Jurnal Matematika MANTIK
Vol. 8, No. 1, June 2022, pp. 36-44

ISSN: 2527-3159 (*print*) 2527-3167 (*online*)

Relation of Morrey Sequence Spaces, Weak Type Morrey Sequence Spaces, and Sequence Spaces

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Article history:
Received Apr 26, 2022
Revised, Jun 5, 2022
Accepted, Jun 28, 2022

Kata Kunci:
Ruang Barisan,
Ruang Morrey
Ruang Morrey Tipe Lemah

Abstrak. Ruang Morrey merupakan ruang yang cukup penting dan banyak dikaji dalam banyak cabang matematika. Ruang Morrey tipe lemah merupakan ruang berorde semu dan memiliki sifat dasar yang sama dengan ruang barisan Morrey.Pada artikel ini diselidiki sifat elementer antara ruang barisan Morrey dan ruang barisan Morrey tipe lemah. Selanjutnya ditunjukkan hubungannya antara ruang barisan Morrey dan ruang barisan Morrey tipe lemah dengan ruang barisan. Berdasarkan hasil pembahasan diperoleh sifat elementer ruang barisan Morrey $\ell^p \subset \ell_q^p$ dengan $1 \le p \le q < \infty$ dan $\ell_q^{p_2} \subseteq \ell_q^{p_1}$ dengan $1 \le p_1 \le p_2 \le q < \infty$. Sifat elementer dari ruang barisan Morrey tipe lemah adalah $\ell_q^p \subseteq \omega\ell_q^p$ dengan $1 \le p \le q < \infty$, $\omega\ell_q^{p_2} \subseteq \omega\ell_q^{p_1}$ dengan $1 \le p_1 \le p_2 \le q < \infty$ dan ruang barisan Morrey tipe lemah merupakan ruang quasi norma. Lebih lanjut hubungan ruang barisan Morrey, ruang barisan Morrey lemah dan ruang barisan adalah $\ell^p \subset \ell_q^p \subseteq \omega\ell_q^p$.

Keywords:
Sequence Spaces,
Morrey Spaces,
Weak Type Morrey Spaces,

Abstract. Morrey space is a space that important spaces and widely studied in many branches of mathematics. Weak type Morrey spaces is a quasinormed spaces and have alike elementary properties with Morrey sequence spaces. This article investigates some properties the elementary properties of Morrey sequence spaces and weak type Morrey sequence space. Next is show relation Morrey sequence spaces and weak type Morrey sequence space with sequence spaces. Based on the results of the discussion, it was obtained that the elementary properties of Morrey sequence spaces is $\ell^p \subset \ell_q^p$ with $1 \le p \le q < \infty$ and $\ell_q^{p_2} \subseteq \ell_q^{p_1}$ with $1 \le p_1 \le p_2 \le q < \infty$. The elementary properties of weak type Morrey sequence spaces $\omega\ell_q^p$ is $\ell_q^p \subseteq \omega\ell_q^p$ with $1 \le p \le q < \infty$, $\omega\ell_q^{p_2} \subseteq \omega\ell_q^{p_1}$ with $1 \le p_1 \le p_2 \le q < \infty$ and weak type Morrey sequence spaces is quasinormed space. Furthermore, the relation of Morrey sequence spaces, weak type Morrey sequence spaces and sequence spaces is $\ell^p \subset \ell_q^p \subseteq \omega\ell_q^p$

How to cite:
P. A. H. Puadi, Eridani, & A. Jaelani, "*Relation of Morrey Sequence Spaces, Weak Type Morrey Sequence Spaces, and Sequence Spaces*". J. Mat Mantik, vol. 8, no. 1, pp. 36-44, Jun. 2022.

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The article can be accessed here. *https://doi.org/10.15642/mantik.2022.8.1.36-44*

1. Introduction

Morrey spaces were first introduced by Charles Bardfield Morrey Jr. (1907-1984) on 1938. Morrey spaces became an important space in many branches of mathematics even though it was first discovered to solve partial differential equations, now there are hundreds of articles and journals that discuss Morrey space to take up the recent development of Morrey Spaces [1], [2].

On 2016, it is defined that Morrey sequence space is denoted by ℓ_q^p with $1 \leq p \leq q < \infty$ is a set of all real sequence $x = \langle x_k \rangle_{k \in \mathbb{Z}}$ that holds

$$\|x\|_{\ell_q^p} = \sup_N |S_N|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k \in S_N} |x_k|^p \right)^{1/p} < \infty$$

with $N \in \mathbb{N}$, $S_N = \{-N, -(N-1), \dots, 0, \dots, N-1, N\}$ and $|S_N|$ denote the cardinality of S_N [3], [4]. Sequence spaces denote by $\ell^p = \ell_p^p$ with $p = q$ is a set of all sequences that holds

$\sum_{k \in \mathbb{Z}} |x_k|^p < \infty$ and p as parameter [5]–[7] .

