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COUPLES OF LOWER AND UPPER SLOPES AND RESONANT SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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Dedicated to Jaroslav Kurzweil, with admiration and friendship, for his ninetieth birthday anniversary

Abstract. A couple (σ,τ) of lower and upper slopes for the resonant second order boundary value problem

$$x'' = f(t,x,x'), \quad x(0)=0, \quad x'(1)=\int_0^1 x'(s)\,\mathrm{d}g(s),$$

with g increasing on $[0,1]$ such that $\int_0^1 dg = 1$, is a couple of functions $\sigma,\tau \in C^1([0,1])$ such that $\sigma(t) \leqslant \tau(t)$ for all $t \in [0,1]$,

$$\begin{aligned} \sigma'(t) &\geqslant f(t,x,\sigma(t)), \quad \sigma(1) \leqslant \int_0^1 \sigma(s)\,\mathrm{d}g(s),\\ \tau'(t) &\leqslant f(t,x,\tau(t)), \quad \tau(1) \geqslant \int_0^1 \tau(s)\,\mathrm{d}g(s), \end{aligned}$$

in the stripe $\int_0^t \sigma(s)\,\mathrm{d}s \leqslant x \leqslant \int_0^t \tau(s)\,\mathrm{d}s$ and $t \in [0,1]$. It is proved that the existence of such a couple (σ,τ) implies the existence and localization of a solution to the boundary value problem. Multiplicity results are also obtained.

Keywords: nonlocal boundary value problem; lower solution; upper solution; lower slope; upper slope; Leray-Schauder degree

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1. Introduction

Let $f\colon [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous and $g\colon [0,1]\to\mathbb{R}$ increasing. We consider the second order boundary value problem with nonlocal boundary conditions

$$x''=f(t,x,x'),\quad x(0)=0,\quad x'(1)=\int_0^1 x'(s)\,\mathrm{d}g(s). \tag{1.1}$$

It is easy to check that the corresponding linear homogeneous problem

$$x''=0,\quad x(0)=0,\quad x'(1)=\int_0^1 x'(s)\,\mathrm{d}g(s)$$

has a nontrivial solution if and only if

$$\int_0^1 \mathrm{d}g=1. \tag{1.2}$$

If condition (1.2) holds, the problem (1.1) is called *resonant*, and *non-resonant* if $\int_0^1 \mathrm{d}g\neq 1$.

Such problems have been considered in the non-resonant case by Gupta-Ntouyas-Tsamatos [9], Gupta-Trofimchuk [10], in the case of multipoint boundary conditions. When the problem is resonant, one can cite the contributions of Gupta [8] for multipoint boundary conditions, and of Karakostas-Tsamatos [11], [12], Xiaojie Lin [14] and the second author [19] for integral boundary conditions. They all deal with situations, where f grows at most linearly in its arguments. The paper [14] considers the slightly more general class of boundary conditions

$$x(0)=ax(b),\quad x'(1)=\int_0^1 x'(s)\,\mathrm{d}g(s).$$

In this paper, we assume that g satisfies the resonance condition (1.2). Notice that, in this case, the second boundary condition in (1.1) means that $x'(1)$ is equal to the weighted average of $x'(t)$ on $[0,1]$ for the measure $\mathrm{d}g$.

In the case of two-point boundary conditions, there is a vast literature associated to the obtention of existence and multiplicity results for the solutions in terms of the concept of lower and upper solutions. We refer to the monographs [6] and [18] for detailed descriptions of the results, history and bibliography. The approach has been extended to some integral boundary conditions by Benchohra-Ouahab [4], Benchohra-Hamani-Nieto [3], and Pang-Lu-Cai [15].

In Section 3 of this paper, we introduce the concept of *couple* (σ,τ) *of lower and upper slopes* for the problem (1.2), which are functions $\sigma,\tau \in C^1([0,1],\mathbb{R})$ such that $\sigma(t) \leqslant \tau(t)$ for all $t \in [0,1]$, and

$$\sigma'(t) \geqslant f(t,x,\sigma(t)), \quad \sigma(1) \leqslant \int_0^1 \sigma(s)\,\mathrm{d}g(s),$$
$$\tau'(t) \leqslant f(t,x,\tau(t)), \quad \tau(1) \geqslant \int_0^1 \tau(s)\,\mathrm{d}g(s)$$

for all (t,x) in the stripe $\int_0^t \sigma(s)\,\mathrm{d}s \leqslant x \leqslant \int_0^t \tau(s)\,\mathrm{d}s$ and $t \in [0,1]$. The concept is compared with the classical ones of lower and upper solutions for the problem (1.2), and of lower and upper solutions for the associated family of first order differential equations.

In Section 4, we prove that the existence of a couple of lower and upper slopes for the problem (1.2) implies the existence of a solution to this problem, as well as some information on its localization and the one of its first derivative (Theorem 4.1). Some examples and special cases are given. Taking the differential inequalities strict for a couple of strictly ordered functions in the definition of couple of lower and upper slopes leads in Section 5 to the concept of couple of strict lower and upper slopes. It is shown there that the existence of such a couple leads to a localization of the solution and its derivative with strict inequalities (Corollary 5.1).

Like in the case of lower and upper solutions, the application of Theorem 4.1 and Corollary 5.1 relies upon the construction of a couple of lower and upper slopes. Some results in this direction are given in Section 6, as well as related examples. Lemmas 6.3 and 6.4 are inspired by Propositions 3.1 and 3.2 of [5] for lower and upper solutions. Theorem 6.1 is motivated by Theorem 1.1 of [7] for lower and upper solutions with periodic boundary conditions.

Finally, a three solutions result in terms of couple of lower and upper slopes, in the spirit of Amann's pioneering result for abstract equations in ordered spaces [1], and Dirichlet boundary problems for elliptic equations [2], and of Rachůnková for periodic solutions of ordinary differential equations [17], [16], is stated and proved in Section 7 (Theorem 7.1). See also [13].

The proof of Theorem 4.1 relies upon some elementary results on linear problems stated and proved in Section 2 for the reader's convenience.

