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MATHIEU-TYPE SERIES BUILT BY (p, q)–EXTENDED GAUSSIAN HYPERGEOMETRIC FUNCTION

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Abstract. The main purpose of this paper is to present closed integral form expressions for the Mathieu-type $\boldsymbol{a}$-series and its associated alternating version whose terms contain a (p, q)–extended Gauss' hypergeometric function. Certain upper bounds for the two series are also given.

1. Introduction and preliminaries

In the recent articles Pogány, either alone and/or with his co-workers Baricz, Butzer, Saxena, Srivastava and Tomovski [1], [10, 11, 12, 13, 14, 15, 16] considered special general Mathieu–type series and their alternating variants whose terms contain the various special functions, for example, Gauss hypergeometric function ${}_2F_1$, generalized hypergeometric ${}_pF_q$, Meijer G-functions and so on. The derived results concern, among others, closed integral form expressions for the considered series and bilateral bounding inequalities. Here we are interested in giving integral expressions for the Mathieu–type series and its alternating variants built by terms which contain the (p, q)–extended Gauss' hypergeometric function which generalizes the so–called p–extension of the p–Gaussian hypergeometric function [4, 9]. These functions are built by first changing some terms in the defining series into Beta-function and then replacing the Beta-function with its p–variant and (p, q)–variant. The above mentioned extensions, generalizations and unifications of Euler's Beta function together with a set of related higher transcendental hypergeometric type special functions have been investigated recently by several authors, for instance, one may refer to [3, 4, 5]. In particular, Chaudhry *et al.* [3, p. 20, Eq. (1.7)] introduced the p–extension of the Eulerian Beta function $\mathrm{B}(x, y)$:

$$\mathrm{B}(x, y; p) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\mathrm{e}^{-\frac{p}{t(1-t)}}\,\mathrm{d}t \qquad (\Re(p) > 0),$$

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whose special case when $p = 0$ reduces to the familiar Beta function $\mathrm{B}(x, y)$ $(\min\{\Re(x), \Re(y)\} > 0)$ (see, *e.g.*, [17, Section 1.1]). They extended Macdonald (or modified Bessel function of the second kind), error and Whittaker functions by using the $\mathrm{B}(x, y; p)$. Also Chaudhry *et al.* [4] used the $\mathrm{B}(x, y; p)$ to extend Gaussian hypergeometric and confluent (Kummer's) hypergeometric functions in the following manner:

$$F_p(a, b; c; z) = \sum_{n\geq 0}^{\infty} (a)_n \frac{\mathrm{B}(b+n,\, c-b\,;\, p)}{\mathrm{B}(b,\, c-b)} \frac{z^n}{n!}, \tag{1}$$

where $p \geq 0$; $\Re(c) > \Re(b) > 0\,; |z| < 1$ and

$$\Phi_p(b; c; z) = \sum_{n\geq 0} \frac{\mathrm{B}(b+n,\, c-b\,;\, p)}{\mathrm{B}(b,\, c-b)} \frac{z^n}{n!}, \tag{2}$$

where $p \geq 0$; $\Re(c) > \Re(b) > 0$, respectively. It is noted that the special case of (1) and (2) when $p = 0$ yield, respectively, the Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$ and the confluent (Kummer's) hypergeometric function ${}_1F_1(b; c; z)$ (see, *e.g.*, [17, Section 1.5]).

Recently, Choi *et al.* [6] have introduced *further* extensions of $\mathrm{B}(x, y; p)$, p–extended Gauss' hypergeometric series $F_p(a, b; c; z)$ and *a fortiori* the p–Kummer (or confluent hypergeometric) $\Phi_p(b; c; z)$ as follows:

$$\mathrm{B}(x, y; p, q) = \int_0^1 t^{x-1}(1-t)^{y-1}\, \mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}}\, \mathrm{d}t \qquad (\min\{\Re(p),\, \Re(q)\} \geq 0).$$

Here $\min\{\Re(x), \Re(y)\} > 0$ if $p = 0 = q$. Next, for all $\Re(c) > \Re(b) > 0$

$$F_{p,q}(a, b; c; z) = \sum_{n\geq 0} (a)_n \frac{\mathrm{B}(b+n,\, c-b\,;\, p, q)}{\mathrm{B}(b,\, c-b)} \frac{z^n}{n!} \qquad (|z| < 1),$$

and

$$\Phi_{p,q}(b; c; z) = \sum_{n\geq 0} \frac{\mathrm{B}(b+n,\, c-b\,;\, p, q)}{\mathrm{B}(b,\, c-b)} \frac{z^n}{n!} \qquad (\Re(c) > \Re(b) > 0). \tag{3}$$

