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Core Potentials: The Consensus Segmentation Conjecture

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Abstract Segmentations are partitions of an ordered set into non-overlapping intervals. The Consensus Segmentation or Segmentation Aggregation problem is a special case of the median problems with applications in time series analysis and computational biology. A wide range of dissimilarity measures for segmentations can be expressed in terms of potentials, a special type of set-functions. In this contribution, we shed more light on the properties of potentials, and how such properties affect the solutions of the Consensus Segmentation problem. In particular, we disprove a conjecture stated in 2021, and we provide further insights into the theoretical foundations of the problem.

Keywords Consensus segmentation · Optimization · Set function · Supermodularity · Metrics · Hypercube

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1 Introduction

Segmentation problems [9] naturally appear in clustering and data mining. In their most general form, they ask for a partition of ground set X given some data associated with the elements or a collection of subsets of X. The one-dimensional version, where X is a totally ordered set, appears naturally in time series analysis [8] as well as computational biology in particular in the analysis of micro-array and high-throughput sequencing data [4,12,17]. The many variants and algorithmic approaches to 1-D segmentation naturally yield similar but distinct results. Likewise, replicate measurements as well as different readouts pertaining to the same stretch of genomic data yield different segmentations. Improved, more robust results can be achieved in many cases by aggregating multiple segmentations into a single consensus segmentation. This Segmentation Aggregation problem [14] is a special case of the Median Partition or `Consensus Clustering` Problem. While this problem is NP-complete for commonly used measures of partition dissimilarity [10,19], efficient dynamic programming solutions are available for a very large class of dissimilarities for partitions into linearly ordered intervals [14,18].

A desirable feature of a consensus segmentation is that it does not introduce novel breakpoints between intervals in the sense that all breakpoints of the consensus segmentation are present in at least one of the input segmentations. This *refinement property* is provably satisfied by segmentations that are the optimal consensus with respect to a particular class of potential-based dissimilarity measures [14,18], which are characterized as *boundedly convex potential functions*. Numerical simulations performed with a variety of other "well-behaved" dissimilarity measures led to the conjecture that the refinement property holds for a very broad class of dissimilarity measures.

In this contribution, we look more specifically into the details of this conjecture, investigating various candidate properties for capturing the idea of a "well-behaved" measure. We show that for most of these properties, the refinement property does not hold. The outline of the paper is as follows.

In Sect. 2, we introduce the basic notions and definitions. In particular, we consider potentials as a means of specifying dissimilarities between segmentations and introduce the novel notion of a core potential, for which the refinement property holds for the associated dissimilarity measure. In Sect. 3, we investigate the conditions under which the dissimilarity measure is a metric. Interestingly, this is closely related to supermodularity, a well-studied property of set-functions (Theorem 7). In Sect. 4, we study the special case of potentials whose values only depend on the size of the set of consideration. We show that, within this restricted framework, supermodularity and convexity are equivalent (Theorem 12). In Sect. 5, we propose a generalization of the notion of supermodularity, from which we derived a companion result to Theorem 12 (Proposition 18). Finally, we show in Sect. 6 that the Consensus Segmentation Problem is directly related to the problem of finding medians in an edge-weighted hypercube (Theorem 25), and we conclude in Sect. 7 with some open questions and potential directions for future work.

2 Preliminaries

Segmentations and Potentials

Let $X = \{1, \dots, n\}$, $n \geq 1$. We call a subset $A \subseteq X$ an *interval* of X if $A = \{x \in X, i \leq x \leq j\}$ for some $1 \leq i \leq j \leq n$. For convenience, we write $A = [i, j]$, and we sometimes write $[i]$ for the interval $[i, i]$. For technical reasons, we also consider the empty set to be an interval of X. We denote by $\mathcal{I}(X) \subseteq 2^X$ the set of intervals of X. We say that two intervals A, B of X *overlap* if $A \cap B \notin \{\emptyset, A, B\}$. Moreover, we write $A < B$ if $\max(A) < \min(B)$ and thus in particular $A \cap B = \emptyset$.

A set $\Sigma \subseteq 2^X$ of subsets of X is a *partition* of X if $X = \bigcup_{A \in \Sigma} A$ and $A_1 \cap A_2 = \emptyset$ or $A_1 = A_2$ holds for all $A_1, A_2 \in \Sigma$. A *segmentation* of X is a partition S of X such that all elements of S are nonempty intervals of X.

For a segmentation S of X, we call an element $A \in S$ a *segment* of S. We say that $i \in X - \{n\}$ is a *breakpoint* of S if S contains a segment A whose maximal element is i (or, equivalently, S contains a segment B whose minimal element is $i + 1$). For $i \in X - \{n\}$, we denote by $S^{(i)}$ the segmentation whose unique breakpoint is i, that is, $S^{(i)} = \{[1, i], [i + 1, n]\}$.

Given two segmentations S_1, S_2 of X, we define the *union segmentation* $S_1 \wedge S_2$ as $\{A \cap B | A \in S_1, B \in S_2\}$. Note that the breakpoints of $S_1 \wedge S_2$ are precisely the union of the breakpoints of S_1 and the breakpoints of S_2. We say that a segmentation S_1 *refines* a segmentation S_2 if $S_1 \wedge S_2 = S_1$, that is, all breakpoints of S_2 are also breakpoints of S_1. In particular, $S_1 \wedge S_2$ refines both S_1 and S_2, and all segmentations S' refining both S_1 and S_2 refine $S_1 \wedge S_2$.

We call a map $\mathfrak{e} : \mathcal{I}(X) \to \mathbb{R}$ a *potential* (on X). For technical reasons, we will always assume that a potential $\mathfrak{e}$ on X satisfies $\mathfrak{e}(\emptyset) = 0$.

We say that a potential $\mathfrak{e}$ is *superadditive* if $\mathfrak{e}(A_1) + \mathfrak{e}(A_2) \leq \mathfrak{e}(A)$ holds for intervals A, A_1, A_2 of X such $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \emptyset$. Note that if $\mathfrak{e}$ is superadditive. then for all intervals A and all partitions of A into $k \geq 1$ intervals $A_1, \dots, A_k$, we have $\mathfrak{e}(A_1) + \dots + \mathfrak{e}(A_k) \leq \mathfrak{e}(A)$. We say that $\mathfrak{e}$ is *supermodular* if $\mathfrak{e}(A_1)+\mathfrak{e}(A_2) \leq \mathfrak{e}(A)+\mathfrak{e}(A_1 \cap A_2)$ holds for all intervals A, A_1, A_2 of X such that $A = A_1 \cup A_2$. Since a potential $\mathfrak{e}$ always satisfies $\mathfrak{e}(\emptyset) = 0$, a supermodular potential is always superadditive. The converse, however, does not hold in general, as exemplified by the potential $\mathfrak{e}$ defined in Eq. 1.

Supermodular potentials are a special case of supermodular set functions. The main difference is that the domain of a potential is restricted to the set of intervals of some set X, rather than encompassing the entire powerset of X. Supermodular set functions, and the analogous notion of submodular set function, defined by reversing the inequality at the heart of the definition of supermodularity, have been studied extensively in the literature, in particular in relation to minimization problems, see [5,7,16] for some surveys.

From a potential $\mathfrak{e}$ on X, one can define the *potential* of a segmentation S of X as:

$$E(S) = \sum_{A \in S} \mathfrak{e}(A).$$

This in turns induces a map $D_{\mathfrak{e}}$ defined, for two segmentations S_1, S_2 of X, as:

$$D_{\mathfrak{e}}(S_1, S_2) = E(S_1) + E(S_2) - 2E(S_1 \wedge S_2).$$

The map $D_{\mathfrak{e}}$ can be interpreted as a dissimilarity between segmentations on X and lies at the heart of the consensus segmentation problem, which we now define.

The Consensus Segmentation Problem

Let $\mathfrak{e}$ be a potential on some set X. Given a set of pairwise distinct segmentations $S_1, \dots, S_k$ of X and a set of positive weights $\omega_1, \dots, \omega_k$, $k \geq 1$, we call the set of pairs $\mathcal{F} = \{(S_1, \omega_1), \dots, (S_k, \omega_k)\}$ a *family of weighted segmentations* of X. Such a family $\mathcal{F}$ induces, together with the map $D_{\mathfrak{e}}$, a map $f_{\mathcal{F}}$ on the set of segmentations of X, defined, for all segmentations S of X, by:

$$f_{\mathcal{F}}(S) = \sum_{i=1}^{k} \omega_i D_{\mathfrak{e}}(S, S_i).$$

The Consensus Segmentation problem asks for a segmentation C minimizing $f_{\mathcal{F}}$. We call such a segmentation C a *consensus segmentation* of $\mathcal{F}$. This problem can be solved by dynamic programming for any given potential $\mathfrak{e}$ and any weighted segmentation family $\mathcal{F}$ [14,18]. The general idea is as follows. Write $S[A]$ for the restriction of a segmentation S to the interval A, set $f_\ell = \min_S f_{\mathcal{F}}(S[1, \ell])$ for the optimal score of the restriction of a consensus segementation to $[1, \ell]$ and write $f_A = \sum_{i=1}^{k} \omega_i D_{\mathfrak{e}}(A, S_i[A])$ for the "score" of the interval A. Then f_ℓ satisfies the recursion

$$f_\ell = \min_{j<\ell}(f_{j-1} + f_{[j,\ell]}) \quad \text{with} \, f_0 = 0.$$

Mielikainen and collaborators [14] showed that for certain potential functions every weighted segmentation family $\mathcal{F} = \{(S_1, \omega_1), \ldots, (S_k, \omega_k)\}$ admits a consensus segmentation C that is refined by the union segmentation $\widehat{S}(\mathcal{F}) = S_1 \wedge \ldots \wedge S_k$. In [18] this result was generalized to so-called boundedly convex potentials. Based on numerical experiments, furthermore, the authors conjectured that this property is shared by a much larger class of potentials.

We formalize this observation as follows:

Definition 1 A potential $\mathfrak{e}$ on X is *core* (short for *COnsensus-REfined*) if for all weighted families of segmentation $\mathcal{F} = \{(S_1, \omega_1), \ldots, (S_k, \omega_k)\}$, there exists a consensus segmentation C such that C is refined by $\widehat{S}(\mathcal{F})$.

Informally speaking, this means that all breakpoints of C are breakpoints of at least one of the segmentations $S_1, \ldots, S_k$. For $m \geq 1$, we say that $\mathfrak{e}$ is *m-core* if $\mathfrak{e}$ satisfies that property for all family $\mathcal{F}$ of size m or less.

Core potentials are of considerable practical interest because they admit much faster algorithms for computing the consensus segmentation. For a core potential, it suffices to consider in the dynamic programming algorithm only the breakpoints of union segmentation instead of all integers in $[1, n]$. Since in practical applications segments are typically long, this yields a practically relevant reduction in both memory consumption and running time.

The first necessary condition for a potential $\mathfrak{e}$ to be core is superadditivity:

Proposition 1 *Let $\mathfrak{e}$ be a potential on X. If $\mathfrak{e}$ is not superadditive, then $\mathfrak{e}$ is not core.*

Proof Suppose that $\mathfrak{e}$ is not superadditive. Then there exist three intervals A, A_1, and A_2 of X such that A_1, A_2 form a partition of A and $\mathfrak{e}(A_1) + \mathfrak{e}(A_2) > \mathfrak{e}(A)$. Without loss of generality, we may choose A such that $\mathfrak{e}(A'_1) + \mathfrak{e}(A'_2) \leq \mathfrak{e}(A')$ for all intervals A' that strictly contain A and all A'_1, A'_2 that form a partition of A'.

Now, let $\mathcal{F} = \{(S_1, \omega_1), \ldots, (S_k, \omega_k)\}$, $k \geq 1$ be a weighted segmentation family such that $A \in S_i$ for all $i \in \{1, \ldots, k\}$.[1] Put $\widehat{S} = \widehat{S}(\mathcal{F})$, and let $\overline{S}$ be such that $\widehat{S}$ refines $\overline{S}$, and $f_{\mathcal{F}}(\overline{S})$ is minimal among all segmentations that are refined by $\widehat{S}$.

We first show that, without loss of generality, we may assume that A is contained in $\overline{S}$. To see this, let S' be a segmentation of X that is refined by $\widehat{S}$, but does not contain A. Then since A is contained in $\widehat{S}$, there exists a (unique) interval A' in S' such that $A \subsetneq A'$. Now, let A^- (*resp.* A^+) be the set of elements of A' that are smaller than $\min(A)$ (*resp.* greater than $\max(A)$). Since $A \neq A'$, at least one of the sets A^- or A^+ is nonempty. We next define the segmentation S'_A of X obtained from S' by removing A', and adding the intervals A^-, A, and A^+ (ignoring the empty interval in $\{A^-, A^+\}$, if any). Note that since $A \in \widehat{S}$, and since $\widehat{S}$ refines S', it follows that $\widehat{S}$ refines S'_A. By definition, we have $E(S') - E(S'_A) = \mathfrak{e}(A') - \mathfrak{e}(A) - \mathfrak{e}(A^-) - \mathfrak{e}(A^+)$. By choice of A, we have $\mathfrak{e}(A^-) + \mathfrak{e}(A) \leq \mathfrak{e}(A^- \cup A)$ and $\mathfrak{e}(A^- \cup A) + \mathfrak{e}(A^+) \leq \mathfrak{e}(A^- \cup A \cup A^+) = \mathfrak{e}(A')$, so $\mathfrak{e}(A) - \mathfrak{e}(A^-) - \mathfrak{e}(A^+) \leq \mathfrak{e}(A')$ holds and $E(S') - E(S'_A) \geq 0$ follows. Moreover, for all $i \in \{1, \ldots, k\}$, we have $A \in S_i$, so $S' \wedge S_i = S'_A \wedge S_i$. It follows that $D_{\mathfrak{e}}(S', S_i) - D_{\mathfrak{e}}(S_A, S_i) = E(S') - E(S'_A) \geq 0$, so we have $D_{\mathfrak{e}}(S'_A, S_i) \leq D_{\mathfrak{e}}(S', S_i)$, and finally $f_{\mathcal{F}}(S'_A) \leq f_{\mathcal{F}}(S')$.

