Rigidity of Convex Surfaces in Homogeneous Spaces

by

Harold Rosenberg and Renato Tribuzy*

Abstract

We prove rigidity of oriented isometric immersions of complete surfaces in the homogeneous 3-manifolds $E(k, \tau)$ (different from the space forms) having the same positive extrinsic curvature.

Introduction.

An isometric immersion $f: M \to N$ is rigid if given any other isometric immersion $g: M \to N$, there is an isometry $h: N \to N$ such that $hf = g$. An isometric immersion $f: M \to N$ is locally rigid if whenever $f(t): M \to N$ is a smooth family of isometric immersions with $f(0) = f$, then there are isometries $h(t): N \to N$ such that $h(t)f(t) = f$.

Strictly convex compact surfaces in $\mathbb{R}^3$ are rigid [C], and there are complete strictly convex surfaces in $\mathbb{R}^3$ that are not rigid [O], [P]. A beautiful open problem is to decide if there is a smooth closed surface M in $\mathbb{R}^3$ that is not locally rigid; i.e., is there a continuous one parameter family of isometric immersions of M into $\mathbb{R}^3$ that are not congruent?

In this paper we consider local rigidity of convex surfaces in the 3-dimensional simply connected homogeneous 3-manifolds $E(k, \tau)$, $k - 4\tau^2 \neq 0$. After the space forms (isometry group of dimension 6), they are the most symmetric 3-manifolds (isometry group of dimension 4). $E(k, \tau)$ is a Riemannian submersion over the two dimensional space form $M^2(k)$, of curvature

2010 Math. Subject Classification: Primay: 53C42, Secondary: 53C40.

Key words: Rigidity, Bonnet problem, convex surfaces, homogeneous space.

* Partially supported by CNPq and FAPEAM.

k: $M^2(k) = S^2(k)$ if $k > 0$, $\mathbb{R}^2$ if $k = 0$, $\mathbb{H}^2(k)$ if $k < 0$. The bundle curvature is τ, and the unit tangent field to the fiber ξ is a Killing field. There is a 2-dimensional group of horizontal translations, translation along the ξ-orbits are isometries, and rotations about any vertical fiber. An important discrete group of isometries is generated by rotation by π about any horizontal geodesic.

When $\tau = 0$, $E(k, 0) = S^2(k) \times \mathbb{R}$ if $k > 0$ and $E(k, 0) = \mathbb{H}^2(k) \times \mathbb{R}$, $k < 0$. For $\tau \neq 0$, $k > 0$, they are the Berger spheres. For $\tau \neq 0$, $k = 0$, this gives Nil(3), i.e., Heisenberg space. And $\tau \neq 0$, $k < 0$, $E(k, \tau) = \widetilde{\mathrm{PSL}}(2, \mathbb{R})$; the universal covering space of the unit tangent bundle of $\mathbb{H}^2(k)$.

Convexity in $E = E(k, \tau)$ can be defined in terms of the second fundamental form. The least one needs is the extrinsic curvature K_e (the product of the principal curvatures) should be positive. However when $\tau \neq 0$, one needs the principal curvatures to be at least $|\tau|$ to obtain global theorems. Assuming $K_e > 0$, one has a Hadamard-Stoker theorem in $\mathbb{H}^2 \times \mathbb{R}$: if $f \colon M^2 \to \mathbb{H}^2 \times \mathbb{R}$ is an immersion (complete) with $K_e > 0$, then f is an embedding and $M^2 = S^2$ or $\mathbb{R}^2$. Also one can describe the embedding, [E-G-R]. This theorem is also true in $E(k, \tau)$ provided the principal curvatures are greater than $|\tau|$, [E-R].

In this paper we prove local rigidity of complete surfaces in $E(k, \tau)$ with the same positive extrinsic curvature, and satisfying a three point condition. In $E(k, \tau)$, K and K_e are related by the Gauss equation so knowing both these functions tells us the angle (up to sign) the tangent plane of the surface makes with the vertical fiber ξ.

In a beautiful paper [G-M-M], the authors studied rigidity of surfaces in $E(k, \tau)$ having the same principal curvatures (the Bonnet problem).