The $F_{p,q}(a, b; c; z)$ and the $\Phi_{p,q}(b; c; z)$ are called *(p, q)–extended Gauss'* and *(p, q)–extended Kummer hypergeometric functions*, respectively. For their related properties, integral representations, differentiation formulas, Mellin transform, recurrence relations and certain summations, the interested reader may refer to [6]. For a (p, q)–extended Srivastava's triple generalized $H_{p,q,A}$ function, one may see [9].

Now, by imposing the $F_{p,q}(a, b; c; z)$ input–kernel instead of the originally used ${}_2F_1$ in the summands of the Mathieu–type series in [10], we extend to define the Mathieu–type $\mathbf{a}$–series $\mathfrak{F}_{\lambda,\eta}$ and its alternating variant $\widetilde{\mathfrak{F}}_{\lambda,\eta}$ in the form of series

$$\mathfrak{F}_{\lambda,\eta}(F_{p,q}; \boldsymbol{a}; r) := \sum_{n\geq 1} \frac{F_{p,q}\left(\lambda,\, b;\, c;\, -\frac{r^2}{a_n}\right)}{a_n^\lambda (a_n + r^2)^\eta} \quad \big(p \geq 0,\ q \geq 0;\ \lambda, \eta, r \in \mathbb{R}^+\big) \tag{4}$$

and in the same range of parameters

$$\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_{p,q};\boldsymbol{a};r) := \sum_{n\ge 1} \frac{(-1)^{n-1} F_{p,q}\left(\lambda,\, b;\, c;\, -\frac{r^2}{a_n}\right)}{a_n^{\lambda}(a_n+r^2)^{\eta}}. \tag{5}$$

Here and in what follows, let $\mathbb{R}$ and $\mathbb{R}^+$ be the sets of real and positive real numbers, respectively.

The main purpose of this note is to present integral representations and allied bounding inequalities for these functions in the widest range of the parameters involved.

2. Integral representations of $\mathfrak{F}_{\lambda,\eta}(F_{p,q};\boldsymbol{a};r)$ and $\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_{p,q};\boldsymbol{a};r)$

In this section, we first give closed integral form expressions for the series $\mathfrak{F}_{\lambda,\eta}(F_{p,q};\boldsymbol{a};r)$ and $\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_{p,q};\boldsymbol{a};r)$. Then we give some special cases of our first main result.

Theorem 2.1. *Let $\lambda,\ \eta,\ r \in \mathbb{R}^+$ and let $\boldsymbol{a} = (a_n)_{n\ge 1}$ be a real sequence which increases monotonically and tends to ∞. Then for $\min\{\Re(p), \Re(q)\} \ge 0$ we have*

$$\mathfrak{F}_{\lambda,\eta}(F_{p,q};\boldsymbol{a};r) = \lambda\, \mathscr{I}_{p,q}(\lambda+1,\eta) + \eta\, \mathscr{I}_{p,q}(\lambda,\eta+1) \tag{6}$$

and

$$\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_{p,q};\boldsymbol{a};r) = \lambda\, \widetilde{\mathscr{I}_{p,q}}(\lambda+1,\eta) + \eta\, \widetilde{\mathscr{I}_{p,q}}(\lambda,\eta+1), \tag{7}$$

where

$$\mathscr{I}_{p,q}(\lambda,\eta) = \int_{a_1}^{\infty} \frac{F_{p,q}\left(\lambda,\, b;\, c;\, -\frac{r^2}{x}\right)[a^{-1}(x)]}{x^{\lambda}(x+r^2)^{\eta}}\,\mathrm{d}x \tag{8}$$

and

$$\widetilde{\mathscr{I}_{p,q}}(\lambda,\eta) = \int_{a_1}^{\infty} \frac{F_{p,q}\left(\lambda,\, b;\, c;\, -\frac{r^2}{x}\right)\, \sin^2\left(\frac{\pi}{2}[a^{-1}(x)]\right)}{x^{\lambda}(x+r^2)^{\eta}}\,\mathrm{d}x \tag{9}$$

and $a : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is an increasing function such that $a(x)|_{x\in\mathbb{N}} = \boldsymbol{a}$, $a^{-1}(x)$ denotes the inverse of $a(x)$ and $[a^{-1}(x)]$ stands for the integer part of the quantity $a^{-1}(x)$.

