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# APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE Effects of time and diffusion phase-lags in a thin circular disc with axisymmetric heat supply

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**Abstract:** The present investigation is concerned with an axisymmetric problem of thin circular disc in a thermoelastic diffusive body within the context of dualphase-lag heat transfer and dual-phase-lag diffusion models. The upper and lower surfaces of the thin disc are traction free and subjected to an axisymmetric heat supply. The solution is found using Laplace and Hankel transform technique and a direct approach without the use of potential functions. The analytical expressions of displacement components, stresses and chemical potential are computed in transformed domain. Numerical inversion technique has been applied to obtain the results in the physical domain. Numerically simulated results are depicted graphically. The effect of diffusion and thermal phase-lags are shown on the various components. Some particular cases of result are also deduced from the present investigation.

# Subjects: Science; Earth Sciences; Technology; Engineering & Technology; Aerospace Engineering; History of Engineering & Technology

Keywords: dual phase lag; isotropic thermoelastic; Laplace transform; Hankel transform; plane axisymmetric; diffusion

# ABOUT THE AUTHORS

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S.K. Garg has been involved in teaching for over three decades. He is Professor of Mathematics, Department of Mathematics, Deen Bandhu Chhotu Ram University of Science and Technology, Murthal, Haryana. The problem investigated in the present study is to analyze the effects of thermal and diffusion phase lags due to axisymmetric heat supply for a disc, which is a significant problem of continuum mechanics having various applications in geophysics and engineering.

# PUBLIC INTEREST STATEMENT

The two-dimensional axisymmetric problem of thin circular disc with thermoelastic diffusion has many important applications in the field of material science, designers of new materials, engineering, physicists and industry. It also plays a significant role in geophysics, metal oxide semiconductor improvement in crude oil extraction from oil deposits and neuroradiology. Thermodiffusion plays an important role in the deformation of an elastic body due to thermal and chemical changes.





#### 1. Introduction

Classical Fourier heat conduction law implies an infinitely fast propagation of a thermal signal which is violated in ultra-fast heat conduction system due to its very small dimensions and short time-scales. Catteno (1958) and Vernotte (1958) proposed a thermal wave with a single phase lag in which the temperature gradient after a certain elapsed time was given by  $q + \tau_q \frac{\partial q}{\partial t} = -k\nabla T$ , where  $\tau_q$  denotes the relaxation time required for thermal physics to take account of hyperbolic effect within the medium. Here, when  $\tau_q > 0$ , the thermal wave propagates through the medium with a finite speed of  $\sqrt{\frac{\alpha}{\tau_q}}$ , where  $\alpha$  is thermal diffusivity. when  $\tau_q$  aproaches zero, the thermal wave has an infinite speed and thus the single phase lag model reduces to traditional Fourier model.

The dual-phase-lag model of heat conduction was proposed by Tzou (1995a, 1995b, 1996),  $q + \tau_q \frac{\partial q}{\partial t} = -k(\nabla T + \tau_t \frac{\partial}{\partial t} \nabla T)$ , where the temperature gradient  $\nabla T$  at a point P of the material at time  $t + \tau_t$  corresponds to the heat flux vector q at the same time at the time  $t + \tau_q$  Here K is thermal conductivity of the material. The delay time  $\tau_t$  is interpreted as that caused by the microstructural interactions and is called the phase lag of temperature gradient. The other delay time  $\tau_q$  interpreted as the relaxation time due to the fast transient effects of thermal inertia and is called the phase lag of heat flux. This universal model is claimed to be able to bridge the gap between microscopic and macroscopic approaches, covering a wide range of heat transfer models. If  $\tau_t = 0$ , Tzou (1996) refers to the model as single-phase-model.

Numerous efforts have been invested in the development of an explicit mathematical solution to the heat conduction equation under dual-phase-lag model. Quintanilla and Racke (2006) compared two different mathematical hyperbolic models proposed by Tzou. Kumar and Mukhopadhaya (2010a, 2010b) investigated the propagation of harmonic waves of assigned frequency by employing the thermoelasticity theory with three phase lags. Chou and Yang (2009) discussed two-dimensional dual-phase-lag thermal behaviour in single/multilayer structures using CESE method. Zhou, Zhang, and Chen (2009) proposed an axisymmetric dual-phase-lag bio heat model for laser heating of living tissues. Kumar, Chawla, and Abbas (2012) discussed effect of viscosity on wave propagation in ani-sotropic thermoelastic medium with three-phase-lag model. Ying and Yun (2015) built a fractional dual-phase-lag model and the corresponding bio-heat transfer equation.

Abdallah (2009) used uncoupled thermoelastic model based on dual phase lag to investigate the thermoelastic properties of a semi-infinite medium. Rukolaine (2014) investigated unphysical effects of the dual-phase-lag model of heat conduction. Tripathi, Kedar, and Deshmukh (2015) analysed generalized thermoelastic diffusion problem in a thick circular plate with axisymmetric heat supply. Chen and Gurtin (1968), Chen, Gurtin, and Williams (1968, 1969) have formulated a theory of heat conduction in deformable bodies which depends upon two distinct temperatures, the conductive temperature and the thermodynamically temperature. Diffusion is defined as the spontaneous movement of the particles from high concentration region to the low concentration region, and it occurs in response to a concentration gradient expressed as the change in concentration due to change in position. Thermal diffusion utilizes the transfer of heat across a thin liquid or gas to accomplish isotope separation. The thermodiffusion in elastic solids is due to coupling of fields of temperature, mass diffusion and that of strain in addition to heat and mass exchange with the environment. Nowadays, it has extensive industrial applications, for example, oil companies are interested in the process of thermodiffusion, as it is efficient in extraction of oil from oil deposits.

