A p-adically entire function with integral values on $\mathbb{Q}_p$ and entire liftings of the p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$

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with an Appendix by Maurizio Candilera *

May 14, 2019

Abstract

We give a self-contained proof of the fact, discovered in [1] and proven in [2] with the methods of [16], that, for any prime number p, there exists a power series

$$\Psi = \Psi_p(T) \in T + T^2\mathbb{Z}[[T]]$$

which trivializes the addition law of the formal group of Witt covectors [16], [13, II.4], is p-adically entire and assumes values in $\mathbb{Z}_p$ all over $\mathbb{Q}_p$. We actually generalize, following a suggestion of M. Candilera, the previous facts to any fixed unramified extension $\mathbb{Q}_q$ of $\mathbb{Q}_p$ of degree f, where $q = p^f$. We show that $\Psi = \Psi_q$ provides a quasi-finite covering of the Berkovich affine line $\mathbb{A}^1_{\mathbb{Q}_p}$ by itself. We prove in section 3 new strong estimates for the growth of Ψ, in view of the application [3] to p-adic Fourier expansions on $\mathbb{Q}_p$. We refer to [3] for the proof of a technical corollary (Proposition 3.10) which we apply here to locate the zeros of Ψ and to obtain its product expansion (Corollary 3.12).

We reconcile the present discussion (for $q = p$) with the formal group proof given in [2] which takes place in the Fréchet algebra $\mathbb{Q}_p\{x\}$ of the analytic additive group $\mathbb{G}_{a,\mathbb{Q}_p}$ over $\mathbb{Q}_p$. We show that, for any $\lambda \in \mathbb{Q}_p^\times$, the closure $\mathscr{E}_\lambda^\circ$ of $\mathbb{Z}_p[\Psi(p^i x/\lambda)\,|\,i = 0, 1, \dots]$ in $\mathbb{Q}_p\{x\}$ is a Hopf algebra object in the category of Fréchet $\mathbb{Z}_p$-algebras.

The special fiber of $\mathscr{E}_\lambda^\circ$ is the affine algebra of the p-divisible group $\mathbb{Q}_p/p\lambda\mathbb{Z}_p$ over $\mathbb{F}_p$, while $\mathscr{E}_\lambda^\circ[1/p]$ is dense in $\mathbb{Q}_p\{x\}$.

From $\mathbb{Z}_p[\Psi(\lambda x)\,|\,\lambda \in \mathbb{Q}_p^\times]$ we construct a p-adic analog $\mathcal{AP}_{\mathbb{Q}_p}(\Sigma_\rho)$ of the algebra of Dirichlet series holomorphic in a strip $(-\rho, \rho) \times i\mathbb{R} \subset \mathbb{C}$. We start developing this analogy. It turns out that the Banach algebra of almost periodic functions on $\mathbb{Q}_p$ identifies with the topological ring of germs of holomorphic almost periodic functions on strips around $\mathbb{Q}_p$.

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0 Introduction

0.1 Foreword

An unfortunate feature of p-adic numbers is that there exists no character

$$\psi : (\mathbb{Q}_p, +) \to (\mathbb{C}_p^\times, \cdot) \ , \ \ \psi \neq 1$$

which extends to an entire function $\mathbb{C}_p \to \mathbb{C}_p$. In fact, let $\pi_p \in \mathbb{C}_p^{\circ\circ}$ be such that the radius of convergence of $\exp(\pi_p x)$ equals 1, so that exp and log establish an isomorphism

$$(\pi_p \, \mathbb{C}_p^{\circ\circ}, +) \xrightarrow{\sim} (\exp(\pi_p \, \mathbb{C}_p^{\circ\circ}), \cdot) \ (\subset (1 + \mathbb{C}_p^{\circ\circ}, \cdot)) \ .$$

Now, assume a ψ as above exists, and let n be a positive integer such that $\psi(p^n) \in \exp(\pi_p \, \mathbb{C}_p^{\circ\circ})$ so that ψ restricts to a character $\psi : (p^n \mathbb{Z}_p, +) \to (\exp(\pi_p \, \mathbb{C}_p^{\circ\circ}), \cdot)$. Let $a := \log(\psi(p^n))$. Then, for any $x \in \mathbb{Z}$, $\psi(p^n x) = \psi(p^n)^x = \exp(ax)$. But $x \mapsto \exp(ax)$ has a finite radius of convergence.