They showed that such surfaces, that are also real analytic, are rigid with some exceptions. In $E(k, 0)$ the exceptions are minimal surfaces (they have a 1-parameter family of isometric deformations; like the associated family from the catenoid to the helicoid in $\mathbb{R}^3$) and screw motion helicoidal surfaces. When $\tau \neq 0$, the exceptions are the helicoidal surfaces and Benoit Daniels' CMC twin surfaces [B].

The Main Result.

Let M be a complete Riemannian oriented surface. Given an immersion $f \colon M \to E(k, \tau)$ and an oriented frame (e_1, e_2) of M, we define the unit normal N_f so that $(f_*(e_1), f_*(e_2), N_f)$ is positive in $E(k, \tau)$ (assumed oriented). When $K_e(f) > 0$, the principal curvatures of $f(M)$

have the same sign on M (always positive or always negative). We always choose the orientation so that they are positive. Given two such immersions with the same positive K_e we say $fg^{-1}\colon g(M) \to f(M)$ is positive when fg^{-1} is orientation preserving, i.e., both surfaces have positive principal curvatures. We say that f is strictly convex in $E(k,\tau)$ if $K_e(f) > \tau^2$.

Theorem A. Let $f(t)\colon M \to E = E(k,\tau)$ be a smooth family of isometric immersions with $f(0) = f$. Suppose f is strictly convex, $K_e(f_t(x)) = K_e(f(x))$ for $x \in M$ and all t, and $Hf_t(x) = H(f(x))$ at three distinct points x of M. Then there are isometries $h(t)\colon E \to E$ such that $h(t)f(t) = f$.

We begin with a remark on vertical points of the immersion.

Lemma 1. Let $f\colon M \to E$ be an immersion with $K_e(x) > 0 \ \forall\, x \in M$. Let $g\colon M \to \mathbb{R}$ be the "angle" function: $g(x) = \langle N(x), \xi\rangle$, N the unit normal N_f. If $p \in \Sigma = f(M)$ and $T_p(\Sigma)$ is vertical (i.e. $\xi(p) \in T_p(\Sigma)$), then f is a submersion in a neighborhood of p.

Corollary 1. At a vertical point $p \in \Sigma$, there is a disk neighborhood D of p in Σ such that $g^{-1}(0)$ is a smooth curve β through p; β separates D into 2 components and g has opposite signs on the two components.

Proof of Lemma.

Let $\pi\colon E(k,\tau) \to M^2(k)$ be the Riemannian submersion, and let γ be a geodesic of $M^2(k)$, $\gamma(0) = \pi(p)$, and $d\pi(N(p)) = \gamma'(0)$. Let P be the vertical "plane": $P = \pi^{-1}(\gamma)$; P is isometric to $\mathbb{R}^2$ and totally geodesic when $\tau = 0$ (the extrinsic curvature of P is $-\tau^2$). At p, $N(p)$ is tangent to P, so $\Sigma \cap P$ is a smooth curve $C(s)$, for s near 0; $C(0) = p$, $C'(0) = \xi(p)$.

Since C is a normal section of Σ at p, the curvature of C at p in P is between the two principal curvatures of Σ at p, so $k_d^P(p) > 0$. Let $T(s) = C'(s)$, s arc length along C.

Denote by $N(s)$ the normal to Σ at $C(s)$ and $N_C^P(s)$ the unit normal to $C(s)$ in P. Write

$$N(s) = a(s)\, N^P(s) + E(s),$$

where $E(s)$ is normal to P along $C(s)$. We have $N(0) = N(p) = N_C^P(0)$ so $a(0) = 1$, $E(0) = 0$.

We want $dg_p(\xi) \neq 0$; we calculate

$$dg_p(\xi) = \frac{d}{ds}\bigg|_{s=0} \langle \xi, N(s)\rangle = \langle \widetilde{\nabla}_T \xi, N\rangle(0) + \langle \xi, \widetilde{\nabla}_T N\rangle(0) = \langle \xi, \widetilde{\nabla}_T N\rangle(0).$$

Since

$$\widetilde{\nabla}_T\xi(0) = \tau(T \wedge \xi)(0) = \tau(\xi \wedge \xi)(0) = 0; (T(0) = \xi).$$

