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REVIEW

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A review of radial basis function with applications explored

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Abstract

Partial differential equations are a vital component of the study of mathematical models in science and engineering. There are various tools and techniques developed by the researchers to solve the differential equations. The radial basis functions have proven to be an efficient basis function for approximating the solutions to ordinary and partial differential equations. There are different types of radial basis function methods that have been developed by the researchers to solve various well known differential equation. It has been developed for approximation of the solution with various approaches that lead to the development of hybrid methods. Radial basis function methods are widely used in numerical analysis and statistics because of their ability to deal with meshless domain. In this work, the different radial basis function approaches were investigated along with the focus on the strategies being addressed to find the shape parameter value. The mathematical formulations of the various radial basis function methods are discussed along with the available shape parameters to get the optimal value of the numerical solutions. The present work will lay a foundation to understand the development of the radial basis functions that could lead to a play an important role in development of method thereafter.

Keywords: Partition of unity, Shape parameter, Partial differential equation, Kansa collocation method, Radial basis function

Introduction

An effective numerical strategy for solving partial differential equations (PDEs) is to approximate them by radial basis functions (RBFs), which approximate the numerical solution when implemented that makes the process computationally efficient. The radial basis function was developed for finding the solution of interpolation matrices and later implemented for solutions of partial differential equations. One of the important characteristics of RBF is that these are easy to use and works well by dynamic and irregular domains. In numerical analysis and statistics, RBF approaches have a wide range of applications. Numerical solutions of PDEs, geomodelling, machine learning, price options, neural networks, data mining, and image processing are just a few examples of these applications.

There are several numerical techniques that are available to solve a modelled partial differential equation. But each method has its own advantage and disadvantage, such as finite difference method is a simple and easy to implement technique. But it becomes quite complex for irregular domains; Finite element method is most popular due to most flexible over complex domains but involves a lot of integration; finite volume method is widely used in computational fluid dynamics, surface integral over control volumes but has complexity involved. Some researchers have developed the computational techniques on the base of the above defined techniques that has been implemented for finding the numerical solutions of PDEs, includes quadrature technique [1], B-spline finite element methods [2], RBF methods [3], exponential B-spline with PSO [4], the modified cubic B-spline differential quadrature method [5, 6]. Some other approaches involve finite difference method, Kansa's approach for solution of parabolic, elliptic, and hyperbolic PDEs, RBF collocation method as pseudo-spectral methods, etc. [7–10].

The numerical solutions of fractional ordinary equations (ODEs) and PDEs are most demanding area of today's research for which various numerical methods has been proposed by many researchers. Maayah [11] proposed the multistep Laplace optimized decomposition method for fractional system of ODEs in which Runge–Kutta method of order four applied for testing the efficiency of the proposed method and Arora G. [12] presented residual power series method for fractional relaxation–oscillation equation. Arqub[13] presented the Dirichlet model by computational approach based on the reproducing kernel in a time-fractional sense.

There are a number of the partial differential equations that has been generated as a result of mathematical modelling when the domain is uniformly distributed. In such cases, there is a need to find the solution in the presence of non-uniform data, which is not easy to handle and leads to complexity. To solve this problem, mesh-free methods are used. Radial basis function method is one of useful technique in the mesh-free methods. RBF methods are modern ways to approximate multivariate functions, especially in the absence of grid data. They have been known, tested, and analysed for several years now, and many positive properties have been identified [14]. The implementation of RBF techniques in approaching multivariate scattered data has been highly appreciated.

Hardy[15] introduced the RBF method in context of the quadric surfaces dealing with the topological approach. Hardy was the first to develop the multi-quadric (MQ) approximation technique. Franke [16] experimented with scattered data interpolation. He evaluates methods into the form of time, storage, exactness, and ease of implementation, and also considers multi-quadric (which is a type of RBF) to be one of the best. Micchelli [17]made a step forward in by demonstrating that multi-quadraic surface interpolation is always solvable. The MQ approach has the benefit of obtaining the interpolant using a linear combination of basic functions that are only dependent on the distance from a specific node, which is known as the centre.