Proof. Consider the Laplace transform formula of the extended Kummer's function $t^{\lambda-1}\,\Phi_{p,q}(b;c;z)$. By using the definition (3), for real ω, it follows easily

$$F_{p,q}\left(\lambda, b; c; \frac{\omega}{z}\right) = \frac{z^{\lambda}}{\Gamma(\lambda)} \int_0^{\infty} \mathrm{e}^{-zt} t^{\lambda-1}\, \Phi_{p,q}(b;c;\omega t)\,\mathrm{d}t. \tag{10}$$

Taking $\xi = a_n + r^2$ in the familiar Gamma formula:

$$\Gamma(\eta)\xi^{-\eta} = \int_0^{\infty} e^{-\xi t} t^{\eta-1}\,\mathrm{d}t \qquad (\min\{\Re(\xi),\, \Re(\eta)\} > 0)$$

and after rearrangement by specifying $\omega = -r^2$, $z = a_n$, in (10), the function $\mathfrak{F}_{\lambda,\eta}(F_{p,q}; \boldsymbol{a}; r)$ becomes

$$\mathfrak{F}_{\lambda,\eta}(F_{p,q}; \boldsymbol{a}; r) = \int_0^\infty \int_0^\infty \frac{\mathrm{e}^{-r^2 s}\, t^{\lambda-1} s^{\eta-1}}{\Gamma(\lambda)\Gamma(\eta)} \Big(\sum_{n\geq 1} \mathrm{e}^{-a_n(t+s)} \Big) \Phi_{p,q}(b; c; -r^2 t)\, \mathrm{d}t\, \mathrm{d}s.$$

Using the Cahen formula [2] for summing up the Dirichlet series in the technique developed in [16], we conclude

$$\mathcal{D}_a(t+s) = \sum_{n\geq 1} \mathrm{e}^{-a_n(s+t)} = (s+t) \int_{a_1}^\infty e^{-(t+s)x} [a^{-1}(x)]\, \mathrm{d}x\,.$$

This gives

$$\begin{aligned}\mathfrak{F}_{\lambda,\eta}(F_{p,q}; \boldsymbol{a}; r) &= \frac{1}{\Gamma(\lambda)\Gamma(\eta)} \int_0^\infty \int_0^\infty \int_{a_1}^\infty \mathrm{e}^{-(r^2+x)s-tx} (t+s) t^{\lambda-1} s^{\eta-1} [a^{-1}(x)] \\ &\quad \times\ \Phi_{p,q}(b; c; -r^2 t)\, \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}x =: \mathcal{I}_t + \mathcal{I}_s\,, \end{aligned} \tag{11}$$

where

$$\begin{aligned}\mathcal{I}_t &= \frac{1}{\Gamma(\eta)} \int_0^\infty \left(\int_{a_1}^\infty \left(\int_0^\infty \frac{\mathrm{e}^{-xt} t^\lambda}{\Gamma(\lambda)} \Phi_{p,q}(b; c; -r^2 t)\, \mathrm{d}t \right) \mathrm{e}^{-xs} [a^{-1}(x)]\, \mathrm{d}x \right) \\ &\qquad \times \mathrm{e}^{-r^2 s}\, s^{\eta-1}\, \mathrm{d}s \\ &= \lambda \int_{a_1}^\infty \left(\int_0^\infty \frac{s^{\eta-1}}{\Gamma(\eta)} \mathrm{e}^{-(x+r^2)s}\, \mathrm{d}s \right) \frac{[a^{-1}(x)]}{x^{\lambda+1}}\, F_{p,q}\left(\lambda+1,\, b;\, c;\, -\frac{r^2}{x} \right) \mathrm{d}x \\ &= \lambda \int_{a_1}^\infty \frac{[a^{-1}(x)]}{x^{\lambda+1}(x+r^2)^\eta}\, F_{p,q}\left(\lambda+1,\, b;\, c;\, -\frac{r^2}{x} \right) \mathrm{d}x = \lambda\, \mathscr{I}_{p,q}(\lambda+1, \eta)\,. \end{aligned} \tag{12}$$

