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The effect of phase transformations at the inner core boundary on the Slichter modes

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ABSTRACT

We analyze the effect of phase transformations at the inner core boundary on the period of Slichter modes. We show that the presence of phase transformations can lead to qualitatively new phenomena. In particular, the frequency is *inversely* proportional to the density contrast at the inner core boundary. We offer a thought experiment that demonstrates this effect. The complete analysis combines an instantaneous kinetics model for phase transformations with a simple planetary model: a rigid inner core, an inviscid, incompressible, constant density outer core, a stationary mantle. The reciprocal dependence on the density contrast leads to periods that are an order of magnitude shorter than those predicted by models that disallow phase transformations.

1. Introduction

The translational modes of the Earth's inner core, known as the Slichter modes, were first studied in Slichter (1961). The first analytical treatment of the problem was given by Busse (1974). In the past 15 years, this field saw an explosion of activity primarily due to the emergence of superconducting gravimeters capable of detecting the relative motion of the inner core by measuring the variations in the Earth's gravitational field (Hinderer and Crossley, 2004). Some prominent authors believe that evidence of Slichter modes can be found in the existing gravimeter data (Smylie, 1992). However, a definitive detection has proven to be controversial (Jensen et al., 1995; Hinderer et al., 1995) and the search for Slichter modes continues (Courtier et al., 2000; Rosat et al., 2003; Rogister, 2003). Models that do not account for phase transformations predict Slichter periods for the Earth on the order of hours. We show that the presence of phase transformations shortens the Slichter periods. The limiting case of *instantaneous kinetics*, which assumes that the phase equilibrium is continuously maintained, predicts periods on the order of minutes. If our basic premise of phase dynamics is correct then the true period falls somewhere between the estimates provided here and the existing estimates, the two being the limits of instantaneous and infinitely slow kinetics. Our analysis may therefore provide an explanation of why the Earth's Slichter modes have not been conclusively detected.

Our intention is to study the dramatic effect that phase transformations can have on the dynamics of self-gravitating fluids in general and oscillation frequencies in particular. We therefore purposefully select a simplistic model that brings forward this effect at the expense of other more established phenomena. The main result displays an inversion of the density contrast—it now appears in the denominator of the frequency estimate. This conclusion constitutes a qualitative change from the existing models: it states that the more closely the two densities match the faster the oscillations. Undoubtedly, in order to obtain accurate quantitative estimates for the Slichter periods, one needs to consider finite kinetics phase transformations and to employ a much more refined contemporary model of the planet. From the point of view of phase transformations, the most critical parameter that those models must accurately establish is the density contrast at the inner core boundary.

We adopt Busse's model of a simple three-layer non-rotating spherical Earth (Grinfeld and Wisdom, 2005). The Earth's rotation leads to a splitting in the Slichter frequencies (the Slichter triplet) and a plethora of other dynamic effects (Bush, 1993), but it is beyond the scope of this paper. Numerous other general effects, including compressibility, stratification, elasticity (Mound et al., 2003), viscosity (Mound and Buffett, 2007), mushiness at the inner core boundary (Peng, 1997) and magnetism can play a role in the dynamics of the inner core. All of these effects are ignored here. Critically, *our model permits mass transfer between the inner core*

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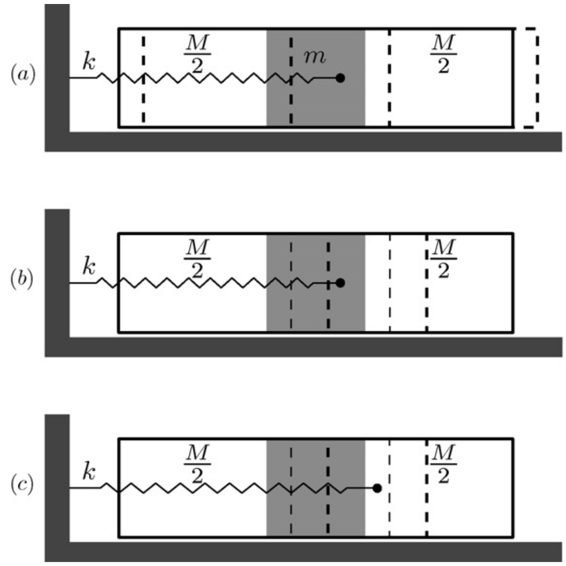

Fig. 1. A thought experiment illustrating the effect of phase transformations. (a) No phase transformations. The body and the ambient fluid move together as a rigid body of mass $m+M$. (b) Phase transformations are present. The body oscillates, solidifying in front and melting the back. The ambient fluid is at rest. The effective amplitude is increased but the mass involved in the motion is m. (c) The gravitational "spring". The restoring gravity force is proportional to *effective* displacement, rather than *material* displacement. The effective "stiffness" is $k\rho_1(\rho_2-\rho_1)^{-1}$.