2. A linear problem

The proof of the existence theorem based upon the existence of a couple of lower and upper slopes requires the following elementary results on the linear non-homogeneous problem

$$x'' = x' + h(t), \quad x(0) = 0, \quad x'(1) = \int_0^1 x'(s)\,\mathrm{d}g(s), \tag{2.1}$$

where $h \in C([0,1])$ and $g\colon [0,1] \to \mathbb{R}$ is increasing and satisfies (1.2). If we write $x(t) = \int_0^t y(s)\,\mathrm{d}s$, the problem (2.1) is reduced to

$$y' = y + h(t), \quad y(1) = \int_0^1 y(s)\,\mathrm{d}g(s).$$

Its unique solution is easily computed and is given by

$$\begin{aligned} y(t) = {} & \frac{\mathrm{e}^{t-1}}{1 - \int_0^1 \mathrm{e}^{s-1}\,\mathrm{d}g(s)} \left[\int_0^1 \int_0^s \mathrm{e}^{s-\tau} h(\tau)\,\mathrm{d}\tau\,\mathrm{d}g(s) - \int_0^1 \mathrm{e}^{1-s} h(s)\,\mathrm{d}s\right] \\ & + \int_0^t \mathrm{e}^{t-s} h(s)\,\mathrm{d}s. \end{aligned} \tag{2.2}$$

Notice that formula (2.2) makes sense, because the increasing character of g and condition (1.2) imply, as $\mathrm{e}^{s-1} < 1$ for $s \in [0,1)$, that

$$\int_0^1 \mathrm{e}^{s-1}\,\mathrm{d}g(s) < \int_0^1 \mathrm{d}g(s) = 1.$$

Formula (2.2) easily implies the existence of a linear operator $S\colon C([0,1]) \to C^1([0,1])$ such that the unique solution x of (2.1) can be written as $x = Sh$, and the existence of $M > 0$ such that

$$\|x\|_\infty \leqslant M\|h\|_\infty, \quad \|x'\|_\infty \leqslant M\|h\|_\infty, \quad \|x''\|_\infty \leqslant M\|h\|_\infty, \tag{2.3}$$

which implies that $S\colon C([0,1]) \to C^1([0,1])$ is compact.

3. Couples of lower and upper slopes

Our existence result is based upon the new concept of couple of lower and upper slopes for the problem (1.1), which plays for this problem the role of the lower and upper solutions for more classical two-point boundary value problems.

Definition 3.1. We say that (σ, τ) is a *couple of lower and upper slopes* to the problem (1.1) if $\sigma, \tau \in C^1([0,1], \mathbb{R})$ are such that for all $t \in [0,1]$,

$$\sigma(t) \leqslant \tau(t),$$

and

$$\sigma'(t) \geqslant f(t, x, \sigma(t)), \quad \sigma(1) \leqslant \int_0^1 \sigma(s)\, \mathrm{d}g(s), \tag{3.1}$$

$$\tau'(t) \leqslant f(t, x, \tau(t)), \quad \tau(1) \geqslant \int_0^1 \tau(s)\, \mathrm{d}g(s) \tag{3.2}$$

for all $(t, x) \in S_{\Sigma, T}$, where

$$\Sigma(t) := \int_0^t \sigma(s)\, \mathrm{d}s, \quad T(t) := \int_0^t \tau(s)\, \mathrm{d}s, \quad \forall\, t \in [0,1] \tag{3.3}$$

and

$$\begin{aligned} S_{\Sigma, T} &= \{(t, x) \in [0,1] \times \mathbb{R} \colon\ \Sigma(t) \leqslant x \leqslant T(t)\} \\ &= \bigcup_{t \in [0,1]} (\{t\} \times [\Sigma(t), T(t)]). \end{aligned} \tag{3.4}$$

In the case of a couple of *constant* lower and upper slopes (σ, τ), the boundary conditions are automatically satisfied because of (1.2), and the conditions of Definition 3.1 reduce to

$$\sigma \leqslant \tau, \quad f(t, x, \sigma) \leqslant 0 \leqslant f(t, x, \tau), \quad \forall\, (t, x) \in [0,1] \times [\sigma, \tau].$$

In the special case where f does not depend upon x, in which case the problem (1.1) reduces to the first order nonlocal problem for $y = x'$

$$y' = f(t, y), \quad y(1) = \int_0^1 y(s)\, \mathrm{d}g(s), \tag{3.5}$$

the definition of lower and upper slopes reduces to the classical definition of ordered lower and upper solutions for the first order boundary value problem (3.5), i.e. of functions σ and τ in $C^1([0,1])$ such that for all $t \in [0,1]$,

$$\begin{gathered}\sigma(t) \leqslant \tau(t),\\ \sigma'(t) \geqslant f(t,\sigma(t)), \quad \sigma(1) \leqslant \int_0^1 \sigma(s)\,\mathrm{d}g(s),\\ \tau'(t) \leqslant f(t,\tau(t)), \quad \tau(1) \geqslant \int_0^1 \tau(s)\,\mathrm{d}g(s).\end{gathered}$$

If (σ,τ) is a couple of lower and upper slopes to the problem (1.1), and if Σ and T are defined by (3.3), then the conditions in Definition 3.1 expressed in terms of Σ and T are

$$\begin{gathered}\Sigma'(t) \leqslant T'(t), \quad \Sigma''(t) \geqslant f(t,x,\Sigma'(t)), \quad T''(t) \leqslant f(t,x,T'(t)), \quad \forall\, t \in [0,1],\\ \Sigma(0) = 0 = T(0), \quad \Sigma'(1) \leqslant \int_0^1 \Sigma'(s)\,\mathrm{d}g(s), \quad T'(1) \geqslant \int_0^1 T'(s)\,\mathrm{d}g(s).\end{gathered}$$

The classical lower and upper solutions $\alpha, \beta \in C^2([0,1])$ to the problem (1.1) are defined by the conditions

$$\begin{gathered}\alpha(t) \leqslant \beta(t), \quad \alpha''(t) \geqslant f(t,\alpha(t),\alpha'(t)), \quad \beta''(t) \leqslant f(t,\beta(t),\beta'(t)), \quad \forall\, t \in [0,1],\\ \alpha(0) \leqslant 0 \leqslant \beta(0), \quad \alpha'(1) \leqslant \int_0^1 \alpha'(s)\,\mathrm{d}g(s), \quad \beta'(1) \geqslant \int_0^1 \beta'(s)\,\mathrm{d}g(s).\end{gathered}$$

Because conditions $\Sigma'(t) \leqslant T'(t)$ and $\Sigma(0) = 0 = T(0)$ imply that $\Sigma(t) \leqslant T(t)$ for all $t \in [0,1]$, one sees immediately that if (σ,τ) is a couple of lower and upper slopes to the problem (1.1), then (Σ, T) is an ordered couple of lower and upper solutions to the problem (1.1). In this sense, Theorem 4.1 below can be seen as a necessary and sufficient condition for the existence of a solution to the problem (1.1).

