such problem statement [8]: natural description of non-stationary and nonlinear features of media, unified definition of the problems inside and outside the inhomogeneity in the media, and inclusion of initial and boundary conditions in the same equation.

The process of generating a separate equation for each mesh point (the vertex of a mesh cell) is very costly; therefore, work [11] presents improvements to the approximating functions method. According to it, each equation is divided into a set of blocks calculated in advance in order to reduce the consumption of computer resources and increase the computational performance of the process of constructing the equations of the system. These blocks have been identified and described, but not explicitly calculated.

The purpose of this article is to find out what explicit form these expressions have in general and for the most common degrees of nonlinearity of polarization, adapt them for fast and convenient programming by user and fast execution by computer, and validate them on several nonlinear electromagnetic problems.

2. FORMULATION OF THE ELECTROMAGNETIC PROBLEM

According to the Volterra integral equation method [1,4], the subject of consideration is the integral equation that describes electromagnetic processes in the one-dimensional space and time domain and has the following form in dimensionless variables [1]

$$\begin{aligned} E(\tau,\xi) = E_0(\tau,\xi) - \frac{1}{2}\frac{\partial}{\partial\tau}\Bigg[& \int_{\tau_{\min}^L}^{\tau} \left(\frac{1}{\varepsilon\varepsilon_0} P(\tau',\xi-\tau+\tau') - \frac{\varepsilon-1}{\varepsilon} E(\tau',\xi-\tau+\tau') \right) d\tau' \\ & + \int_{\tau_{\min}^H}^{\tau} \left(\frac{1}{\varepsilon\varepsilon_0} P(\tau',\xi+\tau-\tau') - \frac{\varepsilon-1}{\varepsilon} E(\tau',\xi+\tau-\tau') \right) d\tau' \Bigg], \end{aligned} \tag{1}$$

where, $E(\tau,\xi)$ is the electric field inside the inhomogeneity located in the spatial interval $\xi\in[0,1]$, $E_0(\tau,\xi)$ is an initial electric field in the media, $P(\tau,\xi)$ is the polarization of the medium inside the inhomogeneity, which has different electromagnetic characteristics than the environment outside of interval [0,1], ε is the permittivity of the medium in the environment, ε_0 is the permittivity of vacuum. Integration limits are $\tau_{\min}^L = \max(0,\tau-\xi)$ and $\tau_{\min}^H = \max(0,\tau+\xi-1)$, the field evolution is in time interval $\tau\in[0,\infty)$. And for convenience the dimensionless variables $\tau = vt/L$, $\xi = x/L$ are introduced, where $v = c/\sqrt{\varepsilon}$, c is the light velocity in vacuum, L is inhomogeneity width, x and t are dimensional space and time variables.

Nonlinearity is introduced to Equation (1) by the features of layer material that are described by polarization that can be written as follows:

$$P(\tau,\xi)=\varepsilon_0(\varepsilon_1-1)E(\tau,\xi)+\sum_{i=2}^{n}\gamma_i E^i(\tau,\xi), \tag{2}$$

where ε_1 is the permittivity and γ_i are nonlinear susceptibilities – the nonlinear features of the *i-th* order.

Let's denote the first and second integration part in Equation (1) as follows:

$$\begin{aligned} I_L(\tau,\xi,F)&=\frac{\partial}{\partial\tau}\int_{\tau_{\min}^L}^{\tau}F(\tau',\xi-\tau+\tau')d\tau',\\ I_H(\tau,\xi,F)&=\frac{\partial}{\partial\tau}\int_{\tau_{\min}^H}^{\tau}F(\tau',\xi+\tau-\tau')d\tau', \end{aligned} \tag{3}$$

where

$$F(\tau,\xi)=\sum_{i=1}^{n}\tilde{\gamma}_i E^i(\tau,\xi) \tag{4}$$

with $\tilde{\gamma}_1=(\varepsilon_1-\varepsilon)/\varepsilon$ and $\tilde{\gamma}_i=\gamma_i/(\varepsilon\varepsilon_0)$ for $i\ge 2$. In this notations Equation (1) will be written in the next short form:

$$E(\tau,\xi)=E_0(\tau,\xi)-\frac{1}{2}\left[I_L(\tau,\xi,F)+I_H(\tau,\xi,F)\right]. \tag{5}$$

3. APPLICATION OF THE APPROXIMATING FUNCTIONS METHOD

By using the approximating functions method, the entire domain of problem D is divided by a mesh of semi-closed squares with a side of the square h

$$D_{ij}=\{ih\le\tau<(i+1)h,\ jh\le\xi<(j+1)h\},\ \ i=\overline{0,\text{n-1}},\ \ j=\overline{0,\text{m-1}}, \tag{6}$$

where $n=\lfloor \mathbf{T}/h\rfloor$ ($\mathbf{T}$ is a some constant - time limit) and $m=\lfloor 1/h\rfloor$ are some constants depending on the size of D and h.