Now

$$\widetilde{\nabla}_T N(s) = a'(s)N_C^P(s) + a(s)\widetilde{\nabla}_T N_C^P(s) + \widetilde{\nabla}_T E(x).$$

Hence

$$\langle\widetilde{\nabla}_T N, \xi\rangle(0) = \langle\widetilde{\nabla}_T N_C^P, \xi\rangle(0) + \langle\widetilde{\nabla}_T E, \xi\rangle(0).$$

Since E is normal to P and ξ is tangent to P, $\langle E(s), \xi\rangle = 0$ along C. So

$$\langle\widetilde{\nabla}_T E, \xi\rangle = -\langle E, \widetilde{\nabla}_T \xi\rangle.$$

At $s = 0$, $E(0) = 0$, so $\langle\widetilde{\nabla}_T E, \xi\rangle(0) = 0$.

Finally we have

$$dg_p(\xi) = \langle \xi, \widetilde{\nabla}_T N_C^P\rangle(0).$$

By the Gauss equation for P:

$$\widetilde{\nabla}_T N_C^P(0) = \nabla_T^P N_C^P(0) + \mathrm{II}^P(\xi, N(0))(N(0) \wedge \xi).$$

Hence

$$\langle\widetilde{\nabla}_T N_C^P(0), \xi\rangle = \langle\nabla_T^P N_C^P, \xi\rangle(0) = k_C^P(0) \neq 0;$$

i.e. $\quad dg_p(\xi) = k_C^P(0) \neq 0$.

Proof of Theorem A. Let $f\colon M \to E(k,\tau)$ be a strictly convex isommetric immersion. We will define a special set of moving frames away from the horizontal points ($\xi \perp TM$) given by $\varepsilon_1 = \frac{P(\varepsilon)}{|P(\varepsilon)|}$, where P denotes the projection into the tangent plane, J is the positive rotation and $\varepsilon_1 = J\varepsilon_2$. This way, we can write

$$\xi = cos(\theta)\varepsilon_1 + sin(\theta)N$$

where θ is the function meauring the angle between the vectors ξ and TM. The function θ and its derivative are defined at least locally.

In order to calculate the second fundamental form, we differentiate $\langle\varepsilon_1, \xi\rangle = cos(\theta)$, getting

$$X\langle\varepsilon_1, \xi\rangle = \langle\nabla_X\varepsilon_1, \xi\rangle + \langle\alpha(\varepsilon_1, X)N, \xi\rangle + \tau\langle\varepsilon_1, X \wedge \xi\rangle = -sin(\theta)d\theta X.$$

Because of $\langle \varepsilon_1, \xi \rangle = 0$, we also get

$$\alpha(\varepsilon_1, X) sin(\theta) + \tau(X, \varepsilon_2\rangle sin(\theta) = -sin(\theta) d\theta X$$

onsequently we have for those points where $sin(\theta) \neq 0$:

$$\alpha(\varepsilon_1, X) = -d\theta X - \tau\langle X, \varepsilon_2\rangle. \tag{1}$$

Now, for the points where $sin(\theta) = 0$, we know they lie in a differential curve as shown in Lemma 1.

By continuity, this equation holds for all points whenever the special moving frame is defined, so this holds for all non-horizontal points.

From (1), we obtain

$$\alpha(\varepsilon_1, \varepsilon_1) = -d\theta \cdot \varepsilon_1$$

$$\alpha(\varepsilon_1, \varepsilon_2) = -d\theta \cdot \varepsilon_2 - \tau$$

Since the immersion is convex, $\alpha(\varepsilon_1, \varepsilon_1) \neq 0$ and thus $d\theta \neq 0$ at each non-horizontal point.

Lemma 2. Every horizontal point of a convex immersion, is isolated.

Proof: The horizontal points are the vanishing points of the field $\xi \in N$. Let $p \in M$ be a point so that $(\xi \times N)_p = 0$ and let $\{v_1, v_2\}$ be an orthonormal positively oriented basis which diagonalizes the Weingarten operator at the point p and is associated to the eigenvalues λ_1 and λ_2 .