On the other hand, we will see in Theorem 4.1 that the existence of a couple of lower and upper slopes to (1.1) implies the existence of a solution to (1.1) with its derivative located between them. On the other hand, without a supplementary condition of Nagumo type upon f with respect to its last variable, the existence of an ordered couple of lower and upper solutions to (1.1) does not guarantee the existence of a solution located between them.

Finally, one should notice that a solution u to the problem (1.1) corresponds to a couple (u',u') of (equal) lower and upper slopes to the problem (1.1).

4. Existence result

We now prove that the existence of a couple (σ, τ) of lower and upper slopes for the problem (1.1) implies the existence and localization of a solution.

Theorem 4.1. *Let $f\colon [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous, and let $g\colon [0,1]\to\mathbb{R}$ be increasing and satisfy condition* (1.2). *If the problem* (1.1) *has a couple (σ,τ) of lower and upper slopes, then it has a solution x such that $\sigma(t)\leqslant x'(t)\leqslant\tau(t)$ and $\Sigma(t)\leqslant x(t)\leqslant T(t)$ for all $t\in[0,1]$.*

P r o o f. Define continuous functions $\gamma\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ and $\delta\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ by

$$\gamma(t,y)=\begin{cases}\sigma(t) & \text{if } y<\sigma(t),\\ y & \text{if } \sigma(t)\leqslant y\leqslant\tau(t),\\ \tau(t) & \text{if } y>\tau(t),\end{cases}$$

$$\delta(t,x)=\begin{cases}\Sigma(t) & \text{if } x<\Sigma(t),\\ x & \text{if } \Sigma(t)\leqslant x\leqslant T(t),\\ T(t) & \text{if } x>T(t).\end{cases}$$

Let us consider the auxiliary boundary value problem

$$\begin{gathered} x''=x'-\gamma(t,x')+f(t,\delta(t,x),\gamma(t,x')),\\ x(0)=0,\quad x'(1)=\int_0^1 x'(s)\,\mathrm{d}g(s). \end{gathered}\tag{4.1}$$

We first prove by contradiction that if $x(t)$ is a solution of (4.1), then

$$\sigma(t)\leqslant x'(t)\leqslant\tau(t),\quad \forall\, t\in[0,1].\tag{4.2}$$

If there is $t_0\in[0,1]$ such that $x'(t_0)<\sigma(t_0)$, then $x'-\sigma$ reaches a negative minimum on $[0,1]$ at some $t_1\in[0,1]$, namely

$$x'(t_1)-\sigma(t_1)<0.\tag{4.3}$$

If $t_1\in(0,1)$, then $x''(t_1)=\sigma'(t_1)$ and hence, using (4.3) and the definition of γ, and the definition of a lower slope,

$$\sigma'(t_1)=x''(t_1)=x'(t_1)-\sigma(t_1)+f\bigl(t_1,\delta(t_1,x(t_1)),\sigma(t_1)\bigr)<\sigma'(t_1),$$

a contradiction. If $t_1=0$, then $x''(t_1)-\sigma'(t_1)\geqslant 0$, and we obtain, in a similar way, the contradiction

$$\sigma'(t_1)\leqslant x''(t_1)<\sigma'(t_1).$$

If $t_1 = 1$, because of the previous cases, we can assume without loss of generality that $x' - \sigma$ does not attain its minimum on $[0, 1)$, i.e. that

$$x'(1) - \sigma(1) < x'(s) - \sigma(s), \quad \forall\, s \in [0, 1).$$

Integrating this inequality over $[0, 1]$ with respect to the measure $\mathrm{d}g$ and using the increasing character of g and (1.2), we obtain

$$x'(1) - \sigma(1) < \int_0^1 x'(s)\, \mathrm{d}g(s) - \int_0^1 \sigma(s)\, \mathrm{d}g(s),$$

which gives

$$\int_0^1 \sigma(s)\, \mathrm{d}g(s) < \sigma(1),$$

a contradiction with the definition of lower slope. A completely similar reasoning shows that $x(t) \leqslant \tau(t)$ for all $t \in [0, 1]$. Thus (4.2) is proved. It follows from (4.2) and the first boundary condition that every solution x of (4.1) satisfies the inequality

$$\Sigma(t) \leqslant x(t) \leqslant T(t), \quad \forall\, t \in [0, 1],$$

so that, by the definition of γ and δ, x is a solution of (1.1).

Now, if we define

$$X = \left\{ x \in C^1([0, 1])\colon\ x(0) = 0,\ x'(1) = \int_0^1 x'(s)\, \mathrm{d}g(s) \right\},$$

with the norm $\|x\| = \|x'\|_\infty$, and the operators $L\colon D(L) \subset X \to C([0, 1])$, $F\colon X \to C([0, 1])$ by

$$D(L) = X \cap C^2([0, 1]), \quad Lx = x'' - x',$$

and

$$Fx = -\gamma(\cdot, x'(\cdot)) + f(\cdot, \delta(\cdot, x(\cdot)), \gamma(\cdot, x'(\cdot))),$$

then it is easy to verify that F is continuous, that there exists $K > 0$ such that $\|Fx\|_\infty \leqslant K$ for all $x \in X$, and that the problem (4.1) is equivalent to the equation $Lx = F(x)$ in X. Consequently, as $L^{-1} = S$ with S defined in Section 2, the problem (4.1) is equivalent to the fixed point problem $x = SFx$, with SF a compact mapping on X sending X to a closed ball of center 0 and radius MK in X with M given in (2.3). Schauder's fixed point theorem implies that SF has a fixed point in X, i.e. that the problem (4.1) has a solution, which is also a solution of (1.1). $\square$

The special case, where σ and τ are constant provides the following existence condition.

Corollary 4.1. *Let* $f\colon [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *be continuous, and let* $g\colon [0,1]\to\mathbb{R}$ *be increasing and satisfy condition* (1.2). *If there exist real numbers* $\sigma\leqslant\tau$ *such that*

$$f(t,x,\sigma)\leqslant 0\leqslant f(t,x,\tau),\quad \forall\,(t,x)\in[0,1]\times[\sigma,\tau],$$

then the problem (1.1) *has at least one solution* x *such that* $\sigma\leqslant x'(t)\leqslant\tau$ *and* $\sigma t\leqslant x(t)\leqslant\tau t$ *for all* $t\in[0,1]$.