Nowacki (1974a, 1974b, 1974c, 1974d) developed the theory of thermoelastic diffusion. The next generalization to the thermoelasticity theory is known as the dual phase lag model developed by Tzou (1995a) and Chandrasekharaiah (1998). Many researchers studied various problems involving dual-phase-lags (e.g. Abbas, 2015a, 2015b, 2015c; Abbas, Kumar, & Reen, 2014; Abbas & Zenkour, 2013, 2014, 2015; Abdallah, 2009; Atwa & Jahangir, 2014; Ezzat & Awad, 2010; Kaushal, Kumar, & Miglani, 2011; Kumar & Gupta, 2014; Kumar & Mukhopadhaya, 2010a, 2010b; Kumar, Sharma, & Garg, 2014; Kaushal, Sharma, & Kumar, 2010; Sharma & Marin, 2013; Youssef, 2006, 2011).

Here in this investigation, a generalized form of mass diffusion equation is introduced instead of classical Fick's diffusion theory using two diffusion phase-lags in axisymmetric form. One-phase-lag of diffusing mass flux vector, represents the delayed time required for the diffusion of the mass flux and the other phase-lag of chemical potential, represents the delayed time required for the establishment of the potential gradient. The basic equations for the isotropic thermoelastic diffusion medium in the context of dual-phase-lag heat transfer (DPLT) and dual-phase-lag diffusion (DPLD) models in axisymmetric form are presented. The components of displacements, stresses and chemical potential, temperature and mass concentration are computed numerically. Numerically computed results are depicted graphically. The effects of diffusion and thermal phase lags are shown on the various components.

## 2. Basic equations

The equations of motion, heat conduction and mass diffusion in a homogeneous isotropic thermoelastic solid with DPLT and DPLD models in the absence of body forces, heat sources and mass diffusion sources are given by

$$(\lambda + \mu)\nabla(\nabla \cdot \vec{u}) + \mu\nabla^2 u - \beta_1 \nabla T - \beta_2 \nabla C = \rho \ddot{u},$$
(1)

$$\left(1+\tau_{t}\frac{\partial}{\partial t}\right)kT_{,ii} = (1+\tau_{q}\frac{\partial}{\partial t}+\tau_{q}^{2}\frac{\partial^{2}}{\partial t^{2}})[\rho C_{E}\dot{T}+\beta_{1}T_{0}\dot{e}_{kk}+aT_{0}\dot{C}],$$
<sup>(2)</sup>

$$\left(1+\tau_{p}\frac{\partial}{\partial t}\right)(D\beta_{2}\nabla^{2}(\nabla.\vec{u})Da\nabla^{2}T-Db\nabla^{2}C+\frac{\partial}{\partial t}(1+\tau_{\eta}\frac{\partial}{\partial t}+\tau_{\eta}^{2}\frac{\partial^{2}}{\partial t^{2}})C=0,$$
(3)

and the constitutive relations are

$$\sigma_{ij} = 2\mu \mathbf{e}_{ij} + \delta_{ij} (\lambda \mathbf{e}_{kk} - \beta_1 \mathbf{T} + \beta_2 \mathbf{C}), \tag{4}$$

$$\rho T_0 S = \left(1 + \tau_q \frac{\partial}{\partial t} + \tau_q^2 \frac{\partial^2}{\partial t^2}\right) \left[\rho C_E T + \beta_1 T_0 e_{kk} + a_0 T_0 C\right],\tag{5}$$

$$P = -\beta_2 \boldsymbol{e}_{kk} - \boldsymbol{a}_0 \boldsymbol{T} - \boldsymbol{b}_0 \boldsymbol{C}, \tag{6}$$

where  $\lambda$ ,  $\mu$  are Lame's constant,  $\rho$ , is the density, D is the diffusivity, P is the chemical potential per unit mass, C is the mass concentration,  $u_i$  are components of displacement vector u, k i the cofficent of thermalconductivity,  $C_E$  is specific heat at a constant strain,  $T = \theta - T_0$  is a small temprature increment,  $\theta$  is absolute temperature of the medium,  $T_0$  is the reference temperature of the body such that  $\left|\frac{\tau}{\tau_0}\right| \ll 1$ ,  $a_0$  is the coefficients describing the measure of thermodiffusion effect,  $b_0$  is the coefficients' describing the measure of mass diffusion effect,  $e_{kk}$  is dilatation, S is the entropy per unit mass,  $\beta_1 = (3\lambda + 2\mu)\alpha_t \beta_2 = (3\lambda + 2\mu)\alpha_c, \alpha_t$  is the coefficient of thermal linear expansion,  $\alpha_c$  is the coefficient of linear diffusion expansion,  $\tau_t$  are phase lag of temperature gradient,  $\tau_q$  are the phase lag of heat flux,  $\tau_p$  are the phase lag of chemical potential and  $\tau_\eta$  are phase lag of diffusing mass flux vector. In above equations, a comma followed by suffix denotes spatial derivative and a superposed dot denotes derivative with respect to time.