In order to solve a PDE, Edward Kansa invented the Kansa method [18] in 1990. It first used the multi-quadraic, a widely supported interpolant. Despite being used in many different applications, the Kansa technique has certain drawbacks, such as asymmetrical traits of the interpolation matrix, which results in a poorly conditioned matrix for a large number of nodes. As an upgradation to the Kansa method, Fasshauer presented a Hermite-based methodology in 1996. The collocation matrices produced by this method are typically more symmetric and have a lower condition number [19]. The symmetric RBF collocation method does have some drawbacks. In comparison to an unsymmetric technique, the symmetric collocation approach is more challenging to implement. Larsson and Fornberg [20] and Power and Barraco [21] compared the symmetric and unsymmetric approaches. Other methods, such as pre-conditioning the interpolation matrix[22], the domain decomposition method [23], etc. have been suggested to overcome the aforesaid difficulties. These techniques can help to some extent by lessening the matrix's poor conditioning. The local approach is another really promising option to address these types of Kansa method issues. Only the local approximation should be taken into consideration for collocation in this strategy rather than all the nodes throughout the entire domain.

Most RBFs have a parameter called the shape parameter, that determines the structure of the RBF. Some RBFs have the best accuracy when the form parameter is set to small values, but this results in improper conditioning of the matrix. An approach for the RBF's stable computation for all values of the form parameter [24] was put forth by Fornberg and Wright in 2004. In 2007, Fornberg and Piret significantly enhanced the method to create the new RBF-QR method, which completely eliminates the matrix's improper condition in cases of nearly flat basis function [25]. In order to maximise the benefits of RBFs, the method continues to be improved by fusing RBF with other widely used approaches. Shu in 2003 gave an approach to combine the meshfree nature of RBF and the high accuracy and simplicity of differential quadrature (DQ) method by proposing a hybrid method known as RBF-DQ method [26]. This technique has been used by researchers to solve PDEs in fluids (such as Navier–stokes, Shallow water problems). Tolstykh in 2003 used local set of nodes to generate the radial basis finite difference approach[27], this hybrid method termed as RBF-FD. Its discretizations are completely mesh-free and very simple to use, even when local refinements are required [28]. Another promising approach is the RBF-PUM to solve PDEs, which combines the partition of unity method with RBF [29]. The idea of RBF-PUM method is to partition the domain into overlapping sub-domains. The local approximation is done on the sub-domains and combines to get the global approximation. RBF-PUM reduces the computational cost while maintaining high accuracy. The approximation done on the points of the sub-domain for finding the solution by the local approximation and whole is used for the global approximation. The computational complexity minimises by the RBF-PUM corresponding the maintenance of the and.

The paper is arranged as follows: In Section 2 "Radial basis function", the radial basis function is discussed. The third section presented a review of radial basis function methods that are used for finding the solutions of PDEs. In last section, conclusion of the paper is presented followed by the discussion of RBF methods. Wherever possible, we attempted to provide the mathematical formulation of the RBF methods. To the best of the authors' understanding, there is no such investigation that presents all the RBF techniques.

Radial basis function

A function $\Phi: R^t \rightarrow R$ is called radial if there exist a function of one variable: $[0, \infty) \rightarrow \mathbb{R}$ such that $\Phi(x) = \varphi(x)$, here Euclidean norm $\|.\|$ is used and $t \in \mathbb{N}$. $\Phi(r)$ is a univariate continuous real valued radial basis function whose value based upon distance value that is measure from any fixed centre point or the origin [14].