Example 4.1. Let us consider the problem

$$x''=a(t,x)\,x'-b(t,x),\quad x(0)=0,\quad x'(1)=\int_0^1 x'(s)\,\mathrm{d}g(s), \tag{4.4}$$

where $g\colon [0,1]\to\mathbb{R}$ is increasing and satisfies (1.2), $a,b\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous and such that

$$a(t,x)\geqslant b(t,x)>0\quad\text{for } (t,x)\in[0,1]\times[0,1].$$

Set $\sigma=0$, $\tau=1$ for $t\in[0,1]$ and observe that

$$f(t,x,0)=-b(t,x)\leqslant 0$$

and

$$f(t,x,1)=a(t,x)-b(t,x)\geqslant 0,$$

for every $(t,x)\in[0,1]\times[0,1]$. Corollary 4.1 implies that there is a solution x to the problem (4.4) such that $0\leqslant x'(t)\leqslant 1$ and $0\leqslant x(t)\leqslant t$ for all $t\in[0,1]$.

The special case where f does not depend upon x immediately leads to the following result for the first order equation.

Corollary 4.2. *Let* $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ *be continuous and* $g\colon [0,1]\to\mathbb{R}$ *an increasing function such that condition* (1.2) *holds. If there exist* $\sigma\in C^1([0,1])$, $\tau\in C^1([0,1])$ *such that* $\sigma(t)\leqslant\tau(t)$ *for all* $t\in[0,1]$, *and*

$$\sigma'(t)\geqslant f(t,\sigma(t)),\quad \sigma(1)\leqslant\int_0^1\sigma(s)\,\mathrm{d}g(s),$$
$$\tau'(t)\leqslant f(t,\tau(t)),\quad \tau(1)\geqslant\int_0^1\tau(s)\,\mathrm{d}g(s)$$

for all $t\in[0,1]$, *then the problem* (3.5) *has at least one solution* y *such that* $\sigma(t)\leqslant y(t)\leqslant\tau(t)$ *for all* $t\in[0,1]$.

5. Couples of strict lower and upper slopes

We now introduce and study the stronger concept of a couple of strict lower and upper slopes.

Definition 5.1. We say that (σ, τ) is a *couple of strict lower and upper slopes* to (1.1) if $\sigma, \tau \in C^1([0,1], \mathbb{R})$ are such that $\sigma(t) \leqslant \tau(t)$ for all $t \in [0,1]$, and the first inequalities in (3.1) and (3.2) are strict.

The following result will imply that, in case of a couple of strict lower and upper slopes, the localization of the solution x in Theorem 4.1 will be given by strict inequalities. We assume throughout that $f\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $g\colon [0,1] \to \mathbb{R}$ is increasing and verifies (1.2).

Lemma 5.1. *Let* (σ, τ) *be a couple of strict lower and upper slopes for* (1.1) *and let* x *be a solution of* (1.1) *such that* $\sigma(t) \leqslant x'(t) \leqslant \tau(t)$ *for all* $t \in [0,1]$. *Then* $\sigma(t) < x'(t) < \tau(t)$ *and* $\Sigma(t) < x(t) < T(t)$ *for all* $t \in (0,1]$.

P r o o f. By assumptions, $\Sigma(t) \leqslant x(t) \leqslant T(t)$ for all $t \in [0,1]$. Let us consider the function $\sigma - x'$ and let $\sigma(t) \leqslant x'(t)$ for all $t \in [0,1]$. We shall show that $\sigma(t) - x'(t) < 0$ for all $t \in [0,1]$. Suppose on the contrary that for some $t_0 \in [0,1]$ we have $\sigma(t_0) - x'(t_0) = 0$. If $t_0 \in [0,1)$, we obtain

$$\begin{aligned} 0 \geqslant \sigma'(t_0) - x''(t_0) &= \sigma'(t_0) - f(t_0, x(t_0), x'(t_0)) \\ &= \sigma'(t_0) - f(t_0, x(t_0), \sigma(t_0)) > 0, \end{aligned}$$

a contradiction. If $t_0 = 1$, we can assume, by what precedes, that $\sigma(1) - x'(1) = 0$; then, for all $t \in [0,1)$, we get

$$\sigma(t) - x'(t) < \sigma(1) - x'(1) = 0, \quad \forall\, t \in [0,1).$$

Consequently, integrating the above inequality with respect to $\mathrm{d}g$, we reach a contradiction with Definition 5.1, namely

$$\sigma(1) > \int_0^1 \sigma(s)\, \mathrm{d}g(s).$$

In the same way one can prove that $x'(t) < \tau(t)$, $t \in [0,1]$. The strict inequalities for $x(t)$ follow immediately. $\square$

From Theorem 4.1 and Lemma 5.1 we get the following result.

Corollary 5.1. *If the problem* (1.1) *has a couple* (σ, τ) *of strict lower and upper slopes, then it has a solution* x *such that* $\sigma(t) < x'(t) < \tau(t)$ *and* $\Sigma(t) < x(t) < T(t)$ *for all* $t \in [0,1]$.

6. How to find a couple of lower and upper slopes

We assume throughout that $f\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $g\colon [0,1] \to \mathbb{R}$ is increasing and verifies condition (1.2), and start with a necessary condition for the existence of a couple of lower and upper slopes.

Lemma 6.1. *If the function* f *has a constant sign on* $[0,1] \times \mathbb{R} \times \mathbb{R}$, *then there is no couple of lower and upper slopes to the problem* (1.1).

Proof. Let the problem (1.1) have a couple (σ, τ) of lower and upper slopes. Assume that $f(t,x,y) > 0$ for $(t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R}$. Then, by the definition of the lower slope (3.1),

$$\sigma'(t) > 0, \ \forall\, t \in [0,1].$$

It means that σ is increasing and we have $\sigma(t) < \sigma(1)$ for $t \in [0,1)$. Consequently,

$$\sigma(1) > \int_0^1 \sigma(s)\, \mathrm{d}g(s),$$

a contradiction with the definition of the lower slope. If f is negative, then using (3.2) we similarly reach a contradiction. $\square$

The argument of the proof of Lemma 6.1 shows infact that the following result holds.

Lemma 6.2. *Let* $\sigma, \tau \in C^1([0,1])$ *be such that* $\sigma(t) \leqslant \tau(t)$ *for all* $t \in [0,1]$. *If* $\sigma(t) < \sigma_M = \sigma(1)$ *or* $\tau(t) > \tau_m = \tau(1)$ *for* $t \in [0,1)$, *then* (σ, τ) *cannot be a couple of lower and upper slopes to the problem* (1.1).

