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ORIGINAL RESEARCH

Independence and domination in divisor graph and mod-difference graphs

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Abstract
We initiate the study of domination and inverse domination in labeled graphs. In this paper, we determined the cardinality of maximal independent and minimum variant dominating (total dominating/independent dominating/co-independent dominating) sets and their inverse in divisor graph and in two new labeling definitions called 0-mod-difference and 1-mod-difference graphs.

Introduction

Consider $G(E, V)$ be a finite, undirected and simple graph. The independence number of G denoted by $\beta(G)$ is the maximum cardinality over all independent sets. The domination number of G denoted by $\gamma(G)$ is the minimum cardinality over all dominating sets. The inverse domination number of G denoted by $\gamma^{-1}(G)$ is the minimum cardinality over all inverse dominating sets.

We consider a finite undirected and simple graph $G(E, V)$ with a set $V(G)$ of vertices and a set $E(G)$ of edges.

A subgraph H of a graph G is said to be *induced* (or *full*) subgraph if, for any pair of vertices x and y of H, xy is an edge of H if and only if xy is an edge of G. If H is an induced subgraph of G and S is a set of its vertices then H is said to be an induced subgraph by S and denoted by $G[S]$.

A set $I \subseteq G$ is an *independent set* or *stable set* in graph G if no two of its vertices are adjacent. An independence number of G denoted by $\beta(G)$ is the maximum cardinality over all independent sets.

A set $D \subseteq V(G)$ is a *dominating set* in G if $N(v) \cap D \neq \varnothing$; for every vertex $v \in V(G) - D$. the *domination number* of G, denoted by $\gamma(G)$, is a minimum cardinality over all dominating sets in G.

A dominating set $D \subseteq V(G)$ is an *independent dominating set* in G if D is an independent set in G.The *independence domination number* of G, denoted by $\gamma_i(G)$, is a minimum cardinality of independent dominating sets in G.

A dominating set $D \subseteq V(G)$ is a *total dominating set* in G if $N(v) \cap D \neq \varnothing$; for every vertex $v \in v(G)$. This means that $G[D]$ has no isolated vertex. A minimum cardinality over all total dominating sets in G is the *total domination number* of G and is denoted by $\gamma_t(G)$ [10].

A dominating set $D \subseteq V(G)$ is a *connected dominating set* in G, if $G[D]$ is connected. The *connected domination number* of G, denoted by $\gamma_c(G)$, is a minimum cardinality over all connected dominating sets in G [8].

A dominating set $D \subseteq V(G)$ is a *co-independent dominating* set in G if the complement of D is an independent set. The *co-independence domination number* of G, denoted by $\gamma_{\text{coi}}(G)$, is a minimum cardinality over all co-independent dominating sets of G [10].

Let $D \subseteq V(G)$ be a minimum dominating (independent dominating/total dominating/connected dominating/co-independent dominating) set in graph G. If $V - D$ contains a dominating (an independence dominating/total dominating/connected dominating/co-independence dominating) set ID of G, where ID is called an inverse dominating (an independent dominating/total dominating/connected dominating/co-independent dominating) set with respect to D. The *inverse* domination (an independence domination/total domination/connected domination/co-independence domination) number of G, denoted by ($\gamma^{-1}(G), {\gamma_i}^{-1}(G), {\gamma_t}^{-1}(G), {\gamma_c}^{-1}(G) and {\gamma_{coi}}^{-1}(G)$) is the minimum cardinality over all inverse dominating (an independent dominating/total dominating/connected dominating/co-independent dominating) sets of G [6].

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Santhosh and Singh [7] call a graph $G(V, E)$ with vertex set V and edge set E a divisor graph if V is labeled by a set of integers and for each edge $uv \in E$ either the label assigned to u divides the label assigned to v or vice versa. We studied the notion "divisor graph" in the sense that its vertices can be labeled with distinct integers $1, 2, \ldots, |V|$ such that for each edge $uv \in E$ either the label assigned to u divides the label assigned to v or vice versa. Also, we introduce two new definitions labeling called 0-mod-difference and 1-mod-difference.

There are more than 75 models of domination listed in the appendix of Haynes [5]. For more details about parameters of domination number, we refer to [2, 3]. In this paper, we study different formulas of cardinality of independence and domination (total domination, independence domination, co-independence domination) in divisor, 0-mod-difference and 1-mod-difference graph. The inverse domination (total domination, independence domination, co-independence domination) number of divisor (0-mod-difference/1-mod-difference graph) graph also determined.

Any notion or definition of graph labeling which is not found here could be found in [1].

Some new methods

In the following sections, we will study three new methods. The following are new notions.