Morrey sequence space ℓ_q^p has a very close relation with sequence space ℓ^p. One of the Morrey sequence space's elementary properties is $\|x\|_{\ell_q^p} \leq \|x\|_{\ell^p}$

for all $x \in \ell^p$, hence $\ell^p \subseteq \ell_q^p$ for $1 \leq p \leq q < \infty$. In other words, Morrey sequence space ℓ_q^p is an extension of sequence space ℓ^p. Morrey sequence space ℓ_q^p can be more extend to weak type Morrey sequence space $\omega\ell_q^p$ [8]–[10]. Weak type Morrey sequence spaces is a set of all real sequence $x = \langle x_k \rangle_{k \in \mathbb{Z}}$ that holds

$$\|x\|_{\omega\, \ell_q^p} = \sup_{N \in \mathbb{N}, \gamma > 0} |S_N|^{\frac{1}{q}-\frac{1}{p}} \gamma |\{k \in S_N : |x_k| > \gamma\}|^{1/p} < \infty.$$

Weak type Morrey sequence spaces is a quasinormed spaces and have alike elementary properties with Morrey sequence spaces. Based on the description above, we will discuss about Morrey sequence spaces' elementary properties and weak type Morrey sequence spaces and also the relation between sequence space ℓ^{p}, Morrey sequence space ℓ_q^p dan weak type Morrey sequence space $\omega\ell_q^p$.

Definition [11]
Quasinorm $\|\cdot\|$ on a vector space V over a field $\mathbb{R}$ is a function from V to $[0, \infty)$ and holds
- (i) $\|u\| = 0$ if and only if $u = 0$
- (ii) $\|ru\| = |r| \|u\|$ for every $r \in \mathbb{R}$ and $u \in V$
- (iii) There exists $C \geq 1$ so that if $u, v \in V$ then $\|u + v\| \leq C(\|u\| + \|v\|)$

if $\| \quad \|$ is a quasinorm and $(V, \|\cdot\|)$ is a quasinormed space.

Theorem [12] (Minkowski inequality)
If $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in \mathbb{R}$ and $p \geq 1$ then
$(\sum_{k=1}^n |x_k + y_k|^p)^{1/p} \leq (\sum_{k=1}^n |x_k|^p)^{1/p} + (\sum_{k=1}^n |y_k|^p)^{1/p}$.

Theorem [12]
If $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in \mathbb{R}$ and $0 < p < 1$ then
$\sum_{k=1}^n |x_k + y_k|^p \leq \sum_{k=1}^n |x_k|^p + \sum_{k=1}^n |y_k|^p$.

Theorem [12] (Hölder inequality)
If $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in \mathbb{R}$ and p, q is an exponent conjugation then
$\sum_{k=1}^n |x_k y_k| \leq (\sum_{k=1}^n |x_k|^p)^{1/p} (\sum_{k=1}^n |y_k|^q)^{1/q}$.

Definition [12]
Suppose that $p \in (0, \infty)$. ℓ^p is a set of all sequence $x : \mathbb{N} \to \mathbb{R}$ that holds $\sum_{i=1}^{\infty} |x_k|^p$ convergent or

$$\ell^p = \{x = \langle x_n \rangle : \sum_{k=1}^{\infty} |x_k|^p < \infty\}.$$

But on this article, ℓ^p sequence space is a set of all sequence $x : \mathbb{Z} \to \mathbb{R}$ that holds

$$\sum_{k \in \mathbb{Z}} |x_k|^p = \sum_{-\infty}^{\infty} |x_k|^p < \infty.$$

Definition [3]
Suppose that $1 \le p \le q < \infty$, $N \in \mathbb{N}$, $S_N = \{-N, -(N-1), \dots, 0, \dots, N-1, N\}$ and $|S_N| = 2N + 1$ denote the cardinality of S_N. Let ℓ_q^p denote Morrey sequence space is a set of all sequence $\langle x_k \rangle_{k \in \mathbb{Z}}$ that holds

$$\|x\|_{\ell_q^p} = \sup_N |S_N|^{\frac{1}{q} - \frac{1}{p}} \left(\sum_{k \in S_N} |x_k|^p\right)^{1/p} < \infty.$$

Definition [3]
Suppose that $1 \le p \le q < \infty$. Let $\omega\ell_q^p$ denote weak type Morrey sequence space is a set of all sequence $\langle x_k \rangle_{k \in \mathbb{Z}}$ that holds

$$\|x\|_{\omega\, \ell_q^p} = \sup_{N \in \mathbb{N}, \gamma > 0} |S_N|^{\frac{1}{q} - \frac{1}{p}} \gamma |\{k \in S_N : |x_k| > \gamma\}|^{1/p} < \infty.$$

2. Methods

We used the properties of definition quasinorm, Minkowski Inequality, Hölder inequality, and characteristics of norm on the set of real [4], [13]–[15] to obtain properties of Morrey sequence spaces, weak type Morrey sequence space, and relation Morrey sequence spaces and weak type Morrey sequence space with sequence spaces.