The following two results are standard and elementary. We give proofs for completeness.

Lemma 6.3. *Let* $\sigma, \tau \in C^1([0,1])$ *and* $L > 0$ *be such that*

$$\sigma'(t) - \tau'(t) \geqslant L[\sigma(t) - \tau(t)], \quad \forall\, t \in [0,1], \ \text{and} \ \sigma(1) \leqslant \tau(1). \tag{6.1}$$

Then $\sigma(t) \leqslant \tau(t)$ *for every* $t \in [0,1]$. *If inequality in* (6.1) *is replaced by*

$$\sigma'(t) - \tau'(t) > L[\sigma(t) - \tau(t)], \quad \forall\, t \in [0,1], \ \text{and} \ \sigma(1) \leqslant \tau(1), \tag{6.2}$$

then $\sigma(t) < \tau(t)$ *for every* $t \in [0,1]$.

P r o o f. Letting $\gamma(t) = \sigma(t) - \tau(t)$ $(t \in [0,1])$, we see, by multiplying both members by e^{-Lt} that inequalities (6.1) and (6.2) are, respectively, equivalent to

$$\begin{aligned}
&(\mathrm{e}^{-Lt}\gamma(t))' \geqslant 0, \quad \forall\, t \in [0,1], \quad \gamma(1) \leqslant 0,\\
&(\mathrm{e}^{-Lt}\gamma(t))' > 0, \quad \forall\, t \in [0,1], \quad \gamma(1) \leqslant 0.
\end{aligned}$$

Integration from t to 1 gives

$$\begin{aligned}
&0 \geqslant \mathrm{e}^{-L}\gamma(1) \geqslant \mathrm{e}^{-Lt}\gamma(t), \quad \forall\, t \in [0,1],\\
&0 \geqslant \mathrm{e}^{-L}\gamma(1) > \mathrm{e}^{-Lt}\gamma(t), \quad \forall\, t \in [0,1],
\end{aligned}$$

and the results follow. □

Lemma 6.4. *Let $h_j\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ $(j = 1,2)$ be continuous and such that there exists $L > 0$ for which*

$$h_j(t,y) - h_j(t,z) \geqslant L(y - z) \tag{6.3}$$

for all $t \in [0,1]$, $y, z \in \mathbb{R}$ and $j = 1,2$. Let σ, τ be solutions to the first order problems

$$y' = h_1(t,y), \quad y(1) = \int_0^1 y(s)\,\mathrm{d}g(s) \tag{6.4}$$

$$y' = h_2(t,y), \quad y(1) = \int_0^1 y(s)\,\mathrm{d}g(s) \tag{6.5}$$

such that $\sigma(1) \leqslant \tau(1)$. Then $\sigma(t) \leqslant \tau(t)$ for all $t \in [0,1]$.

P r o o f. Let σ, τ be, respectively, solutions to problems (6.4) and (6.5). Then we have

$$\sigma'(t) - \tau'(t) = h_1(t,\sigma(t)) - h_2(t,\tau(t)) \geqslant L[\sigma(t) - \tau(t)], \quad (t \in [0,1]).$$

The result follows from Lemma 6.3. □

The special case where the $h_j(t,y)$ are affine functions of y reads as follows.

Corollary 6.1. *Let $L > 0$, $r_1, r_2 \in C([0,1])$ satisfy $r_1(t) \geqslant r_2(t)$, $t \in [0,1]$, and let σ, τ be solutions to the first order problems*

$$y' - Ly = r_1(t), \quad y(1) = \int_0^1 y(s)\,\mathrm{d}g(s) \tag{6.6}$$

and

$$y' - Ly = r_2(t), \quad y(1) = \int_0^1 y(s)\,\mathrm{d}g(s) \tag{6.7}$$

such that $\sigma(1) \leqslant \tau(1)$. Then $\sigma(t) \leqslant \tau(t)$ for all $t \in [0,1]$.

Proof. Letting $h_j(t,y) := Ly + r_j(t)$, $(j = 1,2)$, we have

$$h_1(t,y) - h_2(t,z) = L(y-z) + r_1(t) - r_2(t) \geqslant L(y-z)$$

and we can apply Lemma 6.4. □

Theorem 6.1. *Let σ, τ be, respectively, solutions to the problems* (6.4) *and* (6.5) *such that $\sigma(1) \leqslant \tau(1)$, where h_1, $h_2 \in C([0,1] \times \mathbb{R})$ verify* (6.3)*. Moreover, assume that the following conditions hold:*

$$f(t,x,\sigma(t)) \leqslant h_1(t,\sigma(t)), \quad f(t,x,\tau(t)) \geqslant h_2(t,\tau(t))$$

for all $(t,x) \in S_{\Sigma,T}$. Then the problem (1.1) *has at least one solution x such that $\sigma(t) \leqslant x'(t) \leqslant \tau(t)$ and $\Sigma(t) \leqslant x(t) \leqslant T(t)$ for all $t \in [0,1]$.*

Proof. Let σ, τ be solutions to the defined above problems. Then, by Lemma 6.4, we have $\sigma(t) \leqslant \tau(t)$, and hence $\Sigma(t) \leqslant T(t)$ for all $t \in [0,1]$. Moreover, we have

$$f(t,x,\sigma(t)) \leqslant \sigma'(t), \quad f(t,x,\tau(t)) \geqslant \tau'(t)$$

for all $(t,x) \in S_{\Sigma,T}$. Hence, due to Definition 3.1, (σ,τ) is a couple of lower and upper slopes for (1.1). Consequently, according to Theorem 4.1, there exists a solution x to the problem (1.1) such that $\sigma(t) \leqslant x'(t) \leqslant \tau(t)$ and $\Sigma(t) \leqslant x(t) \leqslant T(t)$ for all $t \in [0,1]$. □

Corollary 6.2. *Let σ, τ be, respectively, solutions to the problems* (6.6) *and* (6.7) *such that $\sigma(1) \leqslant \tau(1)$, where $L > 0$, r_1, $r_2 \in C([0,1])$ and $r_1(t) \geqslant r_2(t)$, $t \in [0,1]$. Moreover, assume that the following conditions hold:*

$$f(t, x, \sigma(t)) \leqslant r_1(t) + L\sigma(t), \quad f(t, x, \tau(t)) \geqslant r_2(t) + L\tau(t)$$

for all $(t, x) \in S_{\Sigma,T}$. Then the problem (1.1) *has at least one solution x such that $\sigma(t) \leqslant x'(t) \leqslant \tau(t)$ and $\Sigma(t) \leqslant x(t) \leqslant T(t)$ for all $t \in [0,1]$.*

