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NONLINEAR PHYSICS AND MECHANICS

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A Special Case of Rolling Tire Vibrations

I. F. Kozhevnikov

We investigate a special case of vibrations of a loaded tire rolling at constant speed without slipping in the contact area. A previously proposed analytical model of a radial tire is considered. The surface of the tire is a flexible tread combined with elastic sidewalls. In the undeformed state, the sidewalls are represented by parts of two tori and consist of incompressible rubber described by the Mooney – Rivlin model. In the undeformed state, the tread is a circular cylinder. The tread is reinforced with inextensible cords. The tread deformations are considered taking into account the exact nonlinear conditions of inextensibility of reinforcing cords. Due to nonlinear geometric constraints in the deformed state, the tread retains its cylindrical shape, which is not circular for a typical configuration. The contact between the wheel and the ground plane occurs by a part of the tread. The previously obtained partial differential equation which describes the tire radial in-plane vibrations about the steady-state regime of rolling is investigated. Analyzing the discriminant of the quartic polynomial, which is the function of the frequency of the tenth degree and the function of the angular velocity of sixth degree, the rare case of two pairs of multiple roots is discovered. If the geometry of the tire and the internal tire pressure are known, then the angular velocity of rotation, the tire speed and the natural frequency, corresponding to this case, are determined analytically. The mode shape of vibration in the neighborhood of the singular point is determined analytically.

Keywords: radial tire, analytical model, rolling, modal analysis, vibrations, multiple roots

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Ivan F. Kozhevnikov
kogevnik@ccas.ru
http://www.ccas.ru/depart/mechanics/TUMUS/Kozhevnikov/Kozhevnikov.html
Dorodnicyn Computing Centre, Federal Research Center “Computer Science and Control”
of Russian Academy of Sciences (CC FRC CSC RAS)
ul. Vavilova 40, Moscow, 119333 Russia

1. Introduction

Natural frequency (NF) is the frequency at which a mechanical system tends to oscillate in the absence of external or damping force. When a system vibrates at a frequency of applied external force and this frequency is equal to the NF, then the system vibration amplitude highly increases. This circumstance could lead to damage. That is why it is very important to know the NF of the structure. For each NF the corresponding mode shape (MS) exists. A mode is a standing wave which oscillates but whose peak amplitude does not move in space.

If we consider the tire rotating with constant angular velocity without contact, then each NF of an unloaded nonrotating tire corresponds to two different NF of an unloaded rotating tire. This is a well-known effect: the two counter-rotating waves, superimposed onto a standing vibration in resting structures, are distinct in speed for a rotating system.

In the case of a nonrotating tire with contact, for each NF of an unloaded nonrotating (UNR) tire there are two different NF of a loaded nonrotating (LNR) tire [1, 2]. The fixed contact points of the tire cause a loss of the circular symmetry and the disturbance of free wave motion. The identical modes split into two not identical ones. The MS subdivides into a symmetric and an antisymmetric shape. The mass center of the tire does not move in the longitudinal direction for the symmetric MS, and it moves for the antisymmetric MS. Thus, the antisymmetric MS “sways” from side to side.

If we consider the tire rotating with constant angular velocity with contact, then the increase in the angular velocity implies that NF decreases. The previously observed split of the NF of an unloaded rotating tire caused by rotation disappears under rolling conditions due to the disturbed symmetry [2, 3]. A phenomenon of frequency loci veering is observed: NF as functions of angular velocity approach each other and then veer away instead of crossing [2, 4]. The MS interact in veering region and, consequently, interchange. Thus, in the dynamical system there are many difficult interesting phenomena.

Current problems of investigating the dynamics require fast calculating models. Therefore, the problem of constructing models which simulate complex dynamic processes and do not require significant computational resources is very important. A model of a reinforced tire was proposed in [5]. In the case of a wheel rolling without slipping in the contact area, an unknown in advance, a complete system of equations of motion was obtained. The steady-state regime of rolling at constant speed was investigated. This tire model was also used in studying the vibrations of an UNR and LNR tire [6] and in studying the steady-state cornering on the plane with slipping [7]. In [2] we study the vibrations of an unloaded and loaded tire rolling at constant speed. Supposing that all the roots of the characteristic equation are different, the NF were determined numerically and MS were determined analytically for a loaded rotating tire. The quantities of the NF obtained for the UNR and LNR tire were compared with experiments: Experiment I[1] (UNR), Experiment II (UNR) and Experiment III[2] (LNR) [8].

The idea to study in detail the roots of a characteristic equation arose. It was observed that for any value of the angular velocity of a wheel there are several frequencies at which

[1]TMPT (Tyre Model Performance Test) was developed by an international group of experts consisting of members of vehicle industry, tire manufacturers, tire model developers, multi-body-system program suppliers and universities. It was organized by: Prof. P. Lugner and Prof. M. Plöchl (Institute of Mechanics and Mechatronics, Div. of Vehicle System Dynamics and Biomechanics, Vienna University of Technology, Austria).

[2]A similar experiment was also performed by the author in 2005 in the laboratory LAMI (Laboratoire d’Analyse des Matériaux et Identification) ENPC (L’Ecole Nationale des Ponts et Chaussées).

two roots can coincide. For these frequencies the frequency function (the function from the frequency equation which defines infinite spectra of NF) tends to zero. This means that they must probably be added to the spectra of NF. If there are multiple roots, then the MS is represented in a different form, and the problem must be solved differently. In the process of research an even rarer special case of two pairs of multiple roots was discovered. This special case is considered in this paper.

The paper is structured as follows. First, we briefly describe the model of a wheel with a reinforced tire proposed in [5]. Next, we recall the main features of the general case of vibrations of a loaded rotating tire [2]. Then we consider the special case of two pairs of multiple roots. In the concluding remarks we discuss the results.

