Solution of Dirac Equations for Cotangent Potential with Coulomb-type Tensor Interaction for Spin and Pseudospin Symmetries Using Romanovski Polynomials

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Abstract

The bound state solutions of Dirac equations for cotangent function potential with the Coulomb-type tensor potential under spin and pseudospin symmetric limits are obtained using finite Romanovski polynomials. The approximate relativistic energy spectra are obtained for spin and pseudospin symmetries exactly. The radial wave functions are obtained in terms of Romanovski polynomials in the limit of spin and pseudospin symmetric conditions. The Coulomb type tensor potential removes the doublet degeneracies for pseudospin and spin symmetric cases. The relativistic energy spectrum for the exact spin symmetric case reduces to the non-relativistic energy spectrum in the non-relativistic limit.

Abstrak

Solusi Persamaan Dirac untuk Potensial Kotangen dengan Tensor Interaksi Tipe-Coulomb untuk Simetri Spin dan Pseudospin Menggunakan Polinomial Romanovski. Solusi keadaan terikat dari persamaan Dirac untuk potensial fungsi kotangen dengan potensial tensor tipe-Coulomb untuk simetri spin dan pseudospin diperoleh menggunakan polinomial Romanovski terbatas. Aproksimasi spektrum energi relativistik diperoleh secara eksak untuk simetri spin dan pseudospin. Fungsi gelombang radial diperoleh dalam bentuk polinomial Romanovski untuk keadaan simetri spin dan pseudospin. Potensial tensor tipe-Coulomb menghilangkan degenerasi doublet untuk kedua kasus simetri spin dan pseudospin. Pada batas non-relativistik, spektrum energi relativistik untuk kasus simetri spin eksak tereduksi menjadi spektrum energi non-relativistik.

Keywords: cotangent potential, Coulomb-type tensor, Dirac equation, pseudospin symmetry, Romanovski polynomials

1. Introduction

The bound state solutions of Dirac equations with a mixture of an attractive scalar potential $S(r)$ and a repulsive vector potential $V(r)$ together with tensor interaction potential have been investigated intensively. Dirac equations for central and non-central potentials with Yukawa-type and Coulomb-type tensor potentials have been applied in quantum chemistry and high-energy physics. They are used to describe the motion of particles governed by a strong force when the relativistic effects are taken into account.

The special conditions of Dirac equations when the vector potential is (nearly) the same as the scalar potential have been investigated recently, such as the non-spherical harmonic oscillator potential [1-3], Makarov potential [4-5], ring-shaped oscillator potential [6], ring-shaped non-spherical harmonic oscillator potentials [7-8], new ring-shaped Coulomb potentials [8], Coulomb potential plus new ring-shaped potential [9], and Hartmann potential plus new ring-shaped potential [10-11]. These potentials are widely used in studying quantum chemistry such as the relativistic effect of the distorted nucleus, the interaction between ring-shaped molecules, and the complex vibration-rotation energy structure of the multi-electron atom.

Dirac equations for central and non-central potentials together with and without tensor potentials have been solved mostly with the Nikiforov-Uvarov method [12-16],

the factorization method and the supersymmetry quantum mechanics method [17], the hypergeometric and confluent hypergeometric methods [18-19], and the asymptotic iteration method [20]. However, only a few potentials are solved exactly such as Coulomb and harmonics oscillator potentials with the Coulomb-type tensor potential; other potentials are solvable only for the *s*-wave. For the *l*-wave, the Dirac equations for central and non-central potentials are solved only approximately due to the contribution of the centrifugal term. An approximation scheme for the centrifugal term was proposed by Greene and Aldrich [21], and this approximation works well for trigonometric, hyperbolic, and exponential potentials.

In addition, Dirac equations for some potentials have been solved in the cases of spin symmetry and pseudospin symmetry [16, 19, 22-29]. Spin symmetry occurs when the difference between the repulsive vector potential with the attractive scalar potential is equal to constant, while the pseudospin symmetry arises when the sum of the scalar potential and the vector potential is equal to constant. Spin symmetric and pseudospin symmetric concepts have been used to study the aspect of deformed and superdeformation nuclei in nuclear physics. Spin symmetry has been applied to the meson and antinucleon spectrum [30], and the pseudospin symmetric concept is used to explain the quasi-degeneracy of nucleon doublets [31], exotic nuclei [32], and superdeformation in nuclei [33], and to establish an affective nuclear shell-model scheme [34].

In this paper, the relativistic energy spectra and wave functions of the trigonometric cotangent potential with the Coulomb-type tensor potential are analyzed using finite Romanovski polynomials. The trigonometric cotangent potential is part of the Rosen-Morse potential, a potential model used to explain nucleon excitation. The Rosen-Morse potential has also been used as a model to describe a fundamental massless gauge theory in addition to the Coulomb potential. The nucleon excitation levels that carry the same degeneracies as the levels of the electron with spin in the hydrogen atom are surprisingly well explained by the model. Compared to the hydrogen atom, the baryon level splitting contain in addition to the Balmer term its inverse but of opposite sign. The surprise explanation is reasonable since the Rosen-Morse potential can be viewed as a combination of the Coulomb potential and the square well potential. The Coulomb-like tensor potential, which is screened Coulomb potential, was originally used to model strong nucleon-nucleon interactions caused by the exchange in nuclear physics [32, 35-40].

Finite Romanovski polynomials are a traditional method that consists of reducing the Schrödinger equation by an appropriate variable substitution to the form of a generalized hypergeometric equation [41]. Romanovski polynomials were discovered by Sir E. J. Routh [42] and

rediscovered 45 years later by V. I. Romanovski [43]. The notion "finite" refers to the observation that, for any given set of parameters (i.e., in any potential), only the finite polynomials appear orthogonal [38]. We apply the finite Romanovski polynomials method since this method is simpler than the Nikiforov-Uvarov method in obtaining the energy spectrum and the wave function although there is a limitation in determining the normalization of the wave function. Until recently, only a few researchers used finite Romanovski polynomials to solve the Schrödinger equation for certain potentials [38-39, 45-47].

This paper is organized as follows. The basic theory of Dirac equations is presented briefly in section 2. Finite Romanovski polynomials as an analytical method is presented in section 3. In section 4, the results and discussion are presented. Finally, a brief conclusion is presented in section 5.