P r o o f. Take $h_j(t, y) = Ly + r_j(t)$ $(j = 1, 2)$ in Theorem 6.1. □

By Lemma 6.3 and Theorem 6.1 we have

Corollary 6.3. *Let $h_1, h_2 \in C([0,1] \times \mathbb{R})$, and $\sigma, \tau \in C^1([0,1])$ be functions defined in Theorem* 6.1*. Assume that*

$$f(t, x, \sigma(t)) < h_1(t, \sigma(t)), \quad f(t, x, \tau(t)) > h_2(t, \tau(t))$$

for all $(t, x) \in S_{\Sigma,T}$, and

$$h_1(t, \sigma(t)) - h_2(t, \tau(t)) > L(\sigma(t) - \tau(t))$$

for all $t \in [0,1]$. Then the problem (1.1) *has at least one solution x such that $\sigma(t) < x'(t) < \tau(t)$ for all $t \in [0,1]$ and $\Sigma(t) < x(t) < T(t)$ for all $t \in (0,1]$.*

P r o o f. We have

$$\sigma'(t) - \tau'(t) = h_1(t, \sigma(t)) - h_2(t, \tau(t)) > L[\sigma(t) - \tau(t)],$$

so that Lemma 6.3 implies that the solution given by Theorem 6.1 satisfies the inequalities $\sigma(t) < x'(t) < \tau(t)$ for all $t \in [0,1]$. □

A special case of Corollary 6.3 is

Corollary 6.4. *Let $r_1, r_2 \in C([0,1])$ and $\sigma, \tau \in C^1([0,1])$ be functions defined in Corollary* 6.2*. Assume that*

$$\sigma(1) < \tau(1) \quad \text{and} \quad r_1(t) > r_2(t), \quad \forall\, t \in [0,1].$$

Moreover, assume that

$$f(t, x, \sigma(t)) < r_1(t) + L\sigma(t), \quad \text{and} \quad f(t, x, \tau(t)) > r_2(t) + L\tau(t)$$

for all $(t,x)\in S_{\Sigma,T}$. *Then the problem* (1.1) *has at least one solution* x *such that* $\sigma(t)<x'(t)<\tau(t)$ *for all* $t\in[0,1]$ *and* $\Sigma(t)<x(t)<T(t)$ *for all* $t\in(0,1]$.

Example 6.1. Set $r_1(t)=0$ and $r_2(t)=-1$ and consider the problem (4.4) from Example 4.1. Then $\sigma(t)=0$, $\tau(t)=1$, $t\in[0,1]$ and $\sigma(1)<\tau(1)$. Moreover, we have

$$f(t,x,\sigma(t))=-b(t,x)<r_1(t)+\sigma(t)=0$$

and

$$f(t,x,\tau(t))\geqslant a_m-b_M>r_2(t)+\tau(t)=0.$$

Hence, from Corollary 6.4, the problem (4.4) has a solution x such that $0<x'(t)<1$ for all $t\in[0,1]$.

Example 6.2. Let $g(s)=s$, so that $\int_0^1 x'(s)\,\mathrm{d}s=x(1)-x(0)$, and consider the two-point boundary value problem

$$x''=x'-ta(x),\quad x(0)=0,\quad x'(1)=x(1)-x(0), \tag{6.8}$$

where $a\colon\mathbb{R}\to\mathbb{R}$ is continuous and positive.

Set $r_1(t)=\mathrm{e}^{2t}$ and $r_2(t)=-t$. Since σ and τ are solutions to first order problems, we obtain

$$\sigma(t)=\Big(\frac12-\frac{\mathrm{e}^2}{2}\Big)\mathrm{e}^t+\mathrm{e}^{2t},\quad \tau(t)=t+1+\frac{\mathrm{e}^t}{2}.$$

Moreover, we have

$$2+\frac{\mathrm{e}}{2}=\tau(1)>\sigma(1)=\frac{\mathrm{e}}{2}-\frac{\mathrm{e}^3}{2}+\mathrm{e}^2.$$

Now, since $r_1(t)>r_2(t)$, Lemma 6.3 implies that $\sigma(t)<\tau(t)$ for every $t\in[0,1]$.

Let us notice that f satisfies the conditions of Corollary 6.2. Indeed, we have

$$\begin{aligned}f(t,x,\sigma(t))&=\Big(\frac12-\frac{\mathrm{e}^2}{2}\Big)\mathrm{e}^t+\mathrm{e}^{2t}-ta(x)\\&<\Big(\frac12-\frac{\mathrm{e}^2}{2}\Big)\mathrm{e}^t+2\mathrm{e}^{2t}=h_\sigma(t)+\sigma(t)\end{aligned}$$

and

$$\begin{aligned}f(t,x,\tau(t))&=t+1+\frac{\mathrm{e}^t}{2}-ta(x)\\&\geqslant t+1+\frac{\mathrm{e}^t}{2}-ta_M\geqslant h_\tau(t)+\tau(t),\end{aligned}$$

when $\Sigma(t)\leqslant x\leqslant T(t)$ and $t\in[0,1]$, and $a_M\leqslant 1$. Hence, from Corollary 6.2, there exists at least one solution x to the problem (6.8) such that

$$\Big(\frac12-\frac{\mathrm{e}^2}{2}\Big)\mathrm{e}^t+\mathrm{e}^{2t}\leqslant x'(t)\leqslant t+1+\frac{\mathrm{e}^t}{2}$$

for all $t\in[0,1]$.

The existence of suitable couples of strict lower and upper slopes implies the existence of at least three solutions. For two continuous functions x_1 and x_2 on $[0,1]$, we defined the relation $x_1 \leqslant x_2$ by $x_1(t) \leqslant x_2(t)$ for all $t \in [0,1]$.

Theorem 7.1. *Let $\sigma_i \in C^1([0,1])$ and $\tau_i \in C^1([0,1])$ $(i=1,2)$ be such that*

$$\sigma_1(t) \leqslant \sigma_2(t) \leqslant \tau_2(t), \quad \sigma_1(t) \leqslant \tau_1(t) \leqslant \tau_2(t), \quad \forall\, t \in [0,1], \ \sigma_2 \nleqslant \tau_1, \tag{7.1}$$

and such that (σ_1, τ_1), (σ_2, τ_2) and (σ_1, τ_2) are couples of strict lower and upper slopes to the problem (1.1). *Then the problem* (1.1) *has at least three solutions x_1, x_2 and x_3 such that*

$$\sigma_1(t) \leqslant x_1'(t) \leqslant \tau_1(t), \quad \sigma_2(t) \leqslant x_2'(t) \leqslant \tau_2(t), \quad \forall\, t \in [0,1],$$

and

$$x_3' \nleqslant \tau_1, \quad x_3' \ngeqslant \sigma_2$$

for $t \in [0,1]$.