2. Tire model

Assume that the wheel with a reinforced tire consists of a disc joined to the sidewalls and of a tread (Fig. 1a). The wheel disc is a rigid body with six degrees of freedom. In the undeformed state, the sidewalls are represented by parts of two tori. The elastic sidewalls of incompressible

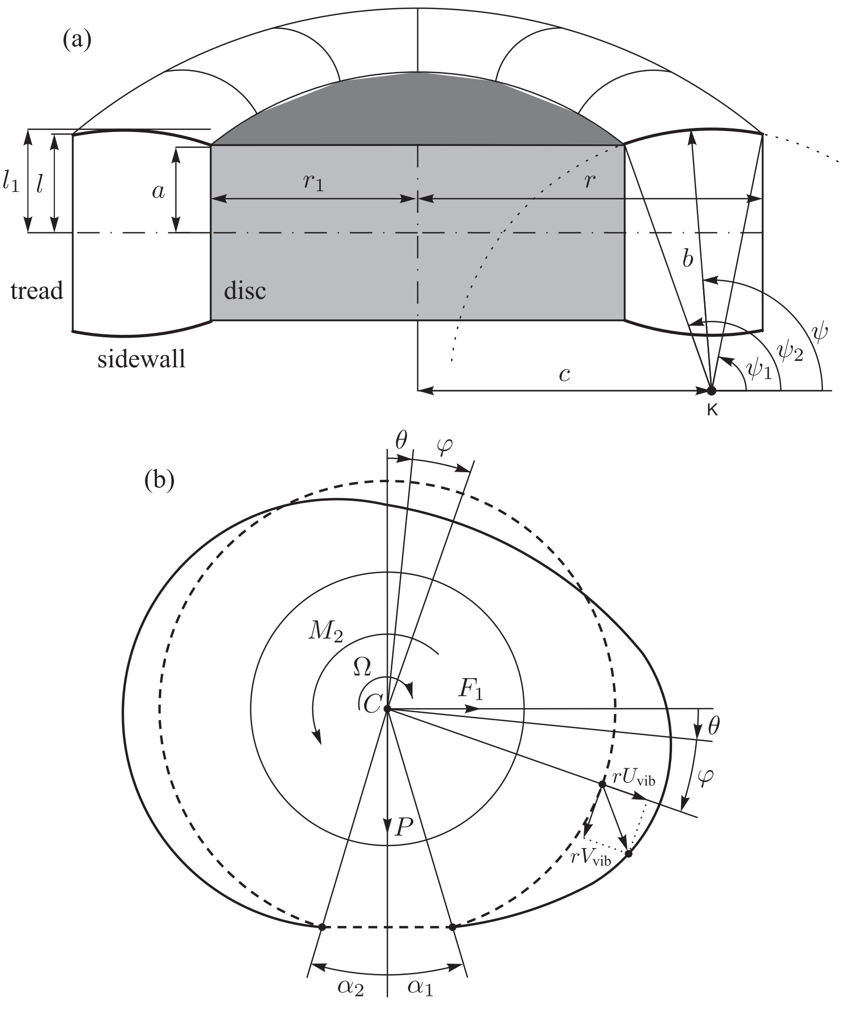

Fig. 1. The model of a wheel with a reinforced tire.

rubber are described by the Mooney – Rivlin model [9]. The tread is reinforced with inextensible cords. In the undeformed state, the tread is a circular cylinder (Fig. 1a) of radius r and height $2l$ (tread width). The tread deformations are considered taking into account the exact nonlinear conditions of inextensibility of reinforcing cords. Due to nonlinear geometric constraints in the deformed state, the tread retains its cylindrical shape, which is not circular for a typical configuration. The tread is the part of the tire that makes actual contact with the ground plane.

Let (X_1, X_3) denote the coordinates of the mass center of the disc C in the inertial frame (Fig. 1b). Introduce a moving frame (MF1) with its origin C and axes fixed to the disc, θ is a rotation angle. Using the Lagrangian specification we determine the position of median line points by angle φ with respect to a MF1. After two rotations by angle $\theta + \varphi$ we obtain a new moving frame (MF2) and $ru(\varphi, t)$, $rv(\varphi, t)$ are, respectively, the radial and tangential components of the displacement vector of median line points in the MF2. The contact area of the tire and the plane can be represented by a rectangle of constant width $2l$, equal to the tread width, and of variable length $r(\varphi_2(t) - \varphi_1(t))$. The length is defined by two functions of time $\varphi_1(t)$, $\varphi_2(t)$, which are unknown in advance. These functions can be obtained from the equations of motion. Suppose that the wheel rolls without slipping and without jumping. This means that the velocity of points of the tread in the contact area $[\varphi_1, \varphi_2]$ vanishes. The equations of motion and the conditions on the boundary of the contact area were obtained [5] from the Hamilton – Ostrogradsky variational principle for nonconservative systems

$$\int\limits_{t_1}^{t_2} (\delta T + \delta A)\,dt = \int\limits_{t_1}^{t_2} (\delta T_d + \delta T_{\mathrm{t}} + \delta A_{\mathrm{F}} + \delta A_{\mathrm{P}} + \delta N_1 + \delta N_3 + \delta N_6)\,dt = 0.$$

The kinetic energy of the wheel T consists of the kinetic energy of the disc T_d and the kinetic energy of the tire T_{t}, assuming that the whole mass of the tire is distributed uniformly along the plane median line of the tread with linear density ρ. The work δA at virtual displacements has the following structure: 1) the work δA_{F} performed by the external longitudinal force, by the vertical load and by the wheel torque applied to the wheel disc (Fig. 1b), 2) the work δA_{P} performed by the potential forces (it comprises the work performed by the pressure and the variation of the potential energy of the rubber stretching in the Mooney – Rivlin model when the sidewalls and the tread are deformed), 3) the works δN_1, δN_3, δN_6 performed by the reactions of the constraints (rolling without slipping and without jumping in the contact area and the condition of the inextensibility of the median line). Using the Hamilton – Ostrogradsky variational principle

$$\int\limits_{t_1}^{t_2} \Bigg(E_1 \delta X_1 + E_2 \delta X_3 + E_3 \delta\theta + \int\limits_{\varphi_1}^{\varphi_2} \Big[E_4 \delta u + E_5 \delta v \Big]\, d\varphi + \int\limits_{\varphi_2}^{2\pi+\varphi_1} \Big[E_6 \delta u + E_7 \delta v \Big]\, d\varphi + \\ + E_8 \delta u_1 + E_9 \delta u_2 + E_{10} \delta v_1 + E_{11} \delta v_2 \Bigg)\, dt = 0,$$

where $u_k = u(\varphi_k, t)$ and $v_k = v(\varphi_k, t)$, one can obtain [2, 5] a complete system of fourteen equations in fourteen unknowns which has the following structure: three Lagrange's equations of motion ($E_1 = E_2 = E_3 = 0$) with Lagrange multipliers (a feature of these equations is that they contain integral terms), four partial equations of motion ($E_4 = E_5 = E_6 = E_7 = 0$), three constraint equations and four dynamic boundary conditions ($E_8 = E_9 = E_{10} = E_{11} = 0$).

