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JORDAN DERIVATIONS ON A LIE IDEAL OF A SEMIPRIME RING AND THEIR APPLICATIONS IN BANACH ALGEBRAS

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Abstract. Let R be a 3!-torsion free noncommutative semiprime ring, U a Lie ideal of R, and let $D : R \to R$ be a Jordan derivation. If $[D(x), x]D(x) = 0$ for all $x \in U$, then $D(x)[D(x), x]y - yD(x)[D(x), x] = 0$ for all $x, y \in U$. And also, if $D(x)[D(x), x] = 0$ for all $x \in U$, then $[D(x), x]D(x)y - y[D(x), x]D(x) = 0$ for all $x, y \in U$. And we shall give their applications in Banach algebras.

1. Introduction

Throughout, R represents an associative ring and A will be a complex Banach algebra. We write $[x, y]$ for the commutator $xy - yx$ for x, y in a ring. Let rad(R) denote the (*Jacobson*) *radical* of a ring R. And a ring R is said to be *semisimple* if its Jacobson radical rad(R) is zero.

A ring R is called *n-torsion free* if $nx = 0$ implies $x = 0$. Recall that R is *prime* if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semiprime* if $aRa = (0)$ implies $a = 0$. On the other hand, let X be an element of a normed algebra. Then for every $a \in X$ the *spectral radius* of a, denoted by $r(a)$, is defined by $r(a) = \inf\{||a^n||^{\frac{1}{n}} : n \in \mathbb{N}\}$. It is well-known that the following theorem holds: if a is an element of a normed algebra, then $r(a) = \lim_{n\to\infty} ||a^n||^{\frac{1}{n}}$ (see F.F. Bonsall and J. Duncan[1]).

An additive mapping D from R to R is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And an additive mapping D from R to R is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$.

B.E. Johnson and A.M. Sinclair[12] proved that any linear derivation on a semisimple Banach algebra is continuous. A result of I.M. Singer and J. Wermer[13] states that every continuous linear derivation on a commutative Banach algebra maps the

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algebra into its radical. From these two results, we can conclude that there are no nonzero linear derivations on a commutative semisimple Banach algebra.
M.P. Thomas[14] has proved that any linear derivation on a commutative Banach algebra maps the algebra into its radical.

J. Vukman[15] proved the following: let R be a 2-torsion free prime ring. If $D : R \longrightarrow R$ is a derivation such that $[D(x), x]D(x) = 0$ for all $x \in R$, then $D = 0$.

Moreover, using the above result, he proved that the following holds: let A be a noncommutative semisimple Banach algebra. Suppose that $[D(x), x]D(x) = 0$ holds for all $x \in A$. In this case, $D = 0$.

B.D. Kim [6] showed the following: let R be a 3!-torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \to R$ such that

$$[D(x), x]D(x)[D(x), x] = 0$$

for all $x \in R$. In this case, we have $[D(x), x]^5 = 0$ for all $x \in R$.

And, B.D. Kim[7] showed the following:let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \to A$ such that $D(x)[D(x), x]D(x) \in \mathrm{rad}(A)$ for all $x \in A$. In this case, we have $D(A) \subseteq \mathrm{rad}(A)$.

In this paper, our first aim is to prove the following results in the ring theory in order to apply it to the Banach algebra theory:

Let R be a 3!-torsion free noncommutative semiprime ring, U a Lie ideal of R. And let $D : R \longrightarrow R$ be a Jordan derivation on R and U a Lie ideal of R. In this case, we show that if $[D(x), x]D(x) = 0$ holds for all $x \in U$, then $D(x)[D(x), x]y - yD(x)[D(x), x] = 0$ for all $x, y \in U$, and also, if $D(x)[D(x), x] = 0$ holds for all $x \in U$, then $[D(x), x]D(x)y-y[D(x), x]D(x) = 0$ for all $x, y \in U$. In particular, when $U = R$, then we conclude that $[D(x), x]D(x) = 0$ is equivalent to $D(x)[D(x), x] = 0$ for all $x \in R$.

Moreover, using the above results, we shall give their applications in Banach algebra as follows.

(i): Suppose there exists a continuous linear Jordan derivation $D : A \to A$ and U a Lie ideal of A. Then

$$[D(x), x]D(x) \in \mathrm{rad}(A) \iff D(x)[D(x), x] \in \mathrm{rad}(A)$$

for all $x \in U$.

And also, we have their applications in Banach algebras as follows. Of course, the following results are already well-known.

(ii): Suppose there exists a continuous linear Jordan derivation D on a noncommutative Banach algebra A with a Lie ideal U such that

$$[D(x), x]D(x) \in \mathrm{rad}(A)$$

for all $x \in U$.

Then we have $D(x)[D(x), x]y - yD(x)[D(x), x] \in \mathrm{rad}(A)$ for all $x, y \in U$. And

(iii): Suppose there exists a continuous linear Jordan derivation D on a noncommutative Banach algebra A with a Lie ideal U such that

$$D(x)[D(x), x] \in \mathrm{rad}(A)$$

for all $x \in U$.

Then $[D(x), x]D(x)y - y[D(x), x]D(x) \in \mathrm{rad}(A)$ for all $x, y \in U$. In particular, when $U = A$, then we see that

$$[D(x), x]D(x) \in \mathrm{rad}(A) \iff D(x)[D(x), x] \in \mathrm{rad}(A)$$

for all $x \in A$. Moreover, we have $D(A) \subseteq \mathrm{rad}(A)$.

The following lemma is due to L.O. Chung and J. Luh[4].

Lemma 1.1. *Let R be a $n!$-torsion free ring. Suppose there exist elements $y_1, y_2, \cdots, y_{n-1}, y_n$ in R such that $\sum_{k=1}^{n} t^k y_k = 0$ for all $t = 1, 2, \cdots, n$. Then we have $y_k = 0$ for every positive integer k with $1 \leq k \leq n$.*

The following theorem is due to M. Brešar[3].

Theorem 1.2. *Let R be a 2-torsion free semiprime ring and let $D : R \longrightarrow R$ be a Jordan derivation. In this case, D is a derivation.*

We write $Q(A)$ for the set of all quasinilpotent elements in A. M. Brešar [2] has proved the following theorem.

Theorem 1.3. *Let D be a bounded derivation of a Banach algebra A. Suppose that $[D(x), x] \in Q(A)$ for every $x \in A$. Then D maps A into rad(A).*

After this, by S_m we denote the set $\{k \in \mathbb{N} \mid 1 \leq k \leq m\}$ where m is a positive integer.

We need Theorems 2.4 and 2.5 to obtain the main theorems for Banach algebra theory.

2. MAIN RESULTS

Lemma 2.1. *Let R be a noncommutative semiprime ring, and U a Lie ideal of R. And suppose that $aya = 0$ for all $y \in U$ and some $a \in R$. Then $a = 0$.*

Proof. By the assumption, we have

$$aya = 0,\ y \in U \tag{2.1}$$

Replacing $[y, z]$ for y in (2.1), we obtain

$$a[y, z]a = 0,\ y \in U, z \in R. \tag{2.2}$$

Writing zaw for z in (2.2), we get

$$a[y, z]awa + aza[y, w]a + az[y, a]wa = 0, y \in U, w, z \in R. \tag{2.3}$$

Combining (2.2) with (2.3),

$$az[y, a]wa = 0, y \in U, w, z \in R. \tag{2.4}$$

From (2.4), we have

$$[y, a]waz[y, a]wa = 0, y \in U, w, z \in R. \tag{2.5}$$

Since R is semiprime, the relation (2.5) yields

$$[y, a]wa = 0, y \in U, w \in R. \tag{2.6}$$

Substituting aw for w in (2.6),

$$ya^2wa - ayawa = 0, y \in U, w \in R. \tag{2.7}$$

From (2.1) and (2.7), we have

$$ya^2wa = 0, y \in U, w \in R. \tag{2.8}$$

From (2.8), we get

$$ya^2wya^2 = 0, y \in U, w \in R. \tag{2.9}$$

Since R is semiprime, the relation (2.9) yields

$$ya^2 = 0, y \in U. \tag{2.10}$$

Putting $[x, w]$ instead of y in (2.10), we obtain

$$ywa^2 - wya^2, y \in U, w \in R. \tag{2.11}$$

Combining (2.10) with (2.11),

$$ywa^2 = 0, x, y \in U, w \in R. \tag{2.12}$$

From (2.12), we get

$$yw[a^2, y] = 0, y \in U, w \in R. \tag{2.13}$$

From (2.13), we have

$$[a^2, y]w[a^2, y] = 0, y \in U, w \in R. \tag{2.14}$$

Thus by semiprimeness of R, it follows from (2.14) that

$$[a^2, y] = 0, y \in U. \tag{2.15}$$

□

For simplicity, we shall denote the maps $B : R \times R \longrightarrow R,\ f, g : R \longrightarrow R$ by $B(x, y) \equiv [D(x), y] + [D(y), x], f(x) \equiv [D(x), x],\ g(x) \equiv [f(x), x]$ for all $x, y \in R$ respectively. Then we have the basic properties:

$$B(x, y) = B(y, x),\ B(x, yz) = B(x, y)z + yB(x, z) + D(y)[z, x] + [y, x]D(z),$$
$$B(x, x) = 2f(x),\ B(xy, z) = B(y, z)x + zB(y, x) + D(z)[x, y] + [z, y]D(x),$$
$$B(x, x^2) = 2(f(x)x + xf(x)),\ x, y, z \in R.$$

After this, we use the above relations without specific reference.