Hence, we can always choose $\overline{S}$ such that $A \in \overline{S}$. Consider now the segmentation C of X obtained from $\overline{S}$ by replacing the interval A with the pair of intervals A_1, A_2. Since $A \in \widehat{S}$, C is not refined by $\widehat{S}$. Moreover, we have $E(\overline{S}) - E(C) = \mathfrak{e}(A) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2)$, and for all $i \in \{1, \ldots, k\}$, $E(S_i \wedge \overline{S}) - E(S_i \wedge C) = \mathfrak{e}(A) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2)$. It follows that $D_{\mathfrak{e}}(S_i, \overline{S}) - D_{\mathfrak{e}}(S_i, C) = \mathfrak{e}(A_1) + \mathfrak{e}(A_2) - \mathfrak{e}(A)$. By choice of A, A_1 and A_2, the latter is positive. Hence, $D_{\mathfrak{e}}(S_i, \overline{S}) > D_{\mathfrak{e}}(C, \overline{S})$ holds for all $i \in \{1, \ldots, k\}$, from which it follows that $f_{\mathcal{F}}(C) < f_{\mathcal{F}}(\overline{S})$. Finally, by choice of $\overline{S}$, we have $f_{\mathcal{F}}(\overline{S}) \leq f_{\mathcal{F}}(S)$ for all segmentations S of X that are refined by $\widehat{S}$. It follows that $f_{\mathcal{F}}(C) < f_{\mathcal{F}}(S)$ holds for all such segmentations S, so the weighted family $\mathcal{F}$ does not admit a consensus that is refined by $\widehat{S}$. Hence, $\mathfrak{e}$ is not core. □

In [18], it was conjectured that the converse of Proposition 1, held, that is, that all superadditive potentials are core. As it turns out, this is not the case, as the following example shows.

Observation 2 *There exists superadditive potentials that are not core.*

[1] Note that depending on the interval A, there might be only one segmentation S of X with $A \in S$. In particular, $|\mathcal{F}| = 1$ may hold.

As an example, consider the potential $\mathfrak{e}$ on $X = \{1, \ldots, 5\}$ defined, for all intervals A of X, by:

$$\mathfrak{e}(A) = \begin{cases} 0 \text{ if } A = \emptyset. \\ 1 \text{ if } |A| = 1. \\ 3 \text{ if } |A| = 2 \text{ and } \min(A) \geq 3. \\ 6 \text{ if } |A| = 2 \text{ and } \max(A) \leq 2. \\ 8 \text{ if } |A| = 3. \\ 5(|A| - 1) \text{ if } |A| > 3. \end{cases} \tag{1}$$

One can easily check that $\mathfrak{e}(A) > \mathfrak{e}(A_1) + \mathfrak{e}(A_2)$ for all intervals A, A_1, A_2 of X such that A_1, A_2 is a partition of A. More precisely, $\mathfrak{e}$ satisfies the (stronger) property that $\mathfrak{e}(A) > \mathfrak{e}(A_1) + \mathfrak{e}(A_2)$ for all nonempty intervals A, A_1, A_2 of X such that $|A_1| + |A_2| = |A|$. Hence, $\mathfrak{e}$ is superadditive.

To see that $\mathfrak{e}$ is not core, consider the segmentations $S_1 = \{[1, 2], [3, 5]\}$ and $S_2 = \{[1], [2, 4], [5]\}$ of X. The family $\mathcal{F} = \{(S_1, 1), (S_2, 1)\}$ admits two consensus segmentations, $C_1 = \{[1, 3], [4, 5]\}$ and $C_2 = \{[1, 3], [4], [5]\}$, neither of which is refined by $\widehat{S}(\mathcal{F}) = S_1 \wedge S_2 = \{[1], [2], [3, 4], [5]\}$.

Despite not being core in general, superadditive potentials enjoy the following interesting property:

Lemma 3 *Let $\mathfrak{e}$ be a superadditive potential on X, and $\mathcal{F} = \{(S_1, \omega_1), \ldots, (S_k, \omega_k)\}$ be a weighted family of segmentations of X. If there exists $x \in X$ such that x is a breakpoint of S_i for all $i \in \{1, \ldots, k\}$, then for all segmentations S, S' of X such that S' is obtained from S by adding x as a breakpoint, we have $f_{\mathcal{F}}(S') \leq f_{\mathcal{F}}(S)$. In particular, $\mathcal{F}$ admits a consensus C such that x is a breakpoint of C.*

Proof Let $i \in \{1, \ldots, k\}$, and let S be a segmentation of X such that x is not a breakpoint of S. Let S' be the segmentation of X obtained from S by adding x as a breakpoint, that is, by replacing the interval A of S containing x by the intervals $A_1 = \{z \in A, z \leq x\}$ and $A_2 = \{z \in A, z > x\}$. By definition, we have $D_{\mathfrak{e}}(S_i, S) = E(S_i) + E(S) - E(S_i \wedge S)$ and $D_{\mathfrak{e}}(S_i, S') = E(S_i) + E(S') - E(S_i \wedge S')$. Moreover, x is a breakpoint of S_i by assumption, so we have $S_i \wedge S = S_i \wedge S'$. It follows that $D_{\mathfrak{e}}(S_i, S) - D_{\mathfrak{e}}(S_i, S') = E(S) - E(S') = \mathfrak{e}(A) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2)$. Since $\mathfrak{e}$ is superadditive, and A is the disjoint union of A_1 and A_2, we have $\mathfrak{e}(A) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) \geq 0$. Hence, we have $D_{\mathfrak{e}}(S_i, S) \geq D_{\mathfrak{e}}(S_i, S')$. Since this holds for all $i \in \{1, \ldots, k\}$, $f_{\mathcal{F}}(S) \geq f_{\mathcal{F}}(S')$ follows. In particular, if S is a consensus of $\mathcal{F}$, then $f_{\mathcal{F}}(S) = f_{\mathcal{F}}(S')$ and S' is also consensus for $\mathcal{F}$. □

3 Supermodularity and Metric Dissimilarities

Given a potential $\mathfrak{e}$ on X, the map $D_{\mathfrak{e}}$ is by definition symmetric, and satisfies $D_{\mathfrak{e}}(S, S) = 0$ for all segmentations S on X. However, $D_{\mathfrak{e}}$ is, in general, not a metric, as it may be negative, or it may violate the triangle inequality. For example, for $\mathfrak{e}$ the potential defined in Eq. 1, and S_1, S_2, C_1 as defined above, one can verify that we have $8 = D_{\mathfrak{e}}(C_1, S_1) + D_{\mathfrak{e}}(C_1, S_2) < D_{\mathfrak{e}}(S_1, S_2) = 12$. Hence, $D_{\mathfrak{e}}$ does not satisfy the triangle inequality, and is, therefore, not a distance. In this section, we characterize those potentials $\mathfrak{e}$ for which $D_{\mathfrak{e}}$ is a distance. We first start with the nonnegativity property.

Proposition 4 *Let $X = \{1, \ldots, n\}$, and let $\mathfrak{e}$ be a potential on X. The map $D_{\mathfrak{e}}$ satisfies $D_{\mathfrak{e}}(S_1, S_2) \geq 0$ for all segmentations S_1, S_2 of X if and only if $\mathfrak{e}$ is superadditive.*

Proof Suppose first that $\mathfrak{e}$ is superadditive. Then for all segmentations S_1, S_2 of X, we have $E(S_1 \wedge S_2) \leq E(S_1)$ and $E(S_1 \wedge S_2) \leq E(S_2)$, so $D_{\mathfrak{e}}(S_1, S_2) = E(S_1) + E(S_2) - 2E(S_1 \wedge S_2) \geq 0$.

Conversely, suppose that $\mathfrak{e}$ is not superadditive. Then, there exist three intervals A, A_1, A_2 of X such that A_1, A_2 form a partition of A and $\mathfrak{e}(A_1) + \mathfrak{e}(A_2) > \mathfrak{e}(A)$. Calling A^- (*resp.* A^+) the (possibly empty) interval of X containing all elements smaller than $\min(A)$ (*resp.* greater than $\max(A)$), we define $S_1 = \{A^-, A, A^+\}$ and $S_2 = \{A^-, A_1, A_2, A^+\}$. Since $S_1 \wedge S_2 = S_2$, it follows that $D_{\mathfrak{e}}(S_1, S_2) = E(S_1) - E(S_2) = \mathfrak{e}(A) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2)$, so $D_{\mathfrak{e}}(S_1, S_2) < 0$ by choice of A, A_1, A_2. □

We now turn our attention to the triangle inequality. As a first observation, we have:

Lemma 5 *Let $X = \{1, \dots, n\}$, and let $\mathfrak{e}$ be a potential on X. For S, T, U three segmentations of X, the following three inequalities are equivalent:*

(i) $D_\mathfrak{e}(S, T) + D_\mathfrak{e}(T, U) \geq D_\mathfrak{e}(S, U)$.
(ii) $E(T) + E(S \wedge U) \geq E(S \wedge T) + E(T \wedge U)$.
(iii) $D_\mathfrak{e}(T, S \wedge U) \geq D_\mathfrak{e}(S \wedge T, T \wedge U)$.

Proof This can be easily verified using the definition of $D_\mathfrak{e}$. □

Armed with this result, we can now show the following:

Proposition 6 *Let $X = \{1, \dots, n\}$, and let $\mathfrak{e}$ be a potential on X. The map $D_\mathfrak{e}$ satisfies the triangle inequality if and only if $\mathfrak{e}$ is supermodular.*

Proof Suppose first that $D_\mathfrak{e}$ satisfies the triangle inequality. Let A, A_1, A_2 be three intervals of X such that $A = A_1 \cup A_2$, and let $I = A_1 \cap A_2$. We denote by A^- and A^+ the intervals of X (possibly empty) containing all elements smaller than $\min(A)$ and greater than $\max(A)$, respectively.

Consider the segmentations $S_1 = \{A^-, A, A^+\}$, $S_2 = \{A^-, A_1, A_2 \backslash I, A^+\}$ and $S_3 = \{A^-, A_1 \backslash I, A_2, A^+\}$. We have:

$$D_\mathfrak{e}(S_1, S_2) = \mathfrak{e}(A) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2 \setminus I)$$
$$D_\mathfrak{e}(S_1, S_3) = \mathfrak{e}(A) - \mathfrak{e}(A_2) - \mathfrak{e}(A_1 \setminus I)$$
$$D_\mathfrak{e}(S_2, S_3) = \mathfrak{e}(A_1) + \mathfrak{e}(A_2) - \mathfrak{e}(A_1 \setminus I) - \mathfrak{e}(A_2 \setminus I) - 2\mathfrak{e}(I).$$

Since $D_\mathfrak{e}$ satisfies the triangle inequality, we have $D_\mathfrak{e}(S_1, S_2) + D_\mathfrak{e}(S_1, S_3) \geq D_\mathfrak{e}(S_2, S_3)$. Replacing all three distances by their expression, we obtain:

$$2\mathfrak{e}(A) - \mathfrak{e}(A_1) - \mathfrak{e}(A_1 \setminus I) - \mathfrak{e}(A_2) - \mathfrak{e}(A_2 \setminus I) \geq \mathfrak{e}(A_1) + \mathfrak{e}(A_2) - \mathfrak{e}(A_1 \setminus I) - \mathfrak{e}(A_2 \setminus I) - 2\mathfrak{e}(I),$$

which becomes

$$2\mathfrak{e}(A) + 2\mathfrak{e}(I) \geq 2\mathfrak{e}(A_1) + 2\mathfrak{e}(A_2).$$

This directly implies $\mathfrak{e}(A) + \mathfrak{e}(I) \geq \mathfrak{e}(A_1) + \mathfrak{e}(A_2)$. Hence $\mathfrak{e}$ is supermodular.

For the converse, suppose that $\mathfrak{e}$ is supermodular. Let S, T, U be three distinct segmentations of X. We next show that $E(T) + E(S \wedge U) \geq E(S \wedge T) + E(T \wedge U)$. To this end, we consider the multiset Π_0 of intervals $(S \wedge T) \cup (T \wedge U)$. By definition, $E(S \wedge T) + E(T \wedge U) = \sum_{A \in \Pi_0} \mathfrak{e}(A)$. For $i \geq 1$ we can recursively define the multiset Π_i from Π_{i-1} by replacing two overlapping intervals A, B of Π_{i-1} with the intervals $A \cup B$ and $A \cap B$, until we obtain a set Π_k containing no overlapping intervals. Since $\mathfrak{e}$ is supermodular, we have $\sum_{A \in \Pi_i} \mathfrak{e}(A) \geq \sum_{A \in \Pi_{i-1}} \mathfrak{e}(A)$, so $\sum_{A \in \Pi_k} \mathfrak{e}(A) \geq E(S \wedge T) + E(T \wedge U)$.