Similarly we get

$$\begin{aligned}\mathcal{I}_s &= \eta \int_{a_1}^\infty \frac{[a^{-1}(x)]}{(x+r^2)^{\eta+1}} \left(\int_0^\infty \frac{\mathrm{e}^{-xt} t^{\lambda-1}}{\Gamma(\lambda)} \Phi_{p,q}(b; c; -r^2 t)\, \mathrm{d}t \right) \mathrm{d}x \\ &= \eta \int_{a_1}^\infty \frac{[a^{-1}(x)]}{x^\lambda (x+r^2)^{\eta+1}}\, F_{p,q}\left(\lambda,\, b;\, c;\, -\frac{r^2}{x} \right) \mathrm{d}x = \eta\, \mathscr{I}_{p,q}(\lambda, \eta+1). \end{aligned} \tag{13}$$

Now, applying (12) and (13) to (11) we deduce the expression (6).

The derivation of (7) is done with a similar procedure as in getting (6). As to the alternating Dirichlet series $\mathcal{D}_a(x)$ integral form, having in mind again the Cahen formula, we have [16]

$$\widetilde{\mathcal{D}}_a(x) = \sum_{n\geq 1} (-1)^{n-1} \mathrm{e}^{-a_n(x)} = x \int_{a_1}^\infty \mathrm{e}^{-xt} \widetilde{A}(t)\, \mathrm{d}t,$$

and therefore

$$\widetilde{\mathcal{D}}_a(x) = x \int_{a_1}^\infty \mathrm{e}^{-xt} \sin^2\left(\frac{\pi}{2} [a^{-1}(x)] \right) \mathrm{d}t,$$

since the counting function turns out to be

$$\widetilde{A}(t) = \sum_{n:\, a_n\le t} (-1)^{n-1} = \frac{1-(-1)^{[a^{-1}(t)]}}{2} = \sin^2\left(\frac{\pi}{2}[a^{-1}(t)]\right)\,.$$

Hence, because

$$\widetilde{\mathcal{D}}_a(t+s) = (t+s)\int_{a_1}^{\infty} \mathrm{e}^{-(t+s)x}\sin^2\left(\frac{\pi}{2}[a^{-1}(t)]\right)\,\mathrm{d}x,$$

we conclude (7) by carrying out the obvious remaining steps. □

Now, in the case $p=q$, Theorem 2.1 reduces to the following corollary.

Corollary 2.2. *Let $\lambda,\,\eta,\,r\in\mathbb{R}^+$ and let $\boldsymbol{a}=(a_n)_{n\ge1}$ be a real sequence which increases monotonically and tends to ∞. Then for $\Re(p)\ge0$ we have*

$$\mathfrak{F}_{\lambda,\eta}(F_p;\boldsymbol{a};r) = \lambda\;\mathscr{I}_p(\lambda+1,\eta)+\eta\;\mathscr{I}_p(\lambda,\eta+1),$$

and

$$\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_p;\boldsymbol{a};r) = \lambda\;\widetilde{\mathscr{I}_p}(\lambda+1,\eta)+\eta\;\widetilde{\mathscr{I}_p}(\lambda,\eta+1),$$

where

$$\mathscr{I}_p(\lambda,\eta) = \int_{a_1}^{\infty}\frac{F_p\left(\lambda,\,b;\,c;\,-\frac{r^2}{x}\right)[a^{-1}(x)]}{x^\lambda(x+r^2)^\eta}\,dt,$$

and

$$\widetilde{\mathscr{I}_p}(\lambda,\eta) = \int_{a_1}^{\infty}\frac{F_p\left(\lambda,\,b;\,c;\,-\frac{r^2}{x}\right)\,\sin^2\left(\frac{\pi}{2}[a^{-1}(x)]\right)}{x^\lambda(x+r^2)^\eta}\,\mathrm{d}t.$$

Remark 2.3. The special case of Theorem 2.1 when $p=q=0$ is seen to immediately reduce to the Gauss hypergeometric function ${}_2F_1$ result in [10].