Table1 Types of infinitely smooth RBFs with $\lambda > 0$ and $r \in x$

RBFs	Definition
Gaussian function (GS)	$\varphi(r) = e^{-(\lambda r)^2}$
Multi-quadric (MQ)	$\varphi(r) = \sqrt{1 + (\lambda r)^2}$
Inverse multi-quadric (IMQ)	$\varphi(r) = \frac{1}{1+(\lambda r)^2}$
Inverse quadric (IQ)	$\varphi(r) = \frac{1}{\sqrt{1+(\lambda r)^2}}$

Table 2 Some piecewise smooth RBFs with r ∈ x

RBFs	Definition
Thin plate spline (TPS)	$\varphi(r) = r^2 \ln(r)$
Linear radial function (LR)	$\varphi(r) = r$
Cubic function	$\varphi(r) = r^3$
Monomial	$\varphi(r) = r^{2k-1}; k \in \mathbb{N}$

From the definition, it is clear that Φ is a special function, which is radially symmetric and only depends on the distance between points. The application of radial basis function to the high dimensional problem is easy as the interpolation problem is insensitive to the space dimension. In all space dimensions, one can work with the function φ that is univariate instead of using a multivariate function Φ. We are centring on types of radial basis functions that are distinguished by the smoothness-piecewise smooth RBFs which are free from shape parameter λ and infinitely differentiable which have parameters called the shape parameter.

Types of RBFs

There are various types of RBFs. Some recognised RBFs are as follows:

1. Infinitely smooth RBFs—These RBFs are based on the shape parameter $\lambda > 0$ that controls the shape or outline of the RBF. If λ is tending to 0 then form of RBFs becomes flat.Table1 presents different types of infinitely smooth RBFs.
2. Piecewise smooth RBFs—these RBFs have no shape parameter. Different types of piecewise smooth RBFs are shown in Table 2.

The summary of the different types of RBF presented above can also be presented as Fig. 1.

From the analysis of the Fig. 2 and the Fig. 3, it seems that change in the value of the shape parameter results in achange in the shape of the radial function. Figure 2 represents the shape of Gaussian RBF with respect to the different values of shape parameter 1; 0.5; 0.2. And Fig. 3 shows the change in the shape of inverse multiquadric RBF at the value of shape parameter 1; 0.5; 0.