We next claim that, for all elements $x \in X$, there are exactly two elements of Π_k (which can be two copies of the same interval) containing x. To see this, we first remark that this holds for Π_0, since all elements x of X belong to one interval of $S \wedge T$ and one interval of $T \wedge U$. Suppose now that $i \geq 1$ is such that Π_i contains exactly two intervals A_x, B_x containing x, and let A, B be the intervals of Π_i replaced by $A \cup B$ and $A \cap B$ in the construction of Π_{i+1}. If A, B, A_x, B_x are pairwise distinct, neither $A \cup B$ nor $A \cap B$ contains x, so A and B are the only intervals of Π_{i+1} containing x. If (up to a permutation) exactly one of $A_x = A$ or $B_x = B$ holds, say $A_x = A$, then B does not contain x, and B_x and $A \cup B$ are the only intervals of Π_{i+1} containing x. Finally, if both $A_x = A$ and $B_x = B$ holds, then $A \cup B$ and $A \cap B$ are the only intervals of Π_{i+1} containing x. The claim being true, there are no three distinct intervals A, B, C in Π_k such that $A \subsetneq B \subsetneq C$.

Based on this, we now define two segmentations P_1 and P_2 of X as follows. If an interval A has multiplicity two in Π_k, we add A to both P_1 and P_2. By construction, each remaining interval A of Π_k satisfies exactly one of

(i) there exists an interval B in Π_k such that $B \subsetneq A$, and (ii) there exists an interval C in Π_k such that $A \subsetneq C$. We then add A to P_1 if the first case holds, and to P_2 if the second case holds. Because of the previous observations, P_1 and P_2 are well defined segmentations of X, and P_2 is a refinement of P_1. Moreover, we have by construction that $E(P_1) + E(P_2) = \sum_{A \in \Pi_k} \mathfrak{e}(A)$. In particular, we have $E(P_1) + E(P_2) \geq E(S \wedge T) + E(T \wedge U)$.

We conclude by showing that $E(T) + E(S \wedge U) \geq E(P_1) + E(P_2)$. We do this by showing that P_1 is a refinement of T, and P_2 is a refinement of $S \wedge U$. Since $\mathfrak{e}(A_1) + \mathfrak{e}(A_2) \leq \mathfrak{e}(A) + \mathfrak{e}(A_1 \cap A_2)$ for all intervals A, A_1, A_2 of X such that $A = A_1 \cup A_2$, $E(P_1) \leq E(T)$ and $E(P_2) \leq E(S \wedge U)$ follow.

To see that P_1 is a refinement of T, let $x \in X$ be a breakpoint of T. Then x is a breakpoint of both $S \wedge T$ and $T \wedge U$. In particular, there exists $a_2 \leq a_1 \leq x$ such that $[a_2, x]$ and $[a_1, x]$ are elements of Π_0. Clearly, no set in Π_0 overlaps $[a_1, x]$, so $[a_1, x]$ is an element of Π_k. If there is no set in Π_0 overlapping $[a_2, x]$, then $[a_2, x]$ is an element of Π_k. If however, such an interval exists, then that interval does not contain x. Hence, in both cases, there exists $a_3 \leq a_2$ such that $[a_3, x]$ is an element of Π_k. Since $[a_3, x] \supseteq [a_1, x]$, it follows by construction that $[a_3, x]$ is an element of P_1. Hence, x is a breakpoint of P_1.

To see that P_2 is a refinement of $S \wedge U$, let x be a breakpoint of $S \wedge U$. Without loss of generality, we may assume that x is a breakpoint of S. Then, x is a breakpoint of $S \wedge T$, and there exists $a_1 \leq x$ such that $[a_1, x]$ is an element of Π_0. Now, let $[a_2, b_2]$ be the interval of $T \wedge U$ containing x. Three cases may occur. If $[a_1, x]$ is contained in $[a_2, b_2]$, then $[a_1, x]$ does not overlap with any element of Π_0. In particular, $[a_1, x]$ is an element of Π_k, and no elements in Π_k are strictly contained in $[a_1, x]$. By construction, $[a_1, x]$ must then be an interval of P_2, so x is a breakpoint of P_2. If $[a_2, b_2]$ is contained in $[a_1, x]$, then since $[a_2, b_2]$ contains x, $b_2 = x$ must hold. Repeating the arguments from the previous case by permuting the roles of $[a_2, x]$ and $[a_1, x]$, it follows that x is a breakpoint of P_2. Finally, if $[a_2, b_2]$ and $[a_1, x]$ overlap, then $[a_2, b_2] \cap [a_1, x] = [a_2, x]$ is an element of Π_k. Moreover, no elements in Π_k are strictly contained in $[a_2, x]$. By construction, $[a_2, x]$ must then be an interval of P_2, so x is a breakpoint of P_2.

In conclusion, we have $E(T) + E(S \wedge U) \geq E(P_1) + E(P_2) \geq E(S \wedge T) + E(T \wedge U)$. By Lemma 5, it follows that $D_\mathfrak{e}(S, T) + D_\mathfrak{e}(T, U) \geq D_\mathfrak{e}(S, U)$, and hence $D_\mathfrak{e}$ satisfies the triangle inequality. □

From Proposition 6 and Proposition 4, we immediately derive the following:

Theorem 7 *Let $X = \{1, \dots, n\}$, and let $\mathfrak{e}$ be a potential on X. Then, $D_\mathfrak{e}$ is a metric if and only if $\mathfrak{e}$ is supermodular.*

Recall that a potential $\mathfrak{e}$ is said to be m-core for some $m \geq 1$ if $\mathfrak{e}$ satisfies the condition of Definition 1 for all families of size at most m. Interestingly, Proposition 4 and Theorem 7 allow us to derive sufficient conditions for a potential to be 1-core and 2-core:

Proposition 8 *Let $\mathfrak{e}$ be a potential on X.*

- *If $\mathfrak{e}$ is superadditive, then $\mathfrak{e}$ is 1-core.*
- *If $\mathfrak{e}$ is supermodular, then $\mathfrak{e}$ is 2-core.*

Proof To see the first statement, suppose that $\mathfrak{e}$ is superadditive. Let $\mathcal{F} = \{(S_1, \omega_1)\}$ be a weighted family on X. For a segmentation S of X, we have $f_\mathcal{F}(S) = \omega_1 D_\mathfrak{e}(S, S_1)$. Since $\mathfrak{e}$ is superadditive, Proposition 4 implies that $f_\mathcal{F}(S) \geq 0$. Moreover, $f_\mathcal{F}(S_1) = \omega_1 D_\mathfrak{e}(S_1, S_1) = 0$. Hence, S_1 is a consensus for $\mathcal{F}$. Since $\widehat{S}(\mathcal{F}) = S_1$, it follows that $\mathcal{F}$ admits a consensus that refines $\widehat{S}(\mathcal{F})$. Hence, $\mathfrak{e}$ is 1-core.

To see the second statement, suppose that $\mathfrak{e}$ is supermodular. In particular, $\mathfrak{e}$ is superadditive, so given the above, $\mathfrak{e}$ is 1-core. Hence, it suffices to consider the case of families of size 2. Let $\mathcal{F} = \{(S_1, \omega_1), (S_2, \omega_2)\}$ be one such family. Without loss of generality, we may assume that $\omega_1 \geq \omega_2$. For S a segmentation of X, we have $f_\mathcal{F}(S) = \omega_1 D_\mathfrak{e}(S, S_1) + \omega_2 D_\mathfrak{e}(S, S_2) = (\omega_1 - \omega_2) D_\mathfrak{e}(S, S_1) + \omega_2 (D_\mathfrak{e}(S, S_1) + D_\mathfrak{e}(S, S_2))$. Since $\mathfrak{e}$ is superadditive, we have $D_\mathfrak{e}(S, S_1) \geq 0$ by Proposition 4. Moreover, $\mathfrak{e}$ is supermodular, so Theorem 7 implies that $D_\mathfrak{e}(S, S_1) + D_\mathfrak{e}(S, S_2) \geq D_\mathfrak{e}(S_1, S_2)$. Together with the fact that $\omega_2 \geq 0$ and $\omega_1 - \omega_2 \geq 0$, these two inequalities together imply that $f_\mathcal{F}(S) \geq \omega_2 (D_\mathfrak{e}(S_1, S_2)) = f_\mathcal{F}(S_1)$. Hence, S_1 is a consensus for $\mathcal{F}$. Moreover, S_1 is refined by $\widehat{S}(\mathcal{F}) = S_1 \wedge S_2$. This concludes the proof that $\mathfrak{e}$ is 2-core. □

We conclude this section with an example of a potential that is core but not supermodular. Consider the set $X = \{1, 2, 3\}$, and the potential $\mathfrak{e}$ defined, for all intervals A of X, by $\mathfrak{e}(A) = |A|^2(5 - |A|)$. We have $\mathfrak{e}([1, 3]) + \mathfrak{e}([2]) = 22 < 24 = \mathfrak{e}([1, 2]) + \mathfrak{e}([2, 3])$, and hence $\mathfrak{e}$ is not supermodular. It is not difficult to verify that $\mathfrak{e}$ is superadditive. To see that $\mathfrak{e}$ is core, we note that X admits only four distinct segmentations: $S_0 = \{[1, 3]\}$, $S_1 = \{[1], [2, 3]\}$, $S_2 = \{[1, 2], [3]\}$, and $S_3 = \{[1], [2], [3]\}$. Now, let $\mathcal{F}$ be a weighted family of segmentations of X. If $\mathcal{F}$ contains S_3, or if it contains both S_1 and S_2, then $\widehat{S}_{\mathcal{F}} = S_3$. Since S_3 refines all segmentations of X, all consensus for $\mathcal{F}$ are refined by $\widehat{S}_{\mathcal{F}}$. The case of families of size 1 comes directly from the facts that $D_{\mathfrak{e}}(S, S) = 0$ for all segmentations S of X, and that $\mathfrak{e}$ is superadditive, and therefore satisfies $D_{\mathfrak{e}}(S, S') \geq 0$ for all segmentations S, S' of X (Proposition 4). It remains to consider the case where $\mathcal{F}$ is one of $\{(\omega_0, S_0), (\omega_1, S_1)\}$ or $\{(\omega_0, S_0), (\omega_2, S_2)\}$. If $\mathcal{F} = \{(\omega_0, S_0), (\omega_1, S_1)\}$, we have $f_{\mathcal{F}}(S_0) = 2\omega_1$, $f_{\mathcal{F}}(S_1) = 2\omega_0$, $f_{\mathcal{F}}(S_2) = 2\omega_0 + 8\omega_1$ and $f_{\mathcal{F}}(S_3) = 6\omega_0 + 4\omega_1$. Since both ω_0 and ω_1 are positive, it follows that a consensus for $\mathcal{F}$ is one of S_0 or S_1 (depending on which of ω_0 and ω_1 is smallest; both if $\omega_0 = \omega_1$), which are both refined by $\widehat{S}_{\mathcal{F}} = S_0 \wedge S_1 = S_1$. The case $\mathcal{F} = \{(\omega_0, S_0), (\omega_2, S_2)\}$ follows immediately by symmetry.

4 Size-Restricted and Convex Potentials

In practical applications, the potential of an interval A is often chosen as a function of the cardinality of A, often with some rescaling. In [14] $\mathfrak{e}(A) := |A|/n$ is used, interpreting $\mathfrak{e}$ as the probability of hitting an interval at random. Given this, we say that a potential $\mathfrak{e}$ on X is *size-restricted* if there exists a map $e : X \cup \{0\} \to \mathbb{R}$ such that $\mathfrak{e}(A) = e(|A|)$ holds for all intervals A of X. In this case, we call e the *underlying map* of $\mathfrak{e}$. Note that the underlying map e of a potential $\mathfrak{e}$ always satisfies $e(0) = 0$.

Perhaps unsurprisingly, not all size-restricted superadditive potentials are core.

Observation 9 *There exist size-restricted superadditive potentials that are not core.*

As an example, consider the potential $\mathfrak{e}$ on $X = \{1, \dots, 8\}$ defined for all intervals A of X, by:

$$\mathfrak{e}(A) = \begin{cases} 0 & \text{if } A = \emptyset. \\ \frac{5|A|-3}{2} & \text{if } |A| \text{ is odd.} \\ \frac{5|A|-5}{2} & \text{if } |A| \text{ is even and } |A| > 0. \end{cases} \tag{2}$$

We first remark that $\mathfrak{e}$ is superadditive. Indeed for A, A_1, A_2 three pairwise distinct intervals of X such that A_1, A_2 form a partition of A, we have $\mathfrak{e}(A_1)+\mathfrak{e}(A_2) \leq \frac{5|A_1|-3}{2}+\frac{5|A_2|-3}{2} = \frac{5(|A_1|+|A_2|)-6}{2}$. Since $|A_1|+|A_2| = |A|$, it follows that $\mathfrak{e}(A_1)+\mathfrak{e}(A_2) \leq \frac{5(|A|)-6}{2} < \frac{5(|A|)-5}{2} \leq \mathfrak{e}(A)$. However, $\mathfrak{e}$ is not supermodular. Consider for example $A = [1, 4]$, $A_1 = [1, 3]$ and $A_2 = [2, 4]$. We have $\mathfrak{e}(A) = 7.5$, $\mathfrak{e}(A_1) = \mathfrak{e}(A_2) = 6$ and $\mathfrak{e}(A_1 \cap A_2) = \mathfrak{e}([2, 3]) = 2.5$. Hence, $\mathfrak{e}(A_1) + \mathfrak{e}(A_2) = 12 > 10 = \mathfrak{e}(A) + \mathfrak{e}(A_1 \cap A_2)$.