3. Results and Discussion

Proposition
Morrey Sequence Space ℓ_q^p is a vector space over $\mathbb{R}$.
Proof :
Suppose that $x = \langle x_k \rangle_{k \in \mathbb{Z}}$, $y = \langle y_k \rangle_{k \in \mathbb{Z}}$ is an element of ℓ_q^p.
For every $a \in \mathbb{R}$,

$$\begin{aligned} \|ax\|_{\ell_q^p} &= \sup_N |S_N|^{\frac{1}{q} - \frac{1}{p}} \left(\sum_{k \in S_N} |a x_k|^p\right)^{1/p} \\ &= |a| \sup_N |S_N|^{\frac{1}{q} - \frac{1}{p}} \left(\sum_{k \in S_N} |x_k|^p\right)^{1/p} \\ &= |a| \|x\|_{\ell_q^p} < \infty \end{aligned}$$

Then $ax \in \ell_q^p$.
With Minkowski Inequality we got

$$\begin{aligned} \left(\sum_{k \in S_N} |x_k + y_k|^p\right)^{1/p} &\le \left(\sum_{k \in S_N} |x_k|^p\right)^{1/p} + \left(\sum_{k \in S_N} |y_k|^p\right)^{1/p} \\ &\le |S_N|^{\frac{1}{q} - \frac{1}{p}} \left(\sum_{k \in S_N} |x_k|^p\right)^{1/p} + |S_N|^{\frac{1}{q} - \frac{1}{p}} \left(\sum_{k \in S_N} |y_k|^p\right)^{1/p} \\ \|x + y\|_{\ell_q^p} &\le \|x\|_{\ell_q^p} + \|y\|_{\ell_q^p}. \end{aligned}$$

If $\|x\|_{\ell_q^p} < \infty$ and $\|y\|_{\ell_q^p} < \infty$, then $\|x + y\|_{\ell_q^p} < \infty$ so that $x + y \in \ell_q^p$. ℓ_q^p is closed under the operation (+).
Let $z \in \ell_q^p$ then

(i) $x + y = \langle x_k + y_k \rangle_{k \in \mathbb{Z}} = \langle y_k + x_k \rangle_{k \in \mathbb{Z}} = y + x$ for every $x, y \in \ell_q^p$

(ii) $x + (y + z) = (x + y) + z$ for every $x, y, z \in \ell_q^p$
(iii) There exists $0 = \langle \dots, 0,0,0, \dots . \rangle$, then $0 + x = x + 0$ for every $x \in \ell_q^p$
(iv) There exists $-x = \langle -x_k \rangle_{k\in\mathbb{Z}}$, then $(-x) + x = x + (-x) = 0$ for every $x \in \ell_q^p$
(v) Let $b \in \mathbb{R}$.

$$a(x + y) = ax + ay$$
$$(a + b)x = ax + bx$$
$$(ab)x = a(bx)$$

for every $x \in \ell_q^p$
(vi) There exists $1 \in \mathbb{R}$ so that $1x = x$ for every $x \in \ell_q^p$

thus ℓ_q^p is a vector space over $\mathbb{R}$.∎

Example:
Suppose that $1 \le p < q < \infty$. A sequence $x = \langle x_k \rangle_{k\in\mathbb{Z}}$ with $x_k = |k|^{-q/p}$ for $k \neq 0$ and $x_k = 0$ for $k = 0$.

For every $N \in \mathbb{N}$,

$$\sum_{k \in S_N} |x_k|^p = 2\sum_{k=1}^{N} \frac{1}{k^q} < 2\sum_{k=1}^{\infty} \frac{1}{k^q}. \tag{1}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^q} = 1 + \left(\frac{1}{2^q} + \frac{1}{3^q}\right) + \left(\frac{1}{4^q} + \frac{1}{5^q} + \frac{1}{6^q} + \frac{1}{7^q}\right) + \cdots$$
$$< 1 + \frac{2}{2^q} + \frac{4}{4^q} + \frac{8}{8^q} + \cdots = 1 + \frac{1}{2^{q-1}} + \frac{1}{4^{q-1}} + \frac{1}{8^{q-1}} + \cdots$$

If $q > 1$ then $0 < \frac{1}{2^{q-1}} < 1$ implies that

$$\sum_{k=1}^{\infty} \frac{1}{k^q} < \frac{1}{1 - \frac{1}{2^{q-1}}}. \tag{2}$$

From (1) and (2) obtained

$$|S_N|^{\frac{p}{q}-1} \sum_{k \in S_N} |x_k|^p = \left(\frac{1}{|S_N|^{1-\frac{p}{q}}}\right) \sum_{k \in S_N} |x_k|^p < \sum_{k \in S_N} |x_k|^p < \frac{2}{1 - \frac{1}{2^{q-1}}}$$

$$\left(|S_N|^{\frac{p}{q}-1} \sum_{k \in S_N} |x_k|^p\right)^{1/p} < \left(\frac{2^q}{2^{q-1}-1}\right)^{1/q}$$

$$\sup_N |S_N|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k \in S_N} |x_k|^p\right)^{1/p} \le \left(\frac{2^q}{2^{q-1}-1}\right)^{1/q}.$$

Thus $x \in \ell_q^p$. ∎

Theorem
Suppose that $1 \le p \le q < \infty$. $\|x\|_{\ell_q^p} \le \|x\|_{\ell^p}$ for every $x \in \ell^p$.