Theorem 2.2. *Let R be a $3!$-torsion free noncommutative semiprime ring, U a Lie ideal of R with $[R, U] \neq \{0\}$. Then Let $D : R \longrightarrow R$ be a Jordan derivation on a semiprime ring.*

(i): *If $[D(x), x]D(x) = 0$ holds for all $x \in U$, then*

$$D(x)[D(x), x]y - yD(x)[D(x), x] = 0$$

for all $x, y \in U$. And

(ii): *If $D(x)[D(x), x] = 0$ holds for all $x \in U$, then*

$$[D(x), x]D(x)y - y[D(x), x]D(x) = 0$$

for all $x, y \in U$.

In particular, when $U = R$, we see that

$$[D(x), x]D(x) = 0,\ \ x \in R \quad \Longleftrightarrow \quad D(x)[D(x), x] = 0,\ \ x \in R.$$

Proof. (i)($\Longrightarrow$): By Theorem 1.2, we can see that D is a derivation on R. By the assumption,

$$f(x)D(x) = 0,\ x \in U \tag{2.16}$$

Replacing $x + ty$ for x in (2.16), we have

$$\begin{aligned} &[D(x+ty), x+ty]D(x+ty) \equiv +t\{B(x,y)D(x) + f(x)D(y)\} \\ &+t^2 H(x,y) + t^3[D(y),y]D(y) = 0,\ x,y \in U, t \in S_3 \end{aligned} \tag{2.17}$$

where H denotes the term satisfying the identity (2.17).
From (2.16) and (2.17), we obtain

$$t\{B(x,y)D(x) + f(x)D(y)\} + t^2 H(x,y) = 0,\ x,y \in U, t \in S_3. \tag{2.18}$$

Since R is 2!-torsion free by assumption, by Lemma 2.1 the relation (2.18) yields

$$B(x,y)D(x) + f(x)D(y) = 0, x, y \in U. \tag{2.19}$$

Let $y = x^2$ in (2.19). Then using (2.16), we have

$$2(f(x)x + xf(x))D(x) + f(x)(D(x)x + xD(x)) = 0, x \in U. \tag{2.20}$$

From (2.16) and (2.20) ,

$$3f(x)xD(x) + 2xf(x)D(x) + f(x)D(x)x = 3f(x)^2 = 0, x \in U. \tag{2.21}$$

Since R is 3!-torsion free, it follows from (2.21) that

$$f(x)^2 = 0, x \in U. \tag{2.22}$$

From (2.16), we obtain

$$0 = [f(x)D(x), x] = g(x)D(x) + f(x)^2, x \in U. \tag{2.23}$$

From (2.22) and (2.23), we have

$$g(x)D(x) = 0, x \in U. \tag{2.24}$$

Writing yx for y in (2.19), we get

$$(B(x,y)x + 2yf(x) + [y,x]D(x))D(x) + f(x)(D(y)x + yD(x)) = 0, x, y \in U. \tag{2.25}$$

Right multiplication of (2.19) by x leads to

$$B(x,y)D(x)x + f(x)D(y)x = 0, x \in U. \tag{2.26}$$

Combining (2.25) with (2.26),

$$-B(x,y)f(x) + 2yf(x)D(x) + [y,x]D(x)^2 + f(x)yD(x) = 0, x, y \in U. \tag{2.27}$$

From (2.16) and (2.27), we have

$$-B(x,y)f(x)+[y,x]D(x)^2+f(x)yD(x)=0, x,y\in U. \tag{2.28}$$

Substituting x^2 for y in (2.28), we get

$$-2(f(x)x+xf(x))f(x)+f(x)x^2D(x)=0, x\in U. \tag{2.29}$$

Comparing (2.22) and (2.24), we obtain

$$\begin{aligned}&-2f(x)xf(x)-xf(x)^2-f(x)(f(x)x+xf(x))\\ &=-3f(x)xf(x)=-3g(x)f(x)=3f(x)g(x)=0, x\in R.\end{aligned} \tag{2.30}$$

Since R is 3!-torsion free by assumption, the relation (2.30) yields

$$g(x)f(x)=f(x)g(x)=0, x\in R. \tag{2.31}$$

Right multiplication of (2.28) by $D(x)$ leads to

$$-B(x,y)f(x)^2+[y,x]D(x)^2f(x)+f(x)yD(x)f(x)=0, x,y\in R. \tag{2.32}$$

From (2.22) and (2.32), we have

$$[y,x]D(x)^2f(x)+f(x)yD(x)f(x)=0, x,y\in R. \tag{2.33}$$

Substituting xy for y in (2.33), we get

$$x[y,x]D(x)^2f(x)+f(x)xyD(x)f(x)=0, x,y\in R. \tag{2.34}$$

Left multiplication of (2.34) by $D(x)$ leads to

$$-x[y,x]D(x)^2f(x)+xf(x)yD(x)f(x)=0, x,y\in R. \tag{2.35}$$

From (2.34) and (2.35), we have

$$g(x)yD(x)f(x)=0, x,y\in R. \tag{2.36}$$

Replacing yx for y in (2.36), we obtain

$$g(x)yxD(x)f(x)=0, x,y\in R. \tag{2.37}$$

Right multiplication of (2.36) by x leads to

$$g(x)yD(x)f(x)x=0, x,y\in R. \tag{2.38}$$

From (2.22), (2.37) and (2.38), we have

$$g(x)yD(x)g(x)=0, x,y\in R. \tag{2.39}$$

Left multiplication of (2.39) by $D(x)$ leads to

$$D(x)g(x)yD(x)g(x)=0, x,y\in R. \tag{2.40}$$

Thus by semiprimeness of R, it is clear that

$$D(x)g(x) = 0, x \in R. \tag{2.41}$$

Putting xy instead of y in (2.28), we get

$$\begin{aligned} &-(xB(x,y) + 2f(x)y + D(x)[y,x])f(x) + x[y,x]D(x)^2 + f(x)xyD(x) \\ &= 0, x, y \in R. \end{aligned} \tag{2.42}$$

Left multiplication of (2.28) by x leads to

$$-xB(x,y)f(x) + x[y,x]D(x)^2 + xf(x)yD(x) = 0, x \in R. \tag{2.43}$$

Combining (2.42) with (2.43),

$$-2f(x)yf(x) - D(x)[y,x]f(x) + g(x)yD(x) = 0, x, y \in R. \tag{2.44}$$

Right multiplication of (2.44) by $D(x)$ yields

$$-2f(x)yf(x)D(x) - D(x)[y,x]f(x)D(x) + g(x)yD(x)^2 = 0, x \in R. \tag{2.45}$$

Combining (2.16) with (2.45),

$$g(x)yD(x)^2 = 0, x, y \in R. \tag{2.46}$$

Replacing $yD(x)$ for y in (2.45), we get

$$\begin{aligned} &-2f(x)yD(x)f(x) - D(x)[y,x]D(x)f(x) - D(x)yf(x)^2 \\ &+g(x)yD(x)^2 = 0, x \in R. \end{aligned} \tag{2.47}$$

From (2.22) and (2.47), we have

$$-2f(x)yD(x)f(x) - D(x)[y,x]D(x)f(x) + g(x)yD(x)^2 = 0, x \in R. \tag{2.48}$$

Comparing (2.46) and (2.48),

$$2f(x)yD(x)f(x) + D(x)[y,x]D(x)f(x) = 0, x \in R. \tag{2.49}$$

Left multiplication of (2.49) by $D(x)$ leads to

$$2D(x)f(x)yD(x)f(x) + D(x)^2[y,x]D(x)f(x) = 0, x, y \in R. \tag{2.50}$$

Substituting $f(x)y$ for y in (2.49), we obtain

$$\begin{aligned} &2f(x)^2yD(x)f(x) + D(x)f(x)[y,x]D(x)f(x) + D(x)g(x)yD(x)f(x) \\ &= 0, x \in R. \end{aligned} \tag{2.51}$$

