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Some Observations on Koide Formula

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Abstract. The Koide parameter for leptons and quarks is discussed. A probabilistic approach is used to verify if the results obtained in the various cases are purely coincidental.

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In 1982 Koide discovered the following relation among lepton masses of the Standard Model [1]

$$\frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} \equiv K \approx \frac{2}{3}. \tag{1}$$

The value of K lies in the range $1/3 \leqslant K < 1$ for arbitrary masess. In fact, consider the two vectors $u = (\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})$ and $v = (1, 1, 1)$, then from Cauchy–Schwarz inequality one has $\left(\sum_{i=1}^{3} u_i v_i\right)^2 \leqslant \left(\sum_{i=1}^{3} u_i^2\right)\left(\sum_{i=1}^{3} v_i^2\right)$, being equal to $\left(\sum_{i=1}^{3} u_i^2\right) \times 3$, thus obtaining the lower bound of K. The upper bound 1 for K is found when considering the limit of one mass being much larger than the other two.

According to the PDG data [2] when using the measured values for lepton masses one has the following value of K

$$K = \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3} \times (0.999991 \pm 0.000010) \tag{2}$$

which is remarkably close to $2/3$, in the center of the allowed values for K.

The same relation has been observed for the quark masses [3–5], and from [2] one obtains, respectively for the light and heavy quarks, (m_u, m_d, m_s) and (m_c, m_b, m_t), the Koide parameters

$$K^q_{light} = 0.5622 \pm 0.0010, \tag{3}$$

$$K^q_{heavy} = \frac{2}{3} \times (1.0042 \pm 0.0020). \tag{4}$$

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As pointed out by Koide himself [6, 7], there are actually two values for the parameter K, depending on the chosen mass used for the calculation of (1). The renormalization group gives the evolution of the observed mass at a scale μ from the pole mass. For the leptons this function is given by [8]

$$m(\mu) = m_{pole}\left[1 - \frac{\alpha(\mu)}{\pi}\left(1 + \frac{3}{4}\ln\frac{\mu^2}{m_{pole}^2}\right)\right] \tag{5}$$

and at the Z scale $\mu = m_Z$ one obtains the Koide parameter for pole masses

$$K_{pole} = \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3} \times (0.998103 \pm 0.000010), \tag{6}$$

which is different from (2), not even being compatible with the value of 2/3. From (5) one could observe that in absence of the logarithmic term the two Koide parameters obtained from the observed and pole masses could still be compatible. The Sumino model [9] contains a mechanism that cancels out the logarithmic term, but at the expense of having anomalies. In another paper [10] Koide and Yamashita propose a modification of Sumino model in a SUSY scenario in order to avoid some shortcomings of the original model.

We will not purse further the solution of the Koide problem by proposing another model for the electroweak sector. Our approach is to proceed the other way round to verify whether the value of K in (1) is a coincidence. Namely, given the function of (1), find out what is the probability to obtain a value at least close to 2/3 for a generic choice of masses.

Dividing numerator and denominator of (1) by the largest mass, we end up with a function of only two variables, x and y, representing the fraction of two masses with respect to the largest one, $0 \leqslant x, y \leqslant 1$. After this operation we obtain the function f

$$f(x, y) = \frac{x + y + 1}{(\sqrt{x} + \sqrt{y} + 1)^2} \tag{7}$$

which average value is given by the expression

$$\langle f\rangle(x, y) = \frac{1}{xy}\int_0^x \int_0^y f(x', y')dx'dy' \tag{8}$$

and its variance

$$\sigma_f^2(x, y) = \frac{1}{xy}\int_0^x \int_0^y [f(x', y') - \langle f\rangle(x, y)]^2 dx'dy'. \tag{9}$$