Definition 1.1. [9] Let $G(V, E)$ be a simple graph of order n and $f : V \to \{1, 2, \ldots, n\}$ be a bijection. For each edge uv, if either $f(u) \backslash f(v)$ ($f(u)$ divides $f(v)$) or $f(v) \backslash f(u)$ ($f(v)$ divides $f(u)$) then f is called a divisor labeling and G is called a divisor graph. A graph which is not divisor is called a non-divisor graph.

Definition 1.2. Let $G(V, E)$ be a simple graph of order n and $f : V \to \{1, 2, \ldots, n\}$ be a bijection. A graph $G(V, E)$ with vertex set V is said to be 0-mod-difference if for each edge $uv \in E$, $|f(u) - f(v)| \equiv 0(modm)$ where $2 \le m \le \lfloor \frac{n}{2} \rfloor$. A graph which is not 0-mod-difference is called a non-0-mod-difference graph [11].

Definition 1.3. Let $G(V,E)$ be a simple graph of order n and $f : V \to \{1, 2, \ldots, n\}$ be a bijection. A graph $G(V,E)$ with vertex set V is said to be 1-mod-difference if for each edge $uv \in E$, $|f(u) - f(v)| \equiv 1(modm)$ where $2 \le m \le \lfloor \frac{n}{2} \rfloor$. A graph which is not a 1 -mod-difference is called a non-1-mod-difference graph.

Definition 1.4. A maximal divisor /0-mod-difference/1-mod-difference graph of n vertices is a divisor/0-mod-difference/1-mod-difference graph such that adding any new edge yields a non-divisor (0-mod-difference/1-mod-difference) graph. Figure 1 gives a maximal divisor graph of order 10.

Definition 1.5. [4] Let x be a nonnegative real number. The *Gauss' s function* $\pi(x)$ is defined to be the number of primes not exceeding x. *i.e*, $\pi(x) = |\{p : \ p \ is \ prime, \ 2 \le p \le x\}|$.

Note 1.6. In all definitions in this article, we define the labeling function by:

$$f(v_i) = i, i = 1, \ldots, n$$

Divisor graph

Theorem 2.1. *If G is a maximal divisor graph then,*

(i) $\beta(G) = \lceil \frac{n}{2} \rceil$.
(ii) $\gamma(G) = \gamma_i(G) = \gamma_c(G) = 1$
(iii) $\gamma_t(G) = 2$
(iv) $\gamma_{\text{coi}}(G) = \lfloor \frac{n}{2} \rfloor$; $n > 3$

Proof

(i) Consider $I = \{v \in G : f(v) > \lfloor \frac{n}{2} \rfloor\}$. Then, I is an independence set, since for each vertex $v \in I$, the vertex of label $2f(v)$ does not belong to G (see Fig. 1), therefore $\beta(G) \ge |I| = \lceil \frac{n}{2} \rceil$. If we assume that there is a set A such that $|A| > |i|$ then A must contain at least two adjacent vertices, since each vertex v which $f(v) \le \lfloor \frac{n}{2} \rfloor$ is adjacent to a vertex of label $2f(v)$. Thus, $\beta(G) = \lceil \frac{n}{2} \rceil$.
(ii) It is obvious, since the vertex of label one is adjacent to all vertices of G.
(iii) Let $D_1 = \{v_1, v_2\}$. D_1 is a dominating set in G with no isolated vertex and it is clear that it is the minimum total dominating set so $\gamma_t(G) = 2$.
(iv) Consider $D_2 = \{v \in G : f(v) \le \lfloor \frac{n}{2} \rfloor\}$. D_2 contains a vertex of label one therefore it is the dominating set in G and $v - D_2$ is an independent set by (i). Thus,

$\gamma_{\text{coi}}(G) \leq \lfloor \frac{n}{2} \rfloor$. If we assume that there is a set c such that $|c| < |D_2|$ then c may be a dominating set, but $v - c$ cannot be an independent set by (i). Thus, $\gamma_{\text{coi}}(G) = \lfloor \frac{n}{2} \rfloor$.

Note 2.2.

(1) If G is a divisor graph and $\beta(G) > \lceil \frac{n}{2} \rceil, \gamma(G) > 1$ *or* $\gamma_i(G) > 1$ then G is not a maximal divisor graph.
(2) If G is a divisor graph and $\gamma_c(G) > 1$ or there is no connected dominating set in G then G is not a maximal divisor graph.
(3) If G is a divisor graph and $\gamma_t(G) > 2$ or there is no total dominating set in G then G is not a maximal divisor graph.
(4) If $\beta(G) < \lceil \frac{n}{2} \rceil$ then G is a non-divisor graph.