Combining (2.22), (2.41) with (2.51), we obtain

$$D(x)f(x)[y,x]D(x)f(x) = 0, x \in R. \tag{2.52}$$

Substituting $yD(x)^2z$ for y in (2.52), we get

$$\begin{aligned}&D(x)f(x)[y,x]D(x)^2zD(x)f(x) + D(x)f(x)y[D(x)^2,x]zD(x)f(x)\\ &+D(x)f(x)yD(x)^2[z,x]D(x)f(x) = 0, x, y \in R. \qquad (2.53)\end{aligned}$$

From (2.16) and (2.53), we have

$$\begin{aligned}&D(x)f(x)[y,x]D(x)^2zD(x)f(x) + D(x)f(x)yD(x)f(x)zD(x)f(x)\\ &+D(x)f(x)yD(x)^2[z,x]D(x)f(x) = 0, x, y \in R. \qquad (2.54)\end{aligned}$$

Comparing (2.50) and (2.54),

$$\begin{aligned}&D(x)f(x)[y,x]D(x)^2zD(x)f(x) + D(x)f(x)yD(x)f(x)zD(x)f(x)\\ &-2D(x)f(x)yD(x)f(x)zD(x)f(x)\\ &= D(x)f(x)[y,x]D(x)^2zD(x)f(x) - D(x)f(x)yD(x)f(x)zD(x)f(x)\\ &= (D(x)f(x)[y,x]D(x)^2 - D(x)f(x)yD(x)f(x))zD(x)f(x) = 0, x, y \in R. \qquad (2.55)\end{aligned}$$

From (2.55), we obtain

$$\begin{aligned}&(D(x)f(x)[y,x]D(x)^2 - D(x)f(x)yD(x)f(x))z(D(x)f(x)[y,x]D(x)^2\\ &-D(x)f(x)yD(x)f(x)) = 0, x, y \in R. \qquad (2.56)\end{aligned}$$

Thus by semiprimeness of R, it is obvious that

$$D(x)f(x)[y,x]D(x)^2 - D(x)f(x)yD(x)f(x) = 0, x, y \in R. \qquad (2.57)$$

Replacing $x + tz$ for x in (2.46), we have

$$\begin{aligned}&g(x+tz)yD(x+tz)^2 \equiv g(x)yD(x)^2 + t\{([B(x,z),x] + [f(x),z])yD(x)^2\\ &+g(x)y(D(z)D(x) + D(x)D(z))\} + t^2I_1(x,y) + t^3I_2(x,y) + t^4I_3(x,y)\\ &+t^5g(z)yD(z)^2 = 0,\ x, y, z \in R, t \in S_4 \qquad (2.58)\end{aligned}$$

where $I_i, 1 \leq i \leq 3$, denotes the term satisfying the identity (2.58).
From (2.46) and (2.58), we obtain

$$\begin{aligned}&t\{([B(x,z),x] + [f(x),z])yD(x)^2 + g(x)y(D(z)D(x) + D(x)D(z))\}\\ &+t^2I_1(x,y) + t^3I_2(x,y) + t^4I_3(x,y) = 0,\ x, y, z \in R, t \in S_3. \qquad (2.59)\end{aligned}$$

Since R is 3!-torsion free by assumption, by Lemma 2.1 the relation (2.59) yields

$$\begin{aligned}&([B(x,z),x] + [f(x),z])yD(x)^2 + g(x)y(D(z)D(x) + D(x)D(z))\\ &= 0, x, y, z \in R. \qquad (2.60)\end{aligned}$$

Writing $ug(x)y$ for y in (2.60), we get

$$\begin{aligned}&([B(x,z),x]+[f(x),z])ug(x)yD(x)^2\\ &+g(x)ug(x)y(D(z)D(x)+D(x)D(z))=0, u,x,y,z\in R.\end{aligned}\tag{2.61}$$

Combining (2.46) with (2.61),

$$g(x)ug(x)y(D(z)D(x)+D(x)D(z))=0, u,x,y,z\in R.\tag{2.62}$$

Replacing $y(D(z)D(x)+D(x)D(z))u$ for u in (2.62), we get

$$\begin{aligned}&g(x)y(D(z)D(x)+D(x)D(z))ug(x)y(D(z)D(x)+D(x)D(z))\\ &=0, u,x,y,z\in R.\end{aligned}\tag{2.63}$$

And so, by semiprimeness of R, it follows that

$$g(x)y(D(z)D(x)+D(x)D(z))=0, x,y,z\in R.\tag{2.64}$$

Replacing $x+tw$ for x in (2.64), we have

$$\begin{aligned}&g(x+tw)y(D(z)D(x+tw)+D(x+tw)D(z)\equiv g(x)y(D(z)D(x)\\ &+D(x)D(z))+t\{([B(x,w),x]+[f(x),w])y(D(z)D(x)+D(x)D(z))\\ &+g(x)y(D(z)D(w)+D(w)D(z))\}+t^2J_1(x,y)+t^3J_2(x,y)\\ &+t^4g(w)y(D(z)D(w)+D(w)D(z))=0,\ w,x,y,z\in R, t\in S_4\end{aligned}\tag{2.65}$$

where J_1 and J_2 denote the term satisfying the identity (2.65).
From (2.64) and (2.65), we obtain

$$\begin{aligned}&t\{([B(x,w),x]+[f(x),w])y(D(z)D(x)+D(x)D(z))+g(x)y(D(z)D(w)\\ &+D(w)D(z))\}+t^2J_1(x,y)+t^3J_2(x,y)=0,\ w,x,y,z\in R, t\in S_3.\end{aligned}\tag{2.66}$$

Since R is 3!-torsion free by assumption, by Lemma 2.1 the relation (2.66) yields

$$\begin{aligned}&([B(x,w),x]+[f(x),w])y(D(z)D(x)+D(x)D(z))+g(x)y(D(z)D(w)\\ &+D(w)D(z))=0, w,x,y,z\in R.\end{aligned}\tag{2.67}$$

Replacing $ug(x)y$ for y in (2.67), we get

$$\begin{aligned}&([B(x,w),x]+[f(x),w])yg(x)ug(x)y(D(z)D(x)+D(x)D(z))\\ &+g(x)ug(x)y(D(z)D(w)+D(w)D(z))=0, w,x,y,z\in R.\end{aligned}\tag{2.68}$$

Combining (2.64) with (2.68),

$$g(x)ug(x)y(D(z)D(w)+D(w)D(z))=0, w,x,y,z\in R.\tag{2.69}$$

Replacing $y(D(z)D(w)+D(w)D(z))u$ for u in (2.69), we obtain

$$\begin{aligned} & g(x)y(D(z)D(w)+D(w)D(z))ug(x)y(D(z)D(x)+D(x)D(z)) \\ (2.70)\quad & =0, u, w, x, y, z \in R. \end{aligned}$$

And so, by semiprimeness of R, it follows from (2.70) that

$$(2.71)\qquad g(x)y(D(z)D(w)+D(w)D(z))=0, x, y, z \in R.$$

Let $w=z$ in (2.71). Then we get

$$(2.72)\qquad g(x)yD(z)^2=0, x, y, z \in R.$$

Replacing $x+tw$ for x in (2.72), we have

$$\begin{aligned} & g(x+tw)yD(z)^2 \equiv g(x)yD(z)^2+t\{([B(x,w),x]+[f(x),w])yD(z)^2\} \\ (2.73)\quad & +t^2K(x,y)+t^3g(w)yD(z)^2=0,\ w, x, y, z \in R, t \in S_3 \end{aligned}$$

where K denotes the term satisfying the identity (2.73).
From (2.72) and (2.73), we obtain

$$\begin{aligned} & t\{([B(x,w),x]+[f(x),w])yD(z)^2\}+t^2K(x,y) \\ (2.74)\quad & =0,\ w, x, y, z \in R, t \in S_3. \end{aligned}$$

Since R is 2!-torsion free by assumption, by Lemma 2.1 the relation (2.74) yields

$$(2.75)\qquad ([B(x,w),x]+[f(x),w])yD(z)^2=0, w, x, y, z \in R.$$

Replacing wx for w in (2.75), we get

$$\begin{aligned} & ([B(x,w),x]x+3[w,x]f(x)+3wg(x)+[[w,x],x]D(x)+[f(x),w]x)yD(x)^2 \\ (2.76)\quad & =0, w, x, y, z \in R. \end{aligned}$$