Basic Dirac spinor equation. The motion of a nucleon with mass M in a repulsive vector potential $V(r)$ and an attractive scalar potential $S(r)$ and coupled by a tensor potential $U(r)$ is described by the Dirac equation given as [12, 16, 19, 25, 26]

$$\bar{\alpha}.\vec{p} + \beta(M + S(r)) - i\beta\bar{\alpha}.\hat{r}U(r)\}\psi(\vec{r})$$
$$= E - V(r)\}\psi(\vec{r}) \qquad , \qquad (1)$$

where E is the relativistic energy and $\vec{p}$ is the three-dimensional momentum operator, $-i\nabla$,

$$\bar{\alpha} = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \text{ and } \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad (2)$$

where σ is the three-dimensional Pauli matrices, and I is the 2x2 identity matrix. Here, we consider the matrix potential in equation (1) as a spherically symmetric potential, they depend not only on the radial coordinate, $r = |\vec{r}|$ and we have taken $\hbar = 1$, $c = 1$. The Dirac equation expressed in equation (1) is invariant under spatial inversion, and therefore, its eigenstates have definite parity. By writing the spinor as

$$\Psi(\vec{r}) = \begin{pmatrix} \zeta(\vec{r}) \\ \phi(\vec{r}) \end{pmatrix} = \begin{pmatrix} \frac{F_{n\kappa}(r)}{r}Y_{jm}^l(\theta,\varphi) \\ i\frac{G_{n\kappa}(r)}{r}Y_{jm}^{\bar{l}}(\theta,\varphi) \end{pmatrix} \qquad , \qquad (3)$$

where $\zeta(\vec{r})$ is the Dirac spinor of the upper (large) component and $\phi(\vec{r})$ is the Dirac spinor of the lower (small) component, $Y_{jm}^l(\theta,\varphi)$ is the spin spherical harmonics, $Y_{jm}^{\bar{l}}(\theta,\varphi)$ is the pseudospin spherical harmonics, l is orbital quantum number, $\bar{l}$ is the pseudo-the orbital quantum number, and m is the projection of the angular momentum on the *z*-axis. The Dirac Hamiltonian in a spherical field commutes with the total angular momentum operator $\vec{J}$ and the spin-orbit coupling operator $\bar{K}$, where $\kappa = -\beta(\bar{\sigma}.\vec{L} + 1)$ with $\vec{L}$ is the orbital angular momentum. The eigenvalues of the spin-orbit coupling operator are $\kappa = (J + 1/2) > 0$ for the

unaligned spin ($p_{1/2}, d_{3/2}, ...$) and $\kappa = -(J + 1/2) < 0$ for the aligned spin ($s_{1/2}, p_{3/2}, ...$). Therefore, the conservative quantities consist of the set of $H, \bar{\kappa}, \vec{J}, J_z$.

Inserting equations (3) and (2) into equation (1), we have

$$
\begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} p \begin{pmatrix} \zeta(r) \\ \phi(r) \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} (M + S(r)) \begin{pmatrix} \zeta(r) \\ \phi(r) \end{pmatrix}
$$
$$
- i\beta\alpha.\hat{r}U(r) \begin{pmatrix} \zeta(r) \\ \phi(r) \end{pmatrix} = \left(E - V(r) \right) \begin{pmatrix} \zeta(r) \\ \phi(r) \end{pmatrix} \tag{4}
$$

From equation (4), we obtain two coupled first-order differential equations given as

$$
\left[\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right] F_{n\kappa}(r) = \left(M + E_{n\kappa} - V(r) \right.
$$
$$
\left. + S(r) \right) G_{n\kappa}(r) \tag{5}
$$

and

$$
\left[\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right] G_{n\kappa}(r) = \left(M - E_{n\kappa} + V(r) \right.
$$
$$
\left. + S(r) \right) F_{n\kappa}(r) \tag{6}
$$

From equations (5) and (6), we get the upper and lower radial part of the Dirac equations,

$$
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - U^2(r) - \frac{dU}{dr} \right\} F_{n\kappa}(r)
$$
$$
+ \left\{ \frac{d\Delta}{dr} \left[\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right] / \left(M + E_{n\kappa} - \Delta(r) \right) \right\} F_{n\kappa}(r)
$$
$$
+ \left(M + E_{n\kappa} - \Delta(r) \right) \left(E_{n\kappa} - M - \Sigma(r) \right) F_{n\kappa}(r) = 0 \tag{7}
$$

$$
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) - U^2(r) + \frac{dU}{dr} \right\} G_{n\kappa}(r)
$$
$$
+ \left\{ \frac{d\Sigma}{dr} \left[\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right] / \left(M - E_{n\kappa} + \Sigma(r) \right) \right\} G_{nK}(r)
$$
$$
+ \left(M + E_{n\kappa} - \Delta(r) \right) \left(E_{n\kappa} - M - \Sigma(r) \right) G_{n\kappa}(r) = 0 \tag{8}
$$

where the spin-orbit quantum number κ is related to the usual orbital angular momentum with $\kappa(\kappa+1) = l(l+1)$ for the upper spinor component and is related to the pseudo-orbital angular momentum with $\kappa(\kappa+1) = \bar{l}(\bar{l}+1)$ for the lower spinor component, $\Sigma(r) = V(r) + S(r)$ is the sum of the scalar and vector potentials, and $\Delta(r) = V(r) + S(r)$ is the difference between the vector potential and the scalar potential. By choosing the vector and scalar potential as the cotangent function potential and the tensor interaction potential, the Coulomb-type tensor potential is given as

$$
V(r) = -(V_0 \cot ar) \tag{9}
$$
$$
U(r) = -\frac{V_1}{—} \tag{10}
$$

Then the relativistic energy of the system and the Dirac spinor wave function are found. In equation (9), V_0 describes the depth of the potential and is positive, a is a positive parameter that controls the width or the range of the potential, and in equation (10), V_1 is the strength of the nucleon force, $V_1 = \kappa z_1 z_2$, z_1 is the projectile charge, z_2 is the charge of the particle, κ is the electrostatic constant, and $0 < r < \infty$.

Pseudospin symmetry occurs when $\Sigma(r) = V(r) + S(r) = C_{ps}$ with C_{ps} is constant; therefore, $\frac{d\Sigma}{dr} = 0$, and the difference between the vector and scalar potentials $\Delta(r)$ is

$$
\Delta(r) = V(r) - S(r) = -V_0 \cot ar \tag{11}
$$

The Dirac equation for the lower component of the Dirac spinor in equation (8) reduces to

$$
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) - U^2(r) + \frac{dU}{dr} \right\} G_{n\kappa}(r)
$$
$$
+ \left(M + E_{n\kappa} - \Delta(r) \right) \left(E_{n\kappa} - M - C_{ps} \right) G_{n\kappa}(r) = 0 \tag{12}
$$

with $\kappa(\kappa+1) = \bar{l}(\bar{l}+1)$ that leads to $\kappa = -\bar{l} = -(j+1/2)$, when $\kappa < 0$ for the aligned spin, and $\kappa = (\bar{l}+1) = j + 1/2$, when $\kappa > 0$ for the unaligned spin. In general, the pseudo-orbital quantum number is written as $\bar{l} = l - \kappa/|\kappa|$. These conditions imply that the total angular momentum $j = \bar{l} \pm \frac{1}{2}$ causes the state to be degenerated for $\bar{l} \neq 0$.