Proof :

For all $N \in \mathbb{N}$, it's easy to see that $0 < |S_N|^{\frac{1}{q}-\frac{1}{p}} \le 1$ then

$$|S_N|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k \in S_N} |x_k|^p\right)^{1/p} \le \left(\sum_{k \in S_N} |x_k|^p\right)^{1/p}$$

and

$$\sup_N |S_N|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k \in S_N} |x_k|^p\right)^{1/p} \le \sup_N \left(\sum_{k \in S_N} |x_k|^p\right)^{1/p}$$

$$\|x\|_{\ell_q^p} \le \|x\|_{\ell^p}. \blacksquare$$

Example :
Let $1 \le p < q < \infty$ and a sequence $x = \langle x_k \rangle_{k\in\mathbb{Z}}$ with $x_k = |k|^{-1/q}$ for $k \ne 0$ and $x_k = 0$ for $k = 0$.
If $0 < \frac{p}{q} < 1$ then

$$\sum_{k\in\mathbb{Z}} |x_k|^p = 2\sum_{k=1}^{\infty} \frac{1}{k^{p/q}} \ge 2\sum_{k=1}^{\infty} \frac{1}{k}.$$

We know that $\sum_{k=1}^{\infty} \frac{1}{k}$ is harmonic series then it's divergent and $\sum_{k\in S_N} |x_k|^p$ divergent thus $x \notin \ell^p$.
For all $N \in \mathbb{N}$, $\sum_{k\in S_N} |x_k|^p = \sum_{k\in S_N, k\ne 0} \frac{1}{|k|^{p/q}} = 2\sum_{k=1}^{N} \frac{1}{k^{p/q}}$.
For every p and q with $1 \le p < q < \infty$, function $y = x^{-p/q}$ has $y' < 0$ and $y'' > 0$ on $x \ge 1$.

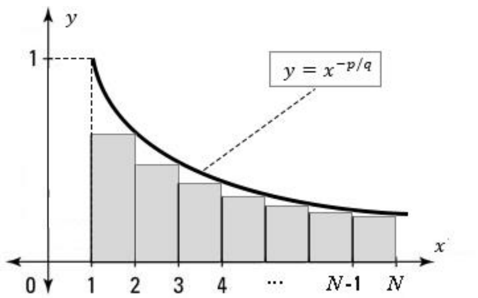

Figure 1. Area of partition below $y = x^{-p/q}$

We know that the sum of areas of partitions equal to

$$\sum_{k=2}^{N} \frac{1}{k^{p/q}}$$

And

$$\sum_{k=2}^{N} \frac{1}{k^{p/q}} \le \sum_{k=1}^{N} \frac{1}{k^{p/q}} = 1 + \sum_{k=2}^{N} \frac{1}{k^{p/q}} \le 1 + \int_1^N \frac{1}{x^{p/q}}\, dx$$
$$\le 1 - \frac{q}{q-p} + \frac{q}{q-p} N^{1-\frac{p}{q}}.$$

Hence,

$$|S_N|^{\frac{p}{q}-1} \sum_{k\in S_N} |x_k|^p \le 2\left(\frac{1-\frac{q}{q-p}+\frac{q}{q-p}N^{1-\frac{p}{q}}}{(2N+1)^{1-\frac{p}{q}}}\right)$$

and

$$\lim_{N\to\infty} 2\left(\frac{1-\frac{q}{q-p}+\frac{q}{q-p}N^{1-\frac{p}{q}}}{(2N+1)^{1-\frac{p}{q}}}\right) = 2\left(\frac{q}{q-p} 2^{\frac{p}{q}-1}\right) = \frac{q}{q-p} 2^{\frac{p}{q}}$$

$$|S_N|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_N} |x_k|^p\right)^{1/p} \le \left(2\left(\frac{1-\frac{q}{q-p}+\frac{q}{q-p}N^{1-\frac{p}{q}}}{(2N+1)^{1-\frac{p}{q}}}\right)\right)^{1/p}$$

$$\sup_N |S_N|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_N} |x_k|^p\right)^{1/p} \le 2^{1/p} \left(\frac{q}{q-p}\right)^{1/p}$$

thus $x \in \ell_q^p$.■

Theorem
Suppose that $1 \le p_1 \le p_2 \le q < \infty$. For all $x \in \ell_q^{p_2}$ applies that