#### 3. Formulation and solution of the problem

Consider a disc of diameter 2*b* occupying the space *D* defined by  $0 \le r \le \infty, -b \le z \le b$ . Let the disc be subjected to an axisymmetric heat supply depending on radial and axial direction of the cylindrical coordinate system  $(r, \theta, z)$  with symmetry about *z*-axis. The initial temperature in the thin disc is given by a constant temperature  $T_0$ , the heat flux  $g_0 F(r, z)$  is prescribed along with vanishing of stress components on the upper and lower boundary surfaces along with traction free boundary  $z = \pm b$ . Under these conditions, the thermoelastic quantities in a thin circular disc are required to be determined. As the problem considered is plane axisymmetric, the field component  $u_{\theta} = 0$  and  $u_r$ ,  $u_r$ , *T* and *C* are independent of  $\theta$  and restrict our analysis to the two-dimensional problem with

$$\mathbf{u} = (u_r, \mathbf{0}, u_z) \tag{7}$$

Equations (1)-(6) with the aid of (7) take the form

$$(\lambda + \mu)\frac{\partial \mathbf{e}}{\partial r} + \mu \left(\nabla^2 - \frac{1}{r^2}\right) u_r - \beta_1 \frac{\partial T}{\partial r} - \beta_2 \frac{\partial \mathbf{c}}{\partial r} = \rho \frac{\partial^2 u_r}{\partial r^2},\tag{8}$$

$$(\lambda + \mu)\frac{\partial e}{\partial r} + \mu \nabla^2 \mathbf{u}_z - \beta_1 \frac{\partial T}{\partial z} - \beta_2 \frac{\partial c}{\partial z} = \rho \frac{\partial^2 \mathbf{u}_z}{\partial t^2},\tag{9}$$

$$\left(1+\tau_{t}\frac{\partial}{\partial t}\right)k\nabla^{2}T = \left(1+\tau_{q}\frac{\partial}{\partial t}+\tau_{\frac{q}{2}}^{2}\frac{\partial^{2}}{\partial t^{2}}\right)\left[\rho C_{E}\dot{T}+\beta_{1}T_{0}\frac{\partial}{\partial t}\operatorname{div}u+aT_{0}\frac{\partial c}{\partial t}\right],\tag{10}$$

$$\left(1+\tau_{p}\frac{\partial}{\partial t}\right)\left(D\beta^{2}\nabla^{2}\operatorname{div} u+Da_{0}\nabla^{2}T-Db_{0}\nabla^{2}C\right)+\frac{\partial}{\partial t}\left(1+\tau_{\eta}\frac{\partial}{\partial t}+\tau_{\eta}^{2}\frac{\partial^{2}}{\partial t^{2}}\right)C=0,$$
(11)

and constitutive relation

$$\sigma_{rr} = 2\mu r_{rr} + \lambda \mathbf{e} - \beta_1 \mathbf{T} - \beta_2 \mathbf{C},\tag{12}$$

$$\sigma_{\theta\theta} = 2\mu r_{\theta\theta} + \lambda e - \beta_1 T - \beta_2, \tag{13}$$

$$\sigma_{zz} = 2\mu r_{zz} + \lambda e \beta_1 T - \beta_2 C, \qquad (14)$$

$$\sigma_{rz} = 2\mu r_{rz}, \ \sigma_{r\theta} = 0, \ \sigma_{z\theta} = 0, \tag{15}$$

$$P = \beta^2 - a_0 T - b_0 C \tag{16}$$

where

$$\mathbf{e} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial r},\tag{17}$$

and  $e_{rr}, e_{_{ heta heta}}, e_{_{zz}}$  and  $e_{_{rz}}$ . The incremental strain components and the rotation are given as

$$\mathbf{e}_{rz} = \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right), \ \mathbf{e}_{rr} = \frac{\partial u_r}{\partial r}, \ \mathbf{e}_{\theta\theta} = \frac{1}{r} \left( \frac{\partial u_{\theta}}{\partial \theta} + u_r \right), \ \mathbf{e}_{zz} = \frac{\partial u_z}{\partial z}, \ \omega_{\theta} = \frac{1}{2} \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right)$$
(18)

To facilitate the solution, the following dimensionless quantities are introduced

$$r' = \frac{w_1^*}{c_1}r, \ z' = \frac{w_1^*}{c_1}z, \ (u'_r, u'_z) = \frac{w_1^*}{c_1}(u_ru_z), \ t' = w_1^*t(\sigma'_{rr}, \sigma'_{\theta\theta}, \sigma'_{zz}, \sigma'_{rz}) = \frac{1}{\beta_1 T_0}(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz}),$$

$$(\tau'_q, \ \tau'_t, \ \tau'_p, \ \tau'_\eta) = w_1^*(\tau_q, \ \tau_t, \ \tau_p, \ \tau_\eta), \ P' = \frac{p}{\beta_2}, \ (T', \ C') = \frac{1}{\rho C_1^2}(\beta_1 T, \beta_2 c)$$
(19)

where

$$W_1^* = \frac{\rho c_{\varepsilon} c_1^2}{k}$$
 and  $C_1^2 = \frac{\lambda + 2\mu}{\rho}$ 

Following Debnath (1995), the Laplace transform of a function  $f(x_1, x_3, t)$  with respect to time variable *t*, with *s* as a Laplace transform variable is defined as

$$\tilde{f}(x_1, x_3, s) = L\{f(x_1, x_3, t)\} = \int_0^\infty e^{-st} f(x_1, x_3, t) dt,$$
(20)

along with the following basic properties:

$$L\left(\frac{\partial f}{\partial t}\right) = s\tilde{f}(x_1, x_3, s) - f(x_1, x_3, 0), \tag{21}$$

$$L\left(\frac{\partial^2 f}{\partial t^2}\right) = s^2 \tilde{f}(x_1, x_3, s) - sf(x_1, x_3, 0) - \left(\frac{\partial f}{\partial t}\right)_{t=0}.$$
(22)