Moreover, the exact pseudospin symmetry arises when $\Sigma(r) = V(r) + S(r) = C_{ps} = 0$, and $\Delta(r) = V(r) - S(r)$ is the cotangent function potential expressed in equation (9); therefore, equation (12) becomes

$$
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) - U^2(r) + \frac{dU}{dr} \right\} G_{n\kappa}(r)
$$
$$
+ \left(E_{nK}{}^2 - M^2 \right) G_{n\kappa}(r) - \Delta(r) \left(E_{n\kappa} - M \right) G_{n\kappa}(r) = 0 \tag{13}
$$

However, spin symmetry occurs when the difference between the vector and scalar potentials is constant, $\Delta(r) = V(r) - S(r) = C_s$, and the sum of the vector and scalar potentials

$$
\Sigma(r) = V(r) + S(r) = -V_0 \cot ar \tag{14}
$$

The upper component of the Dirac spinor obtained from equation (7) is

$$
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - U^2(r) - \frac{dU}{dr} \right\} F_{n\kappa}(r)
$$
$$
+ \left(M + E_{n\kappa} - C_s \right) \left(E_{n\kappa} - M - \Sigma(r) \right) F_{n\kappa}(r) = 0 \tag{15}
$$

with $\kappa(\kappa+1) = l(l+1)$ that leads to $\kappa = l$ and $j = l - \frac{1}{2}$ when $\kappa > 0$ for unaligned spin $\kappa = -(l+1)$ and $j = l + \frac{1}{2}$ when $\kappa < 0$ for the aligned spin. Both Dirac equations for the pseudospin and spin symmetries in equations (12) and (15) are solved using Romanovski polynomials.

2. Methods

The method used to solve the Dirac equation in the limit of spin symmetric and pseudospin symmetric cases is finite Romanovski polynomials since the Dirac equations for the limited condition, when the spin and pseudospin symmetries arise, reduce to one-dimensional Schrödinger-like equations. The one-dimensional second-order differential equation satisfied by Romanovski polynomials is developed based on the hypergeometric differential equation. The one-dimensional Schrödinger equation of the potential of interest reduces to the differential equation of Romanovski polynomials by the appropriate variable and wave function substitutions. The one-dimensional Schrödinger equation is given as

$$-\frac{\hbar^2}{2M}\frac{\partial^2 \Psi(x)}{\partial x^2} + V(x)\Psi(x) = E\Psi(x) \qquad (16)$$

where $V(x)$ is an effective potential, which is mostly the shape invariant potential. By suitable variable substitution of $x = f(s)$, equation (16) changes into the generalized hypergeometric-type equation expressed as

$$\frac{\partial^2 \Psi(s)}{\partial s^2} + \frac{\tilde{\tau}(s)}{\sigma(s)}\frac{\partial \Psi(s)}{\partial s} + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\Psi(s) = 0 \qquad (17)$$

with $\sigma(s)$ and $\tilde{\sigma}(s)$ are mostly polynomials of order two, $\tilde{\tau}(s)$ is a polynomial of order one, s, $\sigma(s)$, $\tilde{\sigma}(s)$, $\tilde{\tau}(s)$ and can have any real or complex values [48]. Equation (17) is solved with the variable separation method. By setting

$$\Psi_n(r) = g_n(s) = (1+s^2)^{\frac{\beta}{2}} e^{\frac{-\alpha}{2}\tan^{-1}s} D_n^{(\beta,\alpha)}(s) \qquad (18)$$

we obtain a hypergeometric-type differential equation, which can be solved using finite Romanovski polynomials [35, 44], expressed as

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \qquad (19)$$

with

$$\sigma(s) = as^2 + bs + c; \tau = fs + h - \{n(n-1) \\ + 2n(1-p)\} = \lambda = \lambda_n \qquad (20)$$

and

$$y_n = D_n^{(\beta,\alpha)}(s) \qquad (21)$$

For Romanovski polynomials, the parameter values in equation (20) are

$$a = 1, b = 0, c = 1, f = 2(1-p) \\ \text{and } h \quad q' \text{ with } p > 0 \qquad (22)$$

Therefore, equation (19) is rewritten as

$$(1+s^2)\frac{\partial^2 R_n^{(p,q')}}{\partial s^2} + \quad 2s(-p+1)+q'\}\frac{\partial R_n^{(p,q')}(s)}{\partial s} \\ - \quad n(n-1)+2n(1-p)\}R_n^{(p,q')}(s) = 0 \qquad (23)$$

$$y(s) = R_n^{(p,q')}(s) \qquad (24)$$

By applying the specific values for Romanovski polynomials expressed in equation (22) to equation (19), then equation (19) reduces to equation (23), which is the second-order differential equation satisfied by Romanovski polynomials. Equation (19) is described in Nikiforov-Uvarov's textbook [44, 48], where the equation is cast into self-adjoint form and its weight function, $w(s)$, satisfies the Pearson differential equation

$$\frac{d(\sigma(s)w(s))}{ds} = \tau(s)w(s) \qquad (25)$$

The weight function, $w(s)$, is obtained by solving the Pearson differential equation expressed in equation (25) and by applying the condition in equations (20) and (22), given as

$$W^{(p,q')}(s) = (1+s^2)^{-p}e^{q'\tan^{-1}(s)} \qquad (26)$$

The corresponding polynomials are classified according to the weight function, and are built up from the Rodrigues representation, which is presented as

$$y_n = \frac{B_n}{w(s)}\frac{d^n}{ds^n}\left\{(as^2+bs+c)^n w(s)\right\} \qquad (27)$$

with B_n is a normalization constant. For $\sigma(s) > 0$ and $w(s) > 0$, $y_n(s)s$ are normalized polynomials and are orthogonal regarding the weight function $w(s)$ within a given interval (s_1, s_2), which is expressed as

$$\int_{s_1}^{s_2} w(s)y_n(s)y_{n'}(s)ds = \delta_{nn'} \qquad (28)$$

This weight function in equation (26) was first reported by Routh [42] and then by Romanovski [43]. The polynomials associated with equation (23) are named after Romanovski and will be denoted by $R_n^{(p,q)}(s)$. Due to the decrease in the weight function by s^{-2p}, an integral of the type

$$\int_{-\infty}^{\infty} w^{(p,q')}R_n^{(p,q')}(s)R_n^{(p,q')}(s)ds \qquad (29)$$

will be convergent only if

$$n' + n < 2p - 1 \qquad (30)$$

The heart of the Romanovski polynomials method is in obtaining equations (17) and (23) from

the one-dimensional Schrödinger equation. The Schrödinger equation of the potential of interest will reduce to a second-order differential equation that is similar to equation (17), which is called the generalized hypergeometric equation by an appropriate transformation of the variable, for example, $r = f(s)$. Then, by substituting a new wave function in equation (18) into equation (17), we get a new equation in the form of equation (23) but with β and α parameters. By comparing equation (23) and the new equation, we get the relation between β and p, and between α and q'. The Romanovski polynomials obtained from the Rodrigues formula expressed in equation (27) corresponding to the weight function in equation (26) are expressed as

$$
R_n^{(p,q')}(s) = D_n^{(\beta,\alpha)}(s) = \frac{1}{(1+s^2)^{-p} e^{q' \tan^{-1}(s)}}
$$
$$
\times \frac{d^n}{ds^n} \left\{ (1+s^2)^n (1+s^2)^{-p} e^{q' \tan^{-1}(s)} \right\} \tag{31}
$$

If the wave function of the *n*th level in equation (18) is rewritten as

$$
\Psi_n(r) = \frac{1}{\sqrt{\frac{df(s)}{ds}}} (1+s^2)^{\frac{-p}{2}} e^{\frac{q'}{2} \tan^{-1}(s)} R_n^{(p,q')}(s) \tag{32}
$$

then the orthogonality integral of the wave function expressed in equation (32) gives rise to the orthogonality integral of the finite Romanovski polynomials, which is given as

$$
\int_0^\infty \Psi_n(r)\Psi_n(r)dr = \int_{-\infty}^\infty w^{(p,q')} R_n^{(p,q')}(s) R_n^{(p,q')}(s) ds \tag{33}
$$

In this case, the p and q' values are not *n*-dependent where n is the degree of the polynomials. However, if either equation (28) or (30) is not fulfilled, then the Romanovski polynomials is infinity [38].