From (2.72) and (2.76), we have

$$\begin{aligned} & \{[B(x,w),x]x+3[w,x]f(x)+[[w,x],x]D(x)+[f(x),w]x\}yD(x)^2 \\ (2.77)\quad & =0, w, x, y, z \in R. \end{aligned}$$

Substituting xy for y in (2.75), we get

$$(2.78)\qquad ([B(x,w),x]x+[f(x),w]x)yD(z)^2=0, w, x, y, z \in R.$$

Combining (2.77) with (2.78),

$$(2.79)\qquad (3[w,x]f(x)+[[w,x],x]D(x))yD(z)^2=0, w, x, y, z \in R.$$

Replacing $D(x)w$ for w in (2.79), we obtain

$$\begin{aligned}&\{3f(x)wf(x)+3D(x)[w,x]f(x)+g(x)wD(x)+2f(x)[w,x]D(x)\\ &+D(x)[[w,x],x]D(x)\}yD(z)^2=0, w,x,y,z\in R.\end{aligned} \tag{2.80}$$

Substituting $D(x)y$ for y in (2.80), we have

$$\begin{aligned}&\{3f(x)wf(x)D(x)+3D(x)[w,x]f(x)D(x)+g(x)wD(x)^2+2f(x)[w,x]D(x)^2\\ &+D(x)[[w,x],x]D(x)^2\}yD(z)^2=0, w,x,y,z\in R.\end{aligned} \tag{2.81}$$

Combining (2.16), (2.66) with (2.81),

$$\begin{aligned}&(2f(x)[w,x]D(x)^2+D(x)[[w,x],x]D(x)^2)yD(z)^2\\ &=0, w,x,y,z\in R.\end{aligned} \tag{2.82}$$

Left multiplication of (2.82) by $D(x)$ leads to

$$\begin{aligned}&(2D(x)f(x)[w,x]D(x)^2+D(x)^2[[w,x],x]D(x)^2)yD(z)^2\\ &=0, w,x,y,z\in R.\end{aligned} \tag{2.83}$$

Substituting $yD(x)^2w$ for y in (2.50), we get

$$\begin{aligned}&2D(x)f(x)yD(x)^2wD(x)f(x)+D(x)^2[y,x]D(x)^2wD(x)f(x)\\ &+D(x)^2yD(x)f(x)wD(x)f(x)+D(x)^2yf(x)D(x)wD(x)f(x)\\ &+D(x)^2yD(x)^2[w,x]D(x)f(x)=0, w,x,y,z\in R.\end{aligned} \tag{2.84}$$

Combining (2.16) with (2.84), we obtain

$$\begin{aligned}&2D(x)f(x)yD(x)^2wD(x)f(x)+D(x)^2[y,x]D(x)^2wD(x)f(x)\\ &+D(x)^2yD(x)f(x)wD(x)f(x)+D(x)^2yD(x)^2[w,x]D(x)f(x)\\ &=0, w,x,y,z\in R.\end{aligned} \tag{2.85}$$

From (2.49) and (2.85), we have

$$\begin{aligned}&(2D(x)f(x)yD(x)^2+D(x)^2[y,x]D(x)^2-D(x)^2yD(x)f(x))\\ &\times wD(x)f(x)=0, w,x,y,z\in R.\end{aligned} \tag{2.86}$$

Replacing $[y,x]$ for y in (2.86), we get

$$\begin{aligned}&(2D(x)f(x)[y,x]D(x)^2+D(x)^2[[y,x],x]D(x)^2-D(x)^2yD(x)f(x))\\ &wD(x)f(x)=0, w,x,y,z\in R.\end{aligned} \tag{2.87}$$

Combining (2.49), (2.57) with (2.87),

$$(4D(x)f(x)yD(x)f(x) + D(x)^2[[y,x],x]D(x)^2)wD(x)f(x)$$

$$= 0, w, x, y, z \in R. \tag{2.88}$$

From (2.16) and (2.83), we arrive at

$$(2D(x)f(x)[y,x]D(x)^2 + D(x)^2[[y,x],x]D(x)^2)w[D(z)^2, z]$$

$$= (2D(x)f(x)[y,x]D(x)^2 + D(x)^2[[y,x],x]D(x)^2)wD(z)f(z)$$

$$= 0, w, x, y, z \in R. \tag{2.89}$$

Let $z = x$ in (2.89). Then

$$(2D(x)f(x)[y,x]D(x)^2 + D(x)^2[[y,x],x]D(x)^2)wD(x)f(x)$$

$$= 0, w, x, y, z \in R. \tag{2.90}$$

Combining (2.88) with (2.90),

$$2(2D(x)f(x)yD(x)f(x) - D(x)f(x)[y,x]D(x)^2)wD(x)f(x)$$

$$= 0, w, x, y, z \in R. \tag{2.91}$$

Since R is 2!-torsion free by assumption, the relation (2.90) yields

$$(2D(x)f(x)yD(x)f(x) - D(x)f(x)[y,x]D(x)^2)wD(x)f(x)$$

$$= 0, w, x, y, z \in R. \tag{2.92}$$

From (2.92), we have

$$(2D(x)f(x)yD(x)f(x) - D(x)f(x)[y,x]D(x)^2)w(2D(x)f(x)yD(x)f(x)$$

$$-D(x)f(x)[y,x]D(x)^2) = 0, w, x, y, z \in R. \tag{2.93}$$

And so, by semiprimeness of R, we get from (2.93)

$$2D(x)f(x)yD(x)f(x) - D(x)f(x)[y,x]D(x)^2 = 0, x, y, z \in R. \tag{2.94}$$

Combining (2.57) with (2.94),

$$D(x)f(x)yD(x)f(x) = 0, x, y \in R. \tag{2.95}$$

And so, by semiprimeness of R, it follows from (2.95) that

$$D(x)f(x) = 0, x \in R.$$

(ii) ($\Longleftarrow$): Suppose that

$$D(x)f(x) = 0, x, y, z \in R. \tag{2.96}$$

Replacing $x+ty$ for x in (2.96), we have

$$D(x+ty)[D(x+ty),x+ty] \equiv +t\{D(y)f(x)+D(x)B(x,y)\}$$
$$+t^2P(x,y)+t^3D(y)f(y)=0,\ x,y\in R, t\in S_3 \tag{2.97}$$

where P denotes the term satisfying the identity (2.97).
From (2.96) and (2.97), we obtain

$$t\{D(y)f(x)+D(x)B(x,y)\}+t^2P(x,y)=0,\ x,y\in R, t\in S_3. \tag{2.98}$$

Since R is 2!-torsion free by assumption, by Lemma 2.1 the relation (2.98) yields

$$D(y)f(x)+D(x)B(x,y)=0, x,y\in R. \tag{2.99}$$

Let $y=x^2$ in (2.213). Then using (2.96), we obtain from (2.99)

$$(D(x)x+xD(x))f(x)+2D(x)(f(x)x+xf(x))$$
$$=3D(x)xf(x)+xD(x)f(x)+2D(x)f(x)x=0, x\in R. \tag{2.100}$$

From (2.96) and (2.100) , we get

$$3D(x)xf(x)=3f(x)^2=-3D(x)g(x)=0, x\in R. \tag{2.101}$$

Since R is 3!-torsion free, it follows from (2.101) that

$$f(x)^2=0, x\in R, \tag{2.102}$$

and

$$D(x)g(x)=0, x\in R. \tag{2.103}$$

Writing xy for y in (2.99),

$$(xD(y)f(x)+D(x)yf(x)+D(x)(xB(x,y)+2f(x)y+D(x)[y,x]$$
$$=0, x,y\in R. \tag{2.104}$$

Left multiplication of (2.104) by $D(x)$ leads to

$$xD(y)f(x)+xD(x)B(x,y)=0, x\in R. \tag{2.105}$$

Combining (2.218) with (2.105),

$$D(x)yf(x)+f(x)B(x,y)+2D(x)f(x)y+D(x)^2[y,x]=0, x,y\in R. \tag{2.106}$$

From (2.96) and (2.106), we have

$$D(x)yf(x)+f(x)B(x,y)+D(x)^2[y,x]=0, x,y\in R. \tag{2.107}$$

Left multiplication of (2.107) by $D(x)$ yields

$$D(x)^2yf(x) + D(x)f(x)B(x,y) + D(x)^3[y,x] = 0, x, y \in R. \tag{2.108}$$

Comparing (2.96) and (2.108), we obtain

$$D(x)^2yf(x) + D(x)^3[y,x] = 0, x, y \in R. \tag{2.109}$$

Putting yx instead of y in (2.99), we get

$$\begin{aligned}&(D(y)x + yD(x))f(x) + D(x)(B(x,y)x + 2yf(x) + [y,x]D(x))\\&= 0, x, y \in R.\end{aligned} \tag{2.110}$$