Using (19) in Equations (8)–(11) and after that suppressing the primes and then applying the Laplace transform define by Equations (20)–(22) on the resulting quantities and after simplification, we obtain

$$\nabla^2 \tilde{T} + \nabla^2 \tilde{C} - \left(\nabla^2 - s^2\right) \tilde{e} = 0,$$
(23)

$$\left(\nabla^2 - \varsigma_{21}\right)\tilde{T} - \varsigma_{22}\tilde{C} - \varsigma_{23}\tilde{e} = \mathbf{0},\tag{24}$$

$$\varsigma_{31}\nabla^2 \tilde{T} - \left(\varsigma_{32}\nabla^2 - \varsigma_{33}\right)\tilde{C} + \varsigma_{34}\nabla^2 \tilde{e} = \mathbf{0},\tag{25}$$

where  $\varsigma_{21}, \varsigma_{22}, \varsigma_{23}, \varsigma_{31}, \varsigma_{32}, \varsigma_{33}, \varsigma_{34}, \tau_q^0, \tau_\eta^0, \tau_P^0$  and  $\tau_t^0$  are given in Appendix 1.

Elimination  $\tilde{T}$ ,  $\tilde{C}$ ,  $\tilde{e}$  From Equations (23)–(25), we obtain

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)(\nabla^2 - k_3^2)(\tilde{T}, \tilde{C}, \tilde{e}) = 0$$
<sup>(26)</sup>

The solution of Equation (26) can be written in the form

$$\tilde{T} = \sum_{i=1}^{3} \tilde{T}_{i}, \ \tilde{e} = \sum_{i=1}^{3} \tilde{e}_{i}, \ \tilde{C} = \sum_{i=1}^{3} \tilde{C}_{i}$$
(27)

where  $\tilde{T}$ ,  $\tilde{e}$ , and  $\tilde{C}$  are the solution of the following equation

$$(\nabla^2 - k_i^2)(\tilde{T}, \tilde{C}, \tilde{e}) = 0, \quad i = 1, 2, 3$$
 (28)

Applying Hankel transform defined by (29),

$$\hat{f}(\varsigma, z) = \int_{0}^{\infty} f(\varsigma, z) r J n(r\varsigma) dr,$$
(29)

on (28), we obtain

$$(D^{2} - \varsigma^{2} - k_{i}^{2})(\tilde{T}_{i}^{*}, \tilde{C}_{i}^{*}, \tilde{e}_{i}^{*}) = 0$$
(30)

The solution of Equation (30) yields

$$\tilde{T}^* = \sum_{i=1}^{3} A_i \cosh q_i z, \tag{31}$$

$$\tilde{C}^* = \sum_{i=1}^3 d_i A_i \cosh q_i z, \tag{32}$$

$$\tilde{\mathbf{e}}^* = \sum_{i=1}^3 f_i \mathbf{A}_i \cosh q_i \mathbf{z},\tag{33}$$

where  $q_i$  and the coupling constants  $d_i$  and  $f_i$  are given in Appendix 2.

Aplying inversion of Hankel transform on (31)–(33), we get

$$\tilde{T} = \int_{0}^{\infty} \left( \sum_{i=1}^{3} A_i \cosh(q_i z) \right) \zeta J_0(\zeta r) d\zeta,$$
(34)

$$\tilde{C} = \int_{0}^{\infty} \left( \sum_{i=1}^{3} d_i A_i \cosh(q_i z) \right) \zeta J_0(\zeta r) d\zeta,$$
(35)

$$\tilde{\mathbf{e}} = \int_0^\infty \left(\sum_{i=1}^3 f_i A_i \cosh(q_i z)\right) \zeta J_0(\zeta r) d\zeta.$$
(36)

With the aid of Equations (8)–(11), (19) and (34)–(36), we obtain the displacement components in the transformed domain as

$$\tilde{u}_r(r, z, s) = \int_0^\infty \varsigma^2 J_1(\varsigma r) R_2 d\varsigma,$$
(37)

$$\tilde{u}_{z}(r, z, s) = \int_{0}^{\infty} \zeta J_{0}(\zeta r) R_{3} d\zeta, \qquad (38)$$

Substituting the values of  $\tilde{T}$ ,  $\tilde{C}$ ,  $\tilde{e}$  from (34)–(36) in (12)–(16) and with the aid of (19) yield the stress components and chemical potential as

$$\tilde{\sigma}_{\theta\theta} = V_1 \int_0^\infty \varsigma^2 J_1(\varsigma r) R_2 d\varsigma + \int_0^\infty R_4 J_0(\varsigma r) d\varsigma,$$
(39)

$$\tilde{\sigma}_{rr} = V_1 \int_0^\infty \varsigma^3 \left( \frac{1}{\varsigma r} J_1(\varsigma r) - J_0(\varsigma r) \right) R_2 d\varsigma + \int_0^\infty R_4 J_0(\varsigma r) d\varsigma, \tag{40}$$

$$\tilde{\sigma}_{zz} = V_1 \int_0^\infty \varsigma J_0(\varsigma r) R_5 d\varsigma + \int_0^\infty R_4 J_0(\varsigma r) d\varsigma, \qquad (41)$$

$$\tilde{\sigma}_{rz} = V_1 \int_0^\infty \varsigma^2 J_1(\varsigma r) R_6 d\varsigma, \tag{42}$$

$$\tilde{P}(r, z, s) = \int_{0}^{\infty} \sum_{i=1}^{3} \sigma_{i} \cosh(q_{i}z) \varsigma J_{0}(\varsigma r) d\varsigma, \qquad (43)$$

where  $\lambda_i$ ,  $E(\zeta s)$ , q,  $R_1$ ,  $R_2$ ,  $R_3$ ,  $R_4$ ,  $F(\zeta, s)$ ,  $R_5$ ,  $\eta_i$ ,  $R_6$ ,  $V_1$ ,  $\zeta_i$  are given in Appendix 3.