Theorem 2.3. *If G is a maximal divisor graph then*

(i) $\gamma^{-1}(G) = \gamma_i{}^{-1}(G) = \pi(n)$
(ii) G *has no inverse total* (*connected*/*co-independence*) *dominating set.*

Proof

(i) Consider $\text{ID} = \{v_i \in G; f(v_i) = p, p \leq n, wherepisaprimenumber\}$. ID is a dominating set in G and $\text{ID} \subseteq v - D$ where D is a minimum dominating set ($D = \{v_1\}$) in G. Therefore $|\text{ID}| \geq \gamma^{-1}(G)$ (see Fig. 1). If we assume that there is a set A such that $|A| < |\text{ID}|$ then there is at least a vertex of prime label which is not belonging to the set A, so it cannot dominate this vertex. Therefore, , $|\text{ID}| = \gamma^{-1}(G)$. Since ID is an independence set then $\gamma^{-1}(G) = \gamma_i{}^{-1}(G) = \pi(n)$.
(ii) G has no inverse total (connected) dominating set, since there is an isolated vertex in $G[V - D]$ where D is a total (connected) dominating set, and there is no inverse co-independence set in G since all co-independence sets in G contain adjacent vertices.

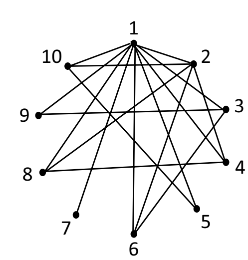

Fig. 1 Maximal divisor graph of order 10

0-mod-difference graph

Theorem 3.1. *Maximal* 0*-mod-difference graph is partitioned into m complete induced subgraphs.*

Proof

Let $S_i = \{v_j \in v; j \equiv i(\text{mod } m), i = 0, 1, \ldots, m-1\}$*. It is clear that* $G[S_i], i = 0, 1, \ldots, m-1$ *are disjoint graphs and* $\cup_{i=0}^{m-1} S_i = v(G)$*. For each* v_{i1} and $v_{i2} \in S_i, i = 0, 1, \ldots, m-1$ *there is an edge* $v_{i1}v_{i2} \in E(G)$ *since* $|f(v_{i_1}) - f(v_{i_2})| \equiv 0(\text{mod } m)$ *so* $G[S_i]$ *is a complete induced subgraph* $\forall i$*.*

Example 3.2.

Figure 2; $n = 9$*;* $m = 3$ *and Fig. 3;* $n = 10, m = 3$ *are illustrate the previous theorem.*

Theorem 3.3. *If* $G(n, q)$ *is a* 0*-mod-difference graph and* $n \equiv r(modm)$ *then*

$$q \leq \frac{1}{2}\lfloor\frac{n}{m}\rfloor\left(m\lfloor\frac{n}{m}\rfloor - m + 2r\right)$$

Proof

Since the maximal 0-mod-difference is partitioned into m complete induced subgraphs (Theorem 3.1),so If $n \equiv r(\text{mod } m)$*, then there is r complete induced subgraphs of order* $\lfloor\frac{n}{m}\rfloor + 1$ *and the others of order* $\lfloor\frac{n}{m}\rfloor$*, since if* $n \equiv 0(\text{mod } m)$*, then* mdividesn *without residue (see Fig. 2), so all complete subgraphs have the same order equal to* n/m*.*

If $n \equiv 1(\text{mod } m)$, then mdividesn with residue one which is the vertex v_n, and it is clear that $v_n \in S_1$, since $n \equiv 1(\text{mod } m)$, therefore S_1 is of order $\lfloor\frac{n}{m}\rfloor + 1$ and the others are of order $\lfloor\frac{n}{m}\rfloor$ (se Fig. 3). If $n \equiv 2(\text{mod } m)$, then mdividesn with residue two which are vertices v_n and v_{n-1}, $v_{n-1} \in S_1$, since $n - 1 \equiv 1(\text{mod } m)$ and $v_n \in S_2$, since $n \equiv 2(\text{mod } m)$, therefore S_1 and S_2 are of order $\lfloor\frac{n}{m}\rfloor + 1$ and the others are of order $\lfloor\frac{n}{m}\rfloor$(see Fig. 4) and so on. The number of edges of any complete graph K_t is $\frac{t(t-1)}{2}$, then the maximal number of edges in G is $r\frac{(\lfloor\frac{n}{m}\rfloor+1)(\lfloor\frac{n}{m}\rfloor)}{2} + (m-r)\frac{\lfloor\frac{n}{m}\rfloor(\lfloor\frac{n}{m}\rfloor-1)}{2} = \frac{1}{2}\lfloor\frac{n}{m}\rfloor\left(m\lfloor\frac{n}{m}\rfloor - m + 2r\right)$. Thus $q \leq \frac{1}{2}\lfloor\frac{n}{m}\rfloor\left(m\lfloor\frac{n}{m}\rfloor - m + 2r\right)$.