The solution to Equation (5) is constructed approximately as a sum of piecewise-smooth functions $\hat{E}(\tau,\xi)=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\hat{E}_{i,j}(\tau,\xi)$ each of which is determined in the corresponding mesh cell D_{ij} and constructed from the approximating polynomials with corresponding weighting coefficients $c_{i,j}$:

$$\hat{E}_{i,j}(\tau,\xi)=\sum_{d_1=0}^{1}\sum_{d_2=0}^{1}c_{i+d_1,j+d_2}\cdot T_{i,j}^{d_1,d_2}(\tau,\xi). \tag{7}$$

where

$$T_{i,j}^{d_1,d_2}(\tau,\xi)=T_i^{d_1}(\tau)\cdot T_j^{d_2}(\xi) \tag{8}$$

$$T_i^d(s)=(1-d)+(-1)^{d+1}\frac{s-ih}{h}, d=\overline{0,1} \tag{9}$$

Substitution of Equation (7) in Equation (5) and calculation of the resulting function at the points (τ_i,ξ_j), $i=\overline{0,\mathrm{n}}, j=\overline{0,\mathrm{m}}$ gives the system of algebraic equations $\{\Psi_{i,j}\}$

$$\Psi_{i,j}:\ c_{i,j}-E_0(\tau_i,\xi_j)+\frac{1}{2}\left[\hat{I}_L(\tau_i,\xi_j)+\hat{I}_H(\tau_i,\xi_j)\right]=0 \tag{10}$$

for the weighting coefficients $c_{i,j}$, which is solved by Newton's method.

Here, $\hat{I}_{L,H}=I_{L,H}(\tau,\xi,\hat{F})$, $\hat{F}$ is the Equation (4) that depends on $\hat{E}(\tau,\xi)$ and $i=\overline{0,n}$, $j=\overline{0,m}$.

According to [11], the explicit expression for $\hat{I}_L(\tau_i,\xi_j)$ is as follows:

$$\hat{I}_L(\tau_i,\xi_j)=\begin{cases}0, & i=0, j=\overline{0,m} \quad or \quad i\ge 1, j=0\\ \hat{I}_L^{tt}(\tau_i,\xi_j), & i=1, j=\overline{1,m} \quad or \quad i\ge 2, j=1,\\ \hat{I}_L^{tct}(\tau_i,\xi_j), & i\ge 2, j=\overline{2,m}\end{cases} \tag{11}$$

where

$$\hat{I}_L^{tt}(\tau_i,\xi_j)=\begin{cases}\Phi_L^{[0,\tau]}(\tau_i,\xi_j), & i\le j\\ \Phi_L^{[\tau-\xi,\tau]}(\tau_i,\xi_j), & i>j\end{cases}, \tag{12}$$

$$\hat{I}_L^{tct}(\tau_i,\xi_j)=\hat{I}_L^{tc}(\tau_i,\xi_j)+\sum_{k=\left(i_{\min}^L+1\right)}^{i-2}\Phi_L^{\left[kh,(k+1)h\right]}(\tau_i,\xi_j)+\Phi_L^{\left[(i-1)h,\tau\right]}(\tau_i,\xi_j), \tag{13}$$

$$\hat{I}_L^{tc}(\tau_i,\xi_j)=\begin{cases}\Phi_L^{\left[0,\left(i_{\min}^L+1\right)h\right]}(\tau_i,\xi_j), & i\le j\\ \Phi_L^{\left[\tau-\xi,\left(i_{\min}^L+1\right)h\right]}(\tau_i,\xi_j), & i>j\end{cases}. \tag{14}$$