#### 4. Boundary conditions

We consider a thermal source and chemical potential source (disc load). The disc load which emanates from origin of the coordinates and expands radically at constant rate c and along with vanishing of stress components at the stress free surface at  $z = \pm b$ . Mathematically these can be written as

$$\frac{\partial T}{\partial z} = \pm g_0 F(r, z), \tag{44}$$

$$\sigma_{zz} = 0, \tag{45}$$

$$\sigma_{rz} = 0, \tag{46}$$

$$P = f(r, t) = \frac{H(t - r)}{\pi (ct)^2},$$
(47)

where H() is Heaviside unit step function and  $g_0$  is the constant temperature applied on the boundary.

Applying Laplace transform and Hankel transform given by the Equations (20) and (29) on the boundary conditions (44)–(47), we obtain at  $z = \pm b$ 

$$\frac{\partial \tilde{T}}{\partial z} = \pm g_0 \tilde{F}(r, z), \tag{48}$$

$$\tilde{\sigma}_{zz} = \mathbf{0},\tag{49}$$

$$\tilde{\sigma}_{rz} = 0,$$
 (50)

$$\tilde{P} = \tilde{f}(\varsigma, s) = \frac{1}{\pi c_{\varsigma} \sqrt{\varsigma^2 + \frac{s^2}{c^2} - \frac{s}{c}}}.$$
(51)

Substitute the value of  $\tilde{T}$  from Equation (34) and  $\tilde{\sigma}_{zz}$ ,  $\tilde{\sigma}_{rz}$  and  $\tilde{P}$  from Equations (41)-(43), in the Equations (48)-(51), we obtain the value of unknown parameters as

$$A_1 = \frac{\Delta_1}{\Delta}, \ A_2 = \frac{\Delta_2}{\Delta}, \ A_3 = \frac{\Delta_3}{\Delta}, \ E\{\varsigma, s\} = \frac{\Delta_4}{\Delta}$$
(52)

where  $\Delta$ ,  $\Delta_{1i'}$ ,  $\Delta_{14}$ ,  $\Delta_{2i'}$ ,  $\Delta_{24}$ ,  $\Delta_{3i'}$ ,  $\Delta_{34}$ ,  $\Delta_{4i'}$ , i = 1, 2, 3 and  $\Delta_i$  (i = 1, 2, 3, 4) are given in Appendix 4.

Substituting the values of  $A_i$  (i = 1, ..., 4) from Equation (52) in the Equations (34)–(43), yield the components of displacement, stress components and chemical potential.

#### 5. Particular cases

#### 5.1. Thermoelastic isotropic half space

Taking  $\beta_2 = a = b = 0$ , in the Equations (34)-(43) along with Equation (52), yield the expressions for components of displacement, stress and temperature distribution in thermoelastic isotropic half space with the changed values of  $f_i$ ,  $d_i$ ,  $\varsigma_{22}$ ,  $\varsigma_{23}$ ,  $\varsigma_{31}$ ,  $\varsigma_{32}$ ,  $\varsigma_{33}$ ,  $\varsigma_{34}$ ,  $\lambda_i$ ,  $R_1$ ,  $R_2$ ,  $R_3$ ,  $R_4$ ,  $E(\varsigma, s)$ ,  $R_5$ ,  $\eta_i$ ,  $R_6$ ,  $\varsigma_i$ 

$$\begin{split} f_{i_0} &= -\frac{\tau_{21}}{\tau_{23}}, \ d_{i_0} = 0, \ \zeta_{22} = \zeta_{23} = \zeta_{31} = \zeta_{32} = \zeta_{33} = \zeta_{34} = 0, \ E(\zeta, s) = 0 \\ \lambda_{i_0} &= \left(\frac{\lambda + \mu}{\rho C_1^2} f_{i_0}\right) A_i, \ R_{1_0} = \frac{\lambda_{i_0}}{\left(\frac{\mu q_i^2}{\rho C_1^2} - s^2\right)}, \ R_{2_0} = \sum_{i=1}^2 R_{1_0} \cosh(q_i z), \ R_{3_0} = \sum_{i=1}^2 R_{1_0} \sinh(q_i z), \\ R_{4_0} &= \sum_{i=1}^2 \eta_{i_0} \cosh(q_i z)\zeta, \ F(\zeta, s) = 0, \ R_{5_0} = \left(\sum_{i=1}^2 R_{1_0} q_i^2 \cosh(q_i z)\right), \ \eta_{i_0} = \left(\frac{f_{i_0} - \rho c^2}{\beta_i T_0}\right) A_i, \\ R_{6_0} &= \left(2\sum_{i=1}^2 R_{1_0} q_i \sinh(q_i z)\right), \ \zeta_{i_0} = \left(-\frac{\rho C_1^2}{\beta_1}\right) A_i(\xi, s) \end{split}$$

as,

$$\tilde{T} = \int_0^\infty \left(\sum_{i=1}^2 A_i \cosh(q_i z)\right) \zeta J_0(\zeta r) d\zeta,$$
(53)