3. Results and Discussion

Pseudospin symmetric case. The Dirac equation for the pseudospin symmetric case presented in equation (12) with the potential and the tensor potential expressed in equations (9) and (10) is given as

$$
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \frac{2\kappa}{r} \frac{V_1(r)}{r} - \frac{V_1^2(r)}{r^2} + \frac{V_1}{r^2} \right\}
$$
$$
G_{n\kappa}(r) + (-V_0 \cot ar)(M - E_{n\kappa} + C_{ps})G_{n\kappa}(r) \tag{34}
$$
$$
= (M + E_{n\kappa})(M - E_{n\kappa} + C_{ps})G_{n\kappa}(r)
$$

Equation (32) cannot be solved exactly except when we use approximation to the $\frac{1}{r^2}$ and $\frac{1}{r}$ terms. For small a where $ra << 1$, then the approximation of $\frac{1}{r^2}$ has the form of [21].

$$
\frac{1}{r^2} \cong \frac{a^2}{(\sin^2 ar)} \tag{35}
$$

Equation (35) is substituted into equation (34). Then we get

$$
\left\{ \frac{d^2}{dr^2} - \frac{(\kappa(\kappa-1) + 2\kappa V_1 + V_1^2 - V_1)a^2}{\sin^2 ar} \right\} G_{n\kappa}(r)
$$
$$
- V_0 \cot ar(M - E_{n\kappa} + C_{ps})G_{n\kappa}(r) \tag{36}
$$
$$
= (M + E_{n\kappa})(M - E_{n\kappa} + C_{ps})G_{n\kappa}(r)
$$

By setting

$$
A_{ps} = (\kappa + V_1 - 1)(\kappa + V_1) \tag{37}
$$

$$
B_{ps} = \frac{V_0}{a^2}(M - E_{n\kappa} + C_{ps}) \tag{38}
$$

$$
E_{ps}' = -\frac{(M + E_{n\kappa})(M - E_{n\kappa} + C_{ps})}{a^2} \tag{39}
$$

in equation (36), equation (36) then reduces to a one-dimensional Schrödinger-type equation

$$
\left\{ \frac{d^2}{dr^2} - \frac{a^2 A_{ps}}{(\sin ar)^2} - a^2 B_{ps} \cot ar \right\} G_{n\kappa}(r)
$$
$$
= -a^2 E_{ps}' G_{n\kappa}(r) \tag{40}
$$

By substituting the spatial variable, $\cot ar = x$ in equation (40) where $-\infty < x < \infty$, we get

$$
(1+x^2)\frac{d^2 G_{n\kappa}}{dx^2} + 2x\frac{dG_{n\kappa}}{dx} - \left(A_{ps} + \frac{B_{ps}x}{1+x^2} \right)
$$
$$
- \frac{E'}{1+x^2} \right\} G_{n\kappa} = 0 \tag{41}
$$

To solve equation (41), we use equation (18) as a new wave function,

$$
G_{n\kappa}(x) = g_n(x) = (1+x^2)^{\frac{\beta}{2}} e^{\frac{-\alpha}{2} \tan^{-1} x} D_n^{(\beta,\alpha)}(x) \tag{42}
$$

After manipulating equations (41) and (42), we obtain

$$
(1+x^2)\frac{\partial^2 D_n^{(\beta,\alpha)}}{\partial x^2} + \left\{ 2x(\beta+1) - \alpha \right\} \frac{\partial D_n^{(\beta,\alpha)}}{\partial x} -
$$
$$
\left\{ \frac{\beta\alpha x - \frac{\alpha^2}{4} + \beta^2 + B_{ps}x - E'}{1+x^2} + A_{ps} - \beta^2 - \beta \right\} \tag{43}
$$
$$
D_n^{(\beta,\alpha)} = 0
$$

Equation (43) reduces to the differential equation that was satisfied by the Romanovski polynomials as in equation (23) if the coefficient of $\frac{1}{1+x^2}$ term is set to be zero, which are

$$
-\frac{\alpha^2}{4} + \beta^2 - E' \quad \text{and} \quad \beta\alpha + \beta_{ps} = 0 \tag{44}
$$

and then equation (43) becomes

$$
(1+x^2)\frac{\partial^2 D}{\partial x^2} + \left\{ 2x(\beta+1) - \alpha \right\} \frac{\partial D}{\partial x} - \left\{ A_{ps} - \right.
$$
$$
\beta^2 - \beta \left\} D = 0 \right. \tag{45}
$$

By comparing the parameters between equations (23) and (45), we obtain the relations

$$A_{ps} - \beta^2 - \beta = n(n-1) + 2n(1-p) \ ;$$
$$2(\beta+1) = 2(-p+1) \ \text{ and } \ \alpha = -q' \tag{46}$$

From equation (46), we have $p = -\beta$; since $p > 0$, then the value of β obtained from equation (46) that has physical meaning is

$$\beta = \beta_n = \sqrt{A_{ps} + \frac{1}{4}} - n - \frac{1}{2} = \kappa + V_1 - n - 1 \tag{47}$$

By using equations (44) and (47), we obtain

$$\alpha = \alpha_n = \frac{B_{ps}}{-\sqrt{A_{ps} + \frac{1}{4}} + n + \frac{1}{2}}$$
$$= \frac{\frac{V_0}{a^2}(M - En\kappa + C_{ps})}{-(\kappa + V_1) + n + 1} \tag{48}$$

By manipulating equations (37)-(39), (44), (47), and (48), we get the relativistic energy as follows. By inserting equations (47) and (48) into equation (44), we have

$$-\frac{B_{ps}^2}{4(-\sqrt{A_{ps} + 1/4} + n + 1/2)^2} +$$
$$\left(-\sqrt{A_{ps} + 1/4} + n + 1/2\right)^2 \tag{49}$$
$$= -\frac{(M + E_{n\kappa})(M - En\kappa + C_{ps})}{a^2}$$

and by setting

$$P = \left(-\sqrt{A_{ps} + 1/4} + n + 1/2\right)^2$$
$$= \left(-(\kappa + V_1) + n + 1\right)^2 \tag{50}$$

in equation (49) we have

$$-\frac{B_{ps}^2}{4} + P^2 = -P\frac{(M + E_{n\kappa})(M - E_{n\kappa} + C_{ps})}{a^2} \tag{51}$$

which gives the energy spectra of the system as

$$E_{n\kappa} = \frac{(2M + C_{ps})\frac{V_0^2}{4a^4} + C_{ps}\left(\frac{V_0^2}{4a^4} + \frac{(-\kappa - V_1 + n + 1)^2}{a^2}\right)}{2\left(\frac{V_0^2}{4a^4} + \frac{(-\kappa - V_1 + n + 1)^2}{a^2}\right)} \pm$$
$$\frac{2(-\kappa - V_1 + n + 1)^2 \sqrt{\left(\frac{V_0^2}{4a^4} + \frac{(-\kappa - V_1 + n + 1)^2}{a^2} + \frac{(C_{ps} + 2M)^2}{4a^4}\right)}}{2\left(\frac{V_0^2}{4a^4} + \frac{(-\kappa - V_1 + n + 1)^2}{a^2}\right)} \tag{52}$$