Right multiplication of (2.110) by x leads to

$$D(y)f(x)x + D(x)B(x,y)x = 0, x \in R. \tag{2.111}$$

Combining (2.110) with (2.111),

$$-D(y)g(x) + yD(x)f(x) + 2D(x)yf(x) + D(x)[y,x]D(x) = 0, x, y \in R. \tag{2.112}$$

From (2.96) and (2.112), we have

$$-D(y)g(x) + 2D(x)yf(x) + D(x)[y,x]D(x) = 0, x, y \in R. \tag{2.113}$$

Writing xy for y in (2.113), we get

$$\begin{aligned}&-xD(y)g(x) - D(x)yg(x) + 2D(x)xyf(x) + D(x)x[y,x]D(x)\\&= 0, x, y \in R.\end{aligned} \tag{2.114}$$

Left multiplication of (2.113) by x leads to

$$-xD(y)g(x) + 2xD(x)yf(x) + xD(x)[y,x]D(x) = 0, x, y \in R. \tag{2.115}$$

Combining (2.114) with (2.115),

$$-D(x)yg(x) + 2f(x)yf(x) + f(x)[y,x]D(x) = 0, x, y \in R. \tag{2.116}$$

Left multiplication of (2.116) by $D(x)$ gives

$$-D(x)^2yg(x) + 2D(x)f(x)yf(x) + D(x)f(x)[y,x]D(x) = 0, x, y \in R. \tag{2.117}$$

Comparing (2.96) and (2.117), we obtain

$$D(x)^2yg(x) = 0, x, y \in R. \tag{2.118}$$

Let $y = D(x)$ in (2.113). Then we get

$$-D^2(x)g(x) + 2D(x)^2f(x) + D(x)f(x)D(x) = 0, x, y \in R. \tag{2.119}$$

Combining (2.96) with (2.119),

$$D^2(x)g(x) = 0, x \in R. \tag{2.120}$$

Writing $yD(x)$ for y in (2.113), we have

$$\begin{aligned} &-D(y)D(x)g(x) - yD^2(x)g(x) + 2D(x)yD(x)f(x) + D(x)[y,x]D(x)^2 \\ &+D(x)yf(x)D(x) = 0, x, y \in R. \end{aligned} \tag{2.121}$$

Combining (2.96), (2.103), (2.120) with (2.121), we arrive at

$$D(x)[y,x]D(x)^2 + D(x)yf(x)D(x) = 0, x, y \in R. \tag{2.122}$$

Left multiplication of (2.122) by $f(x)$ leads to

$$f(x)D(x)[y,x]D(x)^2 + f(x)D(x)yf(x)D(x) = 0, x, y \in R. \tag{2.123}$$

Writing $yD(x)$ for y in (2.116), we get

$$\begin{aligned} &-D(x)yD(x)g(x) + 2f(x)yD(x)f(x) + f(x)[y,x]D(x)^2 + f(x)yf(x)D(x) \\ &= 0, x, y \in R. \end{aligned} \tag{2.124}$$

Comparing (2.96) and (2.103), we obtain from (2.124)

$$f(x)[y,x]D(x)^2 + f(x)yf(x)D(x) = 0, x, y \in R. \tag{2.125}$$

Substituting $D(x)y$ for y in (2.116),

$$\begin{aligned} &-D(x)^2yg(x) + 2f(x)D(x)yf(x) + f(x)D(x)[y,x]D(x) + f(x)^2yD(x) \\ &= 0, x, y \in R. \end{aligned} \tag{2.126}$$

Comparing (2.102) and (2.118) and (2.126), we obtain

$$2f(x)D(x)yf(x) + f(x)D(x)[y,x]D(x) = 0, x, y \in R. \tag{2.127}$$

Right multiplication of (2.127) by $D(x)$ leads to

$$2f(x)D(x)yf(x)D(x) + f(x)D(x)[y,x]D(x)^2 = 0, x, y \in R. \tag{2.128}$$

From (2.237) and (2.128), we have

$$f(x)D(x) = 0, x \in R.$$

□

Corollary 2.3. *Let R be a $3!$-torsion free noncommutative semiprime ring and $D : R \longrightarrow R$ a Jordan derivation. Then*

$$[D(x), x]D(x) = 0 \iff D(x)[D(x), x] = 0$$

for all $x \in R$.

Theorem 2.4. *Let* R *be a* $3!$*-torsion free noncommutative semiprime ring,*U *a Lie ideal of* R*, and let* $D : R \longrightarrow R$ *be a Jordan derivation on* R*. And suppose that* $[D(x), x]D(x) = 0$ *for all* $x \in U$*. Then* $[D(x), x]^2 = 0$ *for all* $x \in U$*.*

Proof. By Theorem 2.2, we can see that D is a derivation on R. By assumption,

$$[D(x), x]D(x) = 0,\ x \in U \tag{2.129}$$

Replacing $x + ty$ for x in (2.129), we have

$$\begin{aligned}&[D(x + ty), x + ty]D(x + ty) \equiv +t\{B(x, y)D(x) + f(x)D(y)\}\\ &+t^2H(x, y) + t^3[D(y), y]D(y) = 0,\ x, y \in U, t \in S_3\end{aligned} \tag{2.130}$$

where H denotes the term satisfying the identity (2.130).
From (2.129) and (2.130), we obtain

$$t\{B(x, y)D(x) + f(x)D(y)\} + t^2H(x, y) = 0,\ x, y \in U, t \in S_3. \tag{2.131}$$

Since R is 2!-torsion free by assumption, by Lemma 2.1 the relation (2.131) yields

$$B(x, y)D(x) + f(x)D(y) = 0, x, y \in U. \tag{2.132}$$

Let $y = x^2$ in (2.132). Then using (2.129), we get

$$2(f(x)x + xf(x))D(x) + f(x)(D(x)x + xD(x)) = 0, x \in U. \tag{2.133}$$

From (2.129) and (2.133) , we arrive at

$$3f(x)xD(x) + 2xf(x)D(x) + f(x)D(x)x = 3f(x)^2 = 0, x \in U. \tag{2.134}$$

Since R is 3!-torsion free, it follows from (2.134) that

$$f(x)^2 = 0, x \in U. \tag{2.135}$$

From (2.129), we obtain

$$0 = [f(x)D(x), x] = g(x)D(x) + f(x)^2, x \in U. \tag{2.136}$$

From (2.135) and (2.136), we have

$$g(x)D(x) = 0, x \in R. \tag{2.137}$$

Writing yx for y in (2.132), we get

$$\begin{aligned}&(B(x, y)x + 2yf(x) + [y, x]D(x))D(x) + f(x)(D(y)x + yD(x))\\ &= 0, x, y \in U.\end{aligned} \tag{2.138}$$

Right multiplication of (2.132) by x leads to

$$B(x,y)D(x)x + f(x)D(y)x = 0, x, y \in U. \tag{2.139}$$

Combining (2.138) with (2.139),

$$-B(x,y)f(x) + 2yf(x)D(x) + [y,x]D(x)^2 + f(x)yD(x) = 0, x, y \in R. \tag{2.140}$$

From (2.129) and (2.140), we have

$$-B(x,y)f(x) + [y,x]D(x)^2 + f(x)yD(x) = 0, x, y \in U. \tag{2.141}$$

Substituting x^2 for y in (2.141), we get

$$-2(f(x)x + xf(x))f(x) + f(x)x^2D(x) = 0, x \in R. \tag{2.142}$$

Comparing (2.135) and (2.137), we obtain

$$\begin{aligned} &-2f(x)xf(x) - xf(x)^2 - f(x)(f(x)x + xf(x)) \\ &= -3f(x)xf(x) = -3g(x)f(x) = 3f(x)g(x) = 0, x \in R. \end{aligned} \tag{2.143}$$