It is possible to express (8) in a closed form

$$\begin{aligned}\langle f\rangle(x, y) = \frac{1}{xy}(&-\sqrt{x}\sqrt{y}\left(\left(4 - 3\sqrt{x}\right)\sqrt{y} + 4\left(x + \sqrt{x} + 1\right) + 4y\right) + \\ &+ 4\left(x\left(x + 2\sqrt{x} + 2\right) + y\left(y + 2\sqrt{y} + 2\right) - 1\right)\ln\left(\sqrt{x} + \sqrt{y} + 1\right) - \\ &- 4\left(x\left(x + 2\sqrt{x} + 2\right) - 1\right)\ln\left(\sqrt{x} + 1\right) - 4\left(y\left(y + 2\sqrt{y} + 2\right) - 1\right)\ln\left(\sqrt{y} + 1\right))\end{aligned} \tag{10}$$

as well as (9), however the latter is a very long and not very enlightning expression.

Calculating the average value of f and its standard deviation in the full mass space for which $0 \leqslant x, y \leqslant 1$ we obtain

$$\langle f\rangle(1, 1) = 36\ln 3 - 32\ln 2 - 17,$$

$$\langle f\rangle(1, 1) \pm \sigma_f(1, 1) = 0.369 \pm 0.041. \tag{11}$$

The value of $\langle f\rangle(1,1)$ obtained is very far from $2/3$ by more than $7\sigma_f$. A random choice for the values of three masses should give therefore a result for the Koide parameter very different from (1).

This result comes out when neglecting the hierarchy of lepton masses and the hierarchies for quark masses. Measured values give the following proportions of masses

$$\begin{aligned} (m_e : m_\mu : m_\tau) &\approx (\varepsilon^2 : \varepsilon : 1), \\ (m_c : m_b : m_t) &\approx (\varepsilon^2 : \varepsilon : 1), \\ (m_u : m_d : m_s) &\approx (\varepsilon : \varepsilon : 1), \end{aligned} \tag{12}$$

showing that the hierarchies among leptons and heavy quarks are similar, and as a consequence so are their Koide parameters. Using the hierarchies found in (12) for evaluating $\langle f\rangle$ and σ_f of leptons, heavy quarks and light quarks one has the following results:

$$\langle f\rangle\left(\frac{m_e}{m_\tau},\frac{m_\mu}{m_\tau}\right)\pm\sigma_f\left(\frac{m_e}{m_\tau},\frac{m_\mu}{m_\tau}\right)=0.752\pm 0.065, \tag{13}$$

$$\langle f\rangle\left(\frac{m_c}{m_t},\frac{m_b}{m_t}\right)\pm\sigma_f\left(\frac{m_c}{m_t},\frac{m_b}{m_t}\right)=0.756\pm 0.051, \tag{14}$$

$$\langle f\rangle\left(\frac{m_u}{m_s},\frac{m_d}{m_s}\right)\pm\sigma_f\left(\frac{m_u}{m_s},\frac{m_d}{m_s}\right)=0.663\pm 0.061. \tag{15}$$

The first two averages, for leptons and heavy quarks, have the approximate value of $3/4$, and are distant from $2/3$ by less than $2\sigma_f$. The average for light quarks is also closer than $2\sigma_f$ to the value obtained in (3).

When the hierarchy of masses like the one illustrated in (12) is taken into account the average value $\langle f\rangle$ is already much closer to the expected result of (1) for the K parameter.

Supposing that the true value of leptonic Koide parameter is in the proximity of (1) we have determined the allowed range for the fraction of the two lighter masses x, y for which (7) assumes this value.