3. Tire vibrations

The steady-state regime of rolling of a loaded tire at constant speed, without slipping in the contact area, was considered in [5]. The problem of vibrations of a tire about this steady-state regime of rolling was investigated in [2]. Suppose that the wheel rotates with constant angular velocity Ω. Then

$$\dot{X}_1 = r\Omega, \quad X_3 = \text{const}, \quad \dot{\theta} = \Omega.$$

Putting $\alpha = \varphi + \Omega t - \pi/2$ we pass from the Lagrangian specification to the Eulerian specification. Now the contact area length $r(\alpha_2(t) - \alpha_1(t))$ is defined by two functions of time $\alpha_1(t)$, $\alpha_2(t)$ (Fig. 1b). We represent the functions determining the shape of the deformed tread and the contact area in the form

$$ru(\varphi, t) = rU(\alpha) + rU_{\text{vib}}(\alpha, t), \quad rv(\varphi, t) = rV(\alpha) + rV_{\text{vib}}(\alpha, t), \quad \alpha_k(t) = \alpha_k^\circ + \alpha_{\text{vib}\,k}(t).$$

The terms $rU(\alpha)$, $rV(\alpha)$, α_k° describe the steady-state regime of rolling without slipping (the dash line in Fig. 1b). The terms $rU_{\text{vib}}(\alpha, t)$, $rV_{\text{vib}}(\alpha, t)$, $\alpha_{\text{vib}\,k}(t)$ describe the vibrations of the tire (the solid heavy line in Fig. 1b) about the steady-state motion. The function V_{vib} satisfies the equation [2]

$$\rho r^3 \ddot{V}''_{\text{vib}} - \rho r^3 \ddot{V}_{\text{vib}} + 2\rho r^3 \Omega \dot{V}'''_{\text{vib}} + 2\rho r^3 \Omega \dot{V}'_{\text{vib}} + a_0 V^{(\text{IV})}_{\text{vib}} + a_1 V''_{\text{vib}} + a_2 V_{\text{vib}} = 0 \tag{3.1}$$

and the boundary conditions

$$\begin{aligned} &V_{\text{vib}}(\alpha_1^\circ + 2\pi + \alpha_{\text{vib}\,1}) = 0, \qquad V_{\text{vib}}(\alpha_2^\circ + \alpha_{\text{vib}\,2}) = 0, \\ &V'_{\text{vib}}(\alpha_1^\circ + 2\pi + \alpha_{\text{vib}\,1}) = 0, \qquad V'_{\text{vib}}(\alpha_2^\circ + \alpha_{\text{vib}\,2}) = 0. \end{aligned} \tag{3.2}$$

Here the coefficients a_0, a_1, a_2 are constant. They are determined analytically by evaluating definite integrals (by integrating over sidewalls and over the tread of the tire) and depend on the geometric parameters of the tire (Fig. 1a) and on the internal tire pressure. For the sake of brevity we omit details of these calculations.

Remark 1. With the phenomenological approach these coefficients are unknown. One must obtain them experimentally.

In determining the frequency of tire vibrations, the length of the contact area is taken as constant, since within the model chosen its variation determines the second order of smallness correction to the frequency. Hence, the boundary conditions in problem (3.2) are equivalent to the following:

$$V_{\text{vib}}(\alpha_1^\circ + 2\pi) = 0, \quad V_{\text{vib}}(\alpha_2^\circ) = 0, \quad V'_{\text{vib}}(\alpha_1^\circ + 2\pi) = 0, \quad V'_{\text{vib}}(\alpha_2^\circ) = 0. \tag{3.3}$$

For simplicity, we will write α_k instead of α_k°. Using the method of separation of variables (the Fourier method), we will represent

$$V_{\text{vib}}(\alpha, t) = \mathrm{e}^{\mathrm{i}\omega t} X(\alpha).$$

Here ω is an angular frequency ($\nu = \omega/(2\pi)$ is a NF in Hertz), $X(\alpha)$ is a MS. Substituting this expression into Eqs. (3.1) and (3.2), we obtain the ordinary differential equation

$$a_0 X^{(\text{IV})} + 2\rho r^3 \Omega\omega \mathrm{i} X''' + (a_1 - \rho r^3 \omega^2) X'' + 2\rho r^3 \Omega\omega \mathrm{i} X' + (a_2 + \rho r^3 \omega^2) X = 0 \tag{3.4}$$

and the boundary conditions

$$X(\alpha_1 + 2\pi) = 0, \quad X(\alpha_2) = 0, \quad X'(\alpha_1 + 2\pi) = 0, \quad X'(\alpha_2) = 0. \tag{3.5}$$

The general solution of (3.4) can be written as

$$X(\alpha) = G\mathrm{e}^{p\alpha}.$$

So, the characteristic equation reads

$$a_0 p^4 + 2\rho r^3 \Omega\omega \mathrm{i} p^3 + (a_1 - \rho r^3 \omega^2) p^2 + 2\rho r^3 \Omega\omega \mathrm{i} p + (a_2 + \rho r^3 \omega^2) = 0. \tag{3.6}$$

Solving this equation one can obtain [2] four roots $p_1(\omega, \Omega)$, $p_2(\omega, \Omega)$, $p_3(\omega, \Omega)$, $p_4(\omega, \Omega)$. The functions $p_j(\omega) = \mathrm{Re}(p_j(\omega)) + \mathrm{i}\,\mathrm{Im}(p_j(\omega))$ are represented in Figs. 2a–2d for $\Omega = 148$ and 157 rad s^{-1}. Most likely, this is a situation close to two pairs of multiple roots.