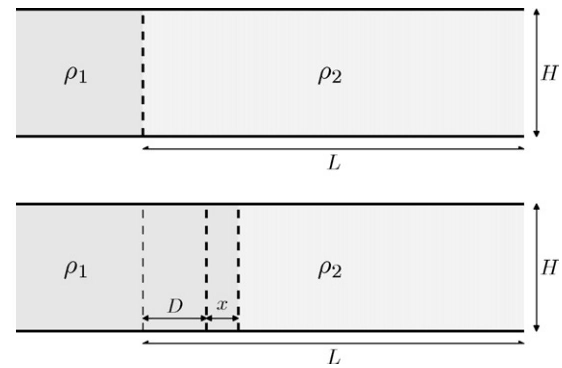

Fig. 2. Mass conservation. When the material particles that initially comprise the phase interface move by D, the actual interface moves by $D+x$. The amount x is found by expressing the mass to the right of the initial interface in two ways: $\rho_2 LH = \rho_1 xH + \rho_2(L-D-x)H$. This equation yields $x = D\rho_2(\rho_1-\rho_2)^{-1}$ and therefore the total advance by the phase interface is $D+x = D\rho_1(\rho_1-\rho_2)^{-1}$.

and the outer core. We consider a phase transformation of the first kind. We focus on the limiting case of *instantaneous kinetics*. That is, we assume the pressure at the inner core boundary instantaneously adjusts to the value P_0 that permits coexistence of the two phases and depends on temperature and material properties of the substances. A static model that postulates the same pressure condition can be found in Ramsey (1950). This idealization certainly leads to further *shortening* of the period since, in actuality, phase equilibrium is attained over a finite amount of time. A finite kinetics analysis would lead to estimates that lie between the values presented here and those given by sans-phase models.

All phase transformations are accompanied by absorption or release of energy. Energy can be transferred in two ways. First is by supplying or drawing energy in the form of heat. This form of energy is known as latent heat. An example of a phase transformation due to latent heat is the boiling of water over a flame. The other way to transfer energy is by performing mechanical work. An example of this mechanism is the melting of ice under a skate. Following Ramsey (1950), we choose this mechanism to be responsible for the phase transformations at the inner core boundary in our model.

2. A thought experiment that illustrates the effect of phase transformations

We offer a thought experiment that demonstrates the qualitative effect that phase transformations can have on the period of oscillations. Consider a rectangular container (Fig. 1) that slides horizontally without friction. A rigid object of mass m and density ρ_1 divides the container in two. The sections on either side of the object are filled with incompressible fluid of density ρ_2 and total mass M. The rigid body is attached to a spring of stiffness k. We assume that $\rho_1 > \rho_2$. This, of course, is the case for the inner core but not all substances. For example, ice is less dense than water.

We designed this thought experiment to gain insight into the counterintuitive placement of the density contrast in the denominator in Eq. (47). The kinematics of this experiment does not capture the more complicated spherical geometry relevant for the dynamics of the inner core, where the ambient fluid can flow around the rigid object. The focus of this experiment is on the boundary and on the qualitative differences between the behaviors of the fluid with and without phase transformations.

In part (a) of Fig. 1, we illustrate the problem without phase transformations. Since the ambient fluid is incompressible, the rigid weight is not able to move within the container and the entire system must move as one on the frictionless substrate. The dashed outline indicates the displacement. It is apparent that when the rigid body is displaced by an amount D, the entire system is displaced by that amount. Therefore, a total mass of $m+M$ participates in the dynamics leading to oscillations whose frequency ω is given by

$$\omega^2 = \frac{k}{m+M}. \tag{1}$$

M is called added mass and in this case the addition is literal. In other geometries, where a body oscillates in an ambient fluid, the net effect of the additional mass involved in the dynamics can often be captured by an effective added mass expression. For example, if a sphere of radius R and mass m oscillates on a spring of stiffness k in infinite space filled with an ideal fluid of density ρ, then the frequency of oscillation is given by $\omega^2 = k/(m+2\pi\rho R^3/3)$—so the effective added mass equals half the mass of a fluid sphere of radius R. Both of these examples demonstrate that added mass naturally leads to longer periods.