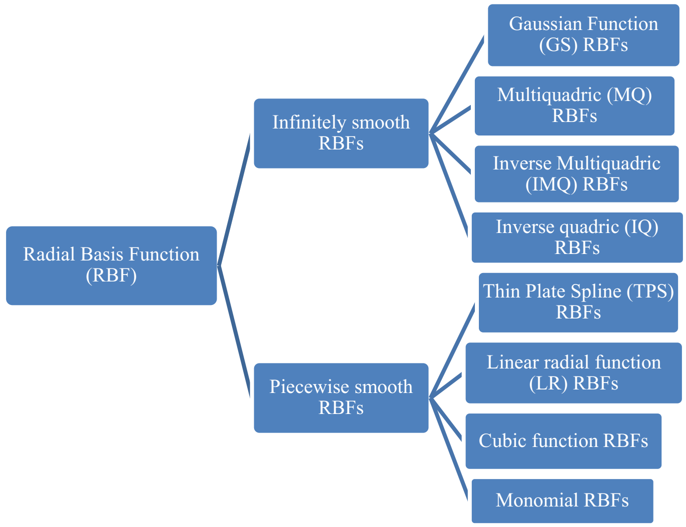

Fig. 1 Various types of RBFs

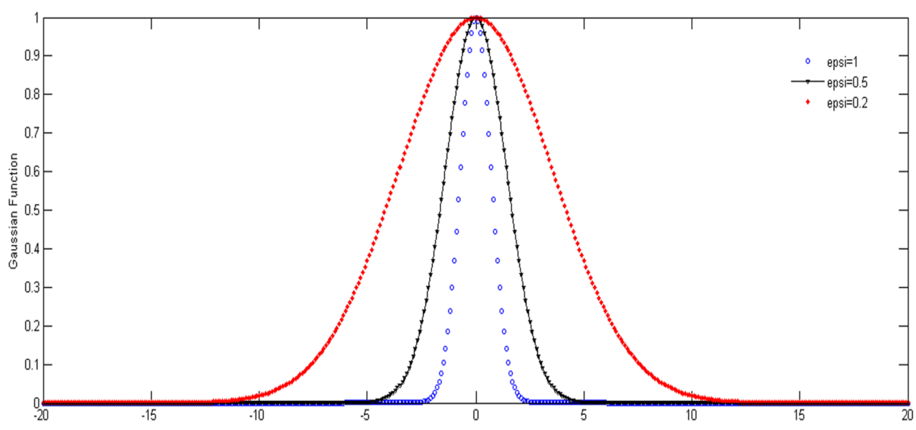

Fig. 2 Gaussian RBF with different values of shape parameter

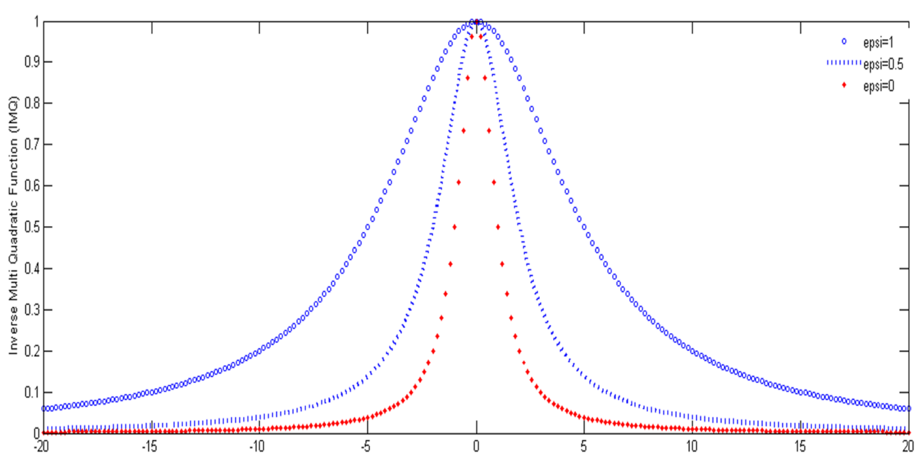

Fig. 3 Inverse multi quadratic RBF with different values of shape parameter

RBF methods for solving PDEs

RBF methods are known for their easy way of implementation and simplicity in approximation of multivariate scattered data. For solving partial differential equations, a recent historic and chronologically development strategy (Fig. 4) of RBF methods has been discussed as follows:

Solutions of PDEs with Kansa collocation method

One of the mesh-free approaches is the Kansa method, often known as the RBF collocation method. Compared to mesh methods, mesh-free methods have a lot of advantages. They are cost saving since they do not require domain or surface discretization. Kansa [18] introduced an asymmetric approach in. Kansa technique is an RBF-based approach for solving PDEs.

By mathematically, Consider $x \in R^d$ and in $\mathbb{R}^{\mathrm{d}}, d \in \mathbb{N}$,consider the norm $\| \cdot \|$ that is Euclidian norm, the radial basis functions of the form $\Phi(\|x - x_i\|)$ that supposed to be strictly positive definite. The RBF approximation can be written by using nodes that are spotted arbitrary in the domain $\Omega \subset R^d$ and assigning a collection of neighbourhood nodes $\mathrm{x_i}$ that are integrated in the supportive domain to every x as follows:

$$u(x) = \sum_{i=1}^{N} \alpha_i \emptyset(x - \mathrm{x_i}) \tag{1}$$

where α_i is unknown coefficients and N represents the numbers of node points. By substituting this solution u(x) in PDE gives the linear system of equations as

$$A\mathbb{X} = B \text{ where } \mathbb{X} = [\alpha(x_1), \alpha(x_2), \alpha(x_3), \ldots\ldots\ldots\ldots \alpha(x_N)]^T \tag{2}$$

The Kansa collocation method can be summarized as:

(i) Consider a PDE with boundary conditions on specific domain.
(ii) Assume its solution as a linear combination of RBFs with node points.