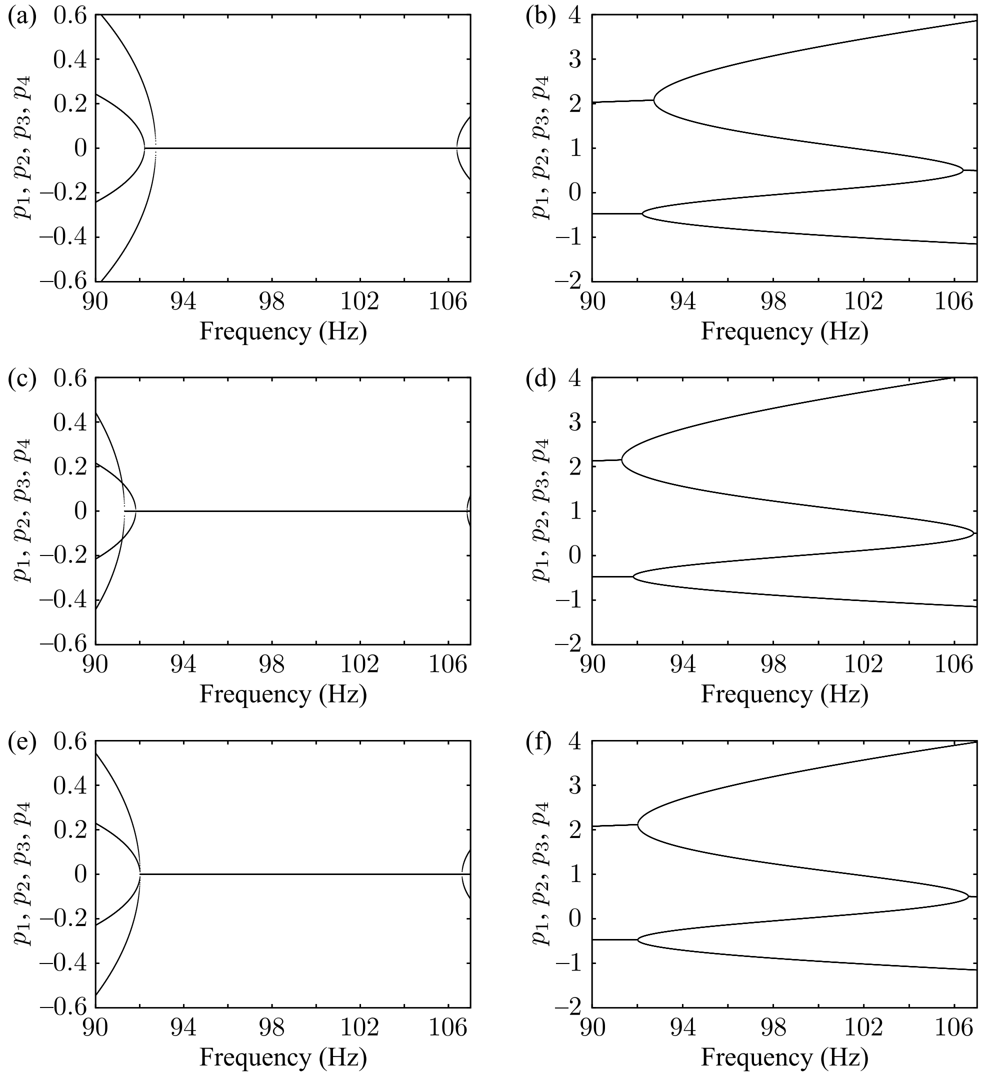

Fig. 2. The functions $p_1(\omega)$, $p_2(\omega)$, $p_3(\omega)$, $p_4(\omega)$, (a) $\mathrm{Re}p_j(\omega)$, $\Omega = 148$ rad s^{-1}, (b) $\mathrm{Im}p_j(\omega)$, $\Omega = 148$ rad s^{-1}, (c) $\mathrm{Re}p_j(\omega)$, $\Omega = 157$ rad s^{-1}, (d) $\mathrm{Im}p_j(\omega)$, $\Omega = 157$ rad s^{-1}, (e) $\mathrm{Re}p_j(\omega)$, $\Omega = 152.55$ rad s^{-1}, (f) $\mathrm{Im}p_j(\omega)$, $\Omega = 152.55$ rad s^{-1}.

Remark 2. Carrying out calculations we used the input data (Input I) as in [2]: the tire size (205/55 R16), the mass of tire and the quantity of internal pressure.

The quartic polynomial (3.6) has the discriminant

$$\begin{aligned}
D(\omega,\Omega) &= a_0^6(p_1-p_2)^2(p_1-p_3)^2(p_1-p_4)^2(p_2-p_3)^2(p_2-p_4)^2(p_3-p_4)^2 = \\
&= D_{10}(\Omega)\omega^{10} + D_8(\Omega)\omega^8 + D_6(\Omega)\omega^6 + D_4(\Omega)\omega^4 + D_2(\Omega)\omega^2 + D_0,\\
D_{10}(\Omega) &= -16\rho^5 r^{15}\left[\rho r^3\Omega^2 - a_0\right], \quad D_0 = 16a_0a_2(a_1^2-4a_0a_2)^2,\\
D_8(\Omega) &= 16\rho^4 r^{12}\left[-44\rho^2 r^6\Omega^4 + \rho r^3\left(3a_1+55a_0-a_2\right)\Omega^2 + a_0a_2 - 8a_0^2 - 4a_0a_1\right],\\
D_6(\Omega) &= -16\rho^3 r^9\left[-16\rho^3 r^9\Omega^6 + \rho^2 r^6\left(24a_0+72a_2-16a_1\right)\Omega^4 + \right.\\
&+ \rho r^3\left(73a_0a_1 - 92a_0a_2 - 3a_1a_2 + 3a_1^2 - 84a_0^2\right)\Omega^2 - \\
&\left. - 6a_0a_1^2 - 16a_0^3 + 16a_0^2a_2 - 16a_0^2a_1 + 4a_0a_1a_2\right],\\
D_4(\Omega) &= 16\rho^2 r^6\left[\rho^2 r^6\left(a_1^2 - 6a_0a_2 + 18a_1a_2 - 27a_0^2 + 18a_0a_1 - 27a_2^2\right)\Omega^4 + \right.\\
&+ \rho r^3\left(36a_0a_2^2 - 3a_1^2a_2 + a_1^3 - 36a_0^2a_1 + 17a_0a_1^2 - 112a_0a_1a_2 + 132a_0^2a_2\right)\Omega^2 + \\
&\left. + 48a_0^3a_2 - 4a_0a_1^3 - 8a_0^2a_2^2 + 6a_0a_1^2a_2 + 32a_0^2a_1a_2 - 8a_0^2a_1^2\right],\\
D_2(\Omega) &= 16\rho r^3\left[\rho r^3\left(a_1^3a_2 + a_0a_1^3 + 20a_0a_1^2a_2 - 36a_0a_1a_2^2 - 36a_0^2a_1a_2 + 48a_0^2a_2^2\right)\Omega^2 - \right.\\
&\left. - 4a_0a_1^3a_2 + a_0a_1^4 + 16a_0^2a_1a_2^2 - 16a_0^2a_1^2a_2 + 48a_0^3a_2^2\right].
\end{aligned}$$