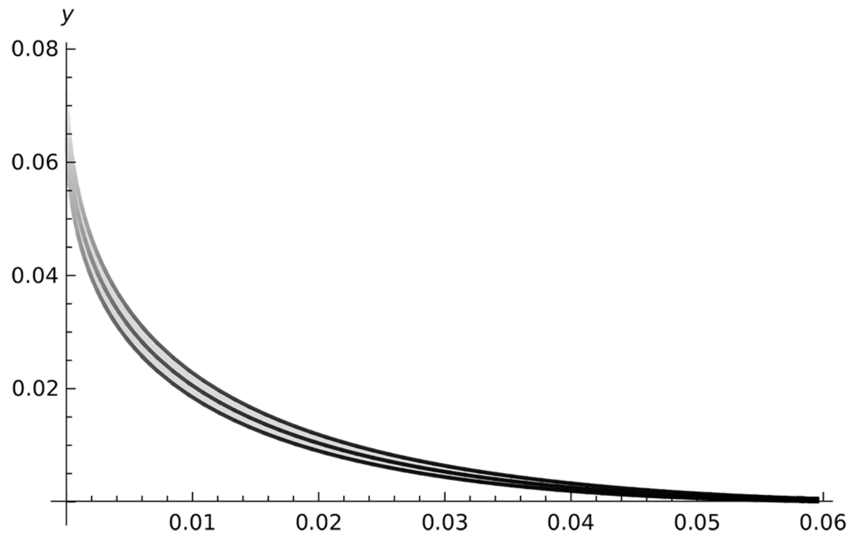

Fig. 1. Fraction of masses range of two lighter leptons for Koide parameter in the $2/3$ region. The x value of the second heaviest particle has the same upper limit found for leptons m_μ/m_τ

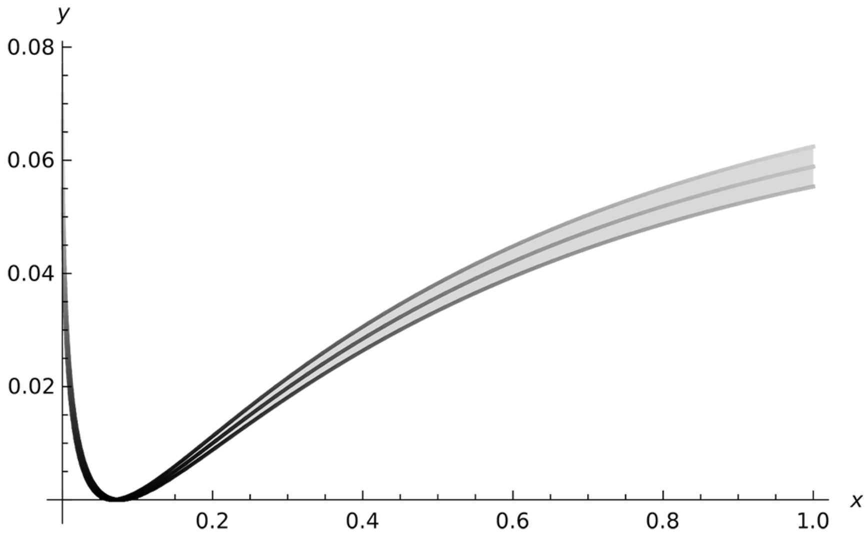

Fig. 2. Fraction of masses range of two lighter leptons for Koide parameter in the 2/3 region

In Figs. 1, 2 we show this x, y range for $K = 2/3 \pm 1\%$. The central curve represents the mean value, the shaded area the allowed fluctuation. In Fig. (1) the upper bound of x is given by the ratio m_μ/m_τ, and is clearly seen how in the allowed gray area the ratio y/x is approximately the same as found in (12). This situation is even more evident in Fig. (2) where the x axis has the $[0, 1]$ range while the y axis range is smaller by an order of magnitude.

We therefore conclude that it is possible to obtain a value close to (1) only if the random choice of masses is also constrained by the hierarchy found in (12). If the hierarchy is neglected and all mass terms are chosen to be completely independent from each other the result of 2/3 is quite different from the average value given in (11). Although there is a real discrepancy between the values found in (2) and (6) for pole and running masses respectively, it cannot be ruled out that those differences are due to other effects like missing perturbative terms of higher orders as well as model choice dependencies.

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Некоторые наблюдения над формулой Койде

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Аннотация. Обсуждается параметр Койде для лептонов и кварков. Вероятностный подход используется для проверки того, являются ли результаты, полученные в различных случаях, чисто случайными.

Ключевые слова: параметр Койде, лептонные массы, теория вероятностей.