The explicit expression for $\hat{I}_H(\tau_i,\xi_j)$ is as follows:

$$\hat{I}_H(\tau_i,\xi_j)=\begin{cases}0, & i=0, j=\overline{0,m} \quad or \quad i\ge 1, j=m\\ \hat{I}_H^{tt}(\tau_i,\xi_j), & i=1, j=\overline{0,m-1} \quad or \quad i\ge 2, j=m-1,\\ \hat{I}_H^{tct}(\tau_i,\xi_j), & i\ge 2, j=\overline{0,m-2}\end{cases} \tag{15}$$

where

$$\hat{I}_H^{tt}\left(\tau_i,\xi_j\right)=\begin{cases}\Phi_H^{[0,\tau]}\left(\tau_i,\xi_j\right), & i\le m-j\\ \Phi_H^{[\tau+\xi-1,\tau]}\left(\tau_i,\xi_j\right), & i>m-j\end{cases}, \tag{16}$$

$$\hat{I}_H^{tct}\left(\tau_i,\xi_j\right)=\hat{I}_H^{tc}\left(\tau_i,\xi_j\right)+\sum_{k=\left(i_{\min}^H+1\right)}^{i-2}\Phi_H^{\left[kh,(k+1)h\right]}\left(\tau_i,\xi_j\right)+\Phi_H^{\left[(i-1)h,\tau\right]}\left(\tau_i,\xi_j\right), \tag{17}$$

$$\hat{I}_H^{tc}\left(\tau_i,\xi_j\right)=\begin{cases}\Phi_H^{\left[0,\left(i_{\min}^H+1\right)h\right]}\left(\tau_i,\xi_j\right), & i\le m-j\\ \Phi_H^{\left[\tau+\xi-1,\left(i_{\min}^H+1\right)h\right]}\left(\tau_i,\xi_j\right), & i>m-j\end{cases}. \tag{18}$$

The general functions Φ have the next form:

$$\Phi_L^{\left[m_1(\tau,\xi),m_2(\tau,\xi)\right]}\left(\tau_i,\xi_j\right)=\frac{\partial}{\partial\tau}\int\limits_{m_1(\tau,\xi)}^{m_2(\tau,\xi)}\hat{F}\left(\tau',\xi-\tau+\tau'\right)d\tau'\Bigg|_{\substack{\tau=ih\\ \xi=jh}}, \tag{19}$$

$$\Phi_H^{\left[m_1(\tau,\xi),m_2(\tau,\xi)\right]}\left(\tau_i,\xi_j\right)=\frac{\partial}{\partial\tau}\int\limits_{m_1(\tau,\xi)}^{m_2(\tau,\xi)}\hat{F}\left(\tau',\xi+\tau-\tau'\right)d\tau'\Bigg|_{\substack{\tau=ih\\ \xi=jh}}. \tag{20}$$

Therefore, the integration limits in Φ_L explicitly can have one of the next values: $m_1(\tau,\xi)\in\{0,ih,\tau-\xi\}$ and $m_2(\tau,\xi)\in\{ih,\tau\}$. For Φ_H they will be: $m_1(\tau,\xi)\in\{0,ih,\tau+\xi-1\}$ and $m_2(\tau,\xi)\in\{ih,\tau\}$. Or in general form:

$$m_{1,2}(\tau,\xi)=a\cdot\tau+b\cdot\xi+c\cdot h, \tag{21}$$

where $a\in\{0,1\}$, $b\in\{-1,0,1\}$, $c\in\{0,i,-m\}$.

4. THE EXPLICIT FORM OF Φ FOR ARBITRARY DEGREE OF NONLINEARITY OF POLARIZATION

For further research, let's introduce Equation (9) for the case when the argument is a multiple of the mesh step *h*:

$$\overline{T}_i^d(c)=T_i^d(c\cdot h)=(1-d)-(-1)^d(c-i), d=\overline{0,1}, c\in\mathbb{Z}, \tag{22}$$

and

$$\overline{T}^d(c)=\overline{T}_i^d(c+i)=(1-d)-(-1)^d c. \tag{23}$$

Rewriting Equation (19) and (20) in general form and substituting Equation (4) with