We partially remedy to the previous inconvenience by showing the existence, for any $\lambda \in \mathbb{Q}_p^\times$, of a representable formal group functor

$$\mathbb{E}_\lambda : \mathcal{ACLM}^u_{\mathbb{Z}_p} \to \mathcal{A}b \tag{0.0.1}$$

(see section 6.1 in Appendix A, for notation) whose generic (resp. special) fiber is the $\mathbb{Q}_p$-analytic group $\mathbb{G}_a$ (resp. the constant p-divisible group $\mathbb{Q}_p/\lambda\mathbb{Z}_p$ over $\mathbb{F}_p$). The idea is the following. Over the complex numbers the formulas

$$e^{iz} = \cos z + i \sin z \ \ , \ \ e^{-iz} = \cos z - i \sin z$$

show that the two (Hopf) algebras $\mathbb{Z}[i][e^{iz}, e^{-iz}]$ and $\mathbb{Z}[i][\sin z, \cos z]$ coincide. The sequence of functions

$$\Psi(x) = \Psi_p(x), \Psi(px), \Psi(p^2x), \dots \tag{0.0.2}$$

plays here the role of the pair $(\cos z, \sin z)$ in that the p-adically entire and integral addition law (0.3.4) holds, and x is a logarithm for that formal group. So, while it is improper to say that Ψ plays the role of an entire character of $\mathbb{Q}_p$, it is suggestive to consider a suitable p-adic completion of the algebra $\mathbb{Z}_p[\Psi(\lambda x) \mid \lambda \in \mathbb{Q}_p^\times]$ and to compare it with the classical algebras of Bohr's almost periodic functions $APH_\mathbb{R}$ and $AP_\mathbb{R}$. We review for convenience the classical definitions of real and complex Fourier analysis in section 7.2 of Appendix B. A closer p-adic analog of those classical constructions, and a generalization of Amice-Fourier theory to p-adic functions on $\mathbb{Q}_p$, will appear in [3].

0.2 The function Ψ

In the paper [2] we introduced, for any prime number p, a power series

$$\Psi(T) = \Psi_p(T) = T + \sum_{i=2}^{\infty} a_i T^i \in \mathbb{Z}[[T]] \ ,$$

which represents an entire p-adic analytic function, *i.e.* is such that

$$\limsup_{i \to \infty} |a_i|_p^{1/i} = 0 \ . \tag{0.0.3}$$

This function has the remarkable property that $\Psi_p(\mathbb{Q}_p) \subset \mathbb{Z}_p$ and that, for any $i \in \mathbb{Z}$ and $x \in \mathbb{Q}_p$, if we write x as in (1.12.2), with x_i defined by (1.12.3), (1.12.4), then

$$x_{-i} = \Psi_p(p^i x) \mod p \ \ \in \mathbb{F}_p \ . \tag{0.0.4}$$

The power series $\Psi(T)$ is defined by the functional relation

$$\sum_{j=0}^{\infty} p^{-j} \Psi(p^j T)^{p^j} = T \ . \tag{0.0.5}$$

Its inverse function $\beta = \beta_p \in T + T^2\mathbb{Z}[[T]]$ was shown to converge exactly in the region

$$|T|_p < p \ \ i.e. \ \ v_p(T) > -1 \ . \tag{0.0.6}$$

One property we had failed to notice in [2] is the following

Proposition 0.1. *The restriction of the function Ψ_p to a map $\mathbb{Q}_p \to \mathbb{Z}_p$ is uniformly continuous. More precisely, for any $j = 0, 1, \dots$ and $x \in \mathbb{Q}_p$,*

$$\Psi_p(x + p^j \mathbb{C}_p^\circ) \subset \Psi_p(x) + p^j \mathbb{C}_p^\circ \ . \tag{0.1.1}$$

This is proven in Corollary 3.4 below. See also the more general Theorem 3.11 whose proof depends on Proposition 3.10, proven in [3].

0.3 Our previous approach [2]

Proofs in [2] were based on Barsotti-Witt algorithms [16]. The most basic notion of topological algebra in [16] is the one of a *simultaneously admissible* family, indexed by $\alpha \in A$, of sequences $i \mapsto x_{\alpha,-i}$ for $i = 0, 1, \dots$ in a Fréchet algebra R over $\mathbb{Z}_p$ (in particular, over $\mathbb{F}_p$) [16, Ch.1, §1]. In case R is a Fréchet algebra over $\mathbb{Q}_p$ the definition of simultaneous admissibility is more restrictive, but the name used in *loc.cit.* is the same. For clarity, the more restrictive notion will be called here (simultaneous) PD-*admissibility*, while the general notion will maintain the name of (simultaneous) *admissibility*.

Using the previous refined terminology, our main technical tool in [2] was a criterion [2, Lemma 1] of *simultaneous* PD-*admissibility* for a family indexed by $\alpha \in A$, of sequences $i \mapsto x_{\alpha,-i}$ for $i = 0, 1, \dots$ in a Fréchet $\mathbb{Q}_p$-algebra. In Barsotti's theory of p-divisible groups one regards an admissible sequence $i \mapsto x_{-i}$ as a *Witt covector* $(\dots, x_{-2}, x_{-1}, x_0)$ [16], [13] with components $x_{-i} \in R$.