The relativistic energy spectra of the cotangent potential combined with the Coulomb-type tensor potential obtained using Romanovski polynomials for pseudospin symmetry is given in equation (52). The energy eigenvalues in equation (52) provide two different energy spectra for each set of quantum numbers (n, κ), but the

energy eigenvalues that satisfied the condition have to be negative values, which are shown in Table 1. Without the presence of the Coulomb-type tensor potential, when $V_1 = 0$, the energy eigenvalues degenerate. In the case of the cotangent function potential, the degeneracy condition is shown by the P term in equation (50) of the energy spectra in equation (52). For different sets of (n, κ) such as (1, -2) and (0, 5), (1, -1) and (0, 4), etc., produce the same values of P such that the energy spectra values result. However, the presence of the Coulomb-type tensor, when $V_1 = 0.6 fm^{-1}$, causes the removal of the degeneracy of the pseudospin doublet as shown in Table 1. Table 1 shows that the degeneracies occur for a pair of quantum numbers (n, l) and $(n-1, l+4)$.

To determine the lower component wave function of the Dirac spinor $G_{n\kappa}$, equations (47) and (48) are inserted into equations (10) and (15) so that we obtain the weight function $w(x)$ and the Romanovski polynomials $R_n^{(p,q')} = R_n^{(-\beta, -\alpha)}(x)$ as

$$w^{(-\beta, -\alpha)} = (1 + x^2)^{\beta_n} e^{-\alpha_n \tan^{-1} x}$$
$$= (1 + \cot^2 ar)^{\beta_n} e^{-\alpha_n \tan^{-1}(\cot ar)} \tag{53}$$

and

$$R_n^{(p,q')}(x) = R_n^{(-\beta, -\alpha)}(x = \cot ar)$$
$$= \frac{1}{(1 + x^2)^{\beta_n} e^{-\alpha_n \tan^{-1}(x)}} \frac{d^n}{dx^n}\left\{(1 + x^2)^{\beta_n + n}\right.$$
$$\left. e^{-\alpha_n \tan^{-1}(x)}\right\} \tag{54}$$

Thus, the lower component of the Dirac spinor wave function obtained from equations (53) and (54) is

$$G_{n\kappa}(f(x)) = (1 + x^2)^{\frac{\beta_n}{2}} e^{\frac{-\alpha_n}{2} \tan^{-1} x} R_n^{(-\beta, -\alpha)}(x)$$
$$= (1 + \cot^2 ar)^{\frac{\beta_n}{2}} e^{\frac{-\alpha_n}{2} \tan^{-1}(\cot ar)} R_n^{(-\beta, -\alpha)}(\cot ar) \tag{55}$$

For the pseudospin symmetry case, the upper component of the Dirac spinor is found from equations (6) and (55) given as

$$F_{n\kappa}(r) = \frac{\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)}{(M - E_{n\kappa} + C_{ps})} G_{n\kappa}(r) \tag{56}$$

The existence of the upper component of the Dirac spinor depends on the values of the energy eigenvalues $E_{n\kappa}$. If the value of C_{ps} is zero, then $M \neq E_{n\kappa}$; therefore, the existence of $F_{n\kappa}$ requires the energy eigenvalues $E_{n\kappa}$ to be negative. The values of the energy eigenvalues that satisfy this condition are shown in Table 1 where the energy of the nucleon under the pseudospin symmetric case is negative.

Since β_n and α_n parameters, expressed in equations (47) and (48), are n-dependent, the orthogonality of the wave

l	n	κ	State	$E_{n\kappa}V_1 = 0$	$E_{n\kappa}V_1 = 0.6fm$	l	n	κ	State	$E_{n\kappa}V_1 = 0$	$E_{n\kappa}V_1 = 0.6fm$
0	1	-1	$1s_{1/2}$	-0.031911539	-0.020460061	4	0	4	$0g_{9/2}$	-0.031911539	-0.04585220
1	1	-2	$1p_{3/2}$	-0.056512901	-0.040930649	5	0	5	$0h_{11/2}$	-0.056512901	-0.074526663
2	1	-3	$1d_{5/2}$	-0.087866279	-0.068254378	6	0	6	$0i_{13/2}$	-0.087866279	-0.109837332
3	1	-4	$1f_{7/2}$	-0.125770864	-0.102254506	7	0	7	$0j_{15/2}$	-0.125770864	-0.15156059
4	1	-5	$1g_{9/2}$	-0.169988595	-0.142714872	8	0	8	$0k_{17/2}$	-0.169988595	-0.19943794

functions may not produce the orthogonality integral of the polynomials [29], as shown in equation (41),

$$\int_0^\infty \chi_n(r)\chi_{n'}(r)dr = \delta_{nn'} \neq \int_1^\infty w^{(-\beta,-\alpha)} R_n^{(-\beta,-\alpha)}(x)$$
$$R_n^{(-\beta,-\alpha)}dx \tag{57}$$

By carrying out the differentiations of equation (54), we find the lowest four un-normalized Romanovski polynomials given as

$$R_0^{(-\beta_0,-\alpha_0)}(x) = 1 \tag{58}$$

$$R_1^{(-\beta_1,-\alpha_1)}(x) = (\beta_1 + 1)2x - \alpha_1 \tag{59}$$

$$R_2^{(-\beta_2,-\alpha_2)}(x) = 2(\beta_2 + 2)(2\beta_2 + 3)x^2 - 2\alpha_2(2\beta_2 + 3)x$$
$$+ \alpha_2^2 + 2\beta_2 + 4 \tag{60}$$

$$R_3^{(-\beta_3,-\alpha_3)}(x) = 4x^3(\beta_3 + 3)(2\beta_3 + 5)(\beta_3 + 2) -$$
$$6\alpha_3 x^2(2\beta_3 + 5) \times (\beta_3 + 2) + 2x(6\beta^2_3 + 3\alpha_3^2\beta_3 + 28\beta_3$$
$$+ 6\alpha_3^2 + 34) - 2\alpha_3(2\beta_3 + 5) - \alpha(\alpha^2 + 2\beta + 6) \tag{61}$$

The first lowest four of the un-normalized radial wave functions for arbitrary values of l are calculated by using equations (55) and (58)-(61). The first lowest two of the un-normalized wave functions for any values obtained from equations (55) and (58)-(59) are

$$G_{0\kappa} = (1 + \cot^2 ar)^{\frac{\kappa+V_1-1}{2}} \times \exp\left(-\frac{V_0}{a^2}\right.$$
$$\left.\frac{(M - E_{n\kappa} + C_{ps})}{2(-\kappa - V_1 + 1)}\right\} \tan^{-1}(\cot ar) \tag{62}$$

$$G_{1\kappa} = (1 + \cot^2 ar)^{\frac{\kappa+V_1-2}{2}} \times \exp\left\{\left(-\frac{V_0}{a^2}\right.\right.$$
$$\left.\frac{(M - E_{n\kappa} + C_{ps})}{2(-\kappa - V_1 + 1)}\right\} \tan^{-1}(\cot ar)\right\} \times \left\{(\kappa + V_1 - 1)\right. \tag{63}$$
$$\left. 2\cot ar - \frac{V_0}{a^2}\frac{(M - E_{n\kappa} + C_{ps})}{(-\kappa - V_1 + 2)}\right\}$$

The lowest upper component wave function of the spinor is obtained from equation (62), and the first excited state of spinor is from equation (63).