Let $\overline{\nabla}$ be the connection of $E(k, \tau)$, then:

$$\begin{aligned}\overline{\nabla}_{v_1}(\xi \times N)_p &= (\overline{\nabla}_{v_1}\xi \times N)_p + (\xi \times \overline{\nabla}_{v_1}N)_p \\ &= \tau(v_1 \times \xi)_p \times N_p - \lambda_1(\xi \times v_1)_p = \tau v_1 - \lambda_1 v_2\end{aligned}$$

and we also have:

$$\overline{\nabla}_{v_2}(\xi \times N)_p = \tau(v_2 \times \xi)_p \times N_p - \lambda_2(\xi \times v_2)_p = -\tau v_2 + \lambda_2 v_1 \, .$$

Let $\{V_1, V_2\}$ be a parallel extension of $\{X_1, X_2\}$ along the geodesic which starts at p. Let us consider the smooth map F defined in a neighborhood of p in M taking values in $\mathbb{R}^2$ given by

$$F(g) = (\langle V_1, \xi \times N\rangle_q \, , , \langle V_2, \xi \times N\rangle_q).$$

Considering that $F'(p) \cdot v_1 = (\tau, -\lambda_1)$ and also $F'(p) \cdot v_2 = (\lambda_2, -\tau)$, we have that $F'(p)$ is an isomorphism.

Therefore F is a diffeomorphism when restricted to a neighborhood of p. Thus p is the unique zero of the function F and consequently the unique zero of $\xi \times N$.

Corollary 2. When M is compact, the set of horizontal points is finite and equals two.

Proof: At every horizontal point we have $cos(\theta) = 0$ and thus either $cos(\theta) \geq 0$ or $cos(\theta) \leq 0$. Consequently we can choose a particular range for the function θ either in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

Let us proceed with the calculation of the second fundamental form. Differentiating the equation $\langle \varepsilon, \xi \rangle = 0$, we obtain

$$\begin{aligned} 0 = X\langle \varepsilon_2, \xi\rangle &= \langle \nabla_X \varepsilon_2, \xi\rangle + \langle \alpha(X, \varepsilon_2)N, \xi\rangle + \tau\langle \varepsilon_2, X \times \xi\rangle \\ &= \langle \nabla_X \varepsilon_2, \varepsilon_1\rangle cos(\theta) + \alpha(X, \varepsilon_2) sin(\theta) + \tau\langle \varepsilon_1, X\rangle sin(\theta). \end{aligned}$$

Consequently $\alpha(X, \varepsilon_2) = cotg(\theta) w_{12}(X) - \tau\langle \varepsilon_1, X\rangle$ and it follows that

$$\begin{aligned} \alpha(\varepsilon_1, \varepsilon_2) &= cotg(\theta) w_{12}(\varepsilon_1) - \tau \\ \alpha(\varepsilon_2, \varepsilon_2) &= cotg(\theta) w_{12}(\varepsilon_2), \end{aligned}$$

where $w_{ij}(X) = \langle \nabla_X \varepsilon_i, \varepsilon_j \rangle$.

Now, we will determine ordinary differential equations satisfied by a certain angle function we now define.

Let $v \in \mathfrak{X}(M)$ be a unit vector such that $d\theta \cdot v = 0$ and chosen in such a way that $Jv = \frac{grad(v)}{|grad(v)|}$.

Let ϕ be the angle between the vectors ε_1 and v. That is,

$$\begin{aligned} v &= cos(\phi)\varepsilon_1 + sin(\phi)\varepsilon_2 \\ Jv &= -sin(\phi)\varepsilon_1 + cos(\phi)\varepsilon_2 \,. \end{aligned}$$

As $\alpha(v, \varepsilon_1) = -d\theta v - \tau\langle v, \varepsilon_2\rangle$, we have $\alpha(v, \varepsilon_1) = -\tau sin(\phi)$ and consequently

$$cos(\phi)\alpha(\varepsilon_1, \varepsilon_2) + sin(\phi)\alpha(\varepsilon_1, \varepsilon_2) = -\tau sin(\phi). \tag{2}$$

In a similar way, $\alpha(Jv, \varepsilon_1) = -d\theta Jv - \tau\langle Jv, \varepsilon_2\rangle = -|grad\theta| - \tau cos(\phi)$ and therefore

$$-sin(\phi)\alpha(\varepsilon_1, \varepsilon_2) + cos(\phi)\alpha(\varepsilon_1, \varepsilon_2) = -|grad\theta| - \tau cos(\phi). \tag{3}$$

From equations (2) and (3), we get

$$\alpha_{11} = |grad\theta| sin(\phi)$$
$$\alpha_{12} = -|grad\theta| cos(\phi) - \tau.$$

In order to obtain the first differential equation satisfied by ϕ, we will calculate $\alpha(\varepsilon_2, v)$ using two different approches.