Phase transformations in the framework of our model mitigate the added mass effect. In our simple sliding container example, the mitigation is complete. According to our model based on instantaneous kinetics, we suppose that the rigid mass and the fluid are two phases of the same substance and that phase equilibrium is continuously maintained. In other words, the phase interface is at pressure P_0 throughout the oscillation. As a result, the ambient fluid remains at rest since the rigid body is unable to apply pressure greater than P_0. Therefore, rather than move when the rigid body is displaced, the fluid instead solidifies at one end (front) and melts at the other (back). When the point of spring attachment shifts by an amount D, indicated by the thin dashed line in part (b) of Fig. 1, the container stays in its original location. However, the rigid body will appear to have moved an even greater amount, indicated by the bold dashed line. Due to the phase transformation, the resulting effective displacement is $D\rho_1(\rho_1-\rho_2)^{-1}$ as explained in Fig. 2. Meanwhile, the spring responds to the material displacement D. Since the surrounding fluid is no longer engaged, the oscillation frequency expression no longer involves an added mass

$$\omega^2 = \frac{k}{m}. \tag{2}$$

When it comes to the oscillation of the inner core, there is an additional effect. The role of the spring is played by gravity. The restoring force is proportional to the *effective* displacement of the center of mass. In our thought experiment, when the point of spring attachment is displaced by D, the interface moves by $D\rho_1(\rho_1-\rho_2)^{-1}$. The effective center of mass moves by the same amount, as indicated in part (c) of Fig. 1: note that the spring is stretched by a greater amount. If we continue to think of gravity as a spring we must set its stiffness to $k\rho_1(\rho_1-\rho_2)^{-1}$. Here, we see the inversion of the density contrast. The resulting frequency is

$$\omega^2 = \frac{k\rho_1}{m(\rho_1-\rho_2)}. \tag{3}$$

Eq. (3) captures the main point of this paper.

A more formal example illustrates the same effect. Consider a flat interface between two heavy fluids of densities ρ_1 and ρ_2 and a wave of frequency ω and wavelength K^{-1} traveling along the interface. In the absence of phase transformations,

$$\omega^2 = \frac{gK(\rho_1-\rho_2)}{\rho_1+\rho_2}, \tag{4}$$

where g is the acceleration of gravity (Lamb, 1993). In the presence of phase transformations (Grinfeld, 1988), the frequency changes to

$$\omega^2 = \frac{gK(\rho_1+\rho_2)}{\rho_1-\rho_2}. \tag{5}$$

3. Model and analysis

We consider a three layer non-rotating planet (as in Grinfeld and Wisdom, 2005) with a rigid inner core of radius R_1 and density ρ_1, an inviscid incompressible fluid outer core of outer radius R_2 and density ρ_2, and a mantle of outer radius R_3 and density ρ_3. We assume that the bulk of the mantle does not participate in the dynamics (Grinfeld and Wisdom (2005) shows the validity of this assumption) nor does it contribute to the forces of gravity within its interior. Therefore, the parameters R_3 and ρ_3 will not appear in the final estimates for frequency. We nevertheless incorporate them in our analysis in case a future question requires considering the motion of the mantle.

Our approach is perturbative and is based on a linearization procedure. Every configuration of the system is treated as a small deviation from the spherically symmetric stable configuration. All velocities are assumed small. The "unperturbed" spherically symmetric gravitational potential is ψ_0. Its rate of change $\partial\psi/\partial t$ is induced by the dynamics of the core. We solve for the fluid velocity field v^R, v^Θ and pressure p consistent with translational motion of the inner core and the stationary mantle.

The motion of the phase interface is specified by its normal velocity C_1 expressed as a harmonic series

$$C_1\left(\theta,\phi\right) = R_1 C_1^{lm} Y_{lm}\left(\theta,\phi\right)\omega e^{i\omega t}, \tag{6}$$

where a summation over l and m is implied. The angle ϕ is longitude and θ is colatitude. The functions $Y_{lm}\left(\theta,\phi\right)$ are spherical harmonics. For reasons of convenience, we do not normalize them to unity as would be customary. We found that it is most convenient to take

$$Y_{1,0}\left(\theta,\phi\right) = \cos\theta \tag{7}$$

whose norm is given by

$$\int_{|z|=R} Y_{1,0}^2\left(\theta,\phi\right) dS = \frac{4}{3}\pi R^2 \tag{8}$$

Note that mass conservation forbids the constant harmonic $Y_{0,0}$, therefore $C_1^{0,0}=0$.