$$\|x\|_{\ell_q^{p_1}} \leq \|x\|_{\ell_q^{p_2}}.$$

Proof :
By Hölder inequality, we have

$$\sum_{k\in S_N}|x_k|^{p_1} \leq \left(\sum_{k\in S_N}|x_k|^{p_2}\right)^{\frac{p_1}{p_2}}\left(\sum_{k\in S_N} 1\right)^{1-\frac{p_1}{p_2}}$$

$$\left(\frac{1}{|S_N|}\sum_{k\in S_N}|x_k|^{p_1}\right)^{\frac{1}{p_1}} \leq \left(\frac{1}{|S_N|}\sum_{k\in S_N}|x_k|^{p_2}\right)^{\frac{1}{p_2}}$$

$$|S_N|^{\frac{1}{q}-\frac{1}{p^1}}\left(\sum_{k\in S_N}|x_k|^{p_1}\right)^{\frac{1}{p_1}} \leq |S_N|^{\frac{1}{q}-\frac{1}{p^2}}\left(\sum_{k\in S_N}|x_k|^{p_2}\right)^{\frac{1}{p_2}}$$

$$\|x\|_{\ell_q^{p_1}} \leq \|x\|_{\ell_q^{p_2}}. \blacksquare$$

From the above theorem, we have $\ell_q^{p_2} \subseteq \ell_q^{p_1}$on $1 \leq p_1 \leq p_2 \leq q < \infty$ but we can't approve $\ell_q^{p_2} \subset \ell_q^{p_1}$ yet [3].

Theorem
If $1 \leq p \leq q < \infty$ then $\|x\|_{\omega\ell_q^p} \leq \|x\|_{\ell_q^p}$
For every $x \in \ell_q^p$.
Proof:

$$|S_N|^{\frac{1}{q}-\frac{1}{p}}\gamma|\{k \in S_N :\ |x_k| > \gamma\}|^{1/p} = |S_N|^{\frac{1}{q}-\frac{1}{p}}|\gamma^p\{k \in S_N :\ |x_k| > \gamma\}|^{1/p}$$

$$= |S_N|^{\frac{1}{q}-\frac{1}{p}}\left(\sum_{k\in S_N,|x_k|>\gamma} \gamma^p\right)^{1/p}$$

$$\leq |S_N|^{\frac{1}{q}-\frac{1}{p}}\left(\sum_{k\in S_N,|x_k|>\gamma} |x_k|^p\right)^{1/p}$$

$$|S_N|^{\frac{1}{q}-\frac{1}{p}}\gamma|\{k \in S_N :\ |x_k| > \gamma\}|^{1/p} \leq |S_N|^{\frac{1}{q}-\frac{1}{p}}\left(\sum_{k\in S_N} |x_k|^p\right)^{1/p}$$

supremum over $N \in \mathbb{N}$ and $\gamma > 0$ on the above inequality, we have

$$\sup_{N\in\mathbb{N},\gamma>0} |S_N|^{\frac{1}{q}-\frac{1}{p}}\gamma|\{k \in S_N :\ |x_k| > \gamma\}|^{1/p} \leq \sup_N |S_N|^{\frac{1}{q}-\frac{1}{p}}\left(\sum_{k\in S_N}|x_k|^p\right)^{1/p}$$

$$\|x\|_{\omega\ell_q^p} \leq \|x\|_{\ell_q^p}.$$

Thus $\ell_q^p \subseteq \omega\ell_q^p$ $\blacksquare$.

Example :
Suppose that $1 \leq p \leq q < \infty$. A sequence $x = \langle x_k\rangle_{k\in\mathbb{Z}}$ with $x_k = |k|^{-1/p}$ for $k \neq 0$ and $x_k = 0$ for $k = 0$.

$$\sum_{k\,\in\,\mathbb{Z}}|x_k|^p = 2\sum_{k=1}^{\infty}\frac{1}{k}.$$

We know that $\sum_{k=1}^{\infty}\frac{1}{k}$ is divergent then $x \notin \ell^p$ but for any $\gamma > 0$,

$$\gamma\left|\{k \in S_N :\ |k|^{-1/p} > \gamma\}\right|^{1/p} = 2\gamma\left|\{k \in \mathbb{N} : 1 \leq k < N, k^{-1/p} > \gamma\}\right|^{1/p}$$

and $k^{-1/p} > \gamma \Longrightarrow k^{-1} > \gamma^p \Longrightarrow k < \frac{1}{\gamma^p}$ then

$$\gamma\left|\{k \in S_N :\ |k|^{-1/p} > \gamma\}\right|^{1/p} < 2\gamma\left(\frac{1}{\gamma^p}\right)^{1/p}$$

$$\gamma\left|\{k \in S_N :\ |k|^{-1/p} > \gamma\}\right|^{1/p} < 2\gamma\left(\frac{1}{\gamma}\right) = 2$$

then $x \in \omega\ell_p^p$ and $\ell^p \subset \omega\ell_p^p = \omega\ell^p$.