This discriminant vanishes if and only if at least two roots are equal. The plot of $D(\omega)$ for angular velocity $\Omega = 152.55$ rad s^{-1} is shown in Fig. 3. So one multiple root (root of multiplicity one, for example, p_1, p_1, p_3, p_4) is located between 106 and 107 Hz. This case is not considered in this paper. Yet, there is a more interesting situation (two pairs of multiple roots) around 92 Hz.

In the general case $p_i \neq p_j$ (this case was investigated in [2]) the solution can be represented in the form

$$X(\alpha) = G_1 e^{p_1\alpha} + G_2 e^{p_2\alpha} + G_3 e^{p_3\alpha} + G_4 e^{p_4\alpha}. \tag{3.7}$$

The coefficients $G_i(p_1,p_2,p_3,p_4) = G_i(\omega,\Omega)$ are determined from the boundary conditions (3.5)

$$\begin{aligned}
G_1 &= e^{-p_1(\alpha_1+2\pi)}\Big((p_4-p_3)e^{p_2(\Delta\alpha-2\pi)} - (p_4-p_2)e^{p_3(\Delta\alpha-2\pi)} + (p_3-p_2)e^{p_4(\Delta\alpha-2\pi)}\Big)G_5^*,\\
G_2 &= e^{-p_2(\alpha_1+2\pi)}\Big(-(p_4-p_3)e^{p_1(\Delta\alpha-2\pi)} + (p_4-p_1)e^{p_3(\Delta\alpha-2\pi)} - (p_3-p_1)e^{p_4(\Delta\alpha-2\pi)}\Big)G_5^*,\\
G_3 &= e^{-p_3(\alpha_1+2\pi)}\Big((p_4-p_2)e^{p_1(\Delta\alpha-2\pi)} - (p_4-p_1)e^{p_2(\Delta\alpha-2\pi)} + (p_2-p_1)e^{p_4(\Delta\alpha-2\pi)}\Big)G_5^*,\\
G_4 &= e^{-p_4(\alpha_1+2\pi)}\Big(-(p_3-p_2)e^{p_1(\Delta\alpha-2\pi)} + (p_3-p_1)e^{p_2(\Delta\alpha-2\pi)} - (p_2-p_1)e^{p_3(\Delta\alpha-2\pi)}\Big)G_5^*.
\end{aligned} \tag{3.8}$$

Here $\Delta\alpha = \alpha_2 - \alpha_1$ determines the length of the contact area, G_5^* is an arbitrary constant. The homogeneous system (3.5) has a nonzero solution (3.8) if its determinant $f(p_1,p_2,p_3,p_4) = f(\omega,\Omega)$ vanishes

$$\begin{aligned}
f = e^{(p_1+p_2+p_3+p_4)(\alpha_1+2\pi)}\Big[&(p_3-p_1)(p_4-p_2)\left(e^{(p_2+p_4)(\Delta\alpha-2\pi)} + e^{(p_1+p_3)(\Delta\alpha-2\pi)}\right) - \\
&-(p_3-p_2)(p_4-p_1)\left(e^{(p_1+p_4)(\Delta\alpha-2\pi)} + e^{(p_2+p_3)(\Delta\alpha-2\pi)}\right) - \\
&-(p_2-p_1)(p_4-p_3)\left(e^{(p_3+p_4)(\Delta\alpha-2\pi)} + e^{(p_1+p_2)(\Delta\alpha-2\pi)}\right)\Big] = 0.
\end{aligned} \tag{3.9}$$

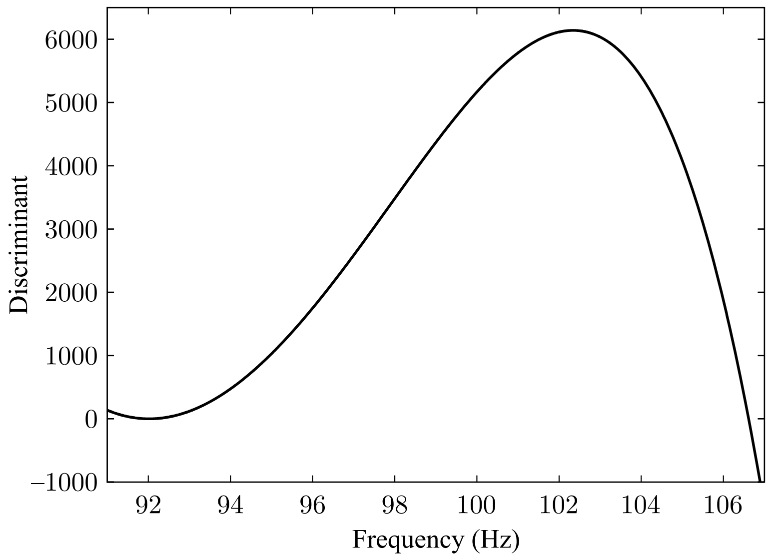

Fig. 3. The discriminant $D(\omega)$ of the quartic polynomial for $\Omega = 152.55$ rad s^{-1}.