The frequency ω is specific to each mode but, for the sake of conciseness, we do not write ω_{lm} which would have been more precise. The multiplicity of ω is $2l+1$. We concentrate on the modes corresponding to $l=1$ since these are the only modes that result in net translation of mass and are therefore most likely to be detectable at the surface of the Earth. In the context of the non-rotating Earth, these modes have multiplicity three. We consider a vertical oscillation described by the harmonic $l=1$, $m=0$. This mode is axially symmetric. This allows us to assume that the azimuthal component of the fluid velocity field vanishes.

We non-dimensionalize our expressions by a length scale R_* and a density ρ_*. The particular choice of R_* and ρ_* can be made later. Introduce the dimensionless densities δ_n and the dimensionless radii Q_n:

$$\delta_n = \frac{\rho_n}{\rho_*} \tag{9a}$$

$$Q_n = \frac{R_n}{R_*} \tag{9b}$$

Introduce a convenient quantity Ψ_* whose dimensions are those of gravitational potential:

$$\Psi_* = \frac{4\pi}{3}G\rho_* R_*^2, \tag{10}$$

where G is the gravitational constant.

4. Gravitational potential

This section outlines the computation of the gravitational potential ψ and its time derivative $\partial\psi/\partial t$ induced by the deformation of the inner core boundary C_1. The analytical framework was constructed in Grinfeld and Wisdom (2005). In this paper, we use slightly different notation and correct a typo in an intermediate Eq. (32) of Grinfeld and Wisdom (2005). A detailed description of the method of moving surfaces and its applications to potential problems can be found in Grinfeld and Wisdom (2006).

The gravitational potential $\psi\left(r,\theta,\phi\right)$ satisfies Poisson's equation

$$\nabla^2\psi = 4\pi G\rho, \tag{11}$$

where ρ is taken to be ρ_1, ρ_2, ρ_3, or 0 depending on the region. The potential ψ is finite at the origin, vanishes at infinity, and is continuous along with its first derivatives across all interfaces. Let $[X]_n$ denote the jump in the enclosed quantity across the interface n (e.g. $[\rho]_1 = \rho_1-\rho_2$). The continuity conditions read

$$\left[\psi\right]_1, \left[\psi\right]_2, \left[\psi\right]_3 = 0 \tag{12a}$$

$$\mathbf{N}\cdot\left[\nabla\psi\right]_1, \mathbf{N}\cdot\left[\nabla\psi\right]_2, \mathbf{N}\cdot\left[\nabla\psi\right]_3 = 0, \tag{12b}$$

where $\mathbf{N}$ is the outward unit normal.

The equilibrium potential $\psi_0(r)$ is given by

$$\psi_0(r) = \Psi_* \begin{cases} \dfrac{\delta_1}{2}\dfrac{r^2}{R_*^2} + A_1, & \text{inner <CE:HSP SP="0.25"/>core} \\ \dfrac{\delta_2}{2}\dfrac{r^2}{R_*^2} + A_2 + B_2\dfrac{r^{-1}}{R_*^{-1}}, & \text{outer <CE:HSP SP="0.25"/>core} \\ \dfrac{\delta_3}{2}\dfrac{r^2}{R_*^2} + A_3 + B_3\dfrac{r^{-1}}{R_*^{-1}}, & \text{mantle} \\ B_4\dfrac{r^{-1}}{R_*^{-1}}, & \text{outside} \end{cases} \tag{13}$$

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where

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}\left[\delta\right]_1 Q_1^2 - \frac{3}{2}\left[\delta\right]_2 Q_2^2 - \frac{3}{2}\left[\delta\right]_3 Q_3^2 \\ -\frac{3}{2}\left[\delta\right]_2 Q_2^2 - \frac{3}{2}\left[\delta\right]_3 Q_3^2 \\ -\frac{3}{2}\left[\delta\right]_3 Q_3^2 \\ -\left[\delta\right]_1 Q_1^3 \\ -\left[\delta\right]_1 Q_1^3 - \left[\delta\right]_2 Q_2^3 \\ -\left[\delta\right]_1 Q_1^3 - \left[\delta\right]_2 Q_2^3 - \left[\delta\right]_3 Q_3^3 \end{bmatrix} \tag{14}$$