Since R is 3!-torsion free by assumption, the relation (2.143) yields

$$g(x)f(x) = f(x)g(x) = 0, x \in U. \tag{2.144}$$

Right multiplication of (2.141) by $D(x)$ leads to

$$-B(x,y)f(x)^2 + [y,x]D(x)^2f(x) + f(x)yD(x)f(x) = 0, x, y \in U. \tag{2.145}$$

From (2.135) and (2.145), we have

$$[y,x]D(x)^2f(x) + f(x)yD(x)f(x) = 0, x, y \in U. \tag{2.146}$$

Substituting xy for y in (2.146), we get

$$x[y,x]D(x)^2f(x) + f(x)xyD(x)f(x) = 0, x, y \in U. \tag{2.147}$$

Left multiplication of (2.147) by $D(x)$ gives

$$-x[y,x]D(x)^2f(x) + xf(x)yD(x)f(x) = 0, x, y \in U. \tag{2.148}$$

From (2.147) and (2.148), we have

$$g(x)yD(x)f(x) = 0, x, y \in U. \tag{2.149}$$

Replacing yx for y in (2.149), we get

$$g(x)yxD(x)f(x) = 0, x, y \in U. \tag{2.150}$$

Right multiplication of (2.149) by x yields

$$g(x)yD(x)f(x)x = 0, x, y \in U. \tag{2.151}$$

From (2.135), (2.150) and (2.151), we have

$$g(x)yD(x)g(x) = 0, x, y \in U. \tag{2.152}$$

Left multiplication of (2.152) by $D(x)$ leads to

$$D(x)g(x)yD(x)g(x) = 0, x, y \in U. \tag{2.153}$$

Thus by semiprimeness of R, it is clear that

$$D(x)g(x) = 0, x \in U. \tag{2.154}$$

Putting xy instead of y in (2.141), we get

$$\begin{aligned} &-(xB(x,y) + 2f(x)y + D(x)[y,x])f(x) + x[y,x]D(x)^2 + f(x)xyD(x) \\ &= 0, x, y \in U. \end{aligned} \tag{2.155}$$

Left multiplication of (2.141) by x yields

$$-xB(x,y)f(x) + x[y,x]D(x)^2 + xf(x)yD(x) = 0, x, y \in U. \tag{2.156}$$

Combining (2.155) with (2.156),

$$-2f(x)yf(x) - D(x)[y,x]f(x) + g(x)yD(x) = 0, x, y \in U. \tag{2.157}$$

Right multiplication of (2.157) by $D(x)$ leads to

$$-2f(x)yf(x)D(x) - D(x)[y,x]f(x)D(x) + g(x)yD(x)^2 = 0, x \in U. \tag{2.158}$$

Combining (2.129) with (2.158),

$$g(x)yD(x)^2 = 0, x, y \in U. \tag{2.159}$$

Replacing $yD(x)$ for y in (2.158), we get

$$\begin{aligned} &-2f(x)yD(x)f(x) - D(x)[y,x]D(x)f(x) - D(x)yf(x)^2 + g(x)yD(x)^2 \\ &= 0, x \in U. \end{aligned} \tag{2.160}$$

From (2.135) and (2.160), we have

$$-2f(x)yD(x)f(x) - D(x)[y,x]D(x)f(x) + g(x)yD(x)^2 = 0, x, y \in U. \tag{2.161}$$

Comparing (2.159) and (2.161),

$$2f(x)yD(x)f(x) + D(x)[y,x]D(x)f(x) = 0, x, y \in U. \tag{2.162}$$

Left multiplication of (2.162) by $D(x)$ gives

$$2D(x)f(x)yD(x)f(x) + D(x)^2[y,x]D(x)f(x) = 0, x, y \in U. \tag{2.163}$$

Substituting $f(x)y$ for y in (2.162), we get

$$2f(x)^2yD(x)f(x) + D(x)f(x)[y,x]D(x)f(x) + D(x)g(x)yD(x)f(x) = 0, x, y \in U. \tag{2.164}$$

Combining (2.135), (2.154) with (2.164), we obtain

$$D(x)f(x)[y,x]D(x)f(x) = 0, x, y \in U. \tag{2.165}$$

Substituting $yD(x)^2z$ for y in (2.165),

$$D(x)f(x)[y,x]D(x)^2zD(x)f(x) + D(x)f(x)y[D(x)^2,x]zD(x)f(x) +D(x)f(x)yD(x)^2[z,x]D(x)f(x) = 0, x, y \in U. \tag{2.166}$$

From (2.129) and (2.166), we have

$$D(x)f(x)[y,x]D(x)^2zD(x)f(x) + D(x)f(x)yD(x)f(x)zD(x)f(x) +D(x)f(x)yD(x)^2[z,x]D(x)f(x) = 0, x, y \in U. \tag{2.167}$$

Comparing (2.163) and (2.167),

$$\begin{aligned} &D(x)f(x)[y,x]D(x)^2zD(x)f(x) + D(x)f(x)yD(x)f(x)zD(x)f(x) \\ &-2D(x)f(x)yD(x)f(x)zD(x)f(x) \\ &= D(x)f(x)[y,x]D(x)^2zD(x)f(x) - D(x)f(x)yD(x)f(x)zD(x)f(x) \\ &= \{D(x)f(x)[y,x]D(x)^2 - D(x)f(x)yD(x)f(x)\}zD(x)f(x) = 0, x, y \in U. \end{aligned} \tag{2.168}$$

From (2.168), we obtain

$$\{D(x)f(x)[y,x]D(x)^2 - D(x)f(x)yD(x)f(x)\}z(D(x)f(x)[y,x]D(x)^2 -D(x)f(x)yD(x)f(x)) = 0, x, y \in U. \tag{2.169}$$

Thus by semiprimeness of R, it is obvious that

$$D(x)f(x)[y,x]D(x)^2 - D(x)f(x)yD(x)f(x) = 0, x, y \in U. \tag{2.170}$$

Replacing $x+tz$ for x in (2.159), we have

$$\begin{aligned} &g(x+tz)yD(x+tz)^2 \equiv g(x)yD(x)^2 + t\{([B(x,z),x] + [f(x),z])yD(x)^2 \\ &+g(x)y(D(z)D(x) + D(x)D(z))\} + t^2L_1(x,y) + t^3L_2(x,y) + t^4L_3(x,y) \\ &+t^5g(z)yD(z)^2 = 0,\ x, y, z \in U, t \in S_4 \end{aligned} \tag{2.171}$$

where $L_i, 1 \leq i \leq 3$, denotes the term satisfying the identity (2.171).
From (2.159) and (2.171), we obtain

$$t\{([B(x,z),x]+[f(x),z])yD(x)^2+g(x)y(D(z)D(x)+D(x)D(z))\}$$
$$+t^2L_1(x,y)+t^3L_2(x,y)+t^4L_3(x,y)=0,\ x,y,z\in U, t\in S_3. \tag{2.172}$$

Since R is 3!-torsion free by assumption, by Lemma 2.1 the relation (2.172) yields

$$([B(x,z),x]+[f(x),z])yD(x)^2+g(x)y(D(z)D(x)+D(x)D(z))$$
$$=0, x,y,z\in U. \tag{2.173}$$

Writing $ug(x)y$ for y in (2.173), we get

$$([B(x,z),x]+[f(x),z])ug(x)yD(x)^2+g(x)ug(x)y(D(z)D(x)+D(x)D(z))$$
$$=0, u,x,y,z\in U. \tag{2.174}$$

Combining (2.159) with (2.174),

$$g(x)ug(x)y(D(z)D(x)+D(x)D(z))=0, u,x,y,z\in U. \tag{2.175}$$

Replacing $y(D(z)D(x)+D(x)D(z))u$ for u in (2.175), we obtain

$$g(x)y(D(z)D(x)+D(x)D(z))ug(x)y(D(z)D(x)+D(x)D(z))$$
$$=0, u,x,y,z\in U. \tag{2.176}$$

And so, by semiprimeness of R, it follows that

$$g(x)y(D(z)D(x)+D(x)D(z))=0, x,y,z\in U. \tag{2.177}$$

Replacing $x+tw$ for x in (2.177), we have

$$g(x+tw)y(D(z)D(x+tw)+D(x+tw)D(z)$$
$$\equiv g(x)y(D(z)D(x)+D(x)D(z))+t\{([B(x,w),x]+[f(x),w])y(D(z)D(x)$$
$$+D(x)D(z))+g(x)y(D(z)D(w)+D(w)D(z))\}+t^2M_1(x,y)+t^3M_2(x,y)$$
$$+t^4g(w)y(D(z)D(w)+D(w)D(z))=0,\ w,x,y,z\in U, t\in S_4 \tag{2.178}$$

where M_1 and M_2 denote the term satisfying the identity (2.178).
From (2.177) and (2.178), we arrive at

$$t\{([B(x,w),x]+[f(x),w])y(D(z)D(x)+D(x)D(z))$$
$$+g(x)y(D(z)D(w)+D(w)D(z))\}+t^2M_1(x,y)+t^3M_2(x,y)$$
$$=0,\ w,x,y,z\in U, t\in S_3. \tag{2.179}$$

Since R is 3!-torsion free by assumption, by Lemma 2.1 the relation (2.179) yields

$$\begin{aligned}&([B(x,w),x]+[f(x),w])y(D(z)D(x)+D(x)D(z))\\ &+g(x)y(D(z)D(w)+D(w)D(z))=0, w,x,y,z\in U.\end{aligned}\tag{2.180}$$