To see that $\mathfrak{e}$ is not core, consider the segmentations $S_1 = \{[1], [2, 5], [6, 8]\}$ and $S_2 = \{[1, 3], [4, 7], [8]\}$ of X. One can check that the family $\mathcal{F} = \{(S_1, 1), (S_2, 1)\}$ admits exactly one consensus, $C = \{[1, 4], [5, 8]\}$,[2] and C is not refined by $\widehat{S}(\mathcal{F}) = S_1 \wedge S_2 = \{[1], [2, 3], [4, 5], [6, 7], [8]\}$.

As stated above, the potential $\mathfrak{e}$ defined in Eq. 2 is not supermodular. In the following, we characterize size-restricted potentials that are supermodular. To do this, we next recall the notion of (discrete) convexity.

We say that a map $e : X \cup \{0\} \to \mathbb{R}$ is *convex* if the discrete 2nd derivative:

$$\partial^2 e : z \mapsto e(z + 1) + e(z - 1) - 2e(z)$$

satisfies $\partial^2 e(z) \geq 0$ for all $z \in \{1, \dots, n - 1\}$. As a first, useful result, we have:

Lemma 10 *A map* $e : X \cup \{0\} \to \mathbb{R}$ *is convex if and only if for all* $0 \leq y \leq x \leq n - 1$, $e(x) - e(y) \leq e(x + 1) - e(y + 1)$.

[2] Although it requires computational help to check in reasonable time that C is indeed a consensus for $\mathcal{F}$, we would like to point out that there exists only $2^4 = 16$ segmentations on X that are refined by $\widehat{S}(\mathcal{F})$. It is therefore possible, although tedious, to verify by hand that for all such segmentations S, we have $f_{\mathcal{F}}(C) < f_{\mathcal{F}}(S)$. This is enough to conclude that $\mathfrak{e}$ is not core.

Proof Suppose first that e is convex, and let $x, y \in \{0, \dots, n-1\}$, with $y \leq x$. Note that the inequality is trivial when $x = y$, so we now assume that $y < x$. We have:

$$e(x) - e(y) = \sum_{i=0}^{x-y-1} e(x-i) - e(x-i-1).$$

Since e is convex, we have $e(x-i) - e(x-i-1) \leq e(x-i+1) - e(x-i)$ for all $0 \leq i \leq x-y+1$. Hence,

$$\sum_{i=0}^{x-y-1} e(x-i) - e(x-i-1) \leq \sum_{i=0}^{x-y-1} e(x-i+1) - e(x-i).$$

The right hand side of this inequality equals $e(x+1)-e(y+1)$. Thus it follows that $e(x)-e(y) \leq e(x+1)-e(y+1)$, as desired.

Conversely, suppose that $e(x) - e(y) \leq e(x+1) - e(y+1)$ holds for all $0 \leq y \leq x \leq n-1$, and let $z \in \{1, \dots, n-1\}$. Replacing x with z and y with $z-1$ in the above inequality, we get $e(z)-e(z-1) \leq e(z+1)-e(z)$, so e is convex. □

As the next result shows, convexity and supermodularity are directly related.

Proposition 11 *Let $\mathfrak{e}$ be a size-restricted potential on X with underlying map e. Then $\mathfrak{e}$ is supermodular if and only if e is convex.*

Proof Suppose first that $\mathfrak{e}$ is supermodular, and let $z \in \{1, \dots, n-1\}$. Consider two intervals A_1, A_2 of X satisfying $|A_1| = |A_2| = z$ and $|A_1 \cap A_2| = z-1$. Note that this necessarily implies $|A_1 \cup A_2| = z+1$. Since $\mathfrak{e}$ is supermodular, we have:

$$\mathfrak{e}(A_1) + \mathfrak{e}(A_2) \leq \mathfrak{e}(A_1 \cup A_2) + \mathfrak{e}(A_1 \cap A_2).$$

Replacing $\mathfrak{e}$ by e and each interval by its cardinality in the above inequality, we obtain $2e(z) \leq e(z+1) + e(z-1)$. Hence, e is convex.

Conversely, suppose that e is convex, and let A, A_1, A_2 be three pairwise distinct intervals of X such that $A = A_1 \cup A_2$. We put $|A_1| = x$, $|A_2| = y$, and $|A_1 \cap A_2| = z$. Note that $z \leq x$ and $z \leq y$ hold. Moreover, $|A| = x + y - z$, so $x + y - z \leq n$ also holds. By Lemma 10, we have $e(x) - e(z) \leq e(x+1) - e(z+1)$. By induction, $e(x) - e(z) \leq e(x+k) - e(z+k)$ holds for all $k \in \{0, \dots, n-x\}$. In particular, choosing $k = y - z$ leads $e(x) - e(z) \leq e(x+y-z) - e(y)$, which by assumption on e and $\mathfrak{e}$, implies:

$$\mathfrak{e}(A_1) - \mathfrak{e}(A_1 \cap A_2) \leq \mathfrak{e}(A_1 \cup A_2) - \mathfrak{e}(A_2).$$

Hence, $\mathfrak{e}$ is supermodular. □

Combining Theorem 7 and Proposition 11 we obtain:

Theorem 12 *Let $\mathfrak{e}$ be a potential on X such that $\mathfrak{e}$ is size-restricted with underlying map e. Then the following are equivalent:*

(i) e is convex.
(ii) $\mathfrak{e}$ is supermodular.
(iii) $D_{\mathfrak{e}}$ is a metric.

For potentials $\mathfrak{e}$ that are not necessarily size-restricted, it is is still interesting to study the convexity of the induced maps $x \mapsto \mathfrak{e}([x, y_0])$ and $y \mapsto \mathfrak{e}([x_0, y])$, for fixed x_0 and y_0, respectively. More precisely, for a potential $\mathfrak{e}$ on X and $z \in X$, we define the two maps

$$\mathfrak{e}_z^- : \{0, 1, \dots, z\} \to \mathbb{R}_{\geq 0} \qquad t \mapsto \begin{cases} 0 & \text{if } t = 0. \\ \mathfrak{e}([z-t+1, z]) & \text{if } t > 0. \end{cases}$$

$$\mathfrak{e}_z^+ : \{0, 1, \dots, n+1-z\} \to \mathbb{R}_{\geq 0} \qquad t \mapsto \begin{cases} 0 & \text{if } t = 0. \\ \mathfrak{e}([z, z+t-1]) & \text{if } t > 0. \end{cases}$$

We say that $\mathfrak{e}$ is *convex* if both maps $\mathfrak{e}_z^-$ and $\mathfrak{e}_z^+$ are convex for all $z \in X$. Intuitively, convexity captures the fact that the growth rate of a potential increases when increasing the size of the interval on which the potential is evaluated. If $\mathfrak{e}(A)$ is size-restricted, one can easily verify that $\mathfrak{e}$ is convex if and only if e is convex. In particular, by Theorem 12, a size-restricted potential $\mathfrak{e}$ is supermodular if and only if $\mathfrak{e}$ is convex. As the next observation shows, this equivalence is not true when the potential $\mathfrak{e}$ is not size-restricted. In fact, there are convex potentials that are not supermodular, and supermodular potentials that are not convex. More interestingly, superadditive potentials satisfying one of these properties but not the other are not necessarily core.

Observation 13 *(i) There exist supermodular, non convex potentials that are not core.*
(ii) There exists superadditive, convex, non-supermodular potentials that are not core.

Consider first the potential $\mathfrak{e}$ on $X = \{1, \dots, 10\}$ defined for all intervals A of X, by:

$$\mathfrak{e}(A) = \begin{cases} 0 & \text{if } A = \emptyset. \\ 1 & \text{if } |A| = 1. \\ 3 & \text{if } |A| = 2 \text{ and } \min(A) \text{ is even.} \\ 6 & \text{if } |A| = 2 \text{ and } \max(A) \text{ is odd.} \\ 9 & \text{if } |A| = 3. \\ 2^{|A|} & \text{if } |A| > 3. \end{cases} \tag{3}$$

We begin by showing that $\mathfrak{e}$ is supermodular and not convex. To see that $\mathfrak{e}$ is supermodular, consider three pairwise distinct intervals A, A_1, A_2 of X such that $A_1 \cup A_2 = A$. A simple case-analysis which we leave to the reader shows that $\mathfrak{e}(A_1) + \mathfrak{e}(A_2) < \mathfrak{e}(A) + \mathfrak{e}(A_1 \cap A_2)$ holds whenever $|A| \leq 4$. Suppose now that $A \geq 5$ and suppose without loss of generality that $|A_1| \geq |A_2|$. In this case, we have $\mathfrak{e}(A) = 2^{|A|}$ and $\mathfrak{e}(A_2) \leq \mathfrak{e}(A_1) \leq \max\{9, 2^{|A_1|}\}$.If $\max\{9, 2^{|A_1|}\} = 9$, then $\mathfrak{e}(A_1) + \mathfrak{e}(A_2) \leq 18 < 32 \leq \mathfrak{e}(A)$. If $\max\{9, 2^{|A_1|}\} = 2^{|A_1|}$, then $\mathfrak{e}(A_1) + \mathfrak{e}(A_2) \leq 2^{|A_1|+1} \leq 2^{|A|} = \mathfrak{e}(A)$. Note that at least one of these inequalities must be strict. Hence, $\mathfrak{e}(A_1) + \mathfrak{e}(A_2) < \mathfrak{e}(A)$ holds, and $\mathfrak{e}(A) \leq \mathfrak{e}(A) + \mathfrak{e}(A_1 \cap A_2)$ always holds, so $\mathfrak{e}$ is supermodular.

To see that $\mathfrak{e}$ is not convex, we consider the map $\mathfrak{e}_1^+$. We have $\mathfrak{e}_1^+(1) = \mathfrak{e}([1]) = 1$, $\mathfrak{e}_1^+(2) = \mathfrak{e}([1, 2]) = 6$, and $\mathfrak{e}_1^+(3) = \mathfrak{e}([1, 3]) = 9$. Hence, $\mathfrak{e}_1^+(1) + \mathfrak{e}_1^+(3) - 2\mathfrak{e}_1^+(2) = -2 < 0$, so $\mathfrak{e}_1^+$ is not convex. By definition, it follows that $\mathfrak{e}$ is not convex.

We next show that $\mathfrak{e}$ is not core. To this aim, consider the segmentations $S_1 = \{[1, 4], [5, 10]\}$, $S_2 = \{[1, 5], [6, 10]\}$, $S_3 = \{[1, 7], [8, 10]\}$, and $S_4 = \{[1, 8], [9, 10]\}$ of X. One can verify that the family $\mathcal{F} = \{(S_1, 1), (S_2, 1), (S_3, 1), (S_4, 1)\}$ admits exactly one consensus, $C = \{[1, 6], [7, 10]\}$, and C is not refined by $\widehat{S}(\mathcal{F}) = \{[1, 4], [5], [6, 7], [8], [9, 10]\}$ (see footnote 2 for more on this claim).

Now, consider the potential $\mathfrak{e}$ on $X = \{1, \dots, 8\}$ defined for all intervals A of X, by:

$$\mathfrak{e}(A) = \begin{cases} 0 & \text{if } A = \emptyset. \\ 1 & \text{if } |A| = 1. \\ 4 & \text{if } |A| = 2 \text{ and } \min(A) \text{ is even.} \\ 2.5 & \text{if } |A| = 2 \text{ and } \min(A) \text{ is odd.} \\ 8.5 & \text{if } |A| = 3. \\ 13.5 & \text{if } |A| = 4 \text{ and } \min(A) \text{ is even.} \\ 16 & \text{if } |A| = 4 \text{ and } \min(A) \text{ is odd.} \\ 2^{|A|} & \text{if } |A| > 4. \end{cases} \tag{4}$$

A simple case analysis shows that $\mathfrak{e}$ is convex. To see that $\mathfrak{e}$ is superadditive, let A, A_1, A_2 be three pairwise distinct intervals of X such that A_1, A_2 form a partition of A. In particular, $|A| = |A_1| + |A_2|$ holds. Without loss of generality, we may assume that $|A_1| \geq |A_2|$. If $|A_2| = 1$, then $|A_1| = |A| - 1$. In this case, one can easily verify that $\mathfrak{e}(A_1) < \mathfrak{e}(A) - 1$ always holds, and since $\mathfrak{e}(A_2) = 1$, $\mathfrak{e}(A_1) + \mathfrak{e}(A_2) < \mathfrak{e}(A)$ follows. If $|A_2| > 1$, then we have $\mathfrak{e}(A_2) \leq \mathfrak{e}(A_1) < 2^{|A_1|} + 1$, so $\mathfrak{e}(A_1) + \mathfrak{e}(A_2) < 2^{|A_1|+1} + 2$. Since $|A_2| \geq 2$ by assumption, we have $|A_1| + 1 \leq |A| - 1$, so $\mathfrak{e}(A_1) + \mathfrak{e}(A_2) < 2^{|A|-1} + 2$. Since $A \geq 4$, $\mathfrak{e}(A) > 2^{|A|-1} + 2$ always holds, so we have $\mathfrak{e}(A_1) + \mathfrak{e}(A_2) < \mathfrak{e}(A)$ as desired. However, $\mathfrak{e}$ is not supermodular. Indeed, for $A_1 = [2, 4]$ and $A_2 = [3, 5]$, we have $\mathfrak{e}(A_1) = \mathfrak{e}(A_2) = 8.5$, $\mathfrak{e}(A_1 \cup A_2) = \mathfrak{e}([2,5]) = 13.5$, and $\mathfrak{e}(A_1 \cap A_2) = \mathfrak{e}([3,4]) = 2.5$. It follows that $\mathfrak{e}(A_1) + \mathfrak{e}(A_2) = 17 > 16 = \mathfrak{e}(A_1 \cup A_2) + \mathfrak{e}(A_1 \cap A_2)$.