Spin symmetric case. The Dirac equation of the cotangent function potential with the Coulomb-type tensor potential for the spin symmetric case that arises from the condition when $\Delta(r) = V(r) - S(r) = C_s$ that gives $\frac{d\Delta(r)}{dr} = 0, \Sigma(r)$ is the cotangent function potential, and $U(r)$ is the Coulomb-type tensor potential, obtained from equations (9), (10), and (15) given as

$$\left\{\frac{d2}{dr2} - \frac{\kappa(\kappa+1)}{r^2} - \frac{2\kappa}{r}\frac{V_1}{r} - \frac{V_1^2}{r^2} - \frac{V_1}{r^2}\right\}F_{n\kappa}(r) -$$
$$(-V_0 \cot ar)(M + E_{n\kappa} - C_s)F_{n\kappa}(r) - (M + E_{n\kappa}$$
$$- C_s)(M - E_{n\kappa})F_{n\kappa}(r) = 0 \tag{64}$$

The solution method of equation (64) is similar to the method used to solve equation (34) for the pseudospin symmetric case. Therefore, by repeating the steps in equations (35)-(52), we have the Dirac equation for the cotangent potential with the Coulomb-type tensor potential for the spin symmetric case as

$$\left\{\frac{d^2}{dr^2} - \frac{(\kappa(\kappa+1) + 2\kappa V_1 + V_1^2 + V_1)a^2}{\sin^2 ar}\right\}F_{n\kappa}(r)$$
$$+ V_0 \cot ar)(M + E_{n\kappa} - C_s)F_{n\kappa}(r) = (M - E_{n\kappa})$$
$$(M + E_{n\kappa} - C_s)F_{n\kappa}(r) \tag{65}$$

By setting

$$A_s = (\kappa + V_1 + 1)(\kappa + V_1) \tag{66}$$

$$B_s = -\frac{V_0}{a^2}(M + E_{n\kappa} - C_s) \tag{67}$$

$$E_s' = -\frac{(M - E_{n\kappa})(M + E_{n\kappa} - C_s)}{a^2} \tag{68}$$

in equation (65), equation (65) then reduces to a one-dimensional Schrödinger-type equation given as

$$\left\{\frac{d^2}{dr^2} - \frac{a^2 A_s}{(\sin ar)^2} - a^2 B_s \cot ar\right\}F_{n\kappa}(r)$$
$$= -a^2 E_s' F_{n\kappa}(r) \tag{69}$$

By substitution of the spatial variable, $\cot ar = x$, in equation (69), as in the pseudospin symmetric case,

equation (69) becomes

$$(1+x^2)\frac{d^2F_{n\kappa}}{dx^2} + 2x\frac{dF_{n\kappa}}{dx} - \left\{A_s + \frac{B_s x}{1+x^2} - \frac{E_s'}{1+x^2}\right\}F_{n\kappa}$$
$$= 0$$

(70)

To solve equation (70), we use equation (18) as a new wave function,

$$F_{n\kappa}(x) = g_n(x) = (1+x^2)^{\frac{\beta}{2}} e^{\frac{-\alpha}{2}\tan^{-1}x} D_n^{(\beta,\alpha)}(x) \qquad (71)$$

and after inserting equation (71) into equation (70), we have

$$(1+x^2)\frac{\partial^2 D_n^{(\beta,\alpha)}}{\partial x^2} + \{2x(\beta+1) - \alpha\}\frac{\partial D_n^{(\beta,\alpha)}}{\partial x} -$$
$$\left\{\frac{\beta\alpha x - \frac{\alpha^2}{4} + \beta^2 + B_s x - E_s'}{1+x^2} + A_s - \beta^2 - \beta\right\}D_n^{(\beta,\alpha)} = 0$$

(72)

Equation (72) reduces to a differential equation that is satisfied by Romanovski polynomials as in equation (23) if the coefficient of $\frac{1}{1+x^2}$ term is set to be zero, which are

$$-\frac{\alpha^2}{4} + \beta^2 - E' = 0 \text{ and } \beta\alpha + B_s = 0 \qquad (73)$$

Then equation (72) becomes

$$(1+x^2)\frac{\partial^2 D_n^{(\beta,\alpha)}}{\partial x^2} + 2x(\beta+1) - \alpha\}\frac{\partial D_n^{(\beta,\alpha)}}{\partial x}$$
$$- A_s - \beta^2 - \beta\}D_n^{(\beta,\alpha)=0}$$

(74)

By comparing the parameters between equations (23) and (74), we obtain the relation

$$A_s - \beta^2 - \beta = n(n-1) + 2n(1-p) \ ;$$
$$2(\beta+1) = 2(-p+1) \text{ and } \alpha = -q'$$

(75)

From equation (75), we have $p = -\beta$. Since $p > 0$, the value of β obtained from equation (75) that has physical meaning is

$$\beta = \beta_n = \sqrt{A_s + \frac{1}{4}} - n - \frac{1}{2} = \kappa + V_1 - n \qquad (76)$$

By using equations (73) and (76), we obtain

$$\alpha_n = -\frac{B_s}{\beta} = \frac{-\frac{V_0}{a^2}(M + E_{n\kappa} - C_s)}{-\kappa - V_1 + n} \qquad (77)$$

By inserting equations (76) and (77) into equation (73),

$$-\frac{B_s^2}{4\left(-\sqrt{A_s + \frac{1}{4}} + n + \frac{1}{2}\right)^2} + \left(-\sqrt{A_s + \frac{1}{4}} + n + \frac{1}{2}\right)$$
$$= -\frac{(M - E_{n\kappa})(M + E_{n\kappa} - C_s)}{a^2}$$

(78)

If in equation (78) C_s is set as $C_s = 0$, then $(M - E_{n\kappa}) = -E_{nR}$, which is the non-relativistic energy of the system; $M + E_{n\kappa} = 2\mu$ with μ is the non-relativistic of mass. If $\hbar = 1$, $V_1 = 0$, and if we set

$$V_0 \rightarrow \frac{V_0}{2\mu}; (\kappa + V_1 + 1)(\kappa + V_1) \rightarrow l(l+1), \qquad (79)$$

then we get

$$B_s = -\frac{V_0}{a^2(2\mu)}(2\mu) \ ; \ E_s' = \frac{E_{nR}2\mu}{a^2} \qquad (80)$$

By inserting equations (79) and (80) into equation (78), we obtain

$$E_{nR} = -\frac{V_0^2}{2\mu a^2(-l+n)^2} + \frac{a^2}{2\mu}(-l+n)^2 \qquad (81)$$

