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On Properties of New Fractional Derivative with Mittag-Leffler Kernel of Two Parameters

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Abstract

In this paper we derive some properties of a new fractional derivative with Mittag-Leffler kernel of two parameters recently introduced by Chinchole and Bhadane. We also describe its application to fractional falling body problem.

Keywords: Fractional calculus, Chinchole-Bhadane fractional derivative, fractional integral.

2020 Mathematics Subject Classifications : 26A33, 34K37

1. INTRODUCTION

The aim of this paper was to describe and prove some properties of new fractional derivative with Mittag-Leffler kernel of two parameters defined by Chinchole and Bhadane (see [2]). In this paper the application of the same derivative to the fractional falling body problem is also explained. We refer the paper [3] on the proprties of a new fractional derivative without singular kernel written by Losada and Nieto.

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2. PRELIMINARIES

Let us recall the well known definition of Atangana and Baleanu fractional derivative [1]. Given $f \in H^1(0,b), 0 < b, \alpha \in [0,1]$, the AB Caputo fractional derivative of f with non-local and non-singular kernel is given by

$$ {}^{ABC}{}_0D_t^{\alpha} f(t) = \frac{N(\alpha)}{1-\alpha} \int_0^t f'(\tau) E_\alpha \left[-\frac{\alpha(t-\tau)^\alpha}{1-\alpha} \right] d\tau \tag{1} $$

with $N(\alpha)$ being a normalisation function mentioned in Caputo fractional time derivative. By changing the kernel $E_\alpha \left[-\frac{\alpha(t-\tau)^\alpha}{1-\alpha} \right]$ with the Mittag-Leffler function $t^{\beta-1} E_{\alpha,\beta} \left[-\frac{(\alpha+\beta-1)t^\alpha}{2-\alpha-\beta} \right]$ of two parameters and $\frac{N(\alpha)}{1-\alpha}$ by $\frac{B(\alpha,\beta)}{2-\alpha-\beta}$, one obtains the Chinchole-Bhadane fractional derivative of order $0 < \alpha, \beta < 1$, which has been recently introduced by Chinchole and Bhadane in [2]. That is,

$$ {}^{SABC}{}_0D_t^{\alpha,\beta} f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta} \int_0^t f'(\tau)(t-\tau)^{\beta-1} E_{\alpha,\beta} \left[-\frac{(\alpha+\beta-1)(t-\tau)^\alpha}{2-\alpha-\beta} \right] d\tau \tag{2} $$

where $B(\alpha,\beta)$ is a normalization function such that $B(\alpha,\beta) = N(\alpha+\beta-1)$. It is well known that Laplace Transform plays an important role in the study of ordinary differential equations. In the case of this new fractional definition, it is also known (see [2]) that, for $0 < \alpha, \beta < 1$,

$$ \mathcal{L} \left\{ {}^{SABC}{}_0D_t^{\alpha,\beta} f(t), p \right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta} \frac{p^{\alpha-\beta} \left[p\mathcal{L}\left\{ f(t), p \right\} - f(0) \right]}{p^\alpha + \frac{\alpha+\beta-1}{2-\alpha-\beta}}. \tag{3} $$

where $\mathcal{L}\{g(t), p\}$ denotes the Laplace transform of function $g(t)$ to the function of another variable p. The fractional integral in two parameters α, β corresponds to the new fractional derivative with non-local and non-singular kernel (see [2]) is defined as

$$ {}^{SAB}{}_0I_t^{\alpha,\beta} f(t) = \begin{cases} \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \int_0^t f(y)(t-y)^{-\beta} dy + & \\ \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \int_0^t f(y)(t-y)^{\alpha-\beta} dy & \beta \neq 1 \\ \frac{1-\alpha}{N(\alpha)} f(t) + \frac{\alpha}{N(\alpha)\Gamma(\alpha)} \int_0^t f(y)(t-y)^{\alpha-1} dy & \beta = 1 \end{cases} \tag{4} $$

3. SOLUTIONS OF SOME FRACTIONAL DIFFERENTIAL EQUATIONS

In this section we study some simple but useful fractional differential equations.

Lemma 1. *Let $0 < \alpha, \beta < 1$ and f be a solution of the following fractional differential equation,*

$$ {}^{SABC}{}_0D_t^{\alpha,\beta} f(t) = 0, \quad t \geq 0. $$

Then, f is a constant function.

Proof. Taking Laplace transform on both sides of differential equation, we get

$$\mathcal{L}\left\{{}^{SABC}{}_0D_t^{\alpha,\beta} f(t), p\right\} = 0, \;\; p > 0. \tag{5}$$

Hence from equation (3), we have

$$\frac{B(\alpha,\beta)}{2-\alpha-\beta} \frac{p^{\alpha-\beta}\left[p\mathcal{L}\left\{f(t),p\right\} - f(0)\right]}{p^{\alpha} + \frac{\alpha+\beta-1}{2-\alpha-\beta}} = 0, \;\; p > 0.$$

$$\Rightarrow \mathcal{L}\left\{{}^{SABC}{}_0D_t^{\alpha,\beta} f(t), p\right\} = \frac{1}{p} f(0).$$

After taking inverse Laplace transform on both sides, we get

$$f(t) = f(0) \quad \forall t \geq 0.$$

This proves that f is a constant function.