Proof. Define $\Sigma_i \colon [0,1] \to \mathbb{R}$ and $T_i \colon [0,1] \to \mathbb{R}$ by

$$\Sigma_i(t) = \int_0^t \sigma_i(s)\,\mathrm{d}s, \quad T_i(t) = \int_0^t \tau_i(s)\,\mathrm{d}s \quad (i=1,2),$$

and define continuous functions $\gamma_i \colon [0,1]\times\mathbb{R} \to \mathbb{R}$ and $\delta_i \colon [0,1]\times\mathbb{R} \to \mathbb{R}$ $(i=1,2,3)$ by

$$\gamma_1(t,y) = \begin{cases} \sigma_1(t) & \text{if } y < \sigma_1(t), \\ y & \text{if } \sigma_1(t) \leqslant y \leqslant \tau_2(t), \\ \tau_2(t) & \text{if } y > \tau_2(t), \end{cases}$$

$$\delta_1(t,x) = \begin{cases} \Sigma_1(t) & \text{if } x < \Sigma_1(t), \\ x & \text{if } \Sigma_1(t) \leqslant x \leqslant T_2(t), \\ T_2(t) & \text{if } x > T_2(t), \end{cases}$$

$$\gamma_2(t,y) = \begin{cases} \sigma_2(t) & \text{if } y < \sigma_2(t), \\ y & \text{if } \sigma_2(t) \leqslant y \leqslant \tau_2(t), \\ \tau_2(t) & \text{if } y > \tau_2(t), \end{cases}$$

$$\delta_2(t,x) = \begin{cases} \Sigma_2(t) & \text{if } x < \Sigma_2(t), \\ x & \text{if } \Sigma_2(t) \leqslant x \leqslant T_2(t), \\ T_2(t) & \text{if } x > T_2(t), \end{cases}$$

$$\gamma_3(t,y) = \begin{cases} \sigma_1(t) & \text{if } y < \sigma_1(t), \\ y & \text{if } \sigma_1(t) \leqslant y \leqslant \tau_1(t), \\ \tau_1(t) & \text{if } y > \tau_1(t), \end{cases}$$

and

$$\delta_3(t,x) = \begin{cases} \Sigma_1(t) & \text{if } x < \Sigma_1(t), \\ x & \text{if } \Sigma_1(t) \leqslant x \leqslant T_1(t), \\ T_1(t) & \text{if } x > T_1(t). \end{cases}$$

Consider three auxiliary boundary value problems

$$\begin{gathered} x'' = x' - \gamma_1(t,x') + f(t, \delta_1(t,x), \gamma_1(t,x')), \\ x(0) = 0, \quad x'(1) = \int_0^1 x'(s)\,\mathrm{d}g(s), \end{gathered} \tag{7.2}$$

$$\begin{gathered} x'' = x' - \gamma_2(t,x') + f(t, \delta_2(t,x), \gamma_2(t,x')), \\ x(0) = 0, \quad x'(1) = \int_0^1 x'(s)\,\mathrm{d}g(s) \end{gathered} \tag{7.3}$$

and

$$\begin{gathered} x'' = x' - \gamma_3(t,x') + f(t, \delta_3(t,x), \gamma_3(t,x')), \\ x(0) = 0, \quad x'(1) = \int_0^1 x'(s)\,\mathrm{d}g(s). \end{gathered} \tag{7.4}$$

Since (σ_1, τ_1) and (σ_2, τ_2) are two couples of strict lower slopes and upper slopes, following the same way as in the proof of Theorem 4.1, one can show that all possible solutions x to the problem (7.2) are such that $\sigma_1(t) < x'(t) < \tau_2(t)$ for all $t \in [0,1]$ and $\Sigma_1(t) < x(t) < T_2(t)$ for all $t \in (0,1]$. Similarly, all solutions x to the problem (7.3) are such that $\sigma_2(t) < x'(t) < \tau_2(t)$ for all $t \in [0,1]$, and $\Sigma_2(t) < x(t) < T_2(t)$ for all $t \in (0,1]$, and all solutions x to the problem (7.4) are such that $\sigma_1(t) < x'(t) < \tau_1(t)$ for all $t \in [0,1]$, and $\Sigma_1(t) < x(t) < T_1(t)$ for all $t \in (0,1]$. Consequently, the solutions to the problems (7.2), (7.3) and (7.4) are also solutions to (1.1).

On the other hand, the operators SF_1, SF_2 and SF_3, where

$$\begin{aligned} F_1 x &= -\gamma_1(\cdot, x'(\cdot)) + f\big(\cdot, \delta_1(\cdot, x(\cdot)), \gamma_1(\cdot, x'(\cdot))\big), \\ F_2 x &= -\gamma_2(\cdot, x'(\cdot)) + f\big(\cdot, \delta_2(\cdot, x(\cdot)), \gamma_2(\cdot, x'(\cdot))\big) \end{aligned}$$

and

$$F_3 x = -\gamma_3(\cdot, x'(\cdot)) + f\big(\cdot, \delta_3(\cdot, x(\cdot)), \gamma_3(\cdot, x'(\cdot))\big),$$

map compactly the space X, respectively, into the closed ball of center 0 and radii MK_1, MK_2 and MK_3 of X, where M is given by (2.3). Set

$$L > \max\{MK_1, MK_2, MK_3, \|\sigma_1\|, \|\tau_2\|\}$$

and

$$\Omega = \{x \in X\colon \ \|x'\|_\infty < L\}.$$

Then, using homotopies $I - \lambda SF_i$, $\lambda \in [0,1]$, we get

$$\deg(I - SF_i, \Omega, 0) = \deg(I, \Omega, 0) = 1 \quad (i = 1, 2, 3). \tag{7.5}$$