The normal derivative ψ_0' of the potential ψ_0 at the inner core boundary $r = R_1$ is given by

$$\psi_0'(R_1) = \frac{\Psi_*}{R_*}\delta_1 Q_1 \tag{15}$$

The system of equations for the potential perturbation ψ_t is obtained by differentiating the gravitational system (11)–(12b) with respect to time. The derivative of the bulk Eq. (11) shows that ψ_t is harmonic at points away from the boundary:

$$\nabla^2\psi_t = 0. \tag{16}$$

The boundary conditions (12a) and (12b) are differentiated in the invariant sense discussed in Grinfeld and Wisdom (2006) and Grinfeld (2009). We obtain that $\partial\psi/\partial t$ is continuous across all interfaces, while the normal derivative of $\partial\psi/\partial t$ jumps by an amount proportional to the velocity of the interface C_1 and the jump in the second normal derivative of the unperturbed potential ψ_0:

$$\left[\psi_t\right]_{1,2,3} = 0 \tag{17a}$$

$$N^i\left[\nabla_i\psi_t\right]_1 = -C_1 N^i N^j\left[\nabla_i\nabla_j\psi_0\right]_1 \tag{17b}$$

$$N^i\left[\nabla_i\psi_t\right]_{2,3} = 0. \tag{17c}$$

The remaining boundary conditions state that ψ_t is finite at the origin and vanishes at infinity.

The resulting system can be solved by separation of variables. We look for a solution of the form

$$\psi_t = \Psi_*\omega s^{lm}(r) Y_{lm} e^{i\omega t} \tag{18}$$

Within the inner core ψ_t is given by

$$\psi_t = -\frac{3\Psi_*\omega\left[\delta\right]_1 e^{i\omega t} C_1^{lm} Y_{lm}}{2l+1} Q_1^{-l+2}\frac{r^l}{R_*^l}, \tag{19}$$

while within the outer core, the mantle and beyond, we have

$$\psi_t = -\frac{3\Psi_*\left[\delta\right]_1\omega e^{i\omega t} C_1^{lm} Y_{lm}}{2l+1} Q_1^{l+3}\frac{r^{-l-1}}{R_*^{-l-1}} \tag{20}$$

In particular, at the inner core boundary $r = R_1$ the two expressions have the same value

$$\psi_t|_{r=R_1} = -\frac{3\Psi_* Q_1^2\left[\delta\right]_1\omega e^{i\omega t}}{2l+1} C_1^{lm} Y_{lm} \tag{21}$$

5. Motion of the fluid

The velocity and pressure fields $\mathbf{v}$ and p are governed by the Euler equations

$$\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho_2}\nabla p - \nabla\psi \tag{22a}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{22b}$$

subject to three, rather than the usual two, boundary conditions. An additional boundary condition is needed at the inner core boundary since the normal velocity of the interface C_1 is an additional unknown.

The first condition is conservation of mass across the phase interface

$$C_1[\rho]_1 - \mathbf{N}\cdot[\rho\mathbf{v}]_1 = 0 \tag{23}$$

Intuitively, this identity states that the amount of matter enveloped by an advancing interface is balanced by the flux of matter across the interface. The velocity field $\mathbf{v}_1$ inside the inner core is a uniform vector field since we assume that the oscillation mode is translational. Let it be given by the form

$$\mathbf{v}_1 = \hat{\mathbf{z}} A\omega e^{i\omega t}, \tag{24}$$

where $\hat{\mathbf{z}}$ is the unit vector in the direction of the oscillation and A is the amplitude of the oscillation. We align the polar coordinates with $\hat{\mathbf{z}}$. The normal component of $\mathbf{v}_1$ is therefore given by

$$\mathbf{v}_1\cdot\mathbf{N} = A\cos\theta\omega e^{i\omega t}. \tag{25}$$

Given our choice of $Y_{1,0}$ (7), this identity can be written as

$$\mathbf{v}_1\cdot\mathbf{N} = AY_{1,0}\left(\theta,\phi\right)\omega e^{i\omega t}. \tag{26}$$

The second condition is instantaneous kinetics. It states that the pressure at the phase interface equals the special value of P_0:

$$p|_{S_1} = P_0 \tag{27}$$

As the third boundary condition, we impose slippage at the mantle boundary S_2—the normal component of the velocity vanishes:

$$\mathbf{v}\cdot\mathbf{N}|_{r=R_2} = 0 \tag{28}$$

Eqs. (22a) and (27) are linearized by differentiation with respect to time and keeping first order terms. This implies that the linearized equations are solved on the equilibrium geometry. Represent the pressure p as a sum of the unperturbed hydrostatically equilibrium pressure p_0 and a small time-dependent correction $\bar{p}$:

$$p(t) = p_0 + \bar{p}(t). \tag{29}$$

Then the linearized Euler equations are

$$\frac{\partial^2\mathbf{v}}{\partial t^2} = -\frac{1}{\rho_2}\nabla\frac{\partial\bar{p}}{\partial t} - \nabla\psi_t \tag{30a}$$

$$\nabla\cdot\mathbf{v} = 0 \tag{30b}$$

and the linearized condition of instantaneous kinetics reads

$$\left.\frac{\partial\bar{p}}{\partial t}\right|_{S_1} = -C_1\mathbf{N}\cdot\nabla p_0 \tag{31}$$