Theorem 1. *Let* $0 < \alpha, \beta < 1$. *Then, the unique solution of the following initial value problem*

$$ {}^{SABC}{}_0D_t^{\alpha,\beta} f(t) = \sigma(t), \;\; t \geq 0. \tag{6}$$

$$f(0) = f_0; \tag{7}$$

is given by

$$f(t) = f_0 + A_{\alpha,\beta} \int_0^t \sigma(y)(t-y)^{-\beta} dy + B_{\alpha,\beta} \int_0^t \sigma(y)(t-y)^{\alpha-\beta} dy \tag{8}$$

where

$$A_{\alpha,\beta} = \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)}, B_{\alpha,\beta} = \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)}. \tag{9}$$

Proof. Suppose that the initial value problem (6)-(7) has two solutions, $f_1(t)$ and $f_2(t)$. In that case, we have

$${}^{SABC}{}_0D_t^{\alpha,\beta} f_1(t) - {}^{SABC}{}_0D_t^{\alpha,\beta} f_2(t) = \sigma(t) - \sigma(t) = 0, \;\; t \geq 0.$$

and

$$f_1(0) - f_2(0) = f_0 - f_0 = 0.$$

That is

$${}^{SABC}{}_0D_t^{\alpha,\beta} \left[f_1 - f_2\right] f(t) = 0, \;\; t \geq 0$$

and

$$\left[f_1 - f_2\right] f(0) = 0.$$

So, by Lemma (1), we have that $f_1 - f_2 = 0$. That is $f_1(t) = f_2(t)$ for all $t \geq 0$. By equation (3), after taking Laplace transform on both sides of differential equation (6), we get

$$\begin{aligned}\mathcal{L}\{f(t), p\} = \frac{1}{p}f(0) &+ \frac{2-\alpha-\beta}{B(\alpha,\beta)}\frac{1}{p^{-\beta+1}}\mathcal{L}\{\sigma(t), p\} \\ &+ \frac{\alpha+\beta-1}{B(\alpha,\beta)}\frac{1}{p^{\alpha-\beta+1}}\mathcal{L}\{\sigma(t), p\}.\end{aligned}$$

After taking inverse Laplace transform on both sides, we get

$$\begin{aligned}f(t) = f_0 &+ \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)}\int_0^t \sigma(y)(t-y)^{-\beta}dy \\ &+ \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)}\int_0^t \sigma(y)(t-y)^{\alpha-\beta}dy.\end{aligned}$$

Remark 1. *For $\beta = 1$, we have*

$$f(t) = f_0 + \frac{1-\alpha}{N(\alpha)}\sigma(t) + \frac{\alpha}{N(\alpha)\Gamma(\alpha)}\int_0^t \sigma(y)(t-y)^{\alpha-1}dy. \tag{10}$$

Remark 2. *For $\alpha = \beta = 1$, we have the solution of (6) is the usual primitive of σ.*

Corollary 1. *Let $0 < \alpha, \beta < 1$. Then, the unique solution of the following initial value problem*

$$\begin{aligned}&{}^{SABC}{}_0D_t^{\alpha,\beta}f(t) = \lambda f(t) + u(t), \;\; t \geq 0 \\ &f(0) = f_0.\end{aligned} \tag{11}$$

where $\lambda \in \mathbb{R}, \lambda \neq 0$ ($\lambda = 0$ corresponds to the case previously studied) is given by

$$\begin{aligned}f(t) - \lambda \;\; &A_{\alpha,\beta}\int_0^t f(y)(t-y)^{-\beta}dy - \lambda B_{\alpha,\beta}\int_0^t f(y)(t-y)^{\alpha-\beta}dy = f_0 \\ &+ \;\; A_{\alpha,\beta}\int_0^t u(y)(t-y)^{-\beta}dy + B_{\alpha,\beta}\int_0^t u(y)(t-y)^{\alpha-\beta}dy.\end{aligned} \tag{12}$$

Proof. From Theorem (1), it is clear that solving equation (11) is equivalent to find a function f such that

$$\begin{aligned}f(t) = f_0 &+ A_{\alpha,\beta}\int_0^t (\lambda f + u)(y)(t-y)^{-\beta}dy \\ &+ B_{\alpha,\beta}\int_0^t (\lambda f + u)(y)(t-y)^{\alpha-\beta}dy, \;\; t \geq 0.\end{aligned}$$

This gives the unique solution as given in the equation (12).

4. APPLICATION TO FRACTIONAL FALLING BODY PROBLEM

Consider a mass m falling due to gravity. The net force acting on the body is equal to the rate of change of the momentum of that body. For constant mass, applying the classical Newton second law, we have

$$mv^{'}(t) = mg - kv(t), \tag{13}$$

where g is the gravitational constant, and the air resistance is proportional to the velocity with proportionality constant k. If air resistance is negligible, then $k = 0$ and the equation simplifies to

$$v^{'}(t) = g. \tag{14}$$

If we replace first order derivative $v^{'}$ by fractional derivative ${}^{SABC}{}_{0}D_{t}^{\alpha,\beta}$ we have the following fractional falling body equation

$${}^{SABC}{}_{0}D_{t}^{\alpha,\beta}v(t) = -\frac{k}{m}v(t) + g. \tag{15}$$

For an initial velocity $v(0) = v_0$ then, according to Corollary (1), it has a unique solution.

5. CONCLUSIONS

In this paper we describe and prove some properties of new fractional derivative with Mittag-Leffler kernel of two parameters defined by Chinchole and Bhadane (see [2]). The application of the same derivative to the fractional falling body problem is also explained.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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REFERENCES

[1] Abdon Atangana and Dumitru Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *arXiv preprint arXiv:1602.03408*, 2016.

[2] S M Chinchole and A P Bhadane, A new definition of fractional derivatives with Mittag-Leffler kernel of two parameters. *Communications in Mathematics and Applications*, v.13, No.1, 2022, Forthcoming.

[3] Jorge Losada and Juan J Nieto, Properties of a new fractional derivative without singular kernel. *Progr. Fract. Differ. Appl*, v.1, No.2, 2015, pp.87–92.