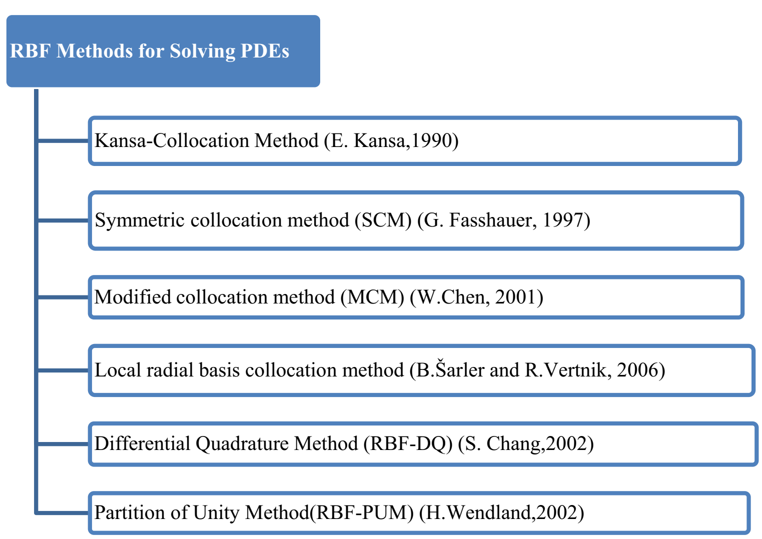

Fig. 4 The various RBF methods

(iii) Implement the assumed solution at given equation and boundary conditions.
(iv) Resultant in the form of linear system of equations.

Various problems have been solved by this approach successfully. By using this approach, Zhou et al.[30] solved shallow water modelling problem, convection diffusion problems solved by Chen et al. [31], and also solved fractional diffusion equation by using Kansa method, Kovacevic et al. [32] solved Stefan problem, time dependent heat conduction problems solved by Chantasiriwan[33], Duan et al. [34] solved electrostatic problems using Kansa method. The Kansa methods have disadvantages due to use in solving various PDEs. Main disadvantage of this method is computational cost that becomes very high due to the unsymmetric interpolation matrix. The accuracy of this method is less in the domain closest to the boundary. To get the better accuracy and hence reduce the errors, the very simplest way is to raise the interpolation points that lead to high condition number matrix. But by increasing the points that are now taken in the entire domain, the resultant matrix turns into ill conditioned. This resulted the need of modification in this method and hence gives rise to following three methods:

Symmetric collocation method (SCM)

After modification in the Kansa method, a new method comes in existence, which is known as Symmetric collocation method. This method is based upon Hermite interpolation and proposed by Fasshauer [19] and also invent the RBF expansion for approximating the function. After applying the collocation conditions, there is a requirement of a non-singular symmetric collocation matrix. Symmetric and non-symmetric techniques had been applied for different applications. These methods are compared by Power &Barraco [21] and find the result as the symmetric collocation technique is surpassing the non-symmetric (Kansa method) technique due to the lower computational cost. But the implementation of Kansa scheme is unproblematic. The symmetric collocation method is also used by Leitao [35] to solve 2-dimensional elastostatic problems.

Modified collocation method (MCM)

As the Modified collocation method is the upgraded structure of symmetric method whose resultant is that the interpolation matrix is symmetric. Chen [36] proposed a method in which Green second identity is used, called modified collocation method.

Ill-conditioned interpolate matrix is the main concern for using Kansa technique to find the results of the various PDEs. To resolve the issue like domain decomposition method, compactly supported RBF and pre-conditioning, numerous techniques were projected. Process of transformation of a set of linear equation into a new system that is constructive approach for iterative solution is called pre-conditioning. This transformation produced by a matrix, which is known as pre-conditioner. Assessment of the conditioned number is also balanced by pre-conditioning, and it also helpful in the improvement of convergence. In the domain decomposition method, a problem with huge point of global domain is divided into sub-domains weather these are overlying or uncorrelated. To avoid the ill-conditioned solutions of the problems on sub-domain except the large domain, this process is very effective.

Local radial basis collocation method

Another method to remove the complex behaviour of the interpolation matrix is local radial basis collocation method (LRBFCM). Local approximation is the main base of this process and depicted by Chen et al. [37].