The non-relativistic energy spectra in equation (81) is in agreement with the energy spectra of the cotangent potential with the centrifugal term obtained using the SUSY QM method. By setting

$$P = \left(-\sqrt{A_{ps} + \frac{1}{4}} + n + \frac{1}{2}\right)^2 = \left(-(\kappa + V_1) + n\right)^2 \quad (82)$$

in equation (78), we obtain the relativistic energy given as

$$E_{n\kappa} = \frac{-(2M - C_s)\frac{V_0^2}{4a^4} + C_s\left[\frac{V_0^2}{4a^4} + \frac{\left(-(\kappa+V_1)+n\right)^2}{a^2}\right]}{2\left[\frac{V_0^2}{4a^4} + \frac{\left(-(\kappa+V_1)+n\right)^2}{a^2}\right]}$$
$$\pm \frac{\left(-(\kappa+V_1)+n\right)^2\sqrt{\left[\frac{V_0^2}{4a^4} + \frac{\left(-(\kappa+V_1)+n\right)^2}{a^2} + \frac{(-C_s+2M)^2}{4a^4}\right]}}{\left[\frac{V_0^2}{4a^4} + \frac{\left(-(\kappa+V_1)+n\right)^2}{a^2}\right]}$$

(83)

The energy eigenvalues in equation (83) provide two different energy spectra for each set of quantum numbers (n,κ); however, the energy spectra for the spin symmetric case have to be positive. Therefore, we put only the list of the positive energy in Table 2. For certain values of M, C_s, V_0, and a, the values of the energy expressed in equation (83) depend only on the values of n, κ, and V_1; therefore, it is expected that the energy of the nucleon is degenerate, as shown in Table 2. In Table 2, for the absence of the tensor potential $U(r)$, the spin doublet occurs for a set of quantum numbers (1, -2) and (0, 3); (1, -3) and (0, 4), etc., but these spin doublets are removed by the presence of the Coulomb-like tensor interaction potential.

As shown in Table 2, for $V_1 = 0.6 fm^{-1}$, the energy for the positive and negative values of κ is different; therefore, the tensor potential removes the energy degeneracy for the case (n, l, j) with $(n-1, l+2, j)$ as shown in Table 2.

Tabel 2: The Energy Spectra for Spin Symmetric Case with $M = 5fm^{-1}, C_s = 5fm^{-1}, \mathbf{V}_0 = 4fm^{-1}, a = 0.05fm$

l	n	κ	State	$E_{n\kappa}V_1 = 0$	$E_{n\kappa}V_1 = 0.6fm$	l	n	κ	State	$E_{n\kappa}V_1 = 0$	$E_{n\kappa}V_1 = 0.6fm$
0	1	-1	$1s_{1/2}$	0.014222253	0.006976804	2	0	2	$0d_{3/2}$	0.014222253	0.023998851
1	1	-2	$1p_{3/2}$	0.031911539	0.020460061	3	0	3	$0f_{5/2}$	0.031911539	0.045852204
2	1	-3	$1d_{5/2}$	0.056512901	0.040930649	4	0	4	$0g_{7/2}$	0.056512901	0.074526663
3	1	-4	$1f_{7/2}$	0.087866279	0.068254378	5	0	5	$0h_{9/2}$	0.087866279	0.109837332
4	1	-5	$1g_{9/2}$	0.125770864	0.102254506	6	0	6	$0i_{11/2}$	0.125770864	0.151560593
5	1	-6	$1h_{11/2}$	0.169988595	0.142714872	7	0	7	$0j_{13/2}$	0.169988595	0.199437944

To determine the upper component of Dirac spinor $F_{n\kappa}$, equations (76) and (77) are inserted into equations (10) and (15) so that we obtain the weight function $w(x)$ and the Romanovski polynomials $R_n^{(p,q')} = R_n^{(-\beta,-\alpha)}(x)$ as

$$w^{(-\beta,-\alpha)} = (1+x^2)^{\beta_n} \exp\left(-\alpha_n \tan^{-1}(x)\right)$$
$$= (1+\cot^2 ar)^{\beta_n} \exp\left(-\alpha_n \tan^{-1}(\cot ar)\right) \tag{84}$$

and

$$R_n^{(p,q')}(x) = R_n^{(-\beta,-\alpha)}(x = \cot ar)$$
$$= \frac{1}{(1+x^2)^{\beta_n} \exp\left(-\alpha_n \tan^{-1}(x)\right)} \times \frac{d^n}{dx^n} \tag{85}$$
$$\left\{(1+x^2)^{\beta_n+n} \exp\left(-\alpha_n \tan^{-1}(x)\right)\right\}$$

By using equations (85), we get the upper component of the Dirac spinor expressed as

$$F_{n\kappa}(f(x)) = (1+\cot^2 ar)^{\frac{\beta_n}{2}}$$
$$\times \exp\left(\frac{-\alpha_n \tan^{-1}(\cot ar)}{2}\right) R_n^{(-\beta,-\alpha)}(\cot ar) \tag{86}$$

For spin symmetry, the lower component of the Dirac spinor wave function is obtained by using equations (5) and (86) as

$$G_{n\kappa}(r) = \frac{\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)}{(M + E_{n\kappa} - C_s)} F_{n\kappa}(r) \tag{87}$$

If the value of $C_s = 0$, the lower component of the Dirac spinor exists only if $M \neq -E_{n\kappa}$, which means the values of $E_{n\kappa}$ for the spin symmetric case are always positive.

Since the β_n and α_n parameters, expressed in equations (76) and (77), are n-dependent, the orthogonality of the wave functions may not produce the orthogonality integral of the polynomials [29],

$$\int_0^\infty \chi_n(r)\chi_{n'}(r)dr = \delta_{nn'} \neq \int_1^\infty w^{(-\beta,-\alpha)}$$
$$R_n^{(-\beta,-\alpha)}(x)R_n^{(-\beta,-\alpha)}dx \tag{88}$$

By carrying out the differentiations of equation (52), we find the lowest four un-normalized Romanovski polynomials given as

$$R_0^{(-\beta_0,-\alpha_0)}(x) = 1 \tag{89}$$

$$R_1^{(-\beta_1,-\alpha_1)}(x) = (\beta_1 + 1)2x - \alpha_1 \tag{90}$$

$$R_2^{(-\beta_2,-\alpha_2)}(x) = 2(\beta_2 + 2)(2\beta_2 + 3)x^2$$
$$- 2\alpha_2(2\beta_2 + 3)x + \alpha_2^2 + 2\beta_2 + 4 \tag{91}$$

$$R_3^{(-\beta_3,-\alpha_3)}(x) = 4x^3(\beta_3 + 3)(2\beta_3 + 5)(\beta_3 + 2) -$$
$$6\alpha_3 x^2(2\beta_3 + 5)(\beta_3 + 2) + 2x(6\beta_3^2 + 3\alpha_3^2\beta_3 + 28\beta_3 \tag{92}$$
$$+ 6\alpha_3^2 + 34) - 2\alpha_3(2\beta_3 + 5) - \alpha(\alpha^2 + 2\beta + 6)$$

The first four lowest of the un-normalized radial wave functions for arbitrary values of l are calculated by using equations (86) and (89)-(92). The Romanovski polynomials are a complex function for odd-degree polynomials, but they are real functions for even-degree polynomials.