So, if the angular velocity $\Omega = \Omega_0$ is fixed, then we obtain the frequency equation $f(\omega, \Omega_0) = = f(\omega) = 0$ which defines an infinite spectrum of NF. The function $f(\omega) = \mathrm{Re}(f(\omega)) + \mathrm{i}\,\mathrm{Im}(f(\omega))$ is a complex-valued function, but it assumes real or purely imaginary values for real-valued arguments. The plots of $\mathrm{Re}(f(\omega))$, $\mathrm{Im}(f(\omega))$ for $\Delta\alpha = 0.3$ rad and for angular velocities $\Omega = 151$ and 154 rad s^{-1} are shown in Figs. 4a and 4b. The right figures zoom the left figures around 92 Hz. The frequencies for which $\mathrm{Re}(f(\omega))$ vanishes in the left neighborhood and is not equal to zero in the right neighborhood, and simultaneously $\mathrm{Im}(f(\omega))$ is not equal to zero in the left neighborhood and vanishes in the right neighborhood (for example, the point around 91.96 Hz in Fig. 4b), correspond to the multiple root (root of multiplicity one, for example, p_1, p_1, p_3, p_4). This case is not considered in this paper. The same holds for frequencies in which $\mathrm{Re}(f(\omega))$ is not equal to zero in the left neighborhood and vanishes in the right neighborhood, and simultaneously $\mathrm{Im}(f(\omega))$ vanishes in the left neighborhood and is not equal to zero in the right neighborhood (for example, the point around 91.79 Hz in Fig. 4b).

3.1. A special case of vibrations of a rolling tire

Let us consider the special case of two pairs of multiple roots, when $p_1 = p_2$ and $p_3 = p_4$. In this case, Eq. (3.6) reads

$$a_0 p^4 + 2\rho r^3 \Omega\omega \mathrm{i} p^3 + (a_1 - \rho r^3\omega^2)p^2 + 2\rho r^3\Omega\omega\mathrm{i}p + (a_2 + \rho r^3\omega^2) = a_0(p - p_1)^2(p - p_3)^2.$$

Multiplying the factors on the right-hand side and identifying the coefficients of each power of p, one might obtain

$$\begin{aligned} &p_1 + p_3 = -\frac{\rho r^3\Omega\omega\mathrm{i}}{a_0}, \quad p_1 p_3(p_1 + p_3) = -\frac{\rho r^3\Omega\omega\mathrm{i}}{a_0}, \\ &p_1^2 + 4p_1p_3 + p_3^2 = \frac{a_1 - \rho r^3\omega^2}{a_0}, \quad p_1^2 p_3^2 = \frac{a_2 + \rho r^3\omega^2}{a_0}. \end{aligned} \tag{3.10}$$

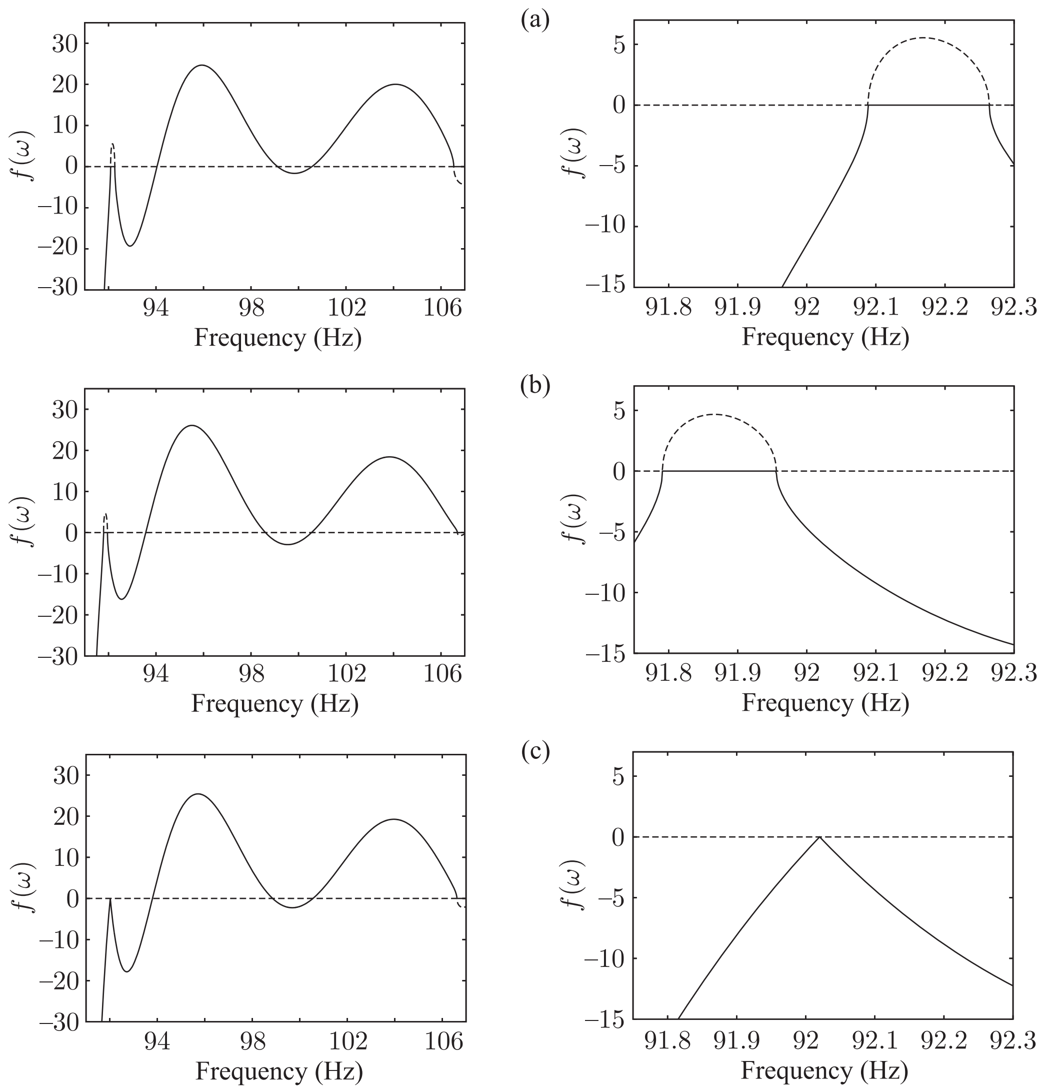

Fig. 4. The function $f(\omega)$, —— $\mathrm{Re}(f(\omega))$, ------ $\mathrm{Im}(f(\omega))$, (a) $\Omega = 151$ rad s^{-1}, (b) $\Omega = 154$ rad s^{-1}, (c) $\Omega = 152.55$ rad s^{-1}. The right figures zoom the left figures around 92 Hz.