To see the procedure of this method, takingthe Elliptic partial differential equation for mathematical formulation with domain D given by $L[u(x)] = f(x), x \in \mathrm{D}$ with boundary conditions $u(x) = g(x), x \in \partial D$.Let the local approximation $u(x^s)$ of the solution $u(x)$ and $\{x^s\}_{s=1}^{N} \in D$ then

$$u\left(x^s\right) = \sum_{k=1}^{n} \alpha_k^s \emptyset(x^s - \mathrm{x}_\mathrm{k}^\mathrm{s}) \tag{3}$$

where x^s is collocation point, n is the neighbourhood of the point x^s including itself, and $\emptyset$ is a radial basis function. Here, coefficients α_k^s to be determined. For the distinct values of collocation nodes, non-singularity will become necessary condition for resultant matrix.

The above discussed LRBFCM is used for finding the solutions of diffusion equations, and this method is intended by using local collocation. As the collocation performed on local domain of influence that minimise the size of the collocation matrix. This approach followed by many authors and applied for finding the solutions of large dimensional problems such as fluid flow and heat transfer problems, convective diffusive solid liquid stages change problems, Darcy-flow, and also for transport phenomena. Further for find the solution of hyperbolic partial differential equation numerically, this method is improved by Siraj [38]. For improving accuracy in this modified approach, multi-quadric RBF is used with consistent related arrangement. For approximate the time derivative, the finite difference formula of first order is numerically used. While comparing with Kansa collocation method, this method was found to be more stable for numerical problems.

Solution of PDEs with differential quadrature method (RBF-DQ)

Bellman et al. [39] proposed an approach—the differential quadrature that approximate derivative of the function rather than function itself. In this technique, a smooth function is considering whose partial derivative is estimated on a node seeing as linear summation of the values of function that lies in the domain, which is similar to the concept of integral quadrature. The derivative at the node $\mathrm{x_i}$ can be written as

$$f^n(x_i) = \sum_{i=1}^{N} a_{ij}^n f\left(x_j\right);\, i = 1, 2, 3, \ldots N \tag{4}$$

Instead of using Lagrange's interpolation, radial basis function is used by Shu and Wu [40] in differential quadrature approach for finding the value of weighting coefficients and hence the method is named as RBF-DQ method. The RBF-DQM can be applied in two different ways to solve the PDEs as Global and Local of RBF-DQ method are given by Shu et al. [41, 42] In Global version, all nodes are used in the whole domain for estimating the derivative at a point. The ill conditioning problem occurs, and the computational cost becomes high by using huge set of nodes. And in local radial basis function differential quadrature (LRBFDQ) method local approach is used which takes all the neighbourhood points of the specific point known as supporting nodes.

2-D Navier–Stokes equations solved by Shu et al. [41]. by the use of LRBFDQ method and then Shu et al. [43] apply it for compressible flows. This method is applied for the boundary level problems by Shen [44]. Two-dimensional transient heat conduction problems are also solved by this method in the work of Soleimani et al. [45]. Integrated radial basis function network used by Shu& Wu [40] with the concept of differential quadrature named as IRBF-DQ method. And one-dimensional burger's equation successively solved with this method. By using LRBFDQ, Dehghan and Nikopour [46] find the solutions of the boundary value problems using multi-quadric (MQ) radial basis function. Dehghan also applied two different methods—OCSP method and OVSP method for finding the value of λ plays a significant role in RBF.

Solution of PDEs with partition of unity method (RBF-PUM)

Babuska and Melenk [47] proposed the partition of unity finite element (PUM) method in 1997 for finding the solutions of PDEs. By the proposal of the partition of unity method the region is fractionalized into intersecting local domains. For choosing a family of compactly supported, a continuous function, this approach is important. The RBF-PUM method is a best way to decrease the computational cost with attaining the higher accuracy. The main advantage of this approach in high dimensional problems is to hold the geometrical flexibility, to overcome computation cost and to facilitate adaptive approximation.