Let

$$\Omega_{\sigma_2} = \{x \in \Omega\colon \ x'(t) > \sigma_2(t), \ t \in [0,1]\}$$

and

$$\Omega_{\tau_1} = \{x \in \Omega\colon \ x'(t) < \tau_1(t), \ t \in [0,1]\}.$$

Observe that, by the definition of Ω_{σ_2}, one has

$$\deg(I - SF_2, \Omega \setminus \overline{\Omega}_{\sigma_2}, 0) = 0.$$

Consequently,

$$\begin{aligned} \deg(I - SF_1, \Omega_{\sigma_2}, 0) &= \deg(I - SF_2, \Omega_{\sigma_2}, 0) \\ &= \deg(I - SF_2, \Omega, 0) + \deg(I - SF_2, \Omega \setminus \overline{\Omega}_{\sigma_2}, 0) = 1. \end{aligned} \tag{7.6}$$

Similarly, by the definition of Ω_{τ_1}, one has

$$\deg(I - SF_3, \Omega \setminus \overline{\Omega}_{\tau_1}, 0) = 0.$$

Hence, we obtain

$$\begin{aligned} \deg(I - SF_1, \Omega_{\tau_1}, 0) &= \deg(I - SF_3, \Omega_{\tau_1}, 0) \\ &= \deg(I - SF_3, \Omega, 0) + \deg(I - SF_3, \Omega \setminus \overline{\Omega}_{\tau_1}, 0) = 1. \end{aligned} \tag{7.7}$$

Now, observe that the sets Ω_{σ_2}, Ω_{τ_1} and $\Omega \setminus \overline{\Omega_{\sigma_2} \cup \Omega_{\tau_1}}$ are nonempty, open and disjoint. Moreover, since (σ_1, τ_1) and (σ_2, τ_2) are couples of strict lower slopes and upper slopes,

$$x \notin (I - SF_1)\partial\Omega_{\sigma_2} \cup (I - SF_1)\partial\Omega_{\tau_1} \cup (I - SF_1)\partial(\Omega \setminus \overline{\Omega_{\sigma_2} \cup \Omega_{\tau_1}}).$$

Hence, by the additivity property of Leray-Schauder's degree, we obtain

$$\begin{aligned}\deg(I - SF_1, \Omega, 0) = \; &\deg(I - SF_1, \Omega_{\sigma_2}, 0) + \deg(I - SF_1, \Omega_{\tau_1}, 0)\\ &+ \deg(I - SF_1, \Omega \setminus \overline{\Omega_{\sigma_2} \cup \Omega_{\tau_1}}, 0).\end{aligned}$$

Therefore, using (7.5) with $i = 1$, (7.6) and (7.7), we obtain

$$\deg(I - SF_1, \Omega \setminus \overline{\Omega_{\sigma_2} \cup \Omega_{\tau_1}}, 0) = -1.$$

Hence, SF_1 has at least three fixed points x_1, x_2 and x_3 such that $x_1 \in \Omega_{\tau_1}$, $x_2 \in \Omega_{\sigma_2}$ and $x_3 \in \Omega \setminus \overline{\Omega_{\sigma_2} \cup \Omega_{\tau_1}}$, which completes the proof. □

Remark 7.1. In the case of classical lower and upper solutions, the fact that (α_1, β_1) and (α_2, β_2) are ordered couples of lower and upper solutions to the problem (1.1) such that $\alpha_1(t) \leqslant \beta_2(t)$ for all $t \in [0,1]$, implies that (α_1, β_2) is an ordered couple of lower and upper solutions to (1.1). On the other hand, when (σ_1, τ_1) and (σ_2, τ_2) are couples of lower and upper slopes to (1.1) such that $\sigma_1(t) \leqslant \tau_2(t)$ for all $t \in [0,1]$, (σ_1, τ_2) need not be a couple of lower and upper slopes to (1.1). This is why the assumption is added in Theorem 7.1 in contrast to the similar three solutions theorems in the frame of lower and upper solutions [6], [17], [16].

Example 7.1. Let us consider the problem (1.1) with the function $f\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of the form $f(t,x,y) = a(t,x)b(y)$, where $a\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ and $b\colon \mathbb{R} \to \mathbb{R}$ are continuous, $a(t,x) \geqslant 1$ for all $(t,x) \in [0,1] \times \mathbb{R}$ and b has the following properties. There are at least three points y such that $b(y) = 0$ belonging, respectively, to the intervals $(-3,-2)$, $(-1,0)$ and $(1,2)$. Other possible zeros of b belong to $(\infty, -4) \cup (3, \infty)$. Assume that in the interval $[-4,3]$, where b is positive, if it has an extremum y_0, then $b(y_0) > 1$. Similarly, if somewhere on $[-4,3]$ b is negative and has an extremum y_0, then $b(y_0) < -1$. Moreover, let

$$b(-4), b(-3), b(0), b(1) < -1, \quad b(-2), b(-1), b(2), b(3) > 1.$$

Set

$$\sigma_1(t) = -t - 3, \quad \sigma_2(t) = -t + 1, \quad \tau_1(t) = t - 2, \quad \tau_2(t) = t + 2.$$

Now, one can easily check that the assumptions of Theorem 7.1 are satisfied. Consequently, there are at least three solutions to the problem (1.1).

For example, the problem

$$x'' = a(t,x)\Big(x' + \frac{5}{2}\Big)\Big(x' + \frac{1}{2}\Big)\Big(x' - \frac{3}{2}\Big), \quad x(0) = 0, \quad x'(1) = \int_0^1 x'(s)\,\mathrm{d}g(s)$$

with $a \in C([0,1] \times \mathbb{R})$, $a(t,x) \geqslant 1$ for all $(t,x) \in [0,1] \times \mathbb{R}$, $g\colon [0,1] \to \mathbb{R}$ increasing and $\int_0^1 \mathrm{d}g = 1$ has at least three solutions.

On the other hand, if we consider the family of couples of lower and upper slopes

$$\sigma_{1,k}(t) = -t - 4k + 1, \quad \tau_{1,k}(t) = t - 4k + 2 \quad (k \in \mathbb{Z}),$$

and define the function $b\colon \mathbb{R} \to \mathbb{R}$ as

$$b(y) = -2 \sin \Big[\frac{1}{2} \pi \Big(y - \frac{1}{2} \Big) \Big],$$

we observe that $f(t,x,y) = a(t,x)b(y)$ with a like above satisfies the above assumptions for each $k \in \mathbb{Z}$. Consequently, Theorem 7.1 implies that the problem

$$x'' = -2a(t,x) \sin \Big[\frac{1}{2} \pi \Big(x' - \frac{1}{2} \Big) \Big], \quad x(0) = 0, \quad x'(1) = \int_0^1 x'(s) \,\mathrm{d}g(s)$$

has infinitely many solutions.

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