The normal derivative $\mathbf{N}\cdot\nabla p_0$ of p_0 is determined from the hydrostatic equilibrium equation

$$\nabla p_0 + \rho_2\nabla\psi_0 = 0 \tag{32}$$

The normal derivative $\mathbf{N}\cdot\nabla p_0$ at the inner core boundary coincides with the ordinary derivative $p_0'(R_1)$ which is easily obtained from Eq. (15):

$$p_0'(R_1) = -\frac{\Psi_*}{R_*}\delta_1\rho_2 Q_1 \tag{33}$$

The system of partial differential Eqs. (30a) and (30b) is solved by separation of variables. We separate time from the spatial variables and r from the angles. The first identity (34a) is a repeat of Eq. (6). The next two give expressions for the pressure perturbation $\bar{p}$, the radial component v^R and the angular component v^Θ of the velocity field with properly chosen phases.

The linearized system (30a) and (30b) is satisfied by

$$C_1\left(t,\theta,\phi\right) = R_1\omega C_1^{lm} Y_{lm}\left(\theta,\phi\right)e^{i\omega t} \tag{34a}$$

$$\bar{p}\left(t,r,\theta,\phi\right) = -i\rho_2 R_*^2\omega^2 p^{lm}(r) Y_{lm}\left(\theta,\phi\right)e^{i\omega t} \tag{34b}$$

$$v^R\left(t,r,\theta,\phi\right)=R_*\omega q^{lm}(r)\,Y_{lm}\left(\theta,\phi\right)e^{i\omega t} \tag{34c}$$

$$v^\Theta\left(t,r,\theta,\phi\right)=\frac{R_*^2\omega}{r^2}u^{lm}(r)\frac{\partial Y_{lm}}{\partial\theta}e^{i\omega t} \tag{34d}$$

where

$$p^{lm}(r)=P_+^{lm}\frac{r^l}{R_*^l}+P_-^{lm}\frac{r^{-l-1}}{R_*^{-l-1}} \tag{35a}$$

$$q^{lm}(r)=R_*\left(\frac{dp^{lm}(r)}{dr}+\frac{\Psi_*}{R_*^2\omega^2}\frac{ds^{lm}(r)}{dr}\right) \tag{35b}$$

$$u^{lm}(r)=p^{lm}(r)+\frac{\Psi_*}{R_*^2\omega^2}s^{lm}(r) \tag{35c}$$

and $s^{lm}(r)$ are defined in (18) and determined in the outer by (20).

We have reduced the system of partial differential equations to an algebraic system with unknowns P_+^{lm}, P_-^{lm}, C_1^{lm}, and A. We have three sets of boundary conditions: mass conservation (23), slippage at the mantle boundary (28), and the linearized condition of instantaneous kinetics (31). The condition that completes our system comes from Newton's second law for the inner core. This is the topic to which we now turn.

5.1. Newton's second law for the inner core

The inner core experiences two forces: the gravitational force exerted at every point inside the inner core and the hydrodynamic pressure applied at the boundary. The gravitational force is proportional to the density of the inner core and the vector gradient of the gravitational potential ψ. The hydrostatic force is proportional to the pressure p at the boundary and points along the normal. Therefore, Newton's second law reads

$$\frac{4\pi}{3}\rho_1R_1^3\mathbf{a}=-\rho_1\int_{\Omega_1}\nabla\psi d\Omega-\int_{S_1}p\mathbf{N}dS, \tag{36}$$

where $\mathbf{a}$ is the acceleration of the inner core's center of mass and $\mathbf{N}$ is the *outward* normal.