The first two lowest of the un-normalized wave functions for any κ values obtained from equations (86) and (89)-(90) are

$$F_{0\kappa}(r) = (1+\cot^2 ar)^{\frac{\kappa+V_1}{2}}$$
$$\times \exp\left\{-\frac{V_0}{a^2}\frac{(M + E_{n\kappa} - C_s)}{2(\kappa + V_1)}\right\} \tan^{-1}(\cot ar) \tag{93}$$

$$F_{1\kappa}(r) = (1+\cot^2 ar)^{\frac{\kappa+V_1-1}{2}}$$
$$\times \exp\left\{-\frac{V_0}{a^2}\frac{(M + E_{1\kappa} - C_s)}{(2\kappa + 2V_1 - 2)}\right\} \tan^{-1}(\cot ar) \tag{94}$$
$$\times \left\{(\kappa + V_1)2\cot ar + \frac{V_0}{a^2}\frac{(M + E_\kappa - C_s)}{(-\kappa - V_1 + 1)}\right\}$$

The ground state and first excited state wave functions of the upper component of the Dirac spinors that correspond to the energy eigenvalues presented in Table 2 are expressed in equations (93) and (94).

4. Conclusions

The relativistic energy spectra and the corresponding wave functions of the lower and upper components of the Dirac spinors for the trigonometric cotangent potential with a Coulomb-like tensor potential under pseudospin and spin symmetry limits are obtained using Romanovski

polynomials. By using the approximation scheme for the centrifugal term, the energy eigenvalues under pseudospin and spin symmetry limits are analytically obtained. The pseudospin and spin doublets formed by a pair of orbital numbers with positive and negative κ values are removed by the presence of the Coulomb-type tensor potential. The Dirac spinors, both the upper and lower components, are expressed in Romanovski polynomials. In the non-relativistic limit, the Dirac equation for the exact spin symmetric case reduces to the Schrödinger equation.

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References

[1] G. Chen, Z.D. Chen, Z.M. Lou, Chin. Phys. 13 (2004) 279.

[2] H. X. Quan, L. Guang, W.Z. Min, N.L. Bin, M. Yan, Commun. Theor. Phys. 53 (2010) 242.

[3] M.C. Zhang, Z.B. Wang, Chin. Phys. Lett. 22 (2005) 2994.

[4] A.A. Smorodinsky, K.H. Valiev, Nuovo Cimento A 52 (1967) 1061.

[5] Z.M. Chang, W.Z. Bang, Chin. Phys. 16/7 (2007) 1863.

[6] W.C. Qing, Chin. Phys. 12 (2003) 136.

[7] B. Gönül, I. Zorba, Phys. Lett. A 269 (2000) 83.

[8] F. Yasuk, C. Berkdemir, A. Berkdemir, J. Phys. A: Math. Gen. 38 (2005) 6579.

[9] C.Y. Chen, S.H. Dong, Phys. Lett. A 335 (2005) 374.

[10] C.Y. Chen, Phys. Lett. A 339 (2005) 283.

[11] C.Y. Chen, C.L. Liu, D.S. Sun, Phys. Lett. A 305 (2002) 341.

[12] M. Hamzavi, A.A. Rajabi, Adv. High Energy Phys. 2013 (2013) 12.

[13] S.M. Ikhdair, R. Sever, arXiv: 1001. 4327v2 [math-ph] (2010) 28.

[14] M. Eshghi, J. Sci. Res. 3/3 (2011) 493.

[15] A.N. Ikot, Commun. Theor. Phys. 59 (2013) 268.

[16] O. Aydogdu, E. Maghsoodhi, H. Hassanabadi, Chin. Phys. B 22 (1) (2013) 010302-1.

[17] Z. Ai-Ping, Q. Wen-Chao, High Energy Phys. Nucl. Phys., 31/11 (2007) 1027.

[18] C.H. Chen, Phys. Lett. A 339 (2005) 283.

[19] O. Aydogdu, R. Sever, Phys. Lett. B 703 (2011) 379.

[20] A. Soylu, O. Bayrak, I. Boztosun, J. Math. Phys. 48 (2007) 082302.

[21] R. L. Greene, C. Aldrich, Phys. Rev. A 14/6 (1976) 2363.

[22] K.T. Hecht, A. Adler, Nucl. Phys. A 137 (1969) 129.

[23] A. Arima, M. Harvey, K. Shimizu, Phys. Lett. B 30/8 (1969) 517.

[24] J.N. Ginocchio, Phys. Rep. C 69 (2004) 034313.

[25] J.N. Ginocchio, Phys. Rev. Lett. 78 (1997) 436.

[26] J. Meng, K. Sugaware-Tanabe, S. Yamaji, A. Arima, Phys. Rev. C 59 (1999) 154.

[27] J. Dudek, W. Nazarewicz, Z. Szymanski, G.A. Leander, Phys. Rev. Lett. 59/13 (1987) 1405.

[28] D. Troltenier, C. Bahri, J.P. Draayer, Nucl. Phys. A 586/1 (1995) 53.

[29] P.R. Page, T. Goldman, J.N. Ginocchio, Phys. Rev. Lett. 86/2-8 (2001) 204.

[30] J.N. Ginocchio, Phys. Rep. 414 (2005) 165.

[31] J.N. Ginocchio, D.G. Madland, Phys. Rev. C 57/3 (1998) 1167.

[32] P. Alberto, M. Fiolhais, M. Malheiro, A. Delfino, M. Chiapparini, Phys. Rev. C 65 (2002) 034307-1.

[33] D. Troltenier, W. Nazarewicz, Z. Szymanski, J.P. Draayer, Nucl. Phys. A 567 (1994) 591.

[34] A.E. Stuchbery, Nucl. Phys. A 700 (2002) 83.

[35] O. Aydogdu, R. Sever, Few-Body Syst. 47 (2010) 193.

[36] H. Akcay, Phys. Lett. 373 (2009) 616.

[37] M. Hamzavi, A.A. Rajabi, H. Hassanabadi, Few-Body Syst. 48 (2010) 171.

[38] D.E. Alvarez-Castillo, M. Kirbach, Rev. Mex. Fis E 53 (2) (2007) 143.

[39] D.E. Alvarez-Castillo, C.B. Compean, M. Kirbach, arXiv 1105. 1354v1 [quant-ph], 2011.

[40] C.B. Compean, M. Kirbach, arXiv: nucl-th/0610001v1, 2006.

[41] W. Koepf, M. Masjed-Jamei, Integral Transform. Special Functions, 17 (2006) 559.

[42] E.J. Routh, Proc. London Math. Soc. 16 (1884) 245.

[43] V.I. Romanovski, Compt. Rend. Ac. Sci. Paris 188 (1929) 1023.

[44] A. Raposo, H.J. Weber, D.E. Alvarez-Castillo, M. Kirchbach, Cent. Eur. Phys. J. 5 (2007) 253.

[45] A. Suparmi, C. Cari, At. Ind. J. 39/3 (2013) In press.

[46] A. Suparmi, C. Cari, the 4th ICANCE 2013, 16-19 September, Bali, Indonesia, 2013, p.61.

[47] A. Suparmi, C. Cari, J. Handhika, J. Phys. Conf. Ser. 423 (2013) 012039. doi: 10.1088/1742-6596/423/1/012039.

[48] A.F. Nikiforov, V.B. Uvarov, Special Functions of Mathematical Physics, Birkhauser, Verlag, Basel, 1988, p.427.