3. Bounding inequalities for the (p,q)–extended Mathieu–type series

Very recently Parmar and Pogány [9] have established an upper bound for the (p,q)–extended Beta function $\mathrm{B}(x,y;p,q)$ (see [9, Lemma 2]). Namely, we have

$$\mathrm{B}(x,y;p,q)\le \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}\,\mathrm{B}(x,y)\qquad(\min\{x,y,p,q\}\ge0),$$

by observing

$$\sup_{0<t<1}\,\mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}} = \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2} =: \mathfrak{C}_{p,q}\qquad(\min\{p,\,q\}\ge0).$$

Here we recall the following results in [9, Theorem 8, Eqs. (3.2) and (3.3)]:

$$|F_{p,q}(a,b;c;z)|\le \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}\;{}_2F_1(a,b;c;|z|) \tag{14}$$

and

$$|\Phi_{p,q}(b;c;z)|\le \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}\,\Phi(b;c;|z|),$$

where $\min\{p,\,q\}\ge0$, $c>b>0$ and $|z|<1$. Also we need to recall a certain Luke's upper bound for the Gaussian hypergeometric function (see [8, p. 52, Eq. (4.7)]): For all $b \in (0,1]$, $c \geq a > 0$ and $z > 0$, the following inequality holds true:

$$ {}_2F_1(a,b;c;-z) < 1 - \frac{2ab(c+1)}{c(a+1)(b+1)}\Big[1 - \frac{2(c+1)}{2(c+1)+(a+1)(b+1)\,z}\Big]. \tag{15}$$

For simplicity, the following notation is introduced:

$$ \mathscr{U}_a(\lambda,\eta) := \int_{a_1}^{\infty} \frac{[a^{-1}(x)]}{x^{\lambda}(x+r^2)^{\eta}}\,\mathrm{d}x. \tag{16}$$

In the sequel we consider Mathieu–type series (4) and (5) in which the defining functions $a\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ behave so that $\mathscr{U}_a(\lambda,\eta)$ converges.

Theorem 3.1. *Let $\lambda \in (0,1]$ and $\eta \in \mathbb{R}^+$ and let $\boldsymbol{a} = (a_n)_{n\geq 1}$ be a real sequence which increases monotonically and tends to ∞. Then, for all $r \in (0, \sqrt{a_1}\,)$, $\min\{p,\,q\} \geq 0$ and $c > b > 0$, we have*

$$\begin{aligned} \mathfrak{F}_{\lambda,\eta}(F_{p,q};\boldsymbol{a};r) \leq \lambda\,\mathfrak{E}_{p,q}\bigg\{ &\left(1 - \frac{2(\lambda+1)b(c+1)}{c(\lambda+2)(b+1)}\right)\mathscr{U}_a(\lambda+1,\eta) \\ &+ \frac{4(\lambda+1)b(c+1)^2\,\mathscr{U}_a(\lambda,\eta)}{c(\lambda+2)(b+1)\,[(\lambda+2)(b+1)r^2+2(c+1)a_1]}\bigg\} \\ &+ \eta\,\mathfrak{E}_{p,q}\bigg\{\left(1 - \frac{2\lambda b(c+1)}{c(\lambda+1)(b+1)}\right)\mathscr{U}_a(\lambda,\eta+1) \\ &+ \frac{4\lambda b(c+1)^2\,\mathscr{U}_a(\lambda-1,\eta+1)}{c(\lambda+1)(b+1)\,[(\lambda+1)(b+1)r^2+2(c+1)a_1]}\bigg\}. \end{aligned} \tag{17}$$

Moreover, for all $\lambda+\eta > 1$, $r \in (0,\sqrt{a_1}\,)$, $\min\{p,\,q\} \geq 0$ and $c > b > 0$, we have