Equation (7) into it, we obtain

$$\Phi_{L,H}^{\left[m_1(\tau,\xi),m_2(\tau,\xi)\right]}\left(\tau_i,\xi_j\right)=\frac{\partial}{\partial\tau}\int_{a_1\cdot\tau+b_1\cdot\xi+c_1\cdot h}^{a_2\cdot\tau+b_2\cdot\xi+c_2\cdot h}\hat{F}\left(\tau',\xi\pm\left(\tau'-\tau\right)\right)d\tau'\Bigg|_{\substack{\tau=ih\\ \xi=jh}}=$$
$$=\sum_{p=1}^{n}\tilde{\gamma}_p\frac{\partial}{\partial\tau}\int_{a_1\cdot\tau+b_1\cdot\xi+c_1\cdot h}^{a_2\cdot\tau+b_2\cdot\xi+c_2\cdot h}\left(\sum_{d_1=0}^{1}\sum_{d_2=0}^{1}c_{i+d_1,j+d_2}\cdot T_i^{d_1}\left(\tau'\right)\cdot T_j^{d_2}\left(\xi\pm\left(\tau'-\tau\right)\right)\right)^p d\tau'\Bigg|_{\substack{\tau=ih\\ \xi=jh}}. \tag{24}$$

Here and after in cases where a double sign is used, the upper one corresponds to Φ_L, and the lower one Φ_H, respectively.

Using Leibniz's formula, we obtain

$$\Phi_{L,H}^{\left[m_1(\tau,\xi),m_2(\tau,\xi)\right]}\left(\tau_i,\xi_j\right)=\sum_{p=1}^{n}\tilde{\gamma}_p\Phi_{L,H}^{\left[m_1^h,m_2^h\right]}\left(\tau_i,\xi_j,p\right), \tag{25}$$

where terms have the following form:

$$\Phi_{L,H}^{\left[m_1^h,m_2^h\right]}\left(\tau_i,\xi_j,p\right)=a_2\left(\sum_{d_1=0}^{1}\sum_{d_2=0}^{1}c_{i+d_1,j+d_2}\cdot\bar{T}^{d_1}\left(m_2^h\right)\cdot\bar{T}^{d_2}\left(\pm m_2^h\right)\right)^p-$$
$$a_1\left(\sum_{d_1=0}^{1}\sum_{d_2=0}^{1}c_{i+d_1,j+d_2}\cdot\bar{T}^{d_1}\left(m_1^h\right)\cdot\bar{T}^{d_2}\left(\pm m_1^h\right)\right)^p\pm \tag{26}$$
$$p\int_{m_1^h}^{m_2^h}\left(\sum_{d_1=0}^{1}\sum_{d_2=0}^{1}c_{i+d_1,j+d_2}\cdot\bar{T}^{d_1}\left(\tau_h''\right)\cdot\bar{T}^{d_2}\left(\pm\tau_h''\right)\right)^{p-1}\sum_{d_1=0}^{1}\sum_{d_2=0}^{1}c_{i+d_1,j+d_2}\left(-1\right)^{d_2}\bar{T}^{d_1}\left(\tau_h''\right)d\tau_h''=$$
$$\sum_{d_1^1=0}^{1}\sum_{d_2^1=0}^{1}\cdots\sum_{d_1^p=0}^{1}\sum_{d_2^p=0}^{1}\prod_{k=1}^{p}c_{i+d_1^k,j+d_2^k}\left[\begin{array}{l}a_2\prod_{k=1}^{p}\bar{T}^{d_1^k}\left(m_2^h\right)\cdot\bar{T}^{d_2^k}\left(\pm m_2^h\right)-\\ a_1\prod_{k=1}^{p}\bar{T}^{d_1^k}\left(m_1^h\right)\cdot\bar{T}^{d_2^k}\left(\pm m_1^h\right)\pm\\ p\left(-1\right)^{d_2^p}\int_{m_1^h}^{m_2^h}\prod_{k=1}^{p-1}\bar{T}^{d_1^k}\left(\tau_h''\right)\cdot\bar{T}^{d_2^k}\left(\pm\tau_h''\right)\cdot\bar{T}^{d_1^p}\left(\tau_h''\right)d\tau_h''\end{array}\right],$$

where $m_1^h=\left(a_1-1\right)i+b_1j+c_1$, $m_2^h=\left(a_2-1\right)i+b_2j+c_2$ and the change of variables $\tau_h''=\tau'/h-i$ is introduced. Because equations for the system of Equations (10) are constructed with respect to unknown coefficients $c_{i,j}$, then the Equation (26) is grouped with respect to them.