Theorem
If $1 \leq p \leq q < \infty$ then $\|\ \|_{\omega\ell_q^p}$ is a quasinorm and $\left(\omega\ell_q^p, \|\ \|_{\omega\ell_q^p}\right)$ is a quasinormed space.

Proof:
From the definition, we know that $\|x\|_{\omega\ell_q^p} \geq 0$ for all $x \in \omega\ell_q^p$.
If $x = 0$ ($x_k = 0$ for all $k \in \mathbb{Z}$) then $\{k \in S_N : |x_k| > \gamma\}$ is an empty space and $|\{k \in S_N : |x_k| > \gamma\}| = 0$,

$$|S_N|^{\frac{1}{q}-\frac{1}{p}}\gamma|\{k \in S_N : |x_k| > \gamma\}|^{1/p} = 0$$

$$\sup_{N\in\mathbb{N},\gamma>0} |S_N|^{\frac{1}{q}-\frac{1}{p}}\gamma|\{k \in S_N : |x_k| > \gamma\}|^{1/p} = 0$$

$$\|x\|_{\omega\ell_q^p} = 0.$$

If $\|x\|_{\omega\ell_q^p} = 0$ then $|\{k \in S_N : |x_k| > \gamma\}| = 0$ for all $N \in \mathbb{N}$ and $\gamma > 0$, hence for every $k \in \mathbb{Z}$ applies that

$$0 \leq |x_k| \leq \gamma$$

$\Longrightarrow x_k = 0$ untuk setiap $k \in \mathbb{Z}$ ($x = 0$).
Let $x \in \omega\ell_q^p$ and $r = 0$ then $\|rx\|_{\omega\ell_q^p} = |r|\|x\|_{\omega\ell_q^p} = 0$. If $r \neq 0$ then applies that

$$\|rx\|_{\omega\ell_q^p} = \sup_{N\in\mathbb{N},\gamma>0} |S_N|^{\frac{1}{q}-\frac{1}{p}}\gamma|\{k \in S_N : |rx_k| > \gamma\}|^{1/p}$$

$$= \sup_{N\in\mathbb{N},\gamma>0} |S_N|^{\frac{1}{q}-\frac{1}{p}}\gamma\left|\left\{k \in S_N : |x_k| > \frac{\gamma}{|r|}\right\}\right|^{1/p}$$

let $\frac{\gamma}{|r|} = a \Longrightarrow \gamma = |r|a$ then

$$\|rx\|_{\omega\ell_q^p} = \sup_{N\in\mathbb{N},a>0} |S_N|^{\frac{1}{q}-\frac{1}{p}}|r|a|\{k \in S_N : |x_k| > a\}|^{1/p}$$

$$= |r| \sup_{N\in\mathbb{N},a>0} |S_N|^{\frac{1}{q}-\frac{1}{p}}a|\{k \in S_N : |x_k| > a\}|^{1/p}$$

$$= |r|\|x\|_{\omega\ell_q^p}.$$

Let $x, y \in \omega\ell_q^p$. For any $N \in \mathbb{N}$, if $k \in S_N$ dan $|x_k| \leq |y_k|$ then

$$|x_k + y_k| \leq |x_k| + |y_k| \leq 2|y_k|$$

else if $k \in S_N$ dan $|x_k| > |y_k|$ then

$$|x_k + y_k| \leq |x_k| + |y_k| \leq 2|x_k|.$$

Thus

$$\{k \in S_N : |x_k + y_k| > \gamma\} \subseteq \{k \in S_N : |x_k| + |y_k| > \gamma\}$$
$$\subseteq \{k \in S_N : 2|x_k| > \gamma\} \cup \{k \in S_N : 2|y_k| > \gamma\}.$$

For any $\gamma > 0$ dan $N \in \mathbb{N}$,

$$|\{k \in S_N : |x_k + y_k| > \gamma\}| \leq |\{k \in S_N : 2|x_k| > \gamma\}| + |\{k \in S_N : 2|y_k| > \gamma\}|$$

Multiply both sides by $\left(|S_N|^{\frac{1}{q}-\frac{1}{p}}\gamma\right)^p$ applies that

$$\left(|S_N|^{\frac{1}{q}-\frac{1}{p}}\gamma\right)^p |\{k \in S_N : |x_k + y_k| > \gamma\}|$$
$$\leq \left(|S_N|^{\frac{1}{q}-\frac{1}{p}}\gamma\right)^p \left|\left\{k \in S_N : |x_k| > \frac{\gamma}{2}\right\}\right| + \left(|S_N|^{\frac{1}{q}-\frac{1}{p}}\gamma\right)^p \left|\left\{k \in S_N : |y_k| > \frac{\gamma}{2}\right\}\right|$$

Let $\frac{\gamma}{2} = \sigma \Longrightarrow \gamma = 2\sigma$

$$|S_N|^{\frac{1}{q}-\frac{1}{p}}\gamma|\{k \in S_N : |x_k + y_k| > \gamma\}|^{1/p}$$
$$\leq 2\left(\left(|S_N|^{\frac{1}{q}-\frac{1}{p}}\sigma\right)^p |\{k \in S_N: |x_k| > \sigma\}| + \left(|S_N|^{\frac{1}{q}-\frac{1}{p}}\sigma\right)^p |\{k \in S_N: |y_k| > \sigma\}|\right)^{1/p}.$$