We now show that $\mathfrak{e}$ is not core. To this aim, consider the segmentations $S_1 = \{[1], [2,5], [6,8]\}$ and $S_2 = \{[1,3], [4,7], [8]\}$ of X. One can verify that the family $\mathcal{F} = \{(S_1, 1), (S_2, 1)\}$ admits exactly one consensus, $C = \{[1,4], [5,8]\}$, and C is not refined by $\widehat{S}(\mathcal{F}) = \{[1], [2,3], [4,5], [6,7], [8]\}$ (see also footnote 2).

We conclude this section by remarking that convexity and supermodularity together are not enough to guarantee that a potential is core.

Observation 14 *There exist supermodular and convex potentials that are not core.*

Consider the potential on $X = \{1, \ldots, 8\}$ defined for all intervals A of X, by:

$$\mathfrak{e}(A) = \begin{cases} 0 & \text{if } A = \emptyset. \\ 1 & \text{if } |A| = 1. \\ 2.5 & \text{if } |A| = 2. \\ 2^{|A|} & \text{if } |A| \geq 3. \end{cases} \tag{5}$$

Clearly, $\mathfrak{e}$ is size-restricted, and the underlying map e of $\mathfrak{e}$ is convex. By Theorem 12, it follows that $\mathfrak{e}$ is supermodular.

Consider now the segmentations $S_1 = \{[1], [2,8]\}$, $S_2 = \{[1,7], [8]\}$, $S_3 = \{[1,3], [4,8]\}$, $S_4 = \{[1,5], [6,8]\}$ and $S_5 = \{[1,3], [4,5], [6,8]\}$ of X. One can verify that the family $\mathcal{F} = \{(S_i, 1), i \in \{1, \ldots, 5\}\}$ admits exactly one consensus, $C = \{[1,4], [5,8]\}$, and C is not refined by $\widehat{S}(\mathcal{F}) = \{[1], [2,3], [4,5], [6,7], [8]\}$ (see footnote 2).

As remarked above, the potential defined in Eq. 5 is size-restricted. However, it is worth noting that Observation 14 is not limited to size-restricted potentials. To see this, consider now the potential $\mathfrak{e}'$ obtained from $\mathfrak{e}$ as follows. For A an interval of X, we put $\mathfrak{e}'(A) = \mathfrak{e}(A)$ if $|A| \neq 2$. Otherwise, we put $\mathfrak{e}'(A) = 2 + \frac{\min(A)}{10}$. Clearly, $\mathfrak{e}'$ is not size-restricted, but the interested reader can verify that $\mathfrak{e}'$ remains supermodular and convex, and retains the property that C is a consensus for $\mathcal{F}$.

5 Expandingly Convex Potentials

We say that a potential $\mathfrak{e}$ on X is *boundedly convex* if $\mathfrak{e}$ is convex and, for all $z \in X$, the maps $\partial^2\mathfrak{e}_z^-$ and $\partial^2\mathfrak{e}_z^+$ are decreasing. As discussed above, a convex potential is a potential for which the growth rate increases when increasing the size of the interval on which the potential is evaluated. Boundedly convex potentials are precisely those convex potentials such that the speed at which the growth rate increases diminishes when the intervals become larger. Examples of boundedly convex potentials include the *α-disagreement potential* $\mathfrak{e} : A \mapsto (|A|/n)^{1+\alpha}/(1+\alpha)$, $0 \leq \alpha \leq 1$, and the *negative entropy* $\mathfrak{e} : A \mapsto -(|A|/n)\ln(|A|/n)$. Paraphrasing [18, Theorem 2], every boundedly convex potential $\mathfrak{e}$ on X is core.

The main part of the proof of this theorem establishes the following property of boundedly convex potentials, which is of interest its own right:

Lemma 15 *Let $\mathfrak{e}$ be a boundedly convex potential on X and let $S_1, \ldots, S_k$ with $k \geq 1$ be segmentations of X. Then, for every segmentation S on X there exists a segmentation S' of X such that S' is refined by $\widehat{S}{:=}S_1 \wedge S_2 \wedge \cdots \wedge S_k$ and $D_\mathfrak{e}(S', S_i) \leq D_\mathfrak{e}(S, S_i)$ for $1 \leq i \leq k$.*

Lemma 15 implies $f_\mathcal{F}(S') \leq f_\mathcal{F}(S)$ for arbitrary non-negative weights ω_i, and immediately implies that $\mathfrak{e}$ is core. The property in Lemma 15, however, is much stronger than just $\mathfrak{e}$ being core.

It is interesting to note that boundedly convex potentials are not necessarily supermodular. As an example, consider the potential $\mathfrak{e}$ on $X = \{1,2,3,4\}$ defined, for all intervals A of X, by $\mathfrak{e}(A) = |A|^{1.1}$ except $\mathfrak{e}([2]) = \mathfrak{e}([3]) = 0.5$ and $\mathfrak{e}([2,3]) = 1.8$. $\mathfrak{e}$. A quick case analysis shows that $\mathfrak{e}$ is boundedly convex, yet we have $\mathfrak{e}([1,3]) + \mathfrak{e}([3,4]) \simeq 5.49 > 5.09 \simeq \mathfrak{e}([1,4]) + \mathfrak{e}([3])$, and thus $\mathfrak{e}$ is not supermodular.

All examples of non-core potentials (Eqs. 1 to 5) discussed so far share the property that there exists $z \in X$ such that (at least) one of $\partial^2\mathfrak{e}_z^-$ and $\partial^2\mathfrak{e}_z^+$ is not monotone. This suggests to study the third case. We call a potential

Table 1 Example of an instance of CONSENSUS SEGMENTATION with expandingly convex potential $\mathfrak{e}(A) = 3^{|A|}$ for $A \neq \emptyset$

.	$D_\mathfrak{e}(., S_1)$	$D_\mathfrak{e}(., S_2)$	$D_\mathfrak{e}(., S_3)$	$\sum$
S	**81**	**69**	**69**	**219**
{[1, 3], [4, 5], [6, 7], [8]}	**87**	69	63	219
{[1, 2], [3], [4, 5], [6, 8]}	57	69	**93**	219
{[1, 3], [4], [5], [6, 7], [8]}	**84**	66	66	216
{[1, 2], [3], [4], [5, 7], [8]}	54	**96**	66	216
{[1, 2], [3], [4], [5], [6, 8]}	54	66	**96**	216
{[1, 2], [3, 4], [5, 7], [8]}	51	**93**	69	213
{[1, 2], [3, 4], [5], [6, 8]}	51	63	**99**	213
{[1, 3], [4, 5], [6, 8]}	72	54	**78**	204
{[1, 3], [4], [5, 7], [8]}	69	**81**	51	201
{[1, 3], [4], [5], [6, 8]}	69	51	**81**	201

For the input segmentations $S_1 = \{[1, 2], [3, 4], [5, 8]\}$, $S_2 = \{[1, 4], [5], [6, 8]\}$, $S_3 = \{[1, 3], [4, 7], [8]\}$ and the segmentation $S = \{[1, 3], [4], [5, 6], [7, 8]\}$, shown are all segmentations with scores not worse than S, their distance D_e to the three input segmentations and the total score Σ

$\mathfrak{e}$ on X *expandingly convex* if both $\partial^2\mathfrak{e}_z^-$ and $\partial^2\mathfrak{e}_z^+$ are increasing for all $z \in X$. Recall that boundedly convex potentials are precisely those convex potentials such that the speed at which the growth rate increases diminishes when the intervals become larger. Expandingly convex potentials have the opposite property, that is, their growth rate accelerates when the intervals become larger. Examples of expandingly convex potentials include $\mathfrak{e} : A \mapsto |A|^\alpha$ with $\alpha > 2$, $\mathfrak{e} : A \mapsto \alpha^{|A|}$ with $\alpha > 1$ and $\mathfrak{e} : A \mapsto \min(A) + \alpha(|A| - 1)$ with $\alpha > n$. The latter serves as an example of a potential that is not size-restricted.

Size-restricted expandingly convex potentials are of practical interest since they encompass the polynomials that were studied in numerically in [18]. Extensive simulations failed to produce a family $\mathcal{F}$ of segmentations for which no consensus segmentation is refined by $\widehat{S}(\mathcal{F})$. Expandingly convex potentials thus are good candidates for being core. Even though we will not be able to settle the conjecture, it is of interest to investigate the properties of expandingly convex potentials in more detail.

We start by noting that the property of Lemma 15 in general does not hold for expandingly convex potentials. As an example, consider the potential $\mathfrak{e}$ on $X = \{1, \dots, 8\}$ defined by $\mathfrak{e}(A) = 0$ if $A = \emptyset$, and $\mathfrak{e}(A) = 3^{|A|}$ otherwise. One can easily verify that $\mathfrak{e}$ is expandingly convex. Let $S_1 = \{[1, 2], [3, 4], [5, 8]\}$, $S_2 = \{[1, 4], [5], [6, 8]\}$, $S_3 = \{[1, 3], [4, 7], [8]\}$, and consider the weighted family $\mathcal{F} = \{(S_1, 1), (S_2, 1), (S_3, 1)\}$. Finally, consider the segmentation $S = \{[1, 3], [4], [5, 6], [7, 8]\}$. Clearly, S is not refined by $\widehat{S}(\mathcal{F})$, since 6 is a breakpoint of S, and is not a breakpoint of any of S_1, S_2, S_3. Using computational help, one can verify that $f_\mathcal{F}(S) = 219$, and that there exist exactly ten segmentations S' of X distinct from S satisfying $f_\mathcal{F}(S') \leq f_\mathcal{F}(S)$. Table 1 shows, for all such segmentations S', there exists $i \in \{1, 2, 3\}$ such that $D_\mathfrak{e}(S', S_i) > D_\mathfrak{e}(S, S_i)$ (indicated in bold face in the table). Moreover, both of the optimal segmentations (with score 201) are refined by $\widehat{S}(\mathcal{F})$.

Since supermodularity seems to play an important role it seems useful to consider potentials that are both supermodular and expandingly convex. A interesting property in this context is the following: We say that a potential $\mathfrak{e}$ on X is *strongly supermodular* if, for all nonempty intervals A, A_1, A_2, and I of X with $A \neq A_1$, $A = A_1 \cup A_2$, and $I = A_1 \cap A_2$ hold

$$\mathfrak{e}(A) + \mathfrak{e}(I) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) \geq \mathfrak{e}(A_1) - \mathfrak{e}(I) - \mathfrak{e}(A_1 \setminus I).$$

The intuition behind this condition is to consider potential for which the curvature increases if an interval A_1 is expanded arbitrarily in either direction. One can thus see this a variation on the notion of expandingly convex potentials.

Examples of strongly supermodular potentials include potentials of the form $\mathfrak{e}(A) = \alpha^{|A|}, \alpha \leq 2$, $\mathfrak{e}(A) = |A|^{\alpha}, \alpha \geq |X|$, or $\mathfrak{e}(A) = |A|^{\alpha - \min(A)}, \alpha \geq |X|$.

First we note that there are potentials that are supermodular and expandingly convex but not strongly supermodular. As an example consider the potential $\mathfrak{e}$ on $X = \{1, \dots, 8\}$ defined, for all intervals A of X, by $\mathfrak{e}(A) = |A|(10 - \min(A))$. One can easily verify that $\mathfrak{e}$ is supermodular and expandingly convex. However, $\mathfrak{e}$ is not strongly supermodular. Indeed, we have $\mathfrak{e}([1, 5]) + \mathfrak{e}([2, 4]) - \mathfrak{e}([1, 4]) - \mathfrak{e}([2, 5]) = 1 < 3 = \mathfrak{e}([1, 4]) - \mathfrak{e}([1]) - \mathfrak{e}([2, 4])$, so the inequality $\mathfrak{e}(A_1 \cup A_2) + \mathfrak{e}(A_1 \cap A_2) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) \geq \mathfrak{e}(A_1) - \mathfrak{e}(A_1 \cap A_2) - \mathfrak{e}(A_1 \backslash (A_1 \cap A_2))$ does not hold for $A_1 = [1, 4]$ and $A_2 = [2, 5]$. However, we have:

Lemma 16 *Let $\mathfrak{e}$ be a potential on X. If $\mathfrak{e}$ is strongly supermodular, then $\mathfrak{e}$ is supermodular.*

Proof Let A, A_1, A_2 be three nonempty intervals of X such that $A = A_1 \cup A_2$, and set $I = A_1 \cap A_2$. We show that $\mathfrak{e}(A) + \mathfrak{e}(I) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) \geq 0$ holds. If $A = A_1$, then $I = A_2$, so $\mathfrak{e}(A) + \mathfrak{e}(I) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) = \mathfrak{e}(A_1) + \mathfrak{e}(A_2) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) = 0$, so the desired inequality holds. Suppose now that $A \neq A_1$. Since $\mathfrak{e}$ is strongly supermodular, we have $\mathfrak{e}(A) + \mathfrak{e}(I) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) \geq \mathfrak{e}(A_1) - \mathfrak{e}(I) - \mathfrak{e}(A_1 \backslash I)$.