4.1. Explicit expressions for linear polarization

Let's express the integrals of Equation (22) in terms of themselves:

$$\int_a^b \overline{T}_i^d(c)\,dc = (-1)^{d+1}\frac{1}{2}\left[\left(\overline{T}_i^d(b)\right)^2 - \left(\overline{T}_i^d(a)\right)^2\right]. \tag{27}$$

For the case when the argument is $c+i$, Equation (27) has the next form:

$$\int_a^b \overline{T}^d(c)\,dc = (-1)^{d+1}\frac{1}{2}\left[\left(\overline{T}^d(b)\right)^2 - \left(\overline{T}^d(a)\right)^2\right]. \tag{28}$$

So, for $p=1$ the explicit expressions for Equation (25) have the next form:

$$\Phi_{L,H}^{[m_1(\tau,\xi),m_2(\tau,\xi)]}\left(\tau_i,\xi_j\right) = \tilde{\gamma}_1\Phi_{L,H}^{[m_1^h,m_2^h]}\left(\tau_i,\xi_j,1\right),$$

$$\Phi_{L,H}^{[m_1^h,m_2^h]}\left(\tau_i,\xi_j,1\right) = \sum_{d_1=0}^{1}\sum_{d_2=0}^{1} c_{i+d_1,j+d_2}\cdot\left[\begin{array}{l} a_2\cdot\overline{T}^{d_1}\left(m_2^h\right)\cdot\overline{T}^{d_2}\left(\pm m_2^h\right) - \\ a_1\cdot\overline{T}^{d_1}\left(m_1^h\right)\cdot\overline{T}^{d_2}\left(\pm m_1^h\right)\mp \\ \dfrac{(-1)^{d_2+d_1}}{2}\left(\left(\overline{T}^{d_1}(m_2)\right)^2 - \left(\overline{T}^{d_1}(m_1)\right)^2\right) \end{array}\right]. \tag{29}$$

4.2. Explicit expressions for second degree polarization

For $p=2$ the explicit expressions for Equation (26) have the next form:

$$\Phi_{L,H}^{[m_1(\tau,\xi),m_2(\tau,\xi)]}\left(\tau_i,\xi_j\right) = \tilde{\gamma}_1\Phi_{L,H}^{[m_1^h,m_2^h]}\left(\tau_i,\xi_j,1\right) + \tilde{\gamma}_2\Phi_{L,H}^{[m_1^h,m_2^h]}\left(\tau_i,\xi_j,2\right), \tag{30}$$

where first term equals to Equation (29) and the second one is

$$\Phi_{L,H}^{[m_1^h,m_2^h]}\left(\tau_i,\xi_j,2\right) =$$

$$\sum_{d_1=0}^{1}\sum_{d_2=0}^{1}\sum_{d_3=0}^{1}\sum_{d_4=0}^{1} c_{i+d_1,j+d_2}c_{i+d_3,j+d_4}\left(\begin{array}{l} a_2\overline{T}^{d_1}\left(m_2^h\right)\overline{T}^{d_2}\left(\pm m_2^h\right)\overline{T}^{d_3}\left(m_2^h\right)\overline{T}^{d_4}\left(\pm m_2^h\right) \\ -a_1\overline{T}^{d_1}\left(m_1^h\right)\overline{T}^{d_2}\left(\pm m_1^h\right)\overline{T}^{d_3}\left(m_1^h\right)\overline{T}^{d_4}\left(\pm m_1^h\right) \\ \pm 2(-1)^{d_4}\displaystyle\int_{m_1^h}^{m_2^h}\overline{T}^{d_1}\left(\tau_h''\right)\overline{T}^{d_2}\left(\pm\tau_h''\right)\overline{T}^{d_3}\left(\tau_h''\right)d\tau_h'' \end{array}\right), \tag{31}$$

where

$$\begin{aligned}
&\int_{m_1^h}^{m_2^h} \bar{T}^{d_1}\left(\tau_h''\right)\cdot \bar{T}^{d_2}\left(\pm\tau_h''\right)\cdot \bar{T}^{d_3}\left(\tau_h''\right) d\tau_h'' = \\
&\mp\frac{1}{4}(-1)^{d_1+d_2+d_3}\,\tau_h''^4 + \tau_h''\left(1-d_1\right)\left(1-d_2\right)\left(1-d_3\right) \\
&-\frac{1}{3}\tau_h''^3\begin{pmatrix} (-1)^{d_1+d_3+1} \mp (-1)^{d_1+d_2} \mp (-1)^{d_2+d_3} \pm (-1)^{d_2+d_3} d_1 + \\ (-1)^{d_1+d_3} d_2 \pm (-1)^{d_1+d_2} d_3 \end{pmatrix} \\
&\left.-\frac{1}{2}\tau_h''^2\begin{pmatrix} (-1)^{d_1} + (-1)^{d_3} \pm (-1)^{d_2} + (-1)^{d_1+1} d_3 \mp (-1)^{d_2} d_3 \\ +d_2\left((-1)^{d_1+1} + (-1)^{d_3+1} + (-1)^{d_1} d_3\right) \\ +d_1\left((-1)^{d_3+1} \mp (-1)^{d_2} + (-1)^{d_3} d_2 \pm (-1)^{d_2} d_3\right) \end{pmatrix}\right|_{m_1^h}^{m_2^h}
\end{aligned} \quad . \tag{32}$$