Theorem 3.4. *If G is a maximal* 0*-m-mod-difference graph, then*

(*i*) $\beta(G) = m$
(*ii*) $\gamma(G) = \gamma_i(G) = m$
(*iii*) $\gamma_{\text{coi}}(G) = n - m$
(*iv*) $\gamma_t(G) = 2m$
(*v*) *G has no connected dominating set.*

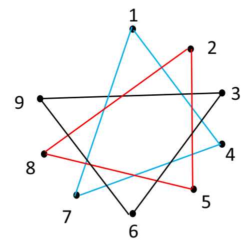

Fig. 2 Case $n = 9$; $m = 3$

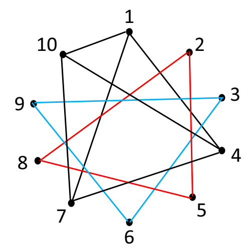

Fig. 3 Case $n = 10$; $m = 3$

Proof

(i) Let $D_1 = \{v_{i0}; v_{i0} \text{onlyonevertexbelongto} S_i, i = 0, 1, \ldots, m-1\}$. By Theorem 3.1 $S_i, i = 0, 1, \ldots, m-1$ are the vertices of complete subgraphs. It is clear that $\beta(G[S_i]) = 1$ therefore $|D_1| \leq \beta(G)$. If we assume that there is a set A such that $|A| > |D_1|$ then A contains at least two vertices belonging to the same set from S_i, since the graph of this set is a complete induced subgraph then these vertices are not independent (see Figs. 2, 3, 4). Thus, $\beta(G) = |D_1| = m$.

(ii) By the same manner in (i) $\{v_{i0}\}$ is a dominating set in $[S_i]$ $\forall i$, so $\gamma(G[S_i]) = 1$. Therefore, $|D_1| \geq \gamma(G)$. If we assume that there is a set B such that $|\mathrm{B}| < |D_1|$ then B does not contain at least one vertex belong to a set from $S_i, i = 0, 1, \ldots, m-1$, so B cannot dominate the vertices of this set, since every $G[S_i]$ is a complete induced subgraph. Thus, $\gamma(G) = |D_1| = m$. Since D_1 is an independent set then $\gamma(G) = \gamma_i(G) = m$.

(iii) Let $D_2 = \{\forall v_j \in S_i \text{exceptonevertex}, i = 0, 1, \ldots, m-1\}$. So D_2 is a dominating set and $v - D_2$ is an independent set therefore $|D_2| \geq \gamma_{\mathrm{coi}}(G)$. If we assume that there is a set c_2 such that $|c_2| < |D_2|$ then $v(G) - c_2$ cannot be an independence set since it contains at least two vertices in the same set from $S_i, i = 0, 1, \ldots, m-1$ then these vertices are adjacent. Thus, $\gamma_{\mathrm{coi}}(G) = |D_2| = n - m$.

(iv) Let $D_3 = \{\text{onlytwoverticesfrom} S_i, i = 0, 1, \ldots, m-1\}$. D_3 is a total dominating set since it is a dominating set and has no isolated vertex so $|D_3| \geq \gamma_t(G)$. If we assume that there is a set c_3 such that $|c_3| < |D_3|$.Then c_3 contains at least one isolated vertex from one set from $S_i, i = 0, 1, \ldots, m-1$ or it has no any vertex from at least one set from $S_i, i = 0, 1, \ldots, m-1$. So c_3 is not a total dominating set in G. Thus, $\gamma_t(G) = |D_2| = 2m$.

(v) G has no connected dominating set since G is a disconnected graph by Theorem 3.1.

Theorem 3.5. *If G is a maximal 0-mod-difference graph, then*

(i) $\gamma^{-1}(G) = {\gamma_i}^{-1}(G) = m$
(ii) ${\gamma_{coi}}^{-1}(G) = m$ *if and only if* $n = 2m$.
(iii) ${\gamma_t}^{-1}(G) = 2m$ *if and only if* $\lfloor \frac{n}{m} \rfloor \geq 4$.