$$\begin{aligned} \widetilde{\mathfrak{F}}_{\lambda,\eta}(F_{p,q};\boldsymbol{a};r) \leq \lambda\,\mathfrak{E}_{p,q}\bigg\{ &\left(1 - \frac{2(\lambda+1)b(c+1)}{c(\lambda+2)(b+1)}\right)\frac{{}_2F_1\left(\eta,\lambda+\eta;\eta+1;-\frac{r^2}{a_1}\right)}{(\lambda+\eta)\,a_1^{\lambda+\eta}} \\ &+ \frac{4(\lambda+1)b(c+1)^2}{c(\lambda+2)(b+1)}\,\frac{a_1^{1-\lambda-\eta}\,{}_2F_1\left(\eta,\lambda+\eta-1;\eta+1;-\frac{r^2}{a_1}\right)}{(\lambda+\eta-1)[(\lambda+2)(b+1)r^2+2(c+1)a_1]}\bigg\} \\ &+ \eta\,\mathfrak{E}_{p,q}\bigg\{\left(1 - \frac{2\lambda b(c+1)}{c(\lambda+1)(b+1)}\right)\frac{{}_2F_1\left(\eta+1,\lambda+\eta;\eta+2;-\frac{r^2}{a_1}\right)}{(\lambda+\eta)\,a_1^{\lambda+\eta}} \\ &+ \frac{4\lambda b(c+1)^2}{c(\lambda+1)(b+1)}\,\frac{a_1^{1-\lambda-\eta}\,{}_2F_1\left(\eta+1,\lambda+\eta-1;\eta+2;-\frac{r^2}{a_1}\right)}{(\lambda+\eta-1)[(\lambda+1)(b+1)r^2+2(c+1)a_1]}\bigg\}. \end{aligned} \tag{18}$$

Proof. Firstly consider relation (6):

$$\mathfrak{F}_{\lambda,\eta}(F_{p,q};\boldsymbol{a};r) = \lambda\,\mathscr{I}_{p,q}(\lambda+1,\eta) + \eta\,\mathscr{I}_{p,q}(\lambda,\eta+1)\,,$$

which is found to be bounded above by the auxiliary integral $\mathscr{I}_{p,q}$ in (8). To do this, we observe that

$$F_{p,q}(a,b;c;z) > 0 \qquad \big(a \in \mathbb{R}^+,\, c > b > 0,\, 0 < z < 1\big)\,. \tag{19}$$

Indeed, it is enough to consider the following known integral expression [6, p. 373, Eq. (8.2)]:

$$F_{p,q}(a,b;c;z) = \frac{1}{\mathrm{B}(b,c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}\, \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) \mathrm{d}t\, > 0,$$

under the given conditions in (19). Therefore, by virtue of (14) and (15), it follows

$$\begin{aligned}
\mathscr{I}_{p,q}(\lambda,\eta) &= \int_{a_1}^\infty \frac{F_{p,q}\left(\lambda,\, b;\, c;\, -\frac{r^2}{x}\right)[a^{-1}(x)]}{x^\lambda (x+r^2)^\eta}\, \mathrm{d}x \\
&\le \mathfrak{C}_{p,q} \int_{a_1}^\infty \frac{{}_2F_1\left(\lambda,\, b;\, c;\, -\frac{r^2}{x}\right)[a^{-1}(x)]}{x^\lambda (x+r^2)^\eta}\, \mathrm{d}x \\
&\le \mathfrak{C}_{p,q}\Bigg\{ \left(1 - \frac{2\lambda b(c+1)}{c(\lambda+1)(b+1)}\right) \int_{a_1}^\infty \frac{[a^{-1}(x)]}{x^\lambda (x+r^2)^\eta}\, \mathrm{d}x \\
&\qquad + \frac{4\lambda b(c+1)^2}{c(\lambda+1)^2(b+1)^2} \int_{a_1}^\infty \frac{[a^{-1}(x)]\, \mathrm{d}x}{x^{\lambda-1}(x+r^2)^\eta \left[r^2 + 2\frac{(c+1)x}{(\lambda+1)(b+1)}\right]} \Bigg\} \\
&\le \mathfrak{C}_{p,q}\Bigg\{ \left(1 - \frac{2\lambda b(c+1)}{c(\lambda+1)(b+1)}\right) \mathscr{U}_a(\lambda,\eta) \\
&\qquad + \frac{4\lambda b(c+1)^2\, \mathscr{U}_a(\lambda-1,\eta)}{c(\lambda+1)(b+1)\left[(\lambda+1)(b+1)r^2 + 2(c+1)a_1\right]} \Bigg\}.
\end{aligned}$$

The rest in deriving (17) is obvious.