4.3. Explicit expressions for third degree polarization

For $p=3$ the explicit expressions for Equation (26) have the next form:

$$\begin{aligned}
&\Phi_{L,H}^{\left[m_1(\tau,\xi),m_2(\tau,\xi)\right]}\left(\tau_i,\xi_j\right) = \\
&\tilde{\gamma}_1\Phi_{L,H}^{\left[m_1^h,m_2^h\right]}\left(\tau_i,\xi_j,1\right) + \tilde{\gamma}_2\Phi_{L,H}^{\left[m_1^h,m_2^h\right]}\left(\tau_i,\xi_j,2\right) + \tilde{\gamma}_3\Phi_{L,H}^{\left[m_1^h,m_2^h\right]}\left(\tau_i,\xi_j,3\right)
\end{aligned}, \tag{33}$$

where first two terms equal to Equation (29) and (31) respectively, and the third one equals to

$$\begin{aligned}
&\Phi_{L,H}^{\left[m_1^h,m_2^h\right]}\left(\tau_i,\xi_j,3\right) = \sum_{d_1=0}^{1}\sum_{d_2=0}^{1}\sum_{d_3=0}^{1}\sum_{d_4=0}^{1}\sum_{d_5=0}^{1}\sum_{d_6=0}^{1} c_{i+d_1,j+d_2}c_{i+d_3,j+d_4}c_{i+d_5,j+d_6}\times \\
&\times\begin{pmatrix} a_2\bar{T}^{d_1}\left(m_2^h\right)\bar{T}^{d_2}\left(\pm m_2^h\right)\bar{T}^{d_3}\left(m_2^h\right)\bar{T}^{d_4}\left(\pm m_2^h\right)\bar{T}^{d_5}\left(m_2^h\right)\bar{T}^{d_6}\left(\pm m_2^h\right) \\ -a_1\bar{T}^{d_1}\left(m_1^h\right)\bar{T}^{d_2}\left(\pm m_1^h\right)\bar{T}^{d_3}\left(m_1^h\right)\bar{T}^{d_4}\left(\pm m_1^h\right)\bar{T}^{d_5}\left(m_1^h\right)\bar{T}^{d_6}\left(\pm m_1^h\right) \\ \pm 3(-1)^{d_6}\int_{m_1^h}^{m_2^h}\bar{T}^{d_1}\left(\tau_h''\right)\bar{T}^{d_2}\left(\pm\tau_h''\right)\bar{T}^{d_3}\left(\tau_h''\right)\bar{T}^{d_4}\left(\pm\tau_h''\right)\bar{T}^{d_5}\left(\tau_h''\right)d\tau_h'' \end{pmatrix}.
\end{aligned} \tag{34}$$

The integral in Equation (34) is easy to calculate and is not included here for brevity.

5. VALIDATION OF THE APPROACH

To validate the developed approach, it was applied to a known problem of interaction of a plane electromagnetic wave with a dielectric layer, which has an exact solution. The following parameters was used: $\varepsilon_1 = 3$, $\varepsilon = 1$, $\tau \in [0, 40]$ and mesh step h=0.02. The median error of the obtained solution is 0.0011%.

5.1. The case of second-degree polarization

To test the accuracy of the approach in the case of second-degree polarization, it was applied to the problem of passing the Gaussian pulse $E_0(\tau, \xi) = -(\tau - \tau_0 - \xi)/\sigma^2 \exp\left(-(\tau - \tau_0 - \xi)^2 / 2\sigma^2\right)$ through the layer with the quadratic nonlinear medium described by the Equation (2) with $\gamma_2 = 0.3$, and the following parameters: $\varepsilon_1 = 11$, $\varepsilon = 9$, $\tau \in [0,10]$ and mesh step h=0.005. The wave propagation starts from the point $(\tau, \xi) = (1,0)$. The correctness of the solution was checked by the energy imbalance (approach from [4] is used), tending to zero.