In this method, local approximation is defined on sub-domains then merge to structure global approximation by using weight functions, which figure out the method of partition of unity. In this method, RBF is employed for the local approximation. The partition of unity method (PUM) combines with RBF by Wendland [29] for solving problems on large extent. Consider elliptic problem of partial differential equation for mathematical interpretation on domain D with the boundary condition as follows:

$$L[u(x)] = f(x), x \in D \tag{5}$$

$$u(x) = g(x), x \in \partial D \tag{6}$$

Algorithm for spherical interpolation proposed by Cavoretto and Rossi [48] for finding the numerical solution of problem using basis function that further projected a method by the use of the partition of unity method. The author uses spherical radial basis function mainly in local approximation. In this process, many operations can be performed equivalently. Further in the extension of this work, Cavoretto and Rossi [49] intended an algorithm of partition of unity method in which domain is partitioning into nodes or cell. This procedure principally based upon cell search. Also the author extended this 2-dimensional algorithm to 3-dimensional by using cube partition searching procedure. Applications of partition of unity method investigate by Safdari et al. [50] for the solutions of parabolic partial differential equation. For this work, 2-D diffusion equations were considered, and pseudo-spectral and finite difference methods are compared with RBF-PUM. After comparison, researcher initiate that RBF-PUM gives more exact

Table 3 Approaches of finding best shape parameter

Approaches	Author	Shape parameter value
Trial and error	Rolland L. Hardy [15]	$0.815\mathrm{d};\ \mathrm{d} = \frac{1}{N}\sum_{k=1}^{N} d_i$ d_i is the distance between point and neighbourhood
	Richard Franke [16]	$\frac{1.25D}{\sqrt{N}}$; D is the diameter of minimal circle
	G. E. Fasshauer (2002) [64]	$\frac{2}{\sqrt{N}}$
The power function	Neyman and Pearson [65]	–
Leave-one-out cross-validation (LOOCV)	Rippa S. [58]	–

Table 4 A chronological scheme of leave-one-out cross-validation (LOOCV) technique

Researcher	Year	Findings
D. M. Allen [56]	1974	For ridge regression
Peter Craven and Grace Wahba [57]	1979	For smoothing splines
S. Rippa [58]	1999	Optimised RBF's shape parameter λ
Gregory E. Fasshauer and Jack G. Zhang [59]	2007	Extensions of LOOCV approach

solution than that of pseudo-spectral method. This method can be applied to irregular shaped domains due to their restricted nature. The constancy of this method was proved by the assistance of abstract and investigational methods. Further improvement in partition of unity method is done by Heryudono et al. [51]. The resultant matrix of this method is ill conditioned, asymmetrical but efficient pre-conditioner is required. Distinct pre-conditioning approaches based on LU factorization are compared and discussed by researcher.

RBF methods with shape parameters

In RBF research field, optimizing the shape parameter λ is continuously a major area. In this regard, a number of studies have been conducted**.** There are some methods for finding best shape parameter λ listed in the Table 3**.**

Huang et al. [52] used arbitrary precision computing to determine the relation among the value of λ and the exactness of the solutions in their investigation. According to their research, method of finding the solution of radial basis functions by 100-digit precision arithmetic is used to avoid the singularity due to round-off error occurs in regular 16-digit precision arithmetic when the parameter value is small. They devised error formulations with respect to the value of λ and grid spacing based on the numerical data obtained. Guo & Jung [53, 54] calculated the best value of λ for discretization approach by Taylor series. The higher-order derivative components that arose in the ideal form parameter that was optimised were calculated using a polynomial reconstruction approach. Homayoon et al. [55] used RBF-based differential quadrature method (RBF-DQ) for finding the results of shallow water and long wave's problems. Here, leave-one-out cross-validation (LOOCV) approach being implemented for getting the optimal value of λ which represented by Table 4.

There is no specific technique or process to controlling the shape parameter λ for RBF kernel methods. The shape parameter can be chosen through numerical evaluations of RBFs to stabilise the solution, that is exceedingly hard and time consuming. Timesli&Saffah [60] build an algorithm for determining the optimal value of shape parameter λ rapidly and instantly determines the appropriate value of λ. The strategy for determining the best possible value of λ is depending upon the idea of combining the RBF method, numerical continuation approach and high-order algorithm Taylor expansion. In this algorithm, author use the description of $\lambda = \alpha \mathrm{ds}$, here any coefficient α is used and ds = domain. So

$$\mathrm{ds} = \frac{1}{\mathrm{N}} \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{d_i}; \tag{7}$$

where distance is calculated by d_i among ith-point & neighbourhood points. Timesli and Saffah [60] aim to reduce the inaccuracy at order 1 of the higher order mesh-free algorithm. Marko Urleb [61] proposed a strategy for finding an optimized value of shape parameter λ for unknown results of PDEs with initial and boundary conditions. In this procedure, Gershgorin's theorem, multi-quadric RBF and the Newton method are implemented for optimal values of diffusion equations and then made a comparison with the results of finite element method. The proposed processwas presented for finding the optimal value λ using Gershgorin's theorem (regarding eigen values of matrix), MQ RBF) and the Newton method for evaluating the zeros of a function.