Replacing $ug(x)y$ for y in (2.180), we get

$$\begin{aligned}&([B(x,w),x]+[f(x),w])yg(x)ug(x)y(D(z)D(x)+D(x)D(z))\\ &+g(x)ug(x)y(D(z)D(w)+D(w)D(z))=0, w,x,y,z\in U.\end{aligned}\tag{2.181}$$

Combining (2.177) with (2.181),

$$g(x)ug(x)y(D(z)D(w)+D(w)D(z))=0, w,x,y,z\in U.\tag{2.182}$$

Replacing $y(D(z)D(w)+D(w)D(z))u$ for u in (2.182), we obtain

$$\begin{aligned}&g(x)y(D(z)D(w)+D(w)D(z))ug(x)y(D(z)D(x)+D(x)D(z))\\ &=0, u,w,x,y,z\in U.\end{aligned}\tag{2.183}$$

And so, by semiprimeness of R, it follows from (2.183) that

$$g(x)y(D(z)D(w)+D(w)D(z))=0, x,y,z\in U.\tag{2.184}$$

Let $w=z$ in (2.184). Then we get

$$g(x)yD(z)^2=0, x,y,z\in U.\tag{2.185}$$

Replacing $x+tw$ for x in (2.185), we have

$$\begin{aligned}&g(x+tw)yD(z)^2\equiv g(x)yD(z)^2+t\{([B(x,w),x]+[f(x),w])yD(z)^2\}\\ &+t^2P(x,y)+t^3g(w)yD(z)^2=0,\ w,x,y,z\in U, t\in S_3\end{aligned}\tag{2.186}$$

where P denotes the term satisfying the identity (2.186).
From (2.185) and (2.186), we obtain

$$\begin{aligned}&t\{([B(x,w),x]+[f(x),w])yD(z)^2\}+t^2P(x,y)\\ &=0,\ w,x,y,z\in U, t\in S_3.\end{aligned}\tag{2.187}$$

Since R is 2!-torsion free by assumption, by Lemma 2.1 the relation (2.187) yields

$$([B(x,w),x]+[f(x),w])yD(z)^2=0, w,x,y,z\in U.\tag{2.188}$$

Replacing wx for w in (2.188), we get

$$\begin{aligned}&([B(x,w),x]x+3[w,x]f(x)+3wg(x)+[[w,x],x]D(x)+[f(x),w]x)yD(x)^2\\ &=0, w,x,y,z\in U.\end{aligned}\tag{2.189}$$

From (2.185) and (2.189), we have

$$([B(x,w),x]x+3[w,x]f(x)+[[w,x],x]D(x)+[f(x),w]x)yD(x)^2 = 0, w,x,y,z\in U. \tag{2.190}$$

Substituting xy for y in (2.188),

$$([B(x,w),x]x+[f(x),w]x)yD(z)^2=0, w,x,y,z\in U. \tag{2.191}$$

Combining (2.190) with (2.191),

$$(3[w,x]f(x)+[[w,x],x]D(x))yD(z)^2=0, w,x,y,z\in U. \tag{2.192}$$

Replacing $D(x)w$ for w in (2.192), we obtain

$$(3f(x)wf(x)+3D(x)[w,x]f(x)+g(x)wD(x)+2f(x)[w,x]D(x) +D(x)[[w,x],x]D(x))yD(z)^2=0, w,x,y,z\in U. \tag{2.193}$$

Substituting $D(x)y$ for y in (2.193), we have

$$(3f(x)wf(x)D(x)+3D(x)[w,x]f(x)D(x)+g(x)wD(x)^2 +2f(x)[w,x]D(x)^2+D(x)[[w,x],x]D(x)^2)yD(z)^2=0, w,x,y,z\in U. \tag{2.194}$$

Combining (2.129), (2.179) with (2.194),

$$(2f(x)[w,x]D(x)^2+D(x)[[w,x],x]D(x)^2)yD(z)^2=0, w,x,y,z\in U. \tag{2.195}$$

Left multiplication of (2.195) by $D(x)$ leads to

$$(2D(x)f(x)[w,x]D(x)^2+D(x)^2[[w,x],x]D(x)^2)yD(z)^2 = 0, w,x,y,z\in U. \tag{2.196}$$

Left multiplication of (2.162) by $D(x)$ yields

$$2D(x)f(x)yD(x)f(x)+D(x)^2[y,x]D(x)f(x)=0, x,y\in U. \tag{2.197}$$

Substituting $yD(x)^2w$ for y in (2.197), we get

$$2D(x)f(x)yD(x)^2wD(x)f(x)+D(x)^2[y,x]D(x)^2wD(x)f(x) +D(x)^2y[D(x)^2,x]wD(x)f(x)+D(x)^2yD(x)^2[w,x]D(x)f(x) = 0, w,x,y,z\in U. \tag{2.198}$$

Combining (2.129) with (2.198), we obtain

$$2D(x)f(x)yD(x)^2wD(x)f(x)+D(x)^2[y,x]D(x)^2wD(x)f(x) +D(x)^2yD(x)f(x)wD(x)f(x)+D(x)^2yD(x)^2[w,x]D(x)f(x) = 0, w,x,y,z\in U. \tag{2.199}$$

From (2.162) and (2.199),

$$(2D(x)f(x)yD(x)^2 + D(x)^2[y,x]D(x)^2 - D(x)^2yD(x)f(x))$$
$$wD(x)f(x) = 0, w, x, y, z \in U. \tag{2.200}$$

Replacing $[y,x]$ for y in (2.200), we have

$$(2D(x)f(x)[y,x]D(x)^2 + D(x)^2[[y,x],x]D(x)^2 - D(x)^2yD(x)f(x))$$
$$wD(x)f(x) = 0, w, x, y, z \in U. \tag{2.201}$$

Combining (2.162), (2.170) with (2.201),

$$(4D(x)f(x)yD(x)f(x) + D(x)^2[[y,x],x]D(x)^2)wD(x)f(x)$$
$$= 0, w, x, y, z \in U. \tag{2.202}$$

From (2.129) and (2.196),

$$(2D(x)f(x)[y,x]D(x)^2 + D(x)^2[[y,x],x]D(x)^2w[D(z)^2,z]$$
$$= (2D(x)f(x)[y,x]D(x)^2 + D(x)^2[[y,x],x]D(x)^2wD(z)f(z)$$
$$= 0, w, x, y, z \in U. \tag{2.203}$$

Let $z = x$ in (2.203). Then

$$(2D(x)f(x)[y,x]D(x)^2 + D(x)^2[[y,x],x]D(x)^2wD(x)f(x)$$
$$= 0, w, x, y, z \in U. \tag{2.204}$$

Combining (2.203) with (2.204),

$$2(2D(x)f(x)yD(x)f(x) - D(x)f(x)[y,x]D(x)^2wD(x)f(x)$$
$$= 0, w, x, y, z \in U. \tag{2.205}$$

Since R is 2!-torsion free by assumption, the relation (2.204) yields

$$(2D(x)f(x)yD(x)f(x) - D(x)f(x)[y,x]D(x)^2wD(x)f(x)$$
$$= 0, w, x, y, z \in U. \tag{2.206}$$

From (2.206), we have

$$(2D(x)f(x)yD(x)f(x) - D(x)f(x)[y,x]D(x)^2)w(2D(x)f(x)yD(x)f(x)$$
$$-D(x)f(x)[y,x]D(x)^2) = 0, w, x, y, z \in U. \tag{2.207}$$

And so, by semiprimeness of R, it follows from (2.207) that

$$2D(x)f(x)yD(x)f(x) - D(x)f(x)[y,x]D(x)^2 = 0, x, y, z \in U. \tag{2.208}$$

Combining (2.170) with (2.208),

$$D(x)f(x)yD(x)f(x) = 0, x, y \in U. \tag{2.209}$$

And so, by semiprimeness of R, obtain from (2.209)

$$D(x)f(x) = 0, x \in U.$$

($\Longleftarrow$): Suppose that

$$D(x)f(x) = 0, x \in R. \tag{2.210}$$

Replacing $x + ty$ for x in (2.210), we have

$$\begin{aligned} &D(x+ty)[D(x+ty), x+ty] \equiv +t\{D(y)f(x) + D(x)B(x,y)\} \\ &+t^2Q(x,y) + t^3D(y)f(y) = 0,\ x, y \in R, t \in S_3 \end{aligned} \tag{2.211}$$

where Q denotes the term satisfying the identity (2.211).
From (2.210) and (2.211), we get

$$t\{D(y)f(x) + D(x)B(x,y)\} + t^2Q(x,y) = 0,\ x, y \in R, t \in S_3. \tag{2.212}$$