On the one hand we have:

$$\alpha(\varepsilon_2, v) = cos(\phi)\alpha(\varepsilon_1, \varepsilon_2) + sin(\phi)\alpha(\varepsilon_2 \varepsilon_2).$$

Denoting K_ε as the extrinsic curvature of M, we have

$$\alpha(\varepsilon_2, \varepsilon_2) = \frac{K_\varepsilon - \alpha(\varepsilon_1, \varepsilon_2)^2}{\alpha(\varepsilon_1, \varepsilon_1)}.$$

Note that since M is convex, $\alpha(\varepsilon_1, \varepsilon_1) \neq 0$, and therefore

$$\begin{aligned}\alpha(\varepsilon_2, v) &= -cos(\phi)(|grad\theta| cos(\phi) + \tau) \\ &\quad + sin(\phi)\,\frac{K_\varepsilon - (|grad\theta| cos(\phi) - \tau)}{|grad\theta| sin(\phi)}.\end{aligned}$$

Note that $sin(\phi) \neq 0$ because of $\alpha(\varepsilon_1 \varepsilon_1) \neq 0$. Consequently,

$$\begin{aligned}\alpha(\varepsilon_2, v) &= cos(\phi)(|grad\theta| cos(\phi) + \tau) \\ &\quad + \frac{K_\varepsilon - (|grad\theta| cos(\phi) - \tau)^2}{|grad\theta|}.\end{aligned} \tag{4}$$

On the other hand, we have $w_{12} = \widetilde{w}_{12} - d\phi$, where $\widetilde{w}_{12}X = \langle \nabla_X v, Jv \rangle$ and considering that $\alpha(v, \varepsilon_2) = cotg(\theta) w_{12}(v) - \tau \langle \varepsilon_1, v \rangle$. We have

$$\alpha(v, \varepsilon_2) = cotg(\theta)(\widetilde{w}_{12}(v) - d\phi, v) - \tau cos(\phi). \tag{5}$$

Equating the equations (4) and (5), we get a differential equaation satisfied by ϕ along the trajectories of v. Namely:

$$\begin{aligned}d\phi \cdot v = \widetilde{w}_{12}(v) - tg(\theta)\{&\tau cos(\phi) + cos(\phi) \cdot [|grad(\theta)| cos(\phi) + \tau] \\ &+ \frac{1}{|grad(\theta)|} \cdot [K_\varepsilon - (|grad(\theta)| cos(\phi) - \tau)^2]\}.\end{aligned}$$

Similarly, we have $\alpha(Jv, \varepsilon_2) = -sin(\phi)\alpha(\varepsilon_1, \varepsilon_2) + cos(\phi)\alpha(\varepsilon_2, \varepsilon_2)$ and therefore

$$\begin{aligned} \alpha(Jv, \varepsilon_2) = {} & sin(\phi)(|grad\theta|cos(\phi) + \tau) \\ & + cotg\, \phi\, \frac{K_\varepsilon - (|grad\theta|cos\phi + \tau)^2}{|grad\theta|}\,. \end{aligned}$$

The following equation also holds:

$$\alpha(Jv, \varepsilon_2) = cotg(\theta)(\widetilde{w}_{12}(Jv) - d\phi \cdot Jv) + \tau sin(\phi). \tag{7}$$

Now we can equate also the equations (6) and (7) to obtain a differential equation satisfied by ϕ along the trajectories of Jv.

$$\begin{aligned} d\phi \cdot Jv = {} & \widetilde{w}_{12}(Jv) - tg(\theta)\{sin(phi) \cdot [|grad(\theta)|cos(\phi) + \tau] \\ & = cotg(\phi)\, \frac{K_\varepsilon - (|grad(\theta)|cos(\phi) - \tau)^2}{|grad(\theta)|} + \tau sin(\phi)\}. \end{aligned}$$

Therefore, if two isometric convex immersions have the same extrinsic curvature, the same function θ and the same function ϕ at a point, then they have the same function ϕ in a neighborhood of that point.