If $p > 1 \Longrightarrow \frac{1}{p} < 1$then

$$|S_N|^{\frac{1}{q}-\frac{1}{p}}\gamma|\{k \in S_N : |x_k + y_k| > \gamma\}|^{1/p}$$

$$\leq 2\left(\left(|S_N|^{\frac{1}{q}-\frac{1}{p}}\sigma\right)^p |\{k\in S_N: |x_k|>\sigma\}| + \left(|S_N|^{\frac{1}{q}-\frac{1}{p}}\sigma\right)^p |\{k\in S_N: |y_k|>\sigma\}|\right)^{1/p}$$
$$\leq 2\left(|S_N|^{\frac{1}{q}-\frac{1}{p}}\sigma|\{k\in S_N: |x_k|>\sigma\}|\right)^{1/p} + 2\left(|S_N|^{\frac{1}{q}-\frac{1}{p}}\sigma|\{k\in S_N: |y_k|>\sigma\}|\right)^{1/p}$$

taking supremum over $\gamma > 0$ dan $N\in\mathbb{N}$ from the above inequality, we get

$$\|x+y\|_{\omega\ell_q^p} \leq 2\|x\|_{\omega\ell_q^p} + 2\|y\|_{\omega\ell_q^p} = 2\left(\|x\|_{\omega\ell_q^p} + \|y\|_{\omega\ell_q^p}\right).$$

Thus $\|\ \ \|_{\omega\ell_q^p}$ is a quasinorm and $\left(\omega\ell_q^p, \|\ \ \|_{\omega\ell_q^p}\right)$ is a quasinormed space.■

Theorem
If $1\leq p_1 \leq p_2 \leq q$ then $\|x\|_{\omega\ell_q^{p_1}} \leq \|x\|_{\omega\ell_q^{p_2}}$
for all $x\in\omega\ell_q^{p_2}$.
Proof :
From the definition, for all $x\in\|x\|_{\omega\ell_q^{p_2}}$ and any $N\in\mathbb{N}$.

$$|S_N|^{\frac{1}{q}-\frac{1}{p_2}}\gamma|\{k\in S_N : |x_k|>\gamma\}|^{1/p_2} \leq \sup_{N\in\mathbb{N},\gamma>0} |S_N|^{\frac{1}{q}-\frac{1}{p_2}}\gamma|\{k\in S_N: |x_k|>\gamma\}|^{1/p_2} = \|x\|_{\omega\ell_q^{p_2}}.$$

And it is equivalent to

$$\gamma \leq \frac{|S_N|^{\frac{1}{p_2}-\frac{1}{q}}}{|\{k\in S_N: |x_k|>\gamma\}|^{1/p_2}}\|x\|_{\omega\ell_q^{p_2}}$$
$$|S_N|^{\frac{1}{q}-\frac{1}{p_1}}\gamma|\{k\in S_N : |x_k|>\gamma\}|^{1/p_1} \leq \frac{|S_N|^{\frac{1}{p_2}-\frac{1}{p_1}}}{|\{k\in S_N: |x_k|>\gamma\}|^{\frac{1}{p_2}-\frac{1}{p_1}}}\|x\|_{\omega\ell_q^{p_2}}$$
$$|S_N|^{\frac{1}{q}-\frac{1}{p_1}}\gamma|\{k\in S_N : |x_k|>\gamma\}|^{1/p_1} \leq \left(\frac{|\{k\in S_N: |x_k|>\gamma\}|}{|S_N|}\right)^{\frac{1}{p_1}-\frac{1}{p_2}}\|x\|_{\omega\ell_q^{p_2}} \leq \|x\|_{\omega\ell_q^{p_2}}$$

taking supremum over $N\in\mathbb{N}$ and $\gamma>0$ on the above inequality, we get

$$\|x\|_{\omega\ell_q^{p_1}} \leq \|x\|_{\omega\ell_q^{p_2}}.\ ■$$

From the above inequality we have $\omega\ell_q^{p_2} \subseteq \omega\ell_q^{p_1}$.