Solving this system, we calculate all the parameters analytically

$$\omega = \sqrt{\frac{a_0 - a_2}{\rho r^3}}, \quad \Omega = \sqrt{\frac{a_0(3a_0 - a_1 - a_2)}{\rho r^3 (a_0 - a_2)}},$$

$$p_{1,3} = \frac{\mathrm{i}}{2}\left[-\frac{\rho r^3 \Omega\omega}{a_0} \pm \sqrt{\left(\frac{\rho r^3 \Omega\omega}{a_0}\right)^2 + 4}\right] = \frac{\mathrm{i}}{2}\left[\sqrt{3 - \frac{a_1 + a_2}{a_0}} \pm \sqrt{7 - \frac{a_1 + a_2}{a_0}}\right].$$

Thus, if the geometry of the tire and the internal tire pressure are fixed (the coefficients a_0, a_1, a_2), then one can immediately identify the angular velocity of rotation ($\Omega = 152.55167\ldots$ rad s^{-1}) and, respectively, the tire speed ($\dot{X}_1 = r\Omega = 174.03705\ldots$ km h^{-1}) and the frequency such that $p_1 = p_2$ and $p_3 = p_4$. The plots of $\mathrm{Re}(f(\omega))$, $\mathrm{Im}(f(\omega))$ for $\Delta\alpha = 0.3$ rad and for angular velocity $\Omega = 152.55$ rad s^{-1} are shown in Fig. 4c. The right figure zooms the left figure around 92 Hz. One can see a singular point of the curve, corresponding to the special case of two pairs of multiple roots. The functions $p_j(\omega) = \mathrm{Re}(p_j(\omega)) + \mathrm{i}\,\mathrm{Im}(p_j(\omega))$ which are the solutions of (3.6) are represented in Figs. 2e and 2f for $\Omega = 152.55$ rad s^{-1}. Now one can really visualize the case of two pairs of multiple roots for $\nu = 92.019909\ldots$ Hz.

In this special case, one cannot use the formulae (3.7). Thus, one must search for the MS in the form

$$X(\alpha) = (G_1 + G_2\alpha)\mathrm{e}^{p_1\alpha} + (G_3 + G_4\alpha)\mathrm{e}^{p_3\alpha}, \tag{3.11}$$

where the coefficients G_1, G_2, G_3, G_4 are determined from the boundary conditions (3.5)

$$\begin{aligned}
&G_1\mathrm{e}^{p_1(\alpha_1+2\pi)} + G_2(\alpha_1+2\pi)\mathrm{e}^{p_1(\alpha_1+2\pi)} + G_3\mathrm{e}^{p_3(\alpha_1+2\pi)} + G_4(\alpha_1+2\pi)\mathrm{e}^{p_3(\alpha_1+2\pi)} = 0,\\
&G_1\mathrm{e}^{p_1\alpha_2} + G_2\alpha_2\mathrm{e}^{p_1\alpha_2} + G_3\mathrm{e}^{p_3\alpha_2} + G_4\alpha_2\mathrm{e}^{p_3\alpha_2} = 0,\\
&G_1p_1\mathrm{e}^{p_1(\alpha_1+2\pi)} + G_2\Big(1 + p_1(\alpha_1+2\pi)\Big)\mathrm{e}^{p_1(\alpha_1+2\pi)} + G_3p_3\mathrm{e}^{p_3(\alpha_1+2\pi)} +\\
&\qquad + G_4\Big(1 + p_3(\alpha_1+2\pi)\Big)\mathrm{e}^{p_3(\alpha_1+2\pi)} = 0,\\
&G_1p_1\mathrm{e}^{p_1\alpha_2} + G_2\Big(1 + p_1\alpha_2\Big)\mathrm{e}^{p_1\alpha_2} + G_3p_3\mathrm{e}^{p_3\alpha_2} + G_4\Big(1 + p_3\alpha_2\Big)\mathrm{e}^{p_3\alpha_2} = 0.
\end{aligned} \tag{3.12}$$

The solution of this system reads

$$\begin{aligned}
&G_1 = \mathrm{e}^{-p_1(\alpha_1+2\pi)}\Big[-\alpha_2\mathrm{e}^{p_1(\Delta\alpha-2\pi)} + \Big(\alpha_2 - (p_3-p_1)(\Delta\alpha-2\pi)(\alpha_1+2\pi)\Big)\mathrm{e}^{p_3(\Delta\alpha-2\pi)}\Big]G_5^*,\\
&G_2 = \mathrm{e}^{-p_1(\alpha_1+2\pi)}\Big[\mathrm{e}^{p_1(\Delta\alpha-2\pi)} - \Big(1 - (p_3-p_1)(\Delta\alpha-2\pi)\Big)\mathrm{e}^{p_3(\Delta\alpha-2\pi)}\Big]G_5^*,\\
&G_3 = \mathrm{e}^{-p_3(\alpha_1+2\pi)}\Big[-\alpha_2\mathrm{e}^{p_3(\Delta\alpha-2\pi)} + \Big(\alpha_2 + (p_3-p_1)(\Delta\alpha-2\pi)(\alpha_1+2\pi)\Big)\mathrm{e}^{p_1(\Delta\alpha-2\pi)}\Big]G_5^*,\\
&G_4 = \mathrm{e}^{-p_3(\alpha_1+2\pi)}\Big[\mathrm{e}^{p_3(\Delta\alpha-2\pi)} - \Big(1 + (p_3-p_1)(\Delta\alpha-2\pi)\Big)\mathrm{e}^{p_1(\Delta\alpha-2\pi)}\Big]G_5^*.
\end{aligned} \tag{3.13}$$

The homogeneous system (3.12) has a nonzero solution (3.13) if its determinant vanishes

$$\begin{aligned}
f &= \mathrm{e}^{(2p_1+2p_3)(\alpha_1+2\pi)}\Big[\Big((p_3-p_1)^2(\Delta\alpha-2\pi)^2 + 2\Big)\mathrm{e}^{(p_1+p_3)(\Delta\alpha-2\pi)} - \mathrm{e}^{2p_1(\Delta\alpha-2\pi)} - \mathrm{e}^{2p_3(\Delta\alpha-2\pi)}\Big] =\\
&= \mathrm{e}^{(p_1+p_3)(\alpha_1+\alpha_2+2\pi)}f_1\Big((p_3-p_1)(\Delta\alpha-2\pi)\Big) = 0, \quad f_1(x) = x^2 + 2 - 2\cosh x.
\end{aligned} \tag{3.14}$$

Substituting (3.13) into (3.12) for verification, one can obtain three identities (the first three equations) and the equation $f\cdot\mathrm{e}^{-(2p_1+2p_3)(\alpha_1+2\pi)} = 0$ (the fourth equation). If f vanishes, then the fourth equation of (3.12) is fulfilled identically. But the multiplier $f_1(x)$ appearing in (3.14) does not vanish except $x = 0$, that is, for $p_1 = p_3$. Thus, this is a case of a root of multiplicity four ($p_1 = p_2 = p_3 = p_4$) which is not considered here. Thus, the homogeneous system (3.12) has only a zero solution $G_1 = G_2 = G_3 = G_4 = 0$ and the corresponding MS $X(\alpha)$ vanishes.