To complete the proof of the Theorem observe:

Fact 1: From the Gauss equation, the set of possibilities for θ is discrete and therefore θ is constant along the deformation.

Fact 2: As the set of critical points of the function θ consists of two points, there exists a point $p \in M$ where $d\theta \neq 0$ and the mean curvature is preserved along the deformation. In a neighborhood of this point, the special moving frame $\{\varepsilon_1(t), \varepsilon_2(t)\}$ is defined for every value of t of the deformation, as well as the functions ϕ_t .

Fact 3: As H and $d\theta$ are both different from zero at p, there exists at most two possible values for $\phi_t(p)$.

Indeed, we have:

$$\alpha(\varepsilon_1(t), \varepsilon_1(t))\{2H - \alpha(\varepsilon_1(t), \varepsilon_1(t))\} - \alpha(\varepsilon_1(t), \varepsilon_2(t))^2 = K_\varepsilon\,.$$

Since $\alpha(\varepsilon_1(t), \varepsilon_1(t)) = |grad(\theta)|sin(\phi_t)$ and $\alpha(\varepsilon_1(t), \varepsilon_2(t)) = -|grad(\theta)|cos(\phi_t) - \tau$, we obtain

$$2H|grad(\theta)|sin(\phi_t) + 2\tau|grad(\theta)|cos(\phi_t) - |grad(\theta)|^2 - \tau^2 - K_\varepsilon = 0.$$

We observe an equation of the type $Asin(\phi_t)+Bsin(\phi_t)+C=0$ yields a quadratic polynomial equation in the variable $cos(\phi_t)$ and consequently has at most two roots, unless all its coefficients are zero, i.e., $A=B=C=0$.

Therefore ϕ is constant along the deformation at the point p. Consequently ϕ is constant along the deformation in a neighborhood of p. Since ϕ is preserved, we have that the second fundamental form is preserved and in particular H is preserved.

Let p_1 and p_2 be points in M where $d\theta=0$ and let $U=M-\{p_1,p_2\}$. As M is connected U is also connected.

Let $X=\{q\in U/H_t(q)=H_0(q)$ and $\phi_t(q)=\phi_0(q)\}$, where H_0 is the mean curvature of f and ϕ_0 is the function ϕ corresponding to f.

As we observed above, X is an open set U. On the other hand the set X is closed since it is intersection of closed sets of U. From the hypothesis of the theorem and the observation above, $p\in X$ and thus $X=U$. Consequently, all immersions f_t of the deformation, have the same function θ, the same second fundamental form and the same horizontal directions in X. Thus, they are equal to f up to some isometry of $E(k,\tau)$. This conclusion can be extended clearly to M.

References

[C] Cohn-Vossen E., *Zwei Satze uber die Starrheit der Eilflchen*, Nachr. Ges. Will. Gottingen (1927), 125-134.

[O] Olovisnishni-Koff, S., *On the bending of infinite convex surfaces*, N.S. 18 (60) (1946), 429-440.

[P] Pogorelov, *On the rigidity of general infinite convex surfaces with integral curvature* 2π, Dokl. Akad. Nauk SSSR (N.S.) 106 (1956), 19-20.

[G-M-M] Gálvez, J., Martínez, A. and Mira, P., *The Bonnet problem for Surfaces in homogeneous* 3*-manifolds*, Communications in Analysis and Geometry, vol. 16, 5, (2008), 907-935.

[B] Benoit Daniel, *Isometric Immersions into* 3*-dimensional homogeneous manifolds*, Comment. Math. Helv. 82 (2007), 87-131.

[E-G-R] Espinar, J., Gálvez, J. and Rosenberg, H., *Complete surfaces of positive extrinsic curvature in product spaces*, Comment. Math. Helv. 84 (1009), 351-389.

[E-R] Espinar, J., Rosenberg, H., *Convex surfaces immersed in homogeneous* 3*-manifolds*, preprint.

– Harold Rosenberg, IMPA, Rio de Janeiro, Brazil
rosen@impa.br

– Renato Tribuzy, UFAM, Manaus, AM, Brazil
tribuzy@pq.cnpq.br