Proof

(i) Let $\mathrm{ID}_1 = \{v_j \in S_i; v_j \in v - D_1, i = 0, 1, \ldots, m-1\}$ where D_1 is a dominating set in G (Theorem 3.4). Similar to manner in Theorem 3.4 (i) ID_1 is a minimum dominating set in G, so $\gamma^{-1}(G) = |ID_1| = m$. And since ID_1 is an independent set in G, then

$$\gamma^{-1}(G) = {\gamma_i}^{-1}(G) = m$$

(ii) If ${\gamma_{\mathrm{coi}}}^{-1}(G) = m$ then there is a minimum co-independent inverse set in G (ID_2) such that $\mathrm{ID}_2 \subseteq v - D_2$ where D_2 is a minimum co-independent dominating set in G (Theorem 3.4) then $ID_2 \cap S_i = 1 \forall i$. Now if $D_2 \cap S_i > 1$ for some i then $V - ID_2$ contains at least two vertices belonging to S_i and these sets are complete subgraphs therefore $V - ID_2$ is not an independent set. Thus, $D_2 \cap S_i = 1$ implies that S_i contain only two vertices $\forall i$ then $n = 2m$ (see Fig. 5).

Conversely If $n = 2m$ then $|S_i| = 2 \forall i$ (see Fig. 5), then ${\gamma_{\mathrm{coi}}}^{-1}(G) = m$.

(iii) If $\lfloor \frac{n}{m} \rfloor \geq 4$ that means $|S_i| \geq 4$. Let $\mathrm{H}_i = \{v_{i1}, v_{i2}\}$ where v_{i1} and v_{i2} are any two vertices belong to S_i and $\mathrm{H}_i \subseteq v - D_3$ where $D_3 i$ s the minimum total dominating set in G (Theorem 3.4) (see Fig. 6). Consider $\mathrm{ID}_3 = \cup\{\mathrm{H}_i, i = 0, 1, \ldots, m-1\}$ as same manner in Theorem 3.4 (iv) ID_3 is the minimum total dominating set in G so ${\gamma_t}^{-1}(G) = 2m$.

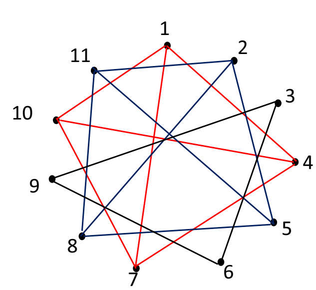

Fig. 4 Case $n = 11;\ m = 4$

Conversely If ${\gamma_t}^{-1}(G) = 2m$ then there is a minimum dominating set in G which contain at least two vertices in S_i and belonging to $v - D_3$ where D_3 is a total dominating set in G (see Fig. 6; $m = 3$), so $|S_i| \geq 4 \forall i$ then $\lfloor \frac{n}{m} \rfloor \geq 4$.

1-mod-difference graph

Lemma 4.1. *If G is a* 1*-mod-difference graph, then* $\Delta(v) \leq \lfloor \frac{n}{m} \rfloor + 1$

Proof

Let $v_j \in G$ *there are two cases as follows:*

(i) (i) If $f(v_j) \leq m$ and $j \equiv i(modm)$, then v_j joins with all vertices of labels which are congruent to $i + 1 \pmod{m}$ and with the vertex v_{j-1} congruent to $i - 1(modm)$, except the vertex v_1, since v_0 does not exist and v_m which are joined with the vertex v_{m-1} and all vertices of labeled in class [1] except{1}. So the maximum number of vertices can be joined with vertex v_j in this case is less than or equal to $\lfloor \frac{n}{m} \rfloor + 1$.

(ii) If $f\left(v_j\right) \geq m + 1$ and $j \equiv i(modm)$, then v_j would join with

(1) All labeled vertices v_r which are congruent to $i - 1 \pmod{m}$ and $f\left(v_j\right) > f(v_r)$, the maximum number of these vertices is less than or equal $\lceil \frac{j}{m} \rceil$.

(2) All labeled vertices $f(v_w)$ congruent to $i + 1 \pmod{m}$ and $f\left(v_j\right) < f(v_w)$, the maximum number of these vertices is $\lceil \frac{n-j}{m} \rceil$.

By 1 and 2, it is clear that $\deg\left(v_j\right) \leq \lceil \frac{j}{m} \rceil + \lceil \frac{n-j}{m} \rceil \leq \lfloor \frac{n}{m} \rfloor + 1$

Theorem 4.2. *If G is a maximal* 1*-mod-difference graph, then* $\gamma(G) = m$.