If $I = \emptyset$, then $A_1 \backslash I = A_1$, and we have $\mathfrak{e}(A_1) - \mathfrak{e}(I) - \mathfrak{e}(A_1 \setminus I) = \mathfrak{e}(A_1) - \mathfrak{e}(A_1) = 0$, so the desired inequality follows. If otherwise, $I \neq \emptyset$, then $A_1 \neq A_1 \backslash I$. In particular, since $\mathfrak{e}$ is strongly supermodular, $\mathfrak{e}(A') + \mathfrak{e}(I') - \mathfrak{e}(A'_1) - \mathfrak{e}(A'_2) \geq \mathfrak{e}(A'_1) - \mathfrak{e}(I') - \mathfrak{e}(A'_1 \backslash I')$ holds for $A'_1 = A_1 \backslash I$, $A'_2 = I$, $A' = A'_1 \cup A'_2$ and $I' = A'_1 \cap A'_2$. In particular, we have $A' = A_1$ and $I' = \emptyset$ in that case, so we have $\mathfrak{e}(A') + \mathfrak{e}(I') - \mathfrak{e}(A'_1) - \mathfrak{e}(A'_2) = \mathfrak{e}(A_1) - \mathfrak{e}(I) - \mathfrak{e}(A_1 \backslash I)$ and $\mathfrak{e}(A'_1) - \mathfrak{e}(I') - \mathfrak{e}(A'_1 \backslash I') = \mathfrak{e}(A_1 \backslash I) - \mathfrak{e}(A_1 \backslash I) = 0$. Hence, the latter inequality becomes $\mathfrak{e}(A_1) - \mathfrak{e}(I) - \mathfrak{e}(A_1 \backslash I) \geq 0$. Putting this together with the first inequality, we obtain $\mathfrak{e}(A) + \mathfrak{e}(I) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) \geq 0$ as desired. □

The following example shows that strongly supermodular potentials are not expandingly convex in general. We shall show below, however, that this implication becomes true if we restrict ourselves to size-restricted potentials.

Observation 17 *The potential $\mathfrak{e}$ on $X = \{1, \dots, 8\}$ defined, for all nonempty intervals A of X, by $\mathfrak{e}(A) = 2^{|A|}$ if $|A| > 1$ or if $A = \{a\}$ with even a, and $\mathfrak{e}(A) = 1$ if $A = \{a\}$ with odd a is strongly supermodular but not expandingly convex.*

Proof We have $\partial^2 \mathfrak{e}_1^+(1) = 2 > \partial^2 \mathfrak{e}_1^+(2) = 1$ and hence $\mathfrak{e}$ is not expandingly convex. To see that $\mathfrak{e}$ is strongly supermodular, let A, A_1, A_2 be three nonempty intervals such that $A = A_1 \cup A_2$ and $A \neq A_1$. In this case we have $|A| \geq 2$ and thus $\mathfrak{e}(A) = 2^{|A|}$. Set $I = A_1 \cap A_2$.

If $I = \emptyset$ we have $\mathfrak{e}(A) + \mathfrak{e}(I) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) \geq 2^{|A|} - 2^{|A_1|} - 2^{|A_2|}$. Since $|A| = |A_1| + |A_2|$, $\mathfrak{e}(A) + \mathfrak{e}(I) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) \geq 0 = \mathfrak{e}(A_1) - \mathfrak{e}(I) - \mathfrak{e}(A_1 \backslash I)$ follows. Assume that $|I| \geq 1$. Since $|A_2| = 1$ would imply $A = A_1$ we have $|A_2| \geq 2$ and thus $\mathfrak{e}(A_2) = 2^{|A_2|}$. If $|A_1| = 1$, then $A_1 = I$ and $A_2 = A$, and one easily verifies that $\mathfrak{e}(A) + \mathfrak{e}(I) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) = 0 = \mathfrak{e}(A_1) - \mathfrak{e}(I) - \mathfrak{e}(A_1 \setminus I)$. If $|A_1| > 1$, we have $\mathfrak{e}(A_1) = 2^{|A_1|}$, and we need to distinguish between the cases $\mathfrak{e}(I) = 2^{|I|}$ and $\mathfrak{e}(I) = 1$.

If $\mathfrak{e}(I) = 2^{|I|}$, then we have $\mathfrak{e}(A) + \mathfrak{e}(I) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) = 2^{|A|} + 2^{|I|} - 2^{|A_1|} - 2^{|A_2|} = 2^{|I|}(2^{|A_1 \backslash I|} - 1)(2^{|A_2 \backslash I|} - 1)$. Since $A_1 \neq A$, $A_2 \neq I$ follows, so $2^{|A_2 \backslash I|} - 1 \geq 1$. In particular, we have $2^{|I|}(2^{|A_1 \backslash I|} - 1)(2^{|A_2 \backslash I|} - 1) \geq 2^{|I|}(2^{|A_1 \backslash I|} - 1) = 2^{|A_1|} - 2^{|I|} = \mathfrak{e}(A_1) - \mathfrak{e}(I)$, and the conclusion follows from the fact that $\mathfrak{e}(A_1 \setminus I) \geq 0$.

If $\mathfrak{e}(I) = 1$, then $|I| = 1$ must hold, so $2^{|I|} = 2$. I particular, we have $\mathfrak{e}(A) + \mathfrak{e}(I) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) = 2^{|A|} + 2^{|I|} - 2^{|A_1|} - 2^{|A_2|} - 1$. As shown in the previous paragraph, we have $2^{|A|} + 2^{|I|} - 2^{|A_1|} - 2^{|A_2|} \geq 2^{|A_1|} - 2^{|I|}$, so we get $\mathfrak{e}(A) + \mathfrak{e}(I) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) \geq 2^{|A_1|} - 2^{|I|} - 1 = \mathfrak{e}(A_1) - \mathfrak{e}(I) - 2$. Finally, we remark that $\mathfrak{e}(A_1 \setminus I) \geq 2$. Note first that $|A_1| > 1$ implies $A_1 \neq I$. If $|A_1 \backslash I| > 1$ then we have $\mathfrak{e}(|A_1 \backslash I|) = 2^{|A_1 \backslash I|} \geq 2$. Otherwise, since $A_1 = I \cup (A_1 \backslash I)$ and A_1 is an interval, the parity of the unique element of $I = \{u\}$ and the parity of the unique element v in $\{v\} = A_1 \setminus I$ must be distinct. Since $\mathfrak{e}(I) = 1$ by assumption, $\mathfrak{e}(I) = 1$, so the unique element u of I is odd. Hence, the unique element v of $A_1 \backslash I$ is even, so $\mathfrak{e}(A_1 \setminus I) = 2^{|A_1 \backslash I|} = 2$. Therefore, $\mathfrak{e}(A_1) - \mathfrak{e}(I) - 2 \geq \mathfrak{e}(A_1) - \mathfrak{e}(I) - \mathfrak{e}(A_1 \setminus I)$, and the conclusion follows. □

Proposition 18 *Let $\mathfrak{e}$ be a strongly supermodular potential on X such that $\mathfrak{e}$ is size-restricted with underlying map e. Then $\mathfrak{e}$ is expandingly convex.*

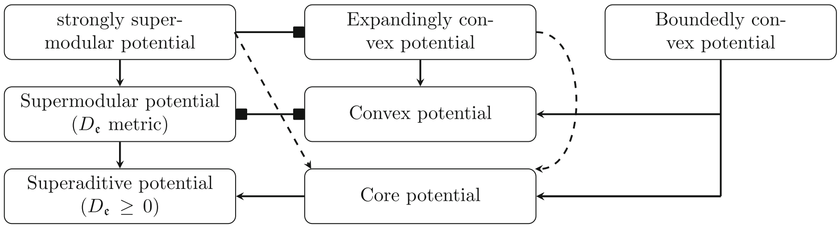

Fig. 1 Containment relations between certain classes of potentials defined above. The square arrows represent containment relations within the subclass of size-restricted potentials and the dashed arrows represent relations of interest for which neither proof nor counterexamples have been found

Proof Suppose first that $\mathfrak{e}$ is strongly supermodular. By Lemma 16, $\mathfrak{e}$ is supermodular, and since $\mathfrak{e}$ is size-restricted, $\mathfrak{e}$ is convex by Proposition 11. Hence, it remains to show that $\partial^2 e$ is increasing. More specifically, we show that $\partial^2 e(z+1) \geq \partial^2 e(z)$ holds for all $z \geq 1$. To this end, let $z \geq 1$, and let A, A_1, A_2 be three intervals of X such that $A = A_1 \cup A_2$, $|A_1| = |A_2| = z+1$ and $|A| = z+2$. Note that these conditions necessary imply $A \neq A_1$ and $|A_1 \cap A_2| = z$. In the following, we put $I = A_1 \cap A_2$ and $A_1' = A_1 \setminus I$. Since $\mathfrak{e}$ is strongly supermodular, we have:

$$\mathfrak{e}(A) + \mathfrak{e}(I) - \mathfrak{e}(A_1) - \mathfrak{e}(A_2) \geq \mathfrak{e}(A_1) - \mathfrak{e}(I) - \mathfrak{e}(A_1'),$$

and therefore:

$$\mathfrak{e}(A) + 2\mathfrak{e}(I) - 2\mathfrak{e}(A_1) - \mathfrak{e}(A_2) + \mathfrak{e}(A_1') \geq 0.$$

By definition of e, we obtain:

$$e(z+2) - 3e(z+1) + 2e(z) + e(1) \geq 0.$$

Moreover, $\mathfrak{e}$ is supermodular, and therefore, superadditive. Hence, we have $e(1) \leq e(z) - e(z-1)$. Putting this inequality back in the previous one, we obtain:

$$e(z+2) - 3e(z+1) + 3e(z) - e(z-1) \geq e(z+2) - 3e(z+1) + 2e(z) + e(1) \geq 0,$$

which becomes:

$$e(z+2) - 2e(z+1) + e(z) \geq e(z+1) - 2e(z) + e(z-1).$$

Therefore, we have $\partial^2 e(z+1) \geq \partial^2 e(z)$. □

The converse of Proposition 18 is not true that is, there exists potentials that are size-restricted and expandingly convex. This is the case, for example, for the potential $\mathfrak{e}$ defined by $\mathfrak{e}(A) = |A|^4$. This potential is is size-restricted and expandingly convex, yet $\mathfrak{e}([1,6]) + \mathfrak{e}([2,5]) - \mathfrak{e}([1,5]) - \mathfrak{e}([2,6]) = 302 < 368 = \mathfrak{e}([1,5]) - \mathfrak{e}([1]) - \mathfrak{e}([2,5])$, so it is not strongly supermodular.

6 Segmentations and Medians on the Hypercube

For a segmentation S on $X = \{1, \ldots, n\}$ we define the vector $v(S) \in \{0,1\}^{n-1}$ as follows: For $i \in \{1, \ldots, n-1\}$, we set the i^{th} coordinate $v(S)_i$ of $v(S)$ as 1 if i is a breakpoint of S, (i.e., the upper bound of an interval of S), and as 0 otherwise. For example, for $X = \{1, \ldots, 9\}$ and $S = \{[1,2], [3], [4,7], [8,9]\}$, we have $v(S) = (0,1,1,0,0,0,1,0)$. For all $n > 1$, the map v clearly is a bijection between the set of segmentations of X, and the set of binary vectors $\{0,1\}^{n-1}$. As an example, the vector $v = (0,0,1,0,1,0,0,1)$ uniquely determines the segmentation $S = \{[1,3], [4,5], [6,8], [9]\}$ of $X = \{1, \ldots, 9\}$. The set $\{0,1\}^{n-1}$ can be interpreted as the vertex set of the $n-1$-dimensional (boolean) hypercube Q_{n-1}, where two vectors $u, v \in \{0,1\}^{n-1}$ are joined by an edge if and only if u and v differ in exactly one coordinate. The hypercube $Q_{|X|-1}$ thus offers a way of visualizing the space of segmentations of some set X.

The refinement property now can be rephrased as follows: Given a weighted family $\mathcal{F} = \{(S_i, \omega) | 1 \leq i \leq k\}$, there is a consensus segmentation S such that $v(S)_j = 1$ only if $v(S_i)_j = 1$ for some $1 \leq i \leq k$.