Let us consider the neighborhood of the singular point. If $p_4 \to p_3$ and $p_2 \to p_1$ it is necessary to use the formulae (3.7), (3.8) and (3.9) obtained in [2]. The frequency function (3.9) tends to zero $f(\omega) \to 0$. Thus, (3.8) practically satisfies the boundary conditions (3.5): the first three equations of (3.5) are the three identities and the left side of the fourth equation $X'(\alpha_2)$ tends to zero. This means that the tangential component of the displacement vector of median line points meets the conditions $rV_{\mathrm{vib}}(\alpha_1 + 2\pi) = 0$, $rV_{\mathrm{vib}}(\alpha_2) = 0$. But the radial component fulfills the conditions $rU_{\mathrm{vib}}(\alpha_1 + 2\pi) = 0$, $rU_{\mathrm{vib}}(\alpha_2) \to 0$ (Fig. 5). As G_i in (3.8) tend to zero, the amplitude of vibrations also tends to zero. The MS in the neighborhood of the singular point

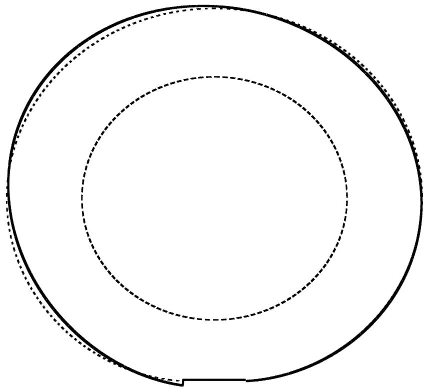

Fig. 5. The MS in the neighborhood of the singular point for $\Omega = 152.55$ rad s^{-1} and $\nu = 92.020198$ Hz.

is represented in Fig. 5. The shape of the deformed median line of the tread in the steady-state regime of rolling of a loaded tire is represented in the figure by the dotted line. The vibrations about the steady-state motion correspond to the solid line.

REMARK 3. If one has the special case of a root of multiplicity four ($p_1 = p_2 = p_3 = p_4$), then the system (3.10) reads

$$2p_1 = -\frac{\rho r^3 \Omega\omega \mathrm{i}}{a_0}, \quad 2p_1^3 = -\frac{\rho r^3 \Omega\omega \mathrm{i}}{a_0}, \quad 6p_1^2 = \frac{a_1 - \rho r^3\omega^2}{a_0}, \quad p_1^4 = \frac{a_2 + \rho r^3\omega^2}{a_0}. \tag{3.15}$$

Solving this system, we calculate all the parameters analytically

$$p_1 = 0, \quad \Omega = 0, \quad \omega = \sqrt{-\frac{a_2}{\rho r^3}}, \quad a_2 = -a_1;$$
$$p_1 = \pm 1, \quad \Omega = \pm\frac{2a_0\omega \mathrm{i}}{a_0 - a_2}, \quad \omega = \sqrt{\frac{a_0 - a_2}{\rho r^3}}, \quad a_1 + a_2 - 7a_0 = 0.$$

Carrying out calculations we used the input data (Input I) as in [2]: $a_0 = -8045.37$ N m, $a_1 = 55661.5$ N m, $a_2 = -58109.6$ N m. Thus, for these parameters a root of multiplicity four is impossible. But theoretically, it is possible to choose such a geometry of the tire that the case $a_2 = -a_1$ can be implemented.

REMARK 4. The system (3.10) has another solution $p_3 = -p_1$. In this case

$$\Omega = 0, \quad \omega^2 = \frac{a_1 + 2a_0 \pm 2\sqrt{a_0(a_0 + a_1 + a_2)}}{\rho r^3}, \quad p_1^2 = \frac{\rho r^3\omega^2 - a_1}{2a_0} = 1 \pm \frac{\sqrt{a_0(a_0 + a_1 + a_2)}}{a_0}. \tag{3.16}$$

If $p_3 = -p_1$ one must search for the MS (3.11) in the form

$$X(\alpha) = (G_1 + G_2\alpha)\mathrm{e}^{p_1\alpha} + (G_3 + G_4\alpha)\mathrm{e}^{-p_1\alpha}.$$

The determinant (3.14) reads

$$f = f_1\Big(2p_1(\Delta\alpha - 2\pi)\Big).$$

The equation $f = 0$ has a unique root $p_1 = 0$. But according to (3.16) p_1 is not equal to zero, and in this case, we again obtain that $X(\alpha)$ vanishes.

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4. Conclusions

A special case of two pairs of multiple roots is considered in this paper. In contrast to [2], if one has a situation of multiple roots, then the solution is represented in a different form, and the problem must be solved differently. If the geometry of the tire and the internal tire pressure are fixed, then the angular velocity of rotation ($152.55167\ldots$ rad s^{-1}) and, respectively, the tire speed ($174.03705\ldots$ km h^{-1}) and the frequency ($92.019909\ldots$ Hz), corresponding to this case, are determined analytically. As for mode shape, here one has an interesting situation. If $p_4 \to p_3$ and $p_2 \to p_1$, it is necessary to use the formulae obtained in [2]. The frequency function (the function from the frequency equation which defines an infinite spectrum of natural frequencies) tends to zero. Thus, the solution practically satisfies the boundary conditions. The amplitude of vibrations tends to zero. If $p_4 = p_3$ and $p_2 = p_1$, one has a special case and cannot use the formulae obtained in [2]. It is necessary to use the formulae obtained in this paper. For the case of two pairs of multiple roots the frequency function is a constant function and has a different structure. This constant function is not equal to zero at the singular point. So the amplitude of vibrations is equal to zero and hence the corresponding mode shape also vanishes.

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