Proof

Let $D = \{v_i, i = 1, 2, \ldots m\}$, $\forall v_i \in D$, v_i *is adjacent to all vertices of labeling that belong to class [i+1]=* $\left\{v_j : f\left(v_j\right) \equiv i + 1(modm)\right\}$ *(see Figs. 7, 8, 9). So D is a dominating set since* $V = \cup_{i=0}^{m-1}[i]$*. Now if there is a set of cardinal equal to* $m - 1$ *then the number of vertices can be dominated by* $m - 1$ *vertices is* $(m - 1)\left(\lfloor \frac{n}{m} \rfloor + 1\right) - (m - 2) < n$*, by Lemma 4.1, since* $m - 2$ *is the minimum number of common edges when we have* $m - 1$ *successive vertices. Thus, D is the minimum dominating set in G.*

Theorem 4.3. *If G is a maximal* 1*-mod-difference graph, then* $\gamma^{-1}(G) = m$.

Proof

Consider $ID = \{v_i, i = m + 1, m + 2, \ldots, 2m\}$*, any m successive vertices constitute minimum dominating set, since these vertices are adjacent to all classes in G, and it is clear that* $ID \subseteq V - D$*, where D is the minimum dominating set in G (Theorem 4.2). Thus,* $\gamma^{-1}(G) = m$.

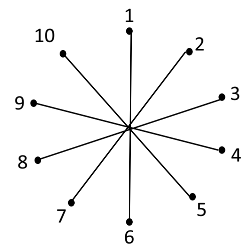

Fig. 5 $n = 2m;\ m = 5$

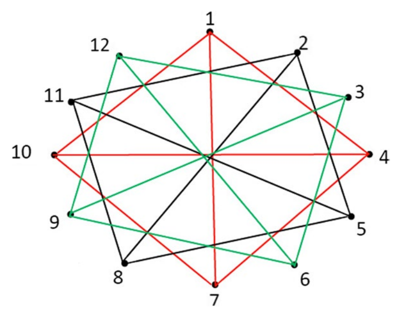

Fig. 6 $m = 3$

Corollary 4.4. *If G is a maximal* 1*-mod-difference graph, then*

$$\gamma_c(G) = \gamma_t(G) = \gamma_c^{-1}(G) = \gamma_t^{-1}(G) = m$$

Proof

It is clear, since the set D in Theorem 4.2 and ID in Theorem 4.3 are connected set.

Theorem 4.5. *If G is a maximal* 1*-mod-difference graph where* $m = 2$*, then* $\gamma_i(G) = \lfloor \frac{n}{2} \rfloor$.

Proof

To get an independent set S, we cannot take any set that contains vertices of odd and even labels together. Since if $v_i, v_j \in D_1$ *such that* v_i *is odd labels and* v_j *is even labels, then* $\left|f(v_j) - f(v_i)\right| \equiv 1(\mathrm{mod}2)$*, these vertices are adjacent. Thus, S is not an independent set. Then, S contains either vertices of odd labels or even labels. The cardinal of all vertices of even labels is less than or equal to the cardinal of odd labels, then let* D_1 *be the set of all vertices of even labels,* D_1 *is a dominating set, since if we take any vertex of* D_1*, this vertex is adjacent to all vertices of odd labels and it is an independent, since* $\forall v_j, v_i \in D_1, \left|f(v_j) - f(v_i)\right| \equiv 0(\mathrm{mod}2)$*, thus* $\gamma_i(G) \leq |D_1| = \lfloor \frac{n}{2} \rfloor$ *(see Figs. 10, 11). Now if there is a set* $A = D_1 - \{v_r\}$*, where* $\{v_r\} \in D_1$*, then A cannot dominate the vertex* v_r*, therefore A cannot be a dominating set in G. Thus,* $\gamma_i(G) = \lfloor \frac{n}{2} \rfloor$.