In the graph theory literature, the *remoteness* function $\delta(u) := \sum_{x \in A} d(u, x)$, i.e., the sum of the path distances of a vertex u from a "profile" $A \subseteq V(G)$ of vertices has been studied [1,11]. A median in G is a vertex minimizing $\delta(u)$. The problem of characterizing medians for profiles in graphs also has been studied extensively [2,3,13]. Some results on this problem for hypercubes can be found in [1,15]. In this section, we investigate the relation between distances in the hypercube and the map $D_{\mathfrak{e}}$ induced by a potential $\mathfrak{e}$ on X. This will lead us to a correspondence between consensus segmentations and medians of profiles in weighted hypercubes.

Let $u = (x_1, \ldots, x_{n-1})$ and $v = (y_1, \ldots, y_{n-1})$ be two vertices of Q_{n-1}. We define $|u| = \sum_{i=1}^{n-1} x_i$, and $d(u, v) = \sum_{i=1}^{n-1} |y_i - x_i|$. Note that $d(u, v)$ is precisely the length of a shortest path between u and v in Q_{n-1}. The next result characterizes those potentials $\mathfrak{e}$ on X such that $D_{\mathfrak{e}}$ coincides, up to a constant factor, with the distance d in the hypercube Q_{n-1}.

Proposition 19 *Let X be a finite set of size $n \geq 2$, let $\mathfrak{e}$ be a potential on X, and let $\beta \geq 0$. We have $D_{\mathfrak{e}}(S_1, S_2) = \beta d(v(S_1), v(S_2))$ for all segmentations S_1, S_2 of X if and only if $\mathfrak{e}(A) = \beta(|A| - 1) + \sum_{x \in A} \mathfrak{e}([x])$ holds for all nonempty intervals A of X.*

Proof Suppose first that $\mathfrak{e}(A) = \beta(|A| - 1) + \sum_{x \in A} \mathfrak{e}([x])$ holds for all nonempty intervals A of X. Then, for all segmentations S of X, we have:

$$E(S) = \sum_{A \in S} (\beta(|A| - 1) + \sum_{x \in A} \mathfrak{e}([x])) = \beta(n - \iota(S)) + \sum_{x \in X} \mathfrak{e}([x]),$$

where $\iota(S)$ is the number of intervals of S. Hence, for S_1, S_2 two segmentations of X, we have:

$$\begin{aligned} D_{\mathfrak{e}}(S_1, S_2) &= E(S_1) + E(S_2) - 2E(S_1 \wedge S_2) \\ &= \beta(n - \iota(S_1)) + \sum_{x \in X} \mathfrak{e}([x]) + \beta(n - \iota(S_2)) \\ &\quad + \sum_{x \in X} \mathfrak{e}([x]) - 2(\beta(n - \iota(S_1 \wedge S_2)) + \sum_{x \in X} \mathfrak{e}([x])) \\ &= \beta(2\iota(S_1 \wedge S_2) - \iota(S_1) - \iota(S_2)). \end{aligned}$$

Next, we note that for all segmentations S of X holds $\iota(S) = b(S) + 1$, where $b(S)$ is the number of breakpoints of S. Since by definition, $b(S) = |v(S)|$, we have $\iota(S) = |v(S)| + 1$. Now consider $v_1 := v(S_1) = (x_1, \ldots, x_{n-1})$ and $v_2 := v(S_2) = (y_1, \ldots, y_{n-1})$. Using the expression for $D_{\mathfrak{e}}$ above, we obtain $D_{\mathfrak{e}} = \beta(2|v(S_1 \wedge S_2)| - |v_1| - |v_2|)$. By definition, $|v(S_1 \wedge S_2)| = |v_1| + |v_2| - c(v_1, v_2)$, where $c(v_1, v_2) = |\{i \in \{1, \ldots, n-1\}, x_i = y_i = 1\}|$. It follows that $D_{\mathfrak{e}} = \beta(|v_1| + |v_2| - 2c(v_1, v_2))$. Since $|v_1| + |v_2| - 2c(v_1, v_2) = d(v_1, v_2)$, we arrive at $D_{\mathfrak{e}}(S_1, S_2) = \beta d(v_1, v_2)$.

Conversely, suppose that $D_{\mathfrak{e}}(S_1, S_2) = \beta d(v(S_1), v(S_2))$ holds for all segmentations S_1, S_2 of X. Let $A = [x_1, x_2, \ldots, x_k]$, $k \geq 1$ be an interval of X, and let A^- (*resp.* A^+) be the set of all elements of X smaller than x_1 (*resp.* greater than x_k). Note that A^- and A^+ may be empty. Put $S_1 = \{A^-, A, A^+\}$ and $S_2 = \{A^-, [x_1], [x_2], \ldots, [x_k], A^+\}$. One easily verify that $S_1 \wedge S_2 = S_2$. In particular, we have:

$$\begin{aligned} D_{\mathfrak{e}}(S_1, S_2) &= E(S_1) - E(S_2) = \mathfrak{e}(A^-) + \mathfrak{e}(A) + \mathfrak{e}(A^+) - \mathfrak{e}(A^-) - \sum_{x \in A} \mathfrak{e}([x]) - \mathfrak{e}(A^+) \\ &= \mathfrak{e}(A) - \sum_{x \in A} \mathfrak{e}([x]). \end{aligned}$$

Since $d(v(S_1), v(S_2)) = |A| - 1$, and $D_{\mathfrak{e}}(S_1, S_2) = \beta d(v(S_1), v(S_2))$ holds by assumption, $\mathfrak{e}(A) - \sum_{x \in A} \mathfrak{e}([x]) = \beta(|A| - 1)$ follows. Therefore, we have $\mathfrak{e}(A) = \beta(|A| - 1) + \sum_{x \in A} \mathfrak{e}([x])$ as desired. □

Note that the equality postulated by Proposition 19 also holds if β is negative. In particular, if $\beta < 0$, it follows directly from Proposition 19 that $D_{\mathfrak{e}}(S_1, S_2) < 0$ for all segmentations S_1, S_2 of X distinct. By Proposition 4, $\mathfrak{e}$ is not superadditive in that case. As a direct consequence of Proposition 19, we have:

Corollary 20 *Let X be a finite set of size $n \geq 2$, let $\alpha, \beta \geq 0$, and let $\mathfrak{e}$ be the potential on X defined, for all nonempty intervals A of X, by $\mathfrak{e}(A) = \alpha|A| - \beta$. Then, $D_{\mathfrak{e}}(S_1, S_2) = \beta d(v(S_1), v(S_2))$ holds for all segmentations S_1, S_2 of X.*

Proof It suffices to note that $\mathfrak{e}(A) = \beta(|A| - 1) + \sum_{x \in A} \mathfrak{e}([x])$ holds for all nonempty intervals A of X. We can then apply Proposition 19. □

For potentials $\mathfrak{e}$ that satisfy the conditions of Proposition 19, the problem of finding consensus segmentations therefore coincides with the problem of identifying medians in the hypercube. We have:

Theorem 21 *Let X be a finite set of size $n \geq 2$, and let $\mathfrak{e}$ be a potential on X. If there exists $\beta \geq 0$ such that $\mathfrak{e}(A) = \beta(|A| - 1) + \sum_{x \in A} \mathfrak{e}([x])$ holds for all intervals A of X, then $\mathfrak{e}$ is core. In this case, given a weighted family of segmentations $\mathcal{F} = \{(\omega_1, S_1), \ldots, (\omega_k, S_k)\}$ of size $k \geq 1$,, all consensus segmentations for $\mathcal{F}$ can be found in time $O(nk)$.*

Proof Set $\omega = \sum_{i=1}^{k} \omega_i$. Suppose first that $\beta = 0$. By Proposition 19, $f_{\mathcal{F}}(S) = 0$ for all segmentations S of X. Hence, all segmentations of X are consensus for for $\mathcal{F}$, and in particular, $\mathfrak{e}$ is core.

Suppose now that $\beta > 0$. By Proposition 19, a segmentation S of X is a consensus for $\mathcal{F}$ if and only if $v(S)$ minimizes the map $f' : v \mapsto \sum_{i=1}^{k} \omega_i d(v, v_i)$ on $\{0, 1\}^{n-1}$. Hence, finding the set of consensus for $\mathcal{F}$ reduces to characterizing the vectors minimizing f' on $\{0, 1\}^{n-1}$. For $i \in \{1, \ldots, k\}$, set $v_i = v(S_i)$. For $j \in \{1, \ldots, n-1\}$, we denote by x_j^i the j^{th} coordinate of v_i. For $j \in \{1, \ldots, n-1\}$, set $m_j = \frac{1}{\omega} \sum_{i=1}^{k} \omega_i x_j^i$. Note that $0 \leq m_j \leq 1$ holds.

Claim. A vector $v = (y_1, \ldots, y_{n-1})$ minimizes the map f' on $\{0, 1\}^{n-1}$ if and only if, for all $j \in \{1, \ldots, n-1\}$, $|y_j - m_j| \leq \frac{1}{2}$.

To prove this, recall that for all $i \in \{1, \ldots, k\}$ $d(v, v_i) = \sum_{j=1}^{n-1} |y_j - x_j^i|$. In particular, we have:

$$\sum_{i=1}^{k} \omega_i d(v, v_i) = \sum_{i=1}^{k} \omega_i \sum_{j=1}^{n-1} |y_j - x_j^i| = \sum_{j=1}^{n-1} \sum_{i=1}^{k} \omega_i |y_j - x_j^i|.$$

Since the components of the latter sum are independent, minimizing the above quantity on $\{0, 1\}^{n-1}$ reduces to minimizing $\sum_{i=1}^{k} \omega_i |y_j - x_j^i|$ on $\{0, 1\}$ for all $j \in \{1, \ldots, n-1\}$. Consider a fixed j. We have $\sum_{i=1}^{k} \omega_i |y_j - x_j^i| = \sum_{i=1}^{k} \omega_i x_j^i = \omega m_j$ if $y_j = 0$, and $\sum_{i=1}^{k} \omega_i |y_j - x_j^i| = \sum_{i=1}^{k} \omega_i (1 - x_j^i) = \omega(1 - m_j)$ if $y_j = 1$. Combining these equalities, we obtain $\sum_{i=1}^{k} \omega_i |y_j - x_j^i| = \omega|y_j - m_j|$. Since $0 \leq m_j \leq 1$, the latter is minimized on $\{0, 1\}$ if and only if $|y_j - m_j| \leq \frac{1}{2}$. Hence the claim is true.

Thus all vectors v that minimize f' on $\{0, 1\}^{n-1}$ can be found in time $O(nk)$ and thus all consensus segmentations for $\mathcal{F}$ can be found in $O(nk)$ time.

It remains to show that $\mathfrak{e}$ is core. To see this, let C be a consensus for $\mathcal{F}$, and let $v(C) = \{y_1, \ldots, y_{n-1}\}$. Let $j \in \{1, \ldots, n-1\}$ be a breakpoint of C. By definition of $v(C)$, we have $y_j = 1$. Moreover, C is a consensus for $\mathcal{F}$, and thus $v(C)$ minimizes f' on $\{0, 1\}$. It follows from the claim that $|y_j - m_j| \leq \frac{1}{2}$. In particular, there exists

$i \in \{1, \ldots, k\}$ such that $x^i_j = 1$, as otherwise, $m_j = 0$ and $|y_j - m_j| = 1 > \frac{1}{2}$. Since $v_i = v(S_i)$, it follows that j is a breakpoint of S_i, and therefore, a breakpoint of $\widehat{S}(\mathcal{F})$. To summarize, if $j \in \{1, \ldots, n-1\}$ is a breakpoint of C, then j is a breakpoint of $\widehat{S}(\mathcal{F})$ and thus C is refined by $\widehat{S}$. Since C is a consensus for $\mathcal{F}$, it follows that $\mathfrak{e}$ is core. □

The claim in the proof of Theorem 21 is a generalization of [15, Proposition 1]). In the latter, the input family of vertices, (the *profile*) is unweighted, i.e., $\omega_i = 1$ for $1 \leq i \leq k$ in the map f', which implies $m_j < 1/2$ for $1 \leq j \leq n-1$ if and only if a strict majority vector has entry 0 at the at j^{th} coordinate and $m_j > 1/2$ if only only if it has entry 1.

Let us now turn to more general potentials. Our aim is again to express $D_{\mathfrak{e}}$ in terms of the distance between breakpoint vectors. We begin with two useful lemmas.