$$\tilde{\mathbf{e}} = \int_0^\infty \left(\sum_{i=1}^2 f_{i_0} A_i \cosh(q_i z)\right) \zeta J_0(\zeta r) d\zeta.$$
(54)

$$\tilde{u}_{r}(r, z, s) = \int_{0}^{\infty} \varsigma^{2} J_{1}(\varsigma r) R_{2_{0}} d\varsigma, \qquad (55)$$

$$\tilde{u}_{z}(r, z, s) = \int_{0}^{\infty} \varsigma J_{0}(\varsigma r) R_{3_{0}} d\varsigma,$$
(56)

$$\tilde{\sigma}_{\theta\theta} = V_1 \int_0^\infty \varsigma^2 J_1(\varsigma r) R_{2_0} d\varsigma + \int_0^\infty R_{4_0} J_0(\varsigma r) d\varsigma,$$
(57)

$$\tilde{\sigma}_{rr} = V_1 \int_0^\infty \varsigma^3 \left( \frac{1}{\varsigma r} J_1(\varsigma r) - J_0(\varsigma r) \right) R_{2_0} d\varsigma + \int_0^\infty R_{4_0} J_0(\varsigma r) d\varsigma,$$
(58)

$$\tilde{\sigma}_{zz} = V_1 \int_0^\infty \zeta J_0(\zeta r) R_{5_0} d\zeta + \int_0^\infty R_{4_0} J_0(\zeta r) d\zeta,$$
(59)

$$\tilde{\sigma}_{rz} = V_1 \int_0^\infty \varsigma^2 J_1(\varsigma r) R_{6_0} d\varsigma, \tag{60}$$

#### 5.2. Coupled thermoelasticity with mass diffusion

Taking  $\tau_q = \tau_t = \tau_p = \tau_\eta = 0$  in the Equations (34)–(43) with the aid of (52), yield the expressions of displacement components, stress components, temperature distribution and chemical potential expression with the changed values of,  $\tau_q^0$ ,  $\tau_\eta^0$ ,  $\tau_p^0 \tau_t^0$ , as,  $\tau_q^0 = 1$ ,  $\tau_\eta^0 = 1$ ,  $\tau_p^0 = 1$ ,  $\tau_t^0 = 1$ , for coupled thermoelasticity with mass diffusion model.

#### 5.3. Dual-phase-lag heat model (DPLT)

Taking  $\tau_p = \tau_\eta = \tau_\eta^2 = 0$ , in the Equations (34)–(43), with the aid of (52), yield the expression for dual-phase-lag heat model with the changed values of  $\tau_p^0$ ,  $\tau_\eta^0$ , as  $\tau_p^0 = 1$ ,  $\tau_\eta^0 = 1$ .

**5.4.** Single-phase-lag heat model (SPLT) and single-phase-lag diffusion model (SPLD) Taking  $\tau_q = \tau_q^2 = \tau_\eta = \tau_\eta^2 = 0$ , in the Equations (34)–(43), with the aid of (52), expression which reduces DPLT and DPLD models to single-phase-lag heat model (SPLT) and single-phase lag diffusion model (SPLD) with the changed values of  $\tau_q^0$ ,  $\tau_\eta^0$  as  $\tau_q^0 = 1$ ,  $\tau_\eta^0 = 1$ .

#### 6. Inversion of transforms

We have obtained the expressions for displacement components, stress components and chemical potential in Equations (37)–(43). These expressions are functions of *z*, parameters of Laplace and Hankel transforms s and  $\varsigma$  and hence are of the form  $\hat{f}(\varsigma, z, s)$ . To get the function f(r, z, t) in physical domain, first we invert the Hankel transform using

$$\tilde{f}(r, z) = \int_0^\infty \varsigma \hat{f}(\varsigma, z) J_n(r, \varsigma) d\varsigma,$$
(61)

The method for evaluating this integral is described by Press, Teukolshy, Vellerling, and Flannery (1986), which involves the use of Romberg's integration with adaptive step size. This also uses the results from successive refinements of the extended trapezoidal rule followed by extrapolation of the results to the limit when the step size tends to zero.

Due to the complexity of the solution in the Laplace transform domain, the inverse of the Laplace transform is obtained using the Gaver-Stehfast algorithm. Graver (1966) and Stehfast (1970a, 1970b) derived the formula given below. By this method, the inverse f(t) of Laplace transform  $f(\tilde{s})$  is approximated by  $f(t) = \frac{\log 2}{t} \sum_{j=1}^{k} D(j, k) F(j \frac{\log 2}{t})$  with  $D(j, k) = (-1)^{j+M} \sum_{n=m}^{\min(j,M)} \left( \frac{n^m (2n)!}{(M-n)!n!(n-1)!(j-n)!(2n-j)!} \right)$ , where *K* is an even integer, whose value depends on the word length of computer used. M = K/2, and *m* is an integer part of (j + 1)/2. The optimal value of *k* was chosen as described in Gaver-Stehfast algorithm, for the fast convergence of results with desired accuracy. The Romberg numerical integration technique [1986] with variable step size used to evaluate the results involved.