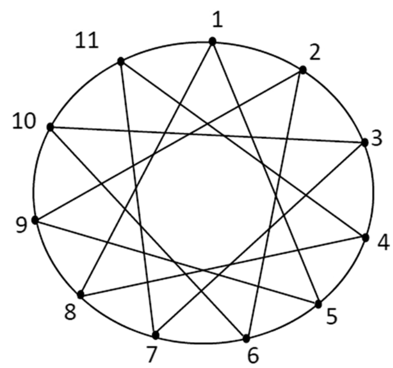

Fig. 7 $|D| = m = 11$

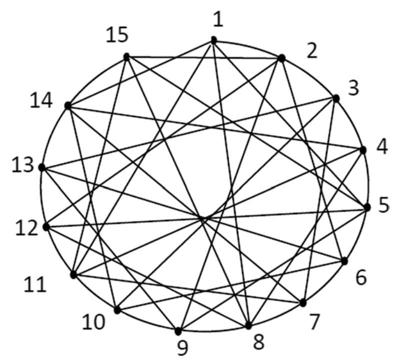

Fig. 8 $|D| = m = 15$

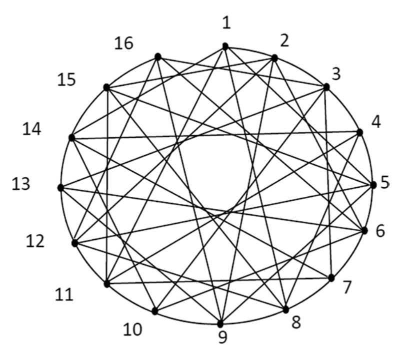

Fig. 9 $|D| = m = 16$

Theorem 4.6. *If G is a maximal* 1*-mod-difference graph where* $m = 2$*, then* $\gamma_i^{-1}(G) = \lceil \frac{n}{2} \rceil$.

Proof

Consider $ID_1 = \{\forall v_j; v_j isanoddvertex\}$*, as the same manner in the previous theorem* ID_1 *is the minimum independent dominating set in* $V - D_1$*, where* D_1 *is an independent dominating set in G (Theorem 4.5),(see Figs. 10, 11). Thus,* $\gamma_i^{-1}(G) = \lceil \frac{n}{2} \rceil$.

Corollary 4.7. *If G is a maximal* 1*-mod-difference graph where* $m = 2$*, then* $\gamma_{\mathrm{coi}}(G) = \lfloor \frac{n}{2} \rfloor$ *and* $\gamma_{\mathrm{coi}}^{-1}(G) = \lceil \frac{n}{2} \rceil$.

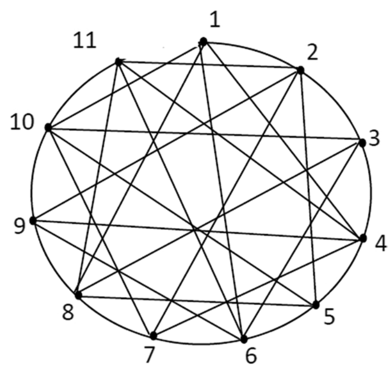

Fig. 10 $|D_1| = n = 11, m = 2$

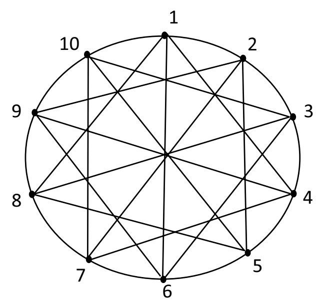

Fig. 11 $|D_1| = n = 10, m = 2$

Proof

We showed that the set D_1 in Theorem 4.5 is the dominating set in G and $V - D_1 = ID_1$ is an independent by Theorem 4.6, if we assume there is a set $A \subseteq V$ with cardinal less than D_1, so A may be still dominating set but $V - A$ is not an independent set since it contains vertices of odd and even labels) (see Figs. 8, 9). Thus, $\gamma_{\text{coi}}(G) = \lfloor \frac{n}{2} \rfloor$. As the same manner with alternate two sets $D_1 and ID_1$, we get $\gamma_{\text{coi}}^{-1}(G) = \lceil \frac{n}{2} \rceil$.

Theorem 4.8. *If G is a maximal 1-mod-difference graph where $m = 3$, then.*

$$\gamma_i(G) = \left\{ \begin{array}{c} \lfloor \frac{n}{3} \rfloor, ifn \equiv 0(mod3) \\ \lfloor \frac{n}{3} \rfloor + 1, ifn \equiv 1, 2(mod3) \end{array} \right\}$$

Proof

Consider $S = \{v_i; f(v_i) \in [1] - \{1\}\}$ and $D = \{v_2\} \cup S$. The vertex v_2 is adjacent to vertex v_1 and all vertices which there labels belong to class 0 ([0]) and the vertex $v_4 \in S$ is adjacent to all vertices which there labels belong to class 2 ([2]) except $\{2\}$, and S covers to all vertices of labeled in $[1] - \{1\}$. Thus, D is the dominating set in G and it is an independent, since $\forall v_i, v_j \in S$, $|f(v_i) - f(v_2)| \equiv 2(mod3)$ and $|f(v_i) - f(v_j)| \equiv 0(mod3)$ (see Figs. 7, 8). Thus, $\gamma_i(G) \leq |D| = \left\{ \begin{array}{c} \lfloor \frac{n}{3} \rfloor, ifn \equiv 0(mod3) \\ \lfloor \frac{n}{3} \rfloor + 1, ifn \equiv 1, 2(mod3) \end{array} \right\}$. If there is an independent set $A \subseteq V$ with $|A| < |D|$, then A is not a dominating set. Thus, we get the result.