Lemma 22 *Let X be a finite set of size $n \geq 2$, and let S_1, S_2 be two segmentations on X. Then, $d(v(S_1), v(S_2)) = d(v(S_1), v(S_1 \wedge S_2)) + d(v(S_2), v(S_1 \wedge S_2))$.*

Proof Set $v(S_1) = (x_1, \ldots, x_{n-1})$, $v(S_2) = (y_1, \ldots, y_{n-1})$ and $v(S_1 \wedge S_2) = (z_1, \ldots, z_{n-1})$. Our aim is to show that $|y_i - x_i| = |x_i - z_i| + |y_i - z_i|$ holds for $1 \leq i \leq n-1$. If $y_i = x_i$, then $z_i = x_i = y_i$ by definition of $S_1 \wedge S_2$, and we have $|y_i - x_i| = |x_i - z_i| + |y_i - z_i| = 0$. Otherwise, i.e., if $y_i \neq x_i$, we have $|y_i - x_i| = 1$. W.l.o.g., we may fix $y_i = 1$ and $x_i = 0$. By definition of $S_1 \wedge S_2$, we then have $z_i = 1$ and thus $|x_i - z_i| = 1$ and $|y_i - z_i| = 0$, and finally $|y_i - x_i| = |x_i - z_i| + |y_i - z_i| = 1$. The conclusion follows from the definition of d. □

Lemma 23 *Let X be a finite set of size $n \geq 2$. Let $\mathfrak{e}$ be a potential on X, and let $S_1, S_2, \ldots, S_k$ be a sequence of segmentations such that, for all $i \in \{1, \ldots, k-1\}$, S_{i+1} is obtained from S_i by adding exactly one breakpoint. Then, $D_{\mathfrak{e}}(S_1, S_k) = \sum_{i=1}^{k-1} D_{\mathfrak{e}}(S_i, S_{i+1})$.*

Proof Let $i \in \{1, \ldots, k-1\}$. Since S_{i+1} is obtained from S_i by adding exactly one breakpoint, we have $S_i \wedge S_{i+1} = S_{i+1}$. In particular, $D_{\mathfrak{e}}(S_i, S_{i+1}) = E(S_i) - E(S_{i+1})$. Hence, we have:

$$\sum_{i=1}^{k-1} D_{\mathfrak{e}}(S_i, S_{i+1}) = \sum_{i=1}^{k-1} E(S_i) - E(S_{i+1}) = E(S_1) - E(S_k).$$

Since by construction $S_1 \wedge S_k = S_k$ we have $E(S_1) - E(S_k) = D_{\mathfrak{e}}(S_1, S_k)$. □

As a consequence, we have:

Proposition 24 *Let $\mathfrak{e}$ be a potential and S, S' be two segmentations on X with $|X| \geq 2$. Let $P : v_1 = v(S), v_2, \ldots, v_k = v(S')$ be a shortest path between $v(S)$ and $v(S')$ in Q_{n-1} containing $v(S \wedge S')$. Then, $D_{\mathfrak{e}}(S, S') = \sum_{i=1}^{k-1} D_{\mathfrak{e}}(S_i, S_{i+1})$.*

Proof By Lemma 22, there is always a shortest path between $v(S)$ and $v(S')$ in Q_{n-1} containing $v(S \wedge S')$. Fixing any such path P, we have $v_\ell = v(S \wedge S')$ for some $1 \leq \ell \leq k$. Since P is a shortest path between $v(S)$ and $v(S')$, the two subpaths $P^- = v_1, \ldots, v_\ell$ and $P^+ = v_\ell, \ldots, v_k$ are also shortest paths in Q_{n-1} between their respective end vertices. Since $S \wedge S'$ refines S, the path P^- induces a sequence $S = S_1, \ldots, \widehat{S} = S_\ell$ of segmentations of X such that S_{i+1} is obtained from S_i by adding exactly one breakpoint. for $1 \leq i \leq \ell - 1$. Analogously, the path $P^+ = v_\ell, \ldots, v_k$ induces a sequence $\widehat{S} = S_\ell, \ldots, S_k = S'$ of segmentations of X such that S_{i+1} is obtained from S_i by removing exactly one breakpoint, and hence S_i is obtained from S_{i+1} by adding exactly one breakpoint. Lemma 23 implies

$$D_{\mathfrak{e}}(S, \widehat{S}) = \sum_{i=1}^{\ell-1} D_{\mathfrak{e}}(S_i, S_{i+1}) \quad \text{and} \quad D_{\mathfrak{e}}(S', \widehat{S})) = \sum_{i=\ell}^{k-1} (D_{\mathfrak{e}}(S_{i+1}, S_i)$$

The statement now follows from $D_{\mathfrak{e}}(S, S \wedge S') + D(S \wedge S', S') = E(S) + E(S \wedge S') - 2E(S \wedge S') + E(S \wedge S') + E(S') - 2E(S \wedge S') = E(S) + E(S') - 2E(S \wedge S') = D_{\mathfrak{e}}(S, S')$. □

Note that the converse of Proposition 24 is not true in general. There may also be shortest paths $P : v_1 = v(S), v_2, \ldots, v_k = v(S')$ between $v(S)$ and $v(S')$ in Q_{n-1} with $D_{\mathfrak{e}}(S, S') = \sum_{i=1}^{k-1} D_{\mathfrak{e}}(S_i, S_{i+1})$ that do not contain $v(S \wedge S')$.

Proposition 24 suggests to associate weights to the edges the edge of the hypercube by setting $w_e(vv') : vv' \to D_{\mathfrak{e}}(v, v')$, where $v = v(S)$, $v' = v(S')$ where S and S' are two segmentations that differ in exactly one breakpoint. If $\mathfrak{e}$ is superadditive, then by Proposition 19 $w_{\mathfrak{e}}(vv') \geq 0$ for all edges of Q_{n-1}. We denote by $d_{\mathfrak{e}}$ the natural distance on the edge-weighted hypercube $(Q_{n-1}, L_{\mathfrak{e}})$. Recall that if $\mathfrak{e}$ is supermodular, $D_{\mathfrak{e}}$ is a metric. In this case there is a close connection between $D_{\mathfrak{e}}$ and $d_{\mathfrak{e}}$.

Theorem 25 *Let X be a finite set of size $n \geq 2$. Let $\mathfrak{e}$ be a potential on X. If $\mathfrak{e}$ is supermodular, then for all segmentations S, S' of X, $D_{\mathfrak{e}}(S, S') = d_{\mathfrak{e}}(v(S), v(S'))$.*

Proof Let S, S' be two segmentations of X. By Lemma 22 and Proposition 24, there exists a path $P : v_1 = v(S), v_2, \ldots, v_k = v(S')$ in Q_{n-1} such that $D_{\mathfrak{e}}(S, S') = \sum_{i=1}^{k-1} D_{\mathfrak{e}}(S_i, S_{i+1})$, where for $i \in \{1, \ldots, k\}$, S_i is the segmentation associated to v_i and, using the definition of the weights, $D_{\mathfrak{e}}(S, S') = \sum_{i=1}^{k-1} w_{\mathfrak{e}}(\{v_i, v_{i+1}\})$ follows. It remains to show that $\sum_{i=1}^{k-1} w_{\mathfrak{e}}(\{v_i, v_{i+1}\}) = d_{\mathfrak{e}}(v(S), v(S'))$ also holds. Since $\mathfrak{e}$ is supermodular, $D_{\mathfrak{e}}$ satisfies the triangle inequality by Proposition 6 and thus $D_{\mathfrak{e}}(S, S') \leq \sum_{i=1}^{k'-1} D_{\mathfrak{e}}(T_i, T_{i+1})$ holds for all paths $u_1 = v(S), u_2, \ldots, u_{k'} = v(S')$ corresponding to segmentations $u_i = v(T_i)$ with $1 \leq i \leq k'$.

Since $D_{\mathfrak{e}}(S, S') = \sum_{i=1}^{k-1} w_{\mathfrak{e}}(\{v_i, v_{i+1}\})$, the latter inequality, together with the definition of $w_{\mathfrak{e}}$, implies $\sum_{i=1}^{k-1} w_{\mathfrak{e}}(\{v_i, v_{i+1}\}) \leq \sum_{i=1}^{k'-1} w_{\mathfrak{e}}(\{u_i, u_{i+1}\})$, and thus P is a also shortest path between v_1 and v_k in the weighted hypercube $(Q_{n-1}, L_{\mathfrak{e}})$, which implies $d_{\mathfrak{e}}(v(S), v(S')) = \sum_{i=1}^{k-1} w_{\mathfrak{e}}(\{v_i, v_{i+1}\})$. □

Theorem 25 establishes a 1–1 correspondence between consensus segmentations and medians on a weighted hypercube. This allows us to rephrase the concept of core potentials in terms of vertices on the hypercube.

The convex hull of a set W of vertices of a graph is the smallest vertex set Q containing W such that all vertices along a shortest paths with endpoints in Q are contained in Q [6]. For a weighted family of segmentations $\mathcal{F} = \{(\omega_1, S_1), \ldots, (\omega_k, S_k)\}$, $k \geq 1$ we denote by $Q[\mathcal{F}]$ the convex hull of the vectors $\{v(S_1), \ldots, v(S_k)\}$ in the hypercube Q_{n-1}. It is not difficult to see that $x = (x_1, \ldots, x_{n-1}) \in V(Q[\mathcal{F}])$ if and only if for every coordinate $1 \leq \ell \leq n-1$ there is a vector $v = v(S)$ with $S \in \mathcal{F}$ such that $x_\ell = v_\ell$. In particular all segmentations S such that $v(S) \in V(Q[\mathcal{F}])$ are refined by $\widehat{S}_{\mathcal{F}}$. As a consequence, we have the following characterization:

Theorem 26 *Let $\mathfrak{e}$ be a superadditive potential on some set X. Then $\mathfrak{e}$ is core if and only if all families $\mathcal{F}$ admit a consensus C such that $v(C) \in V(Q[\mathcal{F}])$.*

Proof The if part follows immediately from the discussion above. To see the converse, suppose there is a consensus C with $x{:=}v(C) \notin V(Q[\mathcal{F}])$ that is refined by $\widehat{S}_{\mathcal{F}}$. Such a vector x exists precisely if $x_\ell = 0$ for some ℓ where all $v \in V(Q[\mathcal{F}])$ have $v_\ell = 1$. Consider now the segmentation C' with $v_i(C') = v_i(C)$ for all $i \neq \ell$ and $v_\ell(C') = 1$. By Lemma 3, C' is also a consensus of $\mathcal{F}$. One can repeat this process until we obtain a consensus C^* such that $v_i(C^*) = 0$ implies that there exists $v \in Q[\mathcal{F}]$ with $v_i = 0$. Since C^*, by construction, is refined by $\widehat{S}_{\mathcal{F}}$, we also have that $v_i(C^*) = 1$ implies that there exists $v \in Q[\mathcal{F}]$ with $v_i = 1$. By definition of $Q[\mathcal{F}]$, $v(C^*) \in V(Q[\mathcal{F}])$ follows. □

A plausible hypothesis is that $\mathfrak{e}$ is core if the convex hull of any input family $\mathcal{F}$ in the weighted hypercube is a subgraph of the convex hull of $\mathcal{F}$ in the unweighted hypercube. The intuition behind this is that a consensus should always be contained in the convex hull in the weighted hypercube. Interestingly, this is not the case; for the potential $\mathfrak{e}$ defined in Eq. 5 and the family $\mathcal{F}$ defined there, the convex hull of the family in the weighted hypercube is precisely $Q[\mathcal{F}] - (0, 0, 0, 0, 0, 0, 0)$, yet $\mathfrak{e}$ is not core.

7 Concluding Remarks and Future Work

In this contribution, we investigated several properties of potentials, which are a key feature in segmentation problems. The properties we focused on here can be divided into three types:

(1) Input/output relation properties: superadditivity, supermodularity, and strongly supermodularity.
(2) Properties of the map $D_{\mathfrak{e}}$: positivity and triangle inequality.
(3) Properties related to the variation rate of $\mathfrak{e}$: convexity, bounded convexity and expanding convexity.

We established several equivalence results between properties of type (1) and (2) (Propositions 4 and Proposition 6, Theorem 7). In the special case where the value of $\mathfrak{e}(A)$ only depends on the size of A, we also established an equivalence result and a one-way relation between properties of type (1) and (3) (Proposition 11 and Proposition 18, respectively), and in one instance, an equivalence result between properties of all three types (Theorem 12). All these relations are summarized in the diagram in Fig. 1. A few open questions remain. First, can we find some equivalence relation between $\mathfrak{e}$ being strongly supermodular and some property of the map $D_{\mathfrak{e}}$? Second, given that the property of being convex means that the second derivative is positive, and the property of being expandingly convex means that the second and third derivative are both positive, what can be said about the fourth derivative and higher?

We also introduced the notion of a core potential, a desirable property in the context of the Consensus Segmentation Problem. Indeed, the space of solutions of the Consensus Segmentation Problem is drastically reduced when the potential under consideration is core. Through a wide range of examples, we showed that most of the properties that we considered are not enough to guarantee that a given potential $\mathfrak{e}$ is core (except in the case of very small families of segmentations, Proposition 8). Lemma 15 provides a sufficient condition for a potential to be core. We know from Theorem 21 that this is not a necessary condition. More interestingly, convexity itself is not a necessary condition, as potentials of the form defined in Theorem 21 are core, but not necessarily convex.

With regards to this specific problem, the following two questions remain:

Open Question 27. *Are increasingly convex potentials core?*

and

Open Question 28. *Are strongly supermodular potentials core?*

A potential way to answer these question may reside in the structure of the edge-weighted hypercube, which can be used as a representation of the distance induced by a supermodular potential (Theorem 25). Whithin this framework, the Consensus Segmentation Problem reduces to the classic problem of identifying the median of a given set of vertices in the hypercube. As a first step in that direction, we were able to use this alternative representation to show that potentials satisfying a given recursive property are core (Theorem 21). It remains an open question, however, which weight functions of Q_{n-1} derive from segmentation problems with (supermodular) potentials. The notion of k-core potentials, which require that the core property holds for families of bounded size, finally, may have connections to medians of "bounded profiles" [1]. These have received some attention for median graphs in general and may be helpful to better understand the median structures on a weighted Q_{n-1} as well.

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