#### 7. Numerical results and discussion

The mathematical model is prepared with copper material for purposes of numerical computation. The material constants for the problem are taken from Dhaliwal and Singh (1980)  $\lambda = 7.76 \times 10^{10} \text{ Nm}^{-2}$ ,  $\mu = 3.86 \times 10^{10} \text{ Nm}^{-2}$ ,  $K = 386 \text{ JK}^{-1} \text{ m}^{-1} \text{s}^{-1}$ ,  $T_0 = 293 \text{ K}$ ,  $\beta_1 = 5.518 \times 10^6 \text{ Nm}^{-2} \text{ deg}^{-1}$ ,  $\beta_2 = 61.38 \times 10^6 \text{ Nm}^{-2} \text{ deg}^{-1}$ ,  $\rho = 8,954 \text{ Kgm}^{-3}$ ,  $a = 1.2 \times 10^4 \text{ m}^2/\text{ s}^2$  K,  $b = 0.9 \times 10^6 \text{ m}^5/\text{ kg s}^2$ ,  $D = 0.88 \times 10^{-8} \text{ Kgsm}^3$ ,  $C_E = 383.1 \text{ Jkg}^{-1} \text{ K}^{-1}$ .

An investigation has been conducted to compare the effect of time on dual phase lag model in heat conduction and diffusion and the graphs have been plotted for the range  $0 \le r \le 10$ , phase lags are taken as  $\tau_n = 0.01$ ,  $\tau_n = 0.03$ ,  $\tau_t = 0.05$  and  $\tau_a = 0.07$ .

In all figures solid line corresponds to the dual-phase-lag of heat transfer and diffusion with nonzero values t = 0.01, small dashed line corresponds to the dual-phase-lag of heat transfer and diffusion with non-zero values t = 0.02, long dashed line corresponds to the dual-phase-lag of heat transfer and diffusion with non-zero values t = 0.03, long dashed line with dots line corresponds to the dual-phase-lag of heat transfer and diffusion with non-zero values t = 0.04.

Figure 1 exhibits variation of displacement component  $u_z$  with respect to distance r. Near the loading surface, there is a sharp decrease in range  $0 \le r \le 2$  and behaviour is oscillatory away from the loading surface for all the cases. Also it is noticed that amplitude of variations decreases as t increases and attains maximum value for t = 0.01.

Figure 2 shows variation of temperature change *T* with respect to distance *r*. We find that in the range  $0 \le r \le 3$  there is a sharp increase for t = 0.01 and small increase for rest of the cases and it is noticed that, there are small variations away from the loading surface.



Figure 3 shows variation of displacement component  $u_r$  with respect to distance r. Here, we observed that near the loading surface, there is a sharp decrease in range  $0 \le r \le 2$ . Also it is evident that trends are oscillatory with decreasing amplitudes values.

Figure 4 shows variation of mass concentration C with respect to distance r. It is noticed that, it decreases in the range  $0 \le r \le 3$  for t = 0.01 and trend is descending oscillatory and small variations near zero are noticed in the range  $3 \le r \le 7$  for the rest of the cases.

Figure 5 gives variation of stress component  $\sigma_{rz}$  with respect to distance r. Here, it is observed that at the loading surface, there is a sharp decrease in range  $0 \le r \le 2.5$ . It is also noticed that in the range  $5 \le r \le 10$ , trends are similar for all the cases.

Figure 6 shows variation in stress component  $\sigma_{\theta\theta}$  with respect to distance r. Here it is noticed that, it decreases in the range  $0 \le r \le 3$  for t = 0.01 sharply and it is observed that, it decreases not





sharply in the rest of the cases. Also it is observed that trends are oscillatory with decreasing amplitudes values.

Figure 7 shows variation of stress component  $\sigma_{zz}$  with respect to distance r. It is observed that, it decreases in the range  $0 \le r \le 3$  for t = .01 sharply. Here for t = 0.01, values of  $\sigma_{zz}$  are almost same than the values for other cases and follows oscillatory pattern, whereas for the rest of the cases, trends are similar with small variations.

Figure 8 exhibits variation of chemical potential function *P* with respect to distance *r*. Near the loading surface, there is a sharp decrease in the range  $0 \le r \le 2$ . Here for t = 0.01, values of *P* are same as compared to other cases and follows oscillatory pattern, whereas for the rest of the cases, trends are similar with same variations.

Figure 8. Variationof chemical potential function *P* w.r.t. distance *r*.



### 8. Conclusion

In this chapter, effects of thermal and diffusion phase lags are investigated due to axi-symmetric heat supply for a disc. The problem is discussed within the context of DPLT and DPLD models. The upper and lower surfaces of the disc are traction free and subjected to an axi-symmetric heat supply. The solution of the problem is found using Laplace and Hankel transforms and a direct approach without the use of potential functions. The analytical expressions of displacements, stresses, the chemical potential, temperature change and mass concentration are computed in transformed domain. Numerical inversion technique has been applied to obtain the results in the physical domain.

Effects of diffusion and thermal phase lags are computed and comparison of variations is made. It is observed that change in phase lags changes the behaviour of deformations of the various components of stresses, displacements, chemical potential function, temperature change and mass concentration. Small difference in phase lags results in big difference of thermal waves. A sound impact of diffusion and thermal phase-lags on the various quantities is observed. The use of diffusion phaselags in the equation of mass diffusion gives more realistic model of thermoelastic diffusion media as it allows a delayed response between the relative mass flux vector and the potential gradient.

Also the behaviour of deformations of the various components of stresses, displacement, chemical potential function, temperature change and mass concentration are dependent on the variation of t. Also, it is observed that near the loading surface, variations are highest. Away from the loading surface, small variations are observed minimum. Also oscillatory trend is observed for t = 0.01 and amplitude of oscillation is decreasing as t increases. The result of the problem is useful in the twodimensional problem of dynamic response due to various sources of thermodiffusion which has various geophysical and industrial applications.