Secondly, here we recall (7) as follows:

$$\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_{p,q};\boldsymbol{a};r) = \lambda\, \widetilde{\mathscr{I}_{p,q}}(\lambda+1,\eta) + \eta\, \widetilde{\mathscr{I}_{p,q}}(\lambda,\eta+1)\,,$$

by positivity of the integrand of (9), we have

$$\widetilde{\mathscr{I}_{p,q}}(\lambda,\eta) \le \int_{a_1}^\infty \frac{F_{p,q}\left(\lambda,\, b;\, c;\, -\frac{r^2}{x}\right)}{x^\lambda (x+r^2)^\eta}\, \mathrm{d}x \le \mathfrak{C}_{p,q} \int_{a_1}^\infty \frac{{}_2F_1\left(\lambda,\, b;\, c;\, -\frac{r^2}{x}\right)}{x^\lambda (x+r^2)^\eta}\, \mathrm{d}x\,.$$

With the aid of (15), we conclude

$$\begin{aligned}
\widetilde{\mathscr{I}_{p,q}}(\lambda,\eta) \le \mathfrak{C}_{p,q}\Bigg\{ &\left(1 - \frac{2\lambda b(c+1)}{c(\lambda+1)(b+1)}\right) \int_{a_1}^\infty \frac{\mathrm{d}x}{x^\lambda (x+r^2)^\eta} \\
&+ \frac{4\lambda b(c+1)^2}{c(\lambda+1)^2(b+1)^2} \int_{a_1}^\infty \frac{[a^{-1}(x)]\, \mathrm{d}x}{x^{\lambda-1}(x+r^2)^\eta \left[r^2 + 2\frac{(c+1)x}{(\lambda+1)(b+1)}\right]} \Bigg\}.
\end{aligned}$$

Using [7, p. 313, Eq. **3.194** 1.] for $\lambda+\eta>1$ we have

$$\int_{a_1}^{\infty} \frac{\mathrm{d}x}{x^{\lambda}(x+r^2)^{\eta}} = \int_0^{\frac{1}{a_1}} \frac{t^{\lambda+\eta-2}}{(1+r^2t)^{\eta}}\,\mathrm{d}t = \frac{{}_2F_1\left(\eta, \lambda+\eta-1; \eta+1; -\frac{r^2}{a_1}\right)}{(\lambda+\eta-1)\,a_1^{\lambda+\eta-1}}\,,$$

which for $\lambda+\eta>2$ implies

$$\begin{aligned}&\int_{a_1}^{\infty} \frac{\mathrm{d}x}{x^{\lambda-1}(x+r^2)^{\eta}\,[(\lambda+1)(b+1)r^2+2(c+1)x]}\\ &\qquad\le \frac{a_1^{2-\lambda-\eta}\,{}_2F_1\left(\eta, \lambda+\eta-2; \eta+1; -\frac{r^2}{a_1}\right)}{(\lambda+\eta-2)[(\lambda+1)(b+1)r^2+2(c+1)a_1]}\,.\end{aligned}$$

Collecting these formulae we get the upper bound

$$\begin{aligned}\widetilde{\mathscr{I}}_{p,q}(\lambda,\eta) \le \mathfrak{E}_{p,q}\Bigg\{&\left(1-\frac{2\lambda b(c+1)}{c(\lambda+1)(b+1)}\right)\frac{{}_2F_1\left(\eta, \lambda+\eta-1; \eta+1; -\frac{r^2}{a_1}\right)}{(\lambda+\eta-1)\,a_1^{\lambda+\eta-1}}\\ &+\frac{4\lambda b(c+1)^2}{c(\lambda+1)(b+1)}\,\frac{a_1^{2-\lambda-\eta}\,{}_2F_1\left(\eta, \lambda+\eta-2; \eta+1; -\frac{r^2}{a_1}\right)}{(\lambda+\eta-2)[(\lambda+1)(b+1)r^2+2(c+1)a_1]}\Bigg\}.\end{aligned}$$

Now, obvious steps lead to the asserted upper bound (18). □

Remark 3.2. First observe $\mathfrak{E}_{p,p}=\mathrm{e}^{-4p}$. Then the special case of the results in Theorem 3.1 can be reduced to yield the simpler upper bound expressions for the respective related Mathieu–type series and its alternating variant $\mathfrak{F}_{\lambda,\eta}(F_p;\boldsymbol{a};r)$ and $\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_p;\boldsymbol{a};r)$. Yet their detailed descriptions are left to the interested reader.

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