The condition of instantaneous kinetics states that pressure has a constant value P_0 at the phase interface boundary S_1. Since P_0 is assumed constant, the pressure term in Newton's second law vanishes:

$$\frac{4\pi}{3}\rho_1R_1^3\mathbf{a}=-\rho_1\int_{\Omega_1}\nabla\psi d\Omega \tag{37}$$

We linearize this equation by applying a time derivative. The key to differentiating the volume integral is the general formula

$$\frac{d}{dt}\int_\Omega Fd\Omega=\int_\Omega\frac{\partial F}{\partial t}d\Omega+\int_S CFdS, \tag{38}$$

where F is an arbitrary scalar or vector field, S is the boundary of Ω and C is the velocity of the interface with respect to the outward normal. An application of this formula to Eq. (37) yields

$$\frac{4\pi}{3}\rho_1R_1^3\frac{d\mathbf{a}}{dt}=-\rho_1\int_{\Omega_1}\nabla\frac{\partial\psi}{\partial t}d\Omega-\rho_1\int_{S_1}C\nabla\psi dS. \tag{39}$$

The volume integral term in (39) can be converted to a surface integral by the divergence theorem, yielding

$$\frac{4\pi}{3}\rho_1R_1^3\frac{d\mathbf{a}}{dt}=-\rho_1\int_{S_1}\left(\mathbf{N}\frac{\partial\psi}{\partial t}+C\nabla\psi\right)dS. \tag{40}$$

By linearization, the leading order contribution to the term $C\nabla\psi$ comes from the unperturbed potential ψ_0 whose radial derivative is given in Eq. (15). Therefore, linearized Newton's second law reads

$$\frac{4\pi}{3}\rho_1R_1^3\frac{d\mathbf{a}}{dt}=-\rho_1\int_{S_1}\left(\mathbf{N}\frac{\partial\psi}{\partial t}+C\nabla\psi_0\right)dS. \tag{41}$$

As a final step, we project this equation onto the direction of oscillation $\hat{\mathbf{z}}$. Since

$$\mathbf{N}\cdot\hat{\mathbf{z}}=\cos\theta \tag{42a}$$

$$\nabla\psi_0\cdot\hat{\mathbf{z}}=\frac{\partial\psi_0}{\partial z}=\frac{d\psi_0}{dr}\frac{\partial r}{\partial z}={\psi'}_0\cos\theta \tag{42b}$$

the final form on Newton's second law reads

$$\frac{4\pi}{3}\rho_1R_1^3\frac{da}{dt}=-\rho_1\int_{S_1}\left(\frac{\partial\psi}{\partial t}+C_1\psi_0'\right)\cos\theta dS. \tag{43}$$

We have all the necessary ingredients to convert this equation to algebraic form. From the velocity Eq. (24) we find that

$$\frac{da}{dt}=-A\omega^3e^{i\omega t} \tag{44a}$$

From Eq. (21) for the value of the gravitational potential perturbation we conclude that only the $Y_{1,0}$ harmonic survives the integration. Recalling the normalization (8), we obtain:

$$\int_{S_1}\frac{\partial\psi}{\partial t}\cos\theta dS=-\frac{4\pi}{3}\Psi_*R_1^2Q_1^2\left[\delta\right]_1\omega e^{i\omega t}C_1^{1,0} \tag{44b}$$

In the remaining integral of $C_1\psi_0'\cos\theta$, the $Y_{1,0}$ harmonic is once again the only term to survive. From equations for C_1 (34a), ψ_0 (15), and normalization (8), we obtain

$$\int_{S_1}C_1\psi_0'\cos\theta dS=\frac{4\pi}{3}\Psi_*R_1^2Q_1^2\delta_1\omega e^{i\omega t}C_1^{1,0} \tag{44c}$$

Combining the terms (44a)–(44c), we arrive at the algebraic form of Newton's second law

$$\frac{A}{R_*}=\frac{\Psi_*}{R_*^2\omega^2}\delta_2Q_1C_1^{1,0} \tag{45}$$

6. Expression for the frequency

The boundary conditions (23), (28), and (31) are easily converted to algebraic form by evaluating these expressions at the appropriate boundary. We present the linear system for the harmonic proportional to $Y_{1,0}\left(\theta,\phi\right)$ since the $Y_{1m}\left(\theta,\phi\right)$ are the only modes that result in net translation of mass and is therefore the most easily detectable mode. The best strategy for analyzing the system is to use Eq. (45) to eliminate A from the equation that arises from (23). The result is a system with three equations and three unknowns:

$$\begin{bmatrix}1 & Q_1^{-3} & -\dfrac{\Psi_*\delta_1}{R_*^2\omega^2}\\ -1 & 2Q_1^{-3} & \dfrac{\Psi_*\left(2\delta_2-\delta_1\right)}{R_*^2\omega^2}-\dfrac{\delta_1-\delta_2}{\delta_2}\\ 1 & -2Q_2^{-3} & 2\dfrac{\Psi_*\left(\delta_1-\delta_2\right)}{R_*^2\omega^2}Q_1^3Q_2^{-3}\end{bmatrix}\begin{bmatrix}P_+^{1,0}\\ P_-^{1,0}\\ Q_1C_1^{1,0}\end{bmatrix}=0 \tag{46}$$