4. Conclusions

Based on the results and discussion, we obtained some conclusions:

1) There are some Morrey sequence space's elementary properties :
   i. If $1\leq p\leq q<\infty$ then for all $x\in\ell^p$, we have $\|x\|_{\ell_q^p} \leq \|x\|_{\ell^p}$ and $\ell^p \subset \ell_q^p$.
   ii. If $1\leq p_1\leq p_2\leq q<\infty$ then for all $x\in\ell_q^{p_2}$, we have $\|x\|_{\ell_q^{p_1}} \leq \|x\|_{\ell_q^{p_2}}$ and $\ell_q^{p_2}\subseteq\ell_q^{p_1}$.
2) There are some weak type Morrey sequence space's elementary properties :
   i. If $1\leq p\leq q<\infty$ then for all $x\in\ell_q^p$, we have $\|x\|_{\omega\ell_q^p} \leq \|x\|_{\ell_q^p}$ and $\ell_q^p\subseteq\omega\ell_q^p$.
   ii. If $1\leq p\leq q<\infty$ then $\|\ \ \|_{\omega\ell_q^p}$ is a quasinorm and $\left(\omega\ell_q^p, \|\ \ \|_{\omega\ell_q^p}\right)$ is a quasinormed space.
   iii. If $1\leq p_1\leq p_2\leq q<\infty$ then for all $x\in\omega\ell_q^{p_2}$, we have $\|x\|_{\omega\ell_q^{p_1}} \leq \|x\|_{\omega\ell_q^{p_2}}$ and $\omega\ell_q^{p_2}\subseteq\omega\ell_q^{p_1}$.

3) Based on the results in points 1 and 2 it can be concluded that ℓ^p sequence space is a subset of Morrey Sequence Space and Morrey Sequence Space is a subset of weak type Morrey Sequence Space or equivalent to $\ell^p \subset \ell_q^p \subseteq \omega\ell_q^p$.

References

[1] Y. Sawano, H. Gunawan, V. Guliyev, and H. Tanaka, "Morrey spaces and related function spaces," *Journal of Function Spaces*, vol. 2014. pp. 1–2, 2014, doi: 10.1155/2014/867192.
[2] V. Kokilashvili, A. Meskhi, and H. Rafeiro, "Sublinear operators in generalized weighted Morrey spaces," *Dokl. Math.*, vol. 94, no. 2, pp. 558–560, 2016, doi: 10.1134/S1064562416050203.
[3] H. Gunawan, E. Kikianty, and C. Schwanke, "Discrete Morrey spaces and their inclusion properties," *Math. Nachrichten*, vol. 291, no. 8–9, pp. 1283–1296, 2018, doi: 10.1002/mana.201700054.
[4] S. Fatimah, A. A. Masta, S. Al Hazmy, C. Kustiawan, and I. Rukmana, "Discrete Orlicz-Morrey spaces and their inclusion properties," *J. Eng. Sci. Technol.*, vol. 16, no. 3, pp. 2018–2027, 2021.
[5] H. Roopaei, "Norm of Hilbert operator on sequence spaces," *J. Inequalities Appl.*, vol. 2020, no. 117, pp. 1–13, 2020, doi: 10.1186/s13660-020-02380-2.
[6] D. Carando, M. Mazzitelli, and P. Sevilla-Peris, "A Note on the Symmetry of Sequence Spaces," *Math. Notes*, vol. 110, no. 1–2, pp. 26–40, 2021, doi: 10.1134/S0001434621070038.
[7] K. Boruah and B. Hazarika, "On some generalized geometric difference sequence spaces," *Proyecciones*, vol. 36, no. 3, 2017, doi: 10.4067/S0716-09172017000300373.
[8] N. Tumalun and H. Gunawan, "Morrey Spaces are Embedded Between Weak Morrey Spaces and Stummel Classes," *J. Indones. Math. Soc.*, pp. 203–209, 2019, doi: 10.22342/jims.25.3.817.203-209.
[9] Y. Sawano and S. R. El-Shabrawy, "Weak Morrey spaces with applications," *Math. Nachrichten*, vol. 291, no. 1, pp. 178–186, 2018, doi: 10.1002/mana.201700001.
[10] E. Nakai and Y. Sawano, "Spaces of pointwise multipliers on morrey spaces and weak morrey spaces," *Mathematics*, vol. 9, no. 21, pp. 1–18, 2021, doi: 10.3390/math9212754.
[11] R. Haller and J. Langemets, "Geometry of banach spaces with an octahedral norm," *Acta Comment. Univ. Tartu. Math.*, vol. 18, no. 1, pp. 125–133, 2014, doi: 10.12697/ACUTM.2014.18.13.
[12] S. Ovchinnikov, *Functional Analysis: An Introductory Course*. Switzerland : Springer, 2018.
[13] H. Gunawan, D. I. Hakim, and A. S. Putri, "On Geometric Properties Of Morrey Spaces," *Ufa Math. J.*, vol. 13, no. 1, pp. 131–136, 2021, doi: 10.13108/2021-13-1-131.
[14] U. Kadak, "On multiplicative difference sequence spaces and related dual properties," *Bol. da Soc. Parana. Mat.*, vol. 35, no. 3, 2017, doi: 10.5269/bspm.v35i3.29182.
[15] A. A. Masta, S. Fatimah, and M. Taqiyuddin, "Third version of Weak Orlicz–Morrey spaces and its inclusion properties," *Indones. J. Sci. Technol.*, vol. 4, no. 2, pp. 257–262, 2019, doi: 10.17509/ijost.v4i2.18182.