Since R is 2!-torsion free by assumption, by Lemma 2.1 the relation (2.212) yields

$$D(y)f(x) + D(x)B(x,y) = 0, x, y \in R. \tag{2.213}$$

Let $y = x^2$ in (2.213). Then using (2.210), we obtain from (2.213)

$$\begin{aligned} &(D(x)x + xD(x))f(x) + 2D(x)(f(x)x + xf(x)) \\ &= 3D(x)xf(x) + xD(x)f(x) + 2D(x)f(x)x = 0, x \in R. \end{aligned} \tag{2.214}$$

From (2.210) and (2.214) , we get

$$3D(x)xf(x) = 3f(x)^2 = -3D(x)g(x) = 0, x \in R. \tag{2.215}$$

Since R is 3!-torsion free, it follows from (2.215) that

$$f(x)^2 = 0, x \in R, \tag{2.216}$$

and

$$D(x)g(x) = 0, x \in R. \tag{2.217}$$

Writing xy for y in (2.213), we obtain

$$\begin{aligned} &(xD(y)f(x) + D(x)yf(x) + D(x)(xB(x,y) + 2f(x)y + D(x)[y,x] \\ &= 0, x, y \in R. \end{aligned} \tag{2.218}$$

Left multiplication of (2.218) by $D(x)$ leads to

$$xD(y)f(x) + xD(x)B(x,y) = 0, x \in R. \tag{2.219}$$

Combining (2.218) with (2.219),

$$D(x)yf(x) + f(x)B(x,y) + 2D(x)f(x)y + D(x)^2[y,x] = 0, x, y \in R. \tag{2.220}$$

From (2.210) and (2.220), we have

$$D(x)yf(x) + f(x)B(x,y) + D(x)^2[y,x] = 0, x, y \in R. \tag{2.221}$$

Left multiplication of (2.221) by $D(x)$ yields

$$D(x)^2yf(x) + D(x)f(x)B(x,y) + D(x)^3[y,x] = 0, x, y \in R. \tag{2.222}$$

Comparing (2.210) and (2.222),

$$D(x)^2yf(x) + D(x)^3[y,x] = 0, x, y \in R. \tag{2.223}$$

Putting yx instead of y in (2.213), we get

$$\begin{aligned}&(D(y)x + yD(x))f(x) + D(x)(B(x,y)x + 2yf(x) + [y,x]D(x))\\ &= 0, x, y \in R.\end{aligned} \tag{2.224}$$

Right multiplication of (2.224) by x gives

$$D(y)f(x)x + D(x)B(x,y)x = 0, x \in R. \tag{2.225}$$

Combining (2.224) with (2.225),

$$-D(y)g(x) + yD(x)f(x) + 2D(x)yf(x) + D(x)[y,x]D(x) = 0, x, y \in R. \tag{2.226}$$

From (2.210) and (2.226), we have

$$-D(y)g(x) + 2D(x)yf(x) + D(x)[y,x]D(x) = 0, x, y \in R. \tag{2.227}$$

Writing xy for y in (2.227), we get

$$\begin{aligned}&-xD(y)g(x) - D(x)yg(x) + 2D(x)xyf(x) + D(x)x[y,x]D(x)\\ &= 0, x, y \in R.\end{aligned} \tag{2.228}$$

Left multiplication of (2.227) by x leads to

$$-xD(y)g(x) + 2xD(x)yf(x) + xD(x)[y,x]D(x) = 0, x, y \in R. \tag{2.229}$$

Combining (2.228) with (2.229),

$$-D(x)yg(x) + 2f(x)yf(x) + f(x)[y,x]D(x) = 0, x, y \in R. \tag{2.230}$$

Left multiplication of (2.230) by $D(x)$ yields

$$-D(x)^2yg(x) + 2D(x)f(x)yf(x) + D(x)f(x)[y,x]D(x) = 0, x, y \in R. \tag{2.231}$$

Comparing (2.210) and (2.231), we obtain

$$D(x)^2yg(x) = 0, x, y \in R. \tag{2.232}$$

Let $y = D(x)$ in (2.227). Then we get

$$-D^2(x)g(x) + 2D(x)^2f(x) + D(x)f(x)D(x) = 0, x, y \in R. \tag{2.233}$$

Combining (2.210) with (2.233),

$$D^2(x)g(x) = 0, x \in R. \tag{2.234}$$

Writing $yD(x)$ for y in (2.227), we have

$$\begin{aligned}&-D(y)D(x)g(x) - yD^2(x)g(x) + 2D(x)yD(x)f(x) + D(x)[y,x]D(x)^2\\ &+D(x)yf(x)D(x) = 0, x, y \in R.\end{aligned} \tag{2.235}$$

Combining (2.210), (2.217), (2.234) with (2.235),

$$D(x)[y,x]D(x)^2 + D(x)yf(x)D(x) = 0, x, y \in R. \tag{2.236}$$

Left multiplication of (2.236) by $f(x)$ leads to

$$f(x)D(x)[y,x]D(x)^2 + f(x)D(x)yf(x)D(x) = 0, x, y \in R. \tag{2.237}$$

Writing $yD(x)$ for y in (2.230), we get

$$\begin{aligned}&-D(x)yD(x)g(x) + 2f(x)yD(x)f(x) + f(x)[y,x]D(x)^2 + f(x)yf(x)D(x)\\ &= 0, x, y \in R.\end{aligned} \tag{2.238}$$

Comparing (2.210) and (2.217), we obtain from (2.238)

$$f(x)[y,x]D(x)^2 + f(x)yf(x)D(x) = 0, x, y \in R. \tag{2.239}$$

Substituting $D(x)y$ for y in (2.230), we have

$$\begin{aligned}&-D(x)^2yg(x) + 2f(x)D(x)yf(x) + f(x)D(x)[y,x]D(x) + f(x)^2yD(x)\\ &= 0, x, y \in R.\end{aligned} \tag{2.240}$$

From (2.216), (2.232) and (2.240), we obtain

$$2f(x)D(x)yf(x) + f(x)D(x)[y,x]D(x) = 0, x, y \in R. \tag{2.241}$$

Right multiplication of (2.241) by $D(x)$ leads to

$$2f(x)D(x)yf(x)D(x) + f(x)D(x)[y,x]D(x)^2 = 0, x, y \in R. \tag{2.242}$$

From (2.237) and (2.242),

$$f(x)D(x) = 0, x \in R.$$

□

Theorem 2.5. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$. Then we have*

$$[D(x), x]D(x) \in rad(A) \iff D(x)[D(x), x] \in rad(A)$$

for all $x \in A$.

Proof. By the result of B.E. Johnson and A.M. Sinclair[5] any linear derivation on a semisimple Banach algebra is continuous. Sinclair[12] proved that every continuous linear Jordan derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a linear Jordan derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P,\ \hat{x} = x + P$. Then D is a derivation on A/P. By the assumption that $D(x)^2 f(x) \in \mathrm{rad}(A),\ x \in A$, we obtain $(D_P(\hat{x}))^2[D_P(\hat{x}), \hat{x}] = 0,\ \hat{x} \in A/P$, since all the assumptions of Theorem 2.4 are fulfilled. And since the prime and factor algebra A/P is noncommutative, from Theorem 2.4 we have $[D_P(\hat{x}), \hat{x}]^7 = 0,\ \hat{x} \in A/P$. And for each P, by the elementary properties of the spectral radius r_P in a Banach algebra A/P, it follows that $r_P([D_P(\hat{x}), \hat{x}])^7 = r_P([D_P(\hat{x}), \hat{x}]^7) = 0$ for all $\hat{x} \in A/P$. Hence we obtain $r_P([D_P(\hat{x}), \hat{x}]) = 0$ for all $\hat{x} \in A/P$. Thus $[D_P(\hat{x}), \hat{x}] \in Q(A/P)$ for all $\hat{x} \in A/P$. On the one hand, since D is continuous, we see that D_P is also continuous. Thus by Theorem 2.3, we obtain $D_P(A/P) \subseteq \mathrm{rad}(A/P)$. But since A/P is semisimple, $D_P(A/P) = \{0\}$ for all primitive ideals of A. Hence we see that $D(A) \subseteq P$ for all primitive ideals of A. And so, $D(A) \subseteq \mathrm{rad}(A)$. On the other hand, In case A/P is a commutative Banach algebra, one can conclude that $D_P = 0$ as well, since A/P is semisimple and since we know that there are no nonzero linear derivations on a commutative semisimple Banach algebra. In other words, $D(x) \in P$ for all primitive ideals of A and all $x \in A$. i.e. we get $D(A) \subseteq \mathrm{rad}(A)$. Therefore in any case we have $D(A) \subseteq \mathrm{rad}(A)$. □

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