The frequency ω is determined by the condition that the determinant of this matrix vanishes. We thus arrive at the central result of this paper that gives the frequency of the translational mode of oscillation

$$\omega^2=\frac{4\pi G\rho_2}{3}\frac{3\rho_2R_2^3+\left(\rho_1-\rho_2\right)\left(2R_1^3+R_2^3\right)}{\left(\rho_1-\rho_2\right)\left(2R_1^3+R_2^3\right)} \tag{47}$$

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For comparison, Busse's estimate for the case with no phase transformations is

$$\omega_{Busse}^2 = \frac{4\pi G\rho_2}{3} \frac{2\left(R_2^3 - R_1^3\right)(\rho_1 - \rho_2)}{3\rho_2 R_2^3 + 2\left(R_2^3 - R_1^3\right)(\rho_1 - \rho_2)} \tag{48}$$

The distinguishing characteristic of Eq. (47) is the density contrast $\rho_1 - \rho_2$ in the denominator. This is analogous to Eq. (3) from the thought experiment discussed above. Consequently, when the density contrast is low, the resulting frequencies can be quite high.

7. Potential applications to the Earth

The model that we proposed is deliberately oversimplified with respect to the structure of the planet and the complexity of geophysical processes that take place inside it. As a result, our model permits analytical study and brings forth the effects of phase transformations. It is designed to be able to predict qualitatively novel effects and describe their orders of magnitude. To obtain accurate estimates, one needs a far more refined model of the planet. Nevertheless, we would like to give a few ballpark estimates for the Earth that may be of value in analyzing the qualitative effects introduced by phase transformations.

Assume the following values for the radii and densities of the Earth (Anderson, 1989):

$$\begin{aligned} G &= 6.672 \times 10^{-11}\ \mathrm{m^3/kg\,s^2} \\ R_1 &= 1.221 \times 10^6\ \mathrm{m} \\ R_2 &= 3.480 \times 10^6\ \mathrm{m} \\ \rho_1 &= 12.8 \times 10^3\ \mathrm{kg/m^3} \\ \rho_2 &= 12.2 \times 10^3\ \mathrm{kg/m^3} \end{aligned}$$

These values yield the following estimates for the oscillation periods

$$\begin{aligned} \frac{2\pi}{\omega} &= 7.5\ \mathrm{min} \\ \frac{2\pi}{\omega_{\mathrm{Busse}}} &= 5.42\ \mathrm{h}. \end{aligned}$$

There are a number of questions that need to be answered before the applicability of our model to the Earth can be justified. Our estimates are most sensitive to the density contrast $\rho_1 - \rho_2$ which must therefore be accurately estimated. Furthermore, the condition of *instantaneous* kinetics is a clear simplification and offers a lower bound on the oscillation period. Analysis of *finite* kinetics will necessarily lengthen the period but the precise amount will depend on a new parameter governing the rate of phase transformations. That parameter will also require careful determination. The validity of the model in which phase transformations are driven by mechanical work must also be established for the Earth. Finally, an accurate model of the Earth that accounts for radial variations in density and temperature must be incorporated into analysis.

8. Conclusions

The crucial aspect of our model is the introduction of phase transformations of the first kind into the modeling of the dynamics of the inner core. We adopted a model by Busse because it is the simplest model that illustrates the influence of phase transformations. As a further idealization, we assumed that phase transformations satisfy the assumption of instantaneous kinetics. As a result, we derived expressions for frequency estimates that display the density contrast in the denominator. This is qualitatively different from the estimates given by existing models. Our estimates show that phase transformations are capable of placing the Earth's Slichter modes in a frequency range that is quite different from what is currently believed. Our estimates give about 9 min while Busse's model predicts nearly 6 h. More refined sans-phase models of the Earth yield varying estimates for the oscillation periods, but typically between 2.5 and 6 h (Rogister, 2003). Therefore, the presence of phase transformations and the resulting shortening of the eigenperiod may hold the key to experimental detection of the modes. Modern superconducting gravimeters (Hinderer and Crossley, 2004) are perfectly capable of resolving oscillations on the order of minutes and the findings of this paper indicate that the shorter frequency range needs to be investigated. We stress, however, that the estimates presented here are based on a simplistic model of the planet and that more refined models are needed in order to provide more accurate estimates.

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