As in the findings of the finite element method, other methods those have or have not λ given the exactness of obtained optimal method. The main purpose is to validate the exactness and strength of the procedure comparative to others. In the presented procedure, an iteration algorithm is given by calculating a matrix, which is obtained by the multi-quadric functions outer side of the time loop in which the value of the operator L on the MQ basis function is defined by matrix W over domain $\Omega \backslash \partial\Omega$ and the boundary operator B on the MQ basis function on $\partial\Omega$.

A novel higher-order RBF-FD schema with an optimal variable shape parameter was proposed by Nga Y. L. et al.[62] for the numerical results of various PDEs. For the solutions of partial differential equations, RBFs with multi-quadric kernels have generally been used. A user-friendly shape parameter was used in the MQ kernel and the exactness of the result depends upon the shape parameter value. In proposed approach, RBF finite difference method based on MQ is calculated in a polynomial structure, i.e., the RBF finite difference (RBF-FD) method is used for approximating the second derivative that contains a shape parameter, which affects the accuracy of the PDE solution. The best value of the shape parameter is found by removing the RBF-FD scheme's leading error term, which improves solution accuracy and speeds up convergence. Combined compact differencing and finite difference techniques are used to determine the ideal shape parameter. The best shape parameter is discovered to vary across the domain, according to the analysis. As a result, compared with the RBF-FD method that uses the value of the shape parameter, the accuracy of the solution of PDEs is high when employing the localized shape parameter. Generally, the solutions derived by employing the shape parameter calculated from the combined compact differencing (CCD) scheme give more accuracy, but they come with a high computational cost. However, as the number of iterations of the shape parameter is restricted to two, the current RBF finite differencing (FD) with the shape parameter by the combined compact differencing technique is as effective as applying the FD scheme, according to the cost-effectiveness analysis.

When the RBFs are going to almost flat and the selection of the value of shape parameter is done correctly, RBF approximation is capable to produce an appropriate estimation for huge collection of data points that provides smooth result for specified tangled points. In research work of Kazeem et al. [63], the inverse multi-quadric (IMQ) RBF function was included for writing and implementing a technique for the solution of partial differential equations. Preference is given to the selection of shape parameters, which must be made carefully. The approach as an algorithm that runs a series of interpolation tests while adjusting the range of the shape parameters, and then chooses the optimal shape parameter with the resultant as the smallest root mean square error (RMSE). Matlab was used for all of the computational work. The selected problems of interpolation, and it root-mean-square errors (RMSEs) are tabularised and diagrammed.

Conclusion

This article provides an overview of the radial basis function and the approaches based upon RBFs for finding the solution of the various PDEs. We make an attempt to highlight some of the current developments of the RBF methods and the approaches for finding the optimised value of shape parameter. This review is intended to familiarise the reader with RBF approaches as collocation methods, the local collocation RBF approaches, global approximation, RBF-DQ method, and RBF-PUM. These strategies aid in the lowering of computational costs and are particularly useful in the solution of large-scale problems. The minimal value of shape parameter leads to good accuracy for smooth RBFs, while the near flat radial basis leads to poor conditioning of the interpolation matrices. To overcome this issue, several algorithms were proposed that are listed in this work. The given approaches can be further improved by investigating the optimal value of the shape parameter for the better accuracy and steadiness of RBF approximations. The efficiency of RBF techniques for solving higher-order PDEs are still being investigated.

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Availability of data and materials
The authors confirm that the data supporting the findings of this study are available within the article.

Declarations

Competing interest
The author declares that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

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