The result is graphically presented on Figure 1, which shows the diffusion of the beam after each reflection from the layer boundaries and the distortion of the pulse shape due to the nonlinearity of the layered medium. Its shape after the second reflection becomes similar to the Airy pulse [12]. The maximum energy flow imbalance is 6.09% with a median of 1.037%.

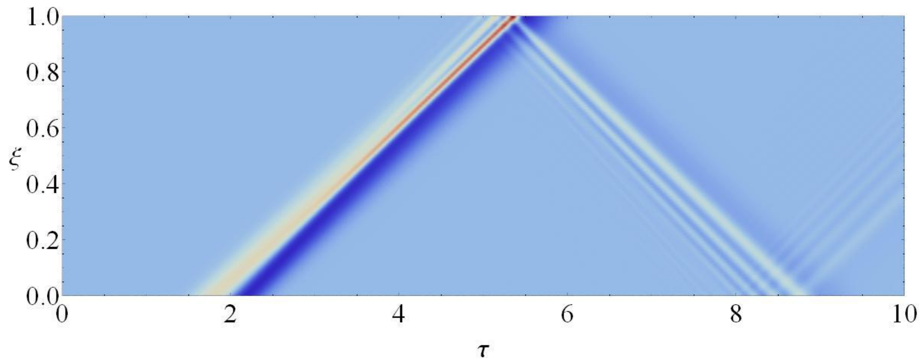

Figure 1. The transformation of the Gaussian pulse by the layer of the second degree of polarization nonlinearity (amplitude decreases from red (positive values) through background color (zero value) to blue (negative values))

5.2. The case of third-degree polarization

The plane electromagnetic wave $E_0(\tau, \xi) = \cos(\eta(\tau - \xi))$ was modeled in time interval $\tau \in [0, 5]$ with nonlinear parameters of the layer medium $\gamma_2 = 0$ and $\gamma_3 \in \{0.01, 0.2, 1\}$, and $\varepsilon_1 = 11$, $\varepsilon = 9$, normalized frequency $\eta = 1$. The wave propagation starts from the point $(\tau, \xi) = (0,0)$.

The result is graphically presented in Figure 2, which shows the behavior of the wave over time at the midpoint $\xi = 0.5$ of the layer. For brevity, the parameter γ_3 is denoted as γ.

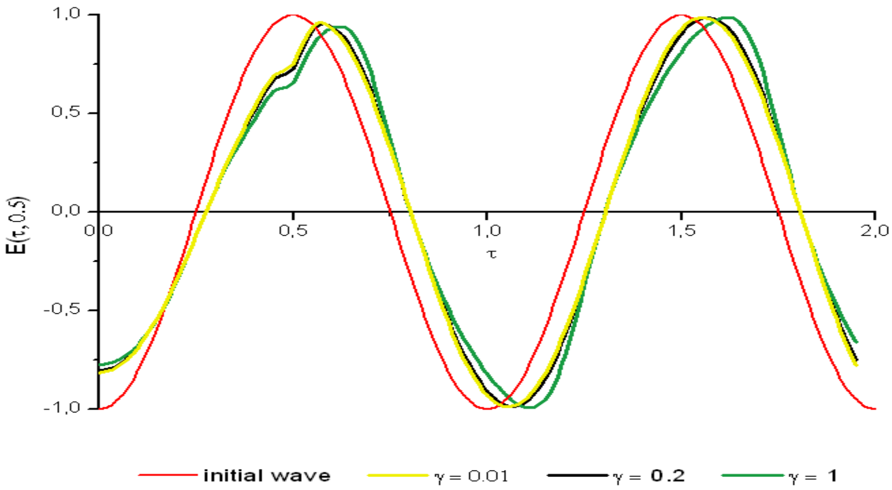

Figure 2. Deformation of a harmonic wave by a layer of medium with third-degree nonlinearity

The results show the deformation of the transformed wave compared to the initial one due to nonlinearity. Figure 3 shows the upper left part of Fig. 2 for easier observation of wave deformations by layer. As expected, an increasing in the degree of nonlinearity from $\gamma_3 = 0.01$ through $\gamma_3 = 0.2$ to $\gamma_3 = 1$ leads to more and more deformation.