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## Appendix 1

$$\begin{split} \varsigma_{21} &= \frac{\tau_q^0}{\tau_t^0} ks, \ \varsigma_{22} = \frac{k a T_0 \tau_q^0}{\rho C_E \beta_2 \tau_t^0}, \ \varsigma_{23} = \frac{k \beta_1^2 T_0 \tau_q^0}{\rho^2 C_E c_1^2 \tau_t^0} s, \\ \varsigma_{31} &= \frac{D a \rho c_1^2}{\beta_1}, \ \varsigma_{32} = D b \frac{\rho c_1^2}{\beta_2}, \\ \varsigma_{33} &= \frac{s \tau_\eta^0 k c_1^2}{\tau_p^0 \beta_2 C_E}, \\ \varsigma_{34} &= D \beta_2 \\ \tau_q^0 &= 1 + s \tau_q + \frac{s^2 \tau_q^2}{2}, \ \tau_\eta^0 = 1 + s \tau_\eta + \frac{s^2 \tau_\eta^2}{2}, \ \tau_p^0 = 1 + s \tau_p, \ \tau_t^0 = 1 + s \tau_t, \end{split}$$

Appendix 2

$$d_{i} = \frac{(\varsigma_{34}q_{i}^{4} - 6)\varsigma_{21}\varsigma_{34}q_{i}^{2} + \varsigma_{23}\varsigma_{31}}{-q_{i}^{2}(\varsigma_{22}\varsigma_{34} + \varsigma_{23}\varsigma_{32}) + \varsigma_{23}\varsigma_{33}},$$
  

$$f_{i} = \frac{-\varsigma_{34}q_{i}^{4} + (\varsigma_{33}\varsigma_{21}\varsigma_{32})q_{i}^{2} - \varsigma_{21}\varsigma_{33}}{-q_{i}^{2}(\varsigma_{22}\varsigma_{34} + \varsigma_{23}\varsigma_{32}) + \varsigma_{23}\varsigma_{33}},$$
  

$$q_{i} = \sqrt{\varsigma^{2} + k_{i}^{2}}, \quad i = 1, 2, 3$$

Appendix 3

$$\begin{split} \lambda_{i} &= (\frac{\lambda + \mu}{\rho c_{1}^{2}} f_{i} - 1 - d_{i})A_{i}, E(\varsigma s) = \frac{\varsigma^{2} E(\varsigma, s)}{q}, q = \sqrt{\varsigma^{2} + \frac{\rho c_{1}^{2}}{\mu} s^{2}}, R_{1} = \frac{\lambda_{i}}{(\frac{\mu q_{1}^{2}}{\rho c_{1}^{2}} - s^{2})}, R_{2} = E(\varsigma, s) \cosh(qz) \\ &+ \sum_{i=1}^{3} R_{1} \cosh(q_{i}z), R_{3} = E(\varsigma, s) \sinh(qz) + \sum_{i=1}^{3} R_{1} \sinh(q_{i}z), R_{4} = \sum_{i=1}^{3} \eta_{i} \cosh(q_{i}z)\varsigma, F(\varsigma, s) = \frac{\varsigma^{2}(E(\varsigma, s))}{q}, R_{5} = \left(E(\varsigma, s)q \cosh(qz) + \sum_{i=1}^{3} R_{1}q_{i}^{2} \cosh(q_{i}z)\right), \eta_{i} = (\frac{f_{i} - \rho c_{1}^{2} - \rho c_{1}^{2}d_{i}}{\beta_{i}T_{0}})A_{i}, \\ R_{6} &= \left(\left(\frac{q^{2} - \varsigma^{2}}{q}\right)E(\varsigma, s)q \cosh(qz) + 2\sum_{i=1}^{3} R_{1}q_{i} \sinh(q_{i}z)\right), V_{1} = \frac{2\mu}{\beta_{1}T_{0}}, \varsigma_{i} = \left(-\beta_{2}f_{i} - \frac{\rho c_{1}^{2}}{\beta_{1}} + \frac{b\rho c_{1}^{2}}{\beta_{2}}\right)A_{i}(\xi, s) \\ \end{split}$$

#### Appendix 4

$\Delta =$	$\Delta_{11}$	$\Delta_{12}$	$\Delta_{13}$	$\Delta_{14}$
	$\Delta_{21}$	$\Delta_{22}$	$\Delta_{23}$	$\Delta_{24}$
	$\Delta_{31}$	$\Delta_{32}$	$\Delta_{33}$	$\Delta_{34}$
	$\Delta_{41}$	$\Delta_{42}$	$\Delta_{43}$	$\Delta_{44}$

where  $\Delta_{1i} = q_i \sinh(q_i b)$ ,  $\Delta_{14} = 0\Delta_{2i} = (\mu_i q_i^2 + \eta_i) \cosh(q_i b)$ ,  $\Delta_{24} = \frac{-2\mu}{\beta_1 T_0} \varsigma^2 q \cos(qb)$ ,  $\Delta_{3i} = (\mu_i q_i \sinh(q_i b), \Delta_{34} = \frac{q^2 - \varsigma^2}{q} \sinh(qb)$ ,  $\Delta_{4i} = \varsigma_i \cosh(q_i b)$ ,  $\Delta_{44} = 0$ , i = 1, 2, 3 and  $\Delta_i (i = 1, 2, 3, 4)$  are obtained from  $\Delta$ , by replacing *i*th column with  $\begin{bmatrix} g_0 F(\varsigma, b) & 0 & 0 & \frac{1}{\pi c_{\varsigma} \sqrt{\varsigma^2 + \frac{s^2}{c^2} - \frac{\varsigma}{c}} \end{bmatrix}^{t_p}$  where  $t_p$  denotes transpose.



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