Theorem 4.9. *If G is a maximal* 1*-mod-difference graph where* $m = 3$, *then*

$$\gamma_i^{-1}(G) = \lfloor \frac{n}{3} \rfloor + 1$$

Proof

Consider $S = \{v_i; f(v_i) \in [3]\}$ *and* $ID = \{v_1\} \cup S$, *it is obvious that* $ID \subseteq V - D$, *where D is the minimum independent dominating set in G (Corollary 4.7). The vertex* v_1 *is adjacent to all vertices which their labels belong to class 2 ([2]); the vertex* $v_3 \in S$ *is adjacent to all vertices which their labels belong to class 1 ([1]) except* $\{1\}$. *S covers all vertices which their labels belong to class 3* [0]. *Thus, ID is the dominating set in G and it is an independent, since* $\forall v_i \in S$, $|f(v_i) - f(v_1)| \equiv 2(mod3)$ *and* $|f(v_i) - f(v_j)| \equiv 0(mod3)$. *Thus,* $\gamma_i^{-1}(G) \le |\mathrm{ID}| = \lfloor \frac{n}{3} \rfloor + 1$ *(see Figs. 7, 8). If there is an independent set* $A \subseteq V - D$ *with* $|A| < |D|$, *then A is not a dominating set. Thus,* $\gamma_i^{-1}(G) = \lfloor \frac{n}{3} \rfloor + 1$.

Corollary 4.10. *If G is a maximal* 1*-mod-difference graph where* $m = 3$, *then*

$$\gamma_{\mathrm{coi}}(G) \le \left\{ \begin{array}{c} n - \lfloor \frac{n}{3} \rfloor, if n \equiv 0(mod3) \\ n - \lfloor \frac{n}{3} \rfloor - 1, if n \equiv 1, 2(mod3) \end{array} \right\}$$

Proof

Consider $M = V - D$, *where D is the set is in Theorem 4.8, it is clear that M is the dominating set and D is an independent set. Thus,*

$$\gamma_{\mathrm{coi}}(G) \le |\mathrm{M}| = \left\{ \begin{array}{c} n - \lfloor \frac{n}{3} \rfloor, if n \equiv 0(mod3) \\ n - \lfloor \frac{n}{3} \rfloor - 1, if n \equiv 1, 2(mod3) \end{array} \right\}$$

Theorem 4.11. *If G is a maximal* 1*-mod-difference graph, then*

$$\beta(G) = \frac{n}{2}, \text{ if } m \text{ is even.}$$

Proof

Consider $I = \{v_i \in G; v_i isanoddlabeledvertex\}$ $\forall v_j, v_k \in I, |f(v_j) - f(v_k)| \equiv w(modm)$ *where w is 0 or even number less than m, then I is an independent set. Now if we add any vertex* $v_h \in V - I$ *to the set I, then* v_h *is an even labeled vertex, then* v_{h-1} *is an odd labeled vertex, so* $v_{h-1} \in I$. *Therefore,* $|f(v_h) - f(v_{h-1})| \equiv 1(mod(m))$ *that means* $I \cup \{v_h\}$ *is not an independent set in G. Thus* $\beta(G) = \lceil \frac{n}{2} \rceil$.

Example 4.12.

The maximal 1-mod-difference graphs of order 11 and 15 where $m = 3$, *as shown in Figs. 7, 8,* $D_1 = \{v_2, v_4, v_7, v_{10}\}$, *is the minimum independent dominating set in* G_1, *and* $ID_1 = \{v_1, v_3, v_6, v_9\}$, *is the minimum independent sets in* $V(G_1) - D$, *so* $\gamma_i(G_1) = \gamma_i^{-1}(G_1) = 4$,, $D_2 = \{v_2, v_4, v_7, v_{10}, v_{13}\}$, *is the minimum independent* *dominating set in* G_2*, and* $ID_1 = \{v_1, v_3, v_6, v_9, v_{12}, v_{15}\}$*, is the minimum independent sets in* $V(G_2) - D$*, so* $\gamma_i(G_2) = 5 and \gamma_i^{-1}(G_2) = 6$*.*

Conclusion and discussion

In this work, we obtain the necessary condition(s) for a graph to be a maximal divisor graph and for a graph to be 0-mod-difference graph, also for a graph to be maximal o-mod-difference graph and finally for a graph to be maximal 1-mod-differnce graph.

These results will lead us to discuss in the future work to the independence and domination in multi-rooted graph.

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Competing interests
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