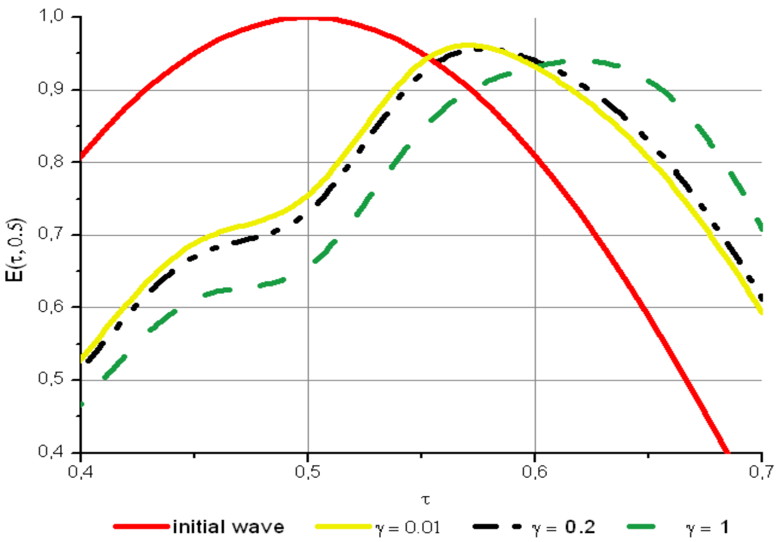

Figure 3. Section $\tau \in [0.4, 0.7]$ of harmonic wave transformation by a layer of medium with third-degree polarization nonlinearity

6. CONCLUSIONS

Explicit expressions that are used to construct a system of equations by improved for fast computations approximating functions method to solve electrodynamics problems in one space dimension and time domain inside nonlinear media are obtained. Since the equations for the system of equations are constructed with respect to unknown coefficients, the resulting expressions are also grouped by them for fast programming. Also, it was found that these expressions have the form of polynomials, so they have the maximum speed of calculation by computer software.

The method using the proposed expressions was validated on the problems of interaction of plane electromagnetic wave incident on a layer with linear and third-order nonlinear medium inserted in linear media. In the second case, several values of the nonlinearity parameter were used, and it was shown that higher nonlinearity leads to greater deformation of the resulting wave. The interaction with a medium of second-order nonlinearity was simulated using the problem of the passage of a Gaussian pulse. In all cases, the results showed good convergence and small error.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] Nerukh A, Zolotariov D, Benson T, 2015. The approximating functions method for nonlinear Volterra integral equations. Opt. Quant. Electron. 47: 2565–2575. https://doi.org/10.1007/s11082-015-0141-2

[2] Zolotariov D, 2021. The new modification of the approximating functions method for cloud computing. Int. Journ. Math. Comp. Res. 9: 2376-2380. https://doi.org/10.47191/ijmcr/v9i9.01

[3] Zolotariov D, Nerukh A, 2021. Development of the approximating functions method for problems in a planar waveguide with constant polarization, Int. Journ. Math. Comp. Res. 9: 2515–2520. https://doi.org/10.47191/ijmcr/v9i12.01

[4] Zolotariov D, Nerukh A, 2011. Extension of the approximation functions method for 2D non-linear Volterra integral equations. Applied Radioelectronics 10: 39-44.

[5] Zolotariov D, Nerukh A, 2010. Method of approximation functions for solution of 2d nonlinear Volterra integral equation. IEEE International Conference on Mathematical Methods in Electromagnetic Theory. https://doi.org/10.1109/mmet.2010.5611393

[6] Ern A, Guermond J-L. Finite Elements I: Approximation and Interpolation, Springer International Publishing (2021). https://doi.org/10.1007/978-3-030-56341-7

[7] Lee J F, Lee R, Cangellaris A, 1997. Time-Domain Finite-Element Methods. IEEE Trans. Antennas Propagat. 45: 430-442.

[8] Nerukh A G, Sakhnenko N K, Benson T, Sewell P. Non-stationary electromagnetics. Pan Stanford Publishing Ltd. (2013).

[9] Nerukh A, Benson T. Non-Stationary Electromagnetics: An Integral Equations Approach (2nd ed.). Jenny Stanford Publishing (2018).

[10] Deuflhard P. Newton Methods for Nonlinear Problems, Springer, (2011). https://doi.org/10.1007/978-3-642-23899-4

[11] D. Zolotariov, About one improvement of approximating functions method for the problems with nonlinear media, 2022 (Submitted for publication).

[12] Nerukh A G, Zolotariov D A, Nerukh D A, 2012. Properties of decelerating non-diffractive electromagnetic Airy pulses. Applied Radio Electronics Sci. Mag. 11: 77–83.