We take here only a short detour on the group functor viewpoint and refer the reader to [13] for precisions. As abelian group functors on a suitable category of topological $\mathbb{Z}_p$-algebras the direct limit $\mathrm{W}_n \to \mathrm{W}_{n+1}$ of the Witt vector groups of length n via the *Verschiebung map*

$$V : (x_{-n}, \dots, x_{-1}, x_0) \to (0, x_{-n}, \dots, x_{-1}, x_0)$$

indeed exists. It is the group functor CW of *Witt covectors.* For a topological $\mathbb{Z}_p$-algebra R on which $\mathrm{CW}(R)$ is defined, it is convenient to denote an element $x \in \mathrm{CW}(R)$ by an inverse sequence

$$x = (\dots, x_{-2}, x_{-1}, x_0)$$

of elements of R, that is a Witt covector with components in R. Two Witt covectors $x = (\dots, x_{-2}, x_{-1}, x_0)$ and $y = (\dots, y_{-2}, y_{-1}, y_0)$ with components R can be summed by taking limits of sums of finite Witt vectors. Namely, let

$$\varphi_i(X_0, \dots, X_i; Y_0, \dots, Y_i) \in \mathbb{Z}[X_0, \dots, X_i, Y_0, \dots, Y_i] \tag{0.1.2}$$

be the i-th (= the last!) entry of the Witt vector $(X_0, \dots, X_i) + (Y_0, \dots, Y_i)$. Then,

$$x + y = z = (\dots, z_{-2}, z_{-1}, z_0)$$

means that, for any $i = 0, -1, \dots,$

$$z_i = \lim_{n \to +\infty} \varphi_n(x_{i-n}, x_{i-n+1}, \dots, x_i; y_{i-n}, y_{i-n+1}, \dots, y_i) \tag{0.1.3}$$

converges in R. The convergence properties on the Witt covectors x and y above for the expressions (0.1.3) to converge, are dictated by the following

Lemma 0.2. *([16, Teorema 1.11]) Notation as above. For $i = 0, 1, 2, \dots,$ let us attribute the weight p^i to the variables X_i, Y_i. Then, for any $i \geq 0$ the polynomial φ_i in* (0.1.2) *is isobaric of weight p^i. Moreover, for any $i \geq 1$,*

$$\begin{gathered}\varphi_i(X_0, X_1, \dots, X_i; Y_0, Y_1, \dots, Y_i) - \varphi_{i-1}(X_1, \dots, X_i; Y_1, \dots, Y_i) \in \\ X_0\, Y_0\, \mathbb{Z}[X_0, X_1, \dots, X_i, Y_0, Y_1, \dots, Y_i]\,.\end{gathered} \tag{0.2.1}$$

So, we equip the polynomial ring $\mathbb{Z}[X_0, X_{-1}, \dots, X_{-i}, \dots; Y_0, Y_{-1}, \dots, Y_{-i}, \dots]$ with the linear topology defined by the powers of the ideals $I_N := (X_{-N}, X_{-N-1}, \dots; Y_{-N}, Y_{-N-1}, \dots)$ and set

$$\mathcal{P} := \varprojlim_{N,M \to +\infty} \mathbb{Z}[X_0, X_{-1}, \dots, X_{-i}, \dots; Y_0, Y_{-1}, \dots, Y_{-i}, \dots]/I_N^M\ .$$

Then, the sequence

$$i \longmapsto \varphi_i(X_{-i}, \dots, X_{-1}, X_0; Y_{-i}, \dots, Y_{-1}, Y_0) \tag{0.2.2}$$

converges to an element

$$\Phi(X_0, X_{-1}, \dots, X_{-i}, \dots; Y_0, Y_{-1}, \dots, Y_{-i}, \dots) \in \mathcal{P} \ . \tag{0.2.3}$$

So, (0.1.3) is expressed more compactly as

$$z_i = \Phi(x_i, x_{i-1}, \dots; y_i, y_{i-1}, \dots) \tag{0.2.4}$$

Remark 0.3. The projective limit

$$\mathrm{W}_{n+1} \to \mathrm{W}_n \quad , \quad (x_0, x_1, \dots, x_{n+1}) \mapsto (x_0, x_1, \dots, x_n) \ ,$$

produces instead the algebraic group W of *Witt vectors.*

The approach of Barsotti [16] is more flexible and easier to apply to analytic categories. If R is complete, for two simultaneously admissible Witt covectors $x = (\dots, x_{-2}, x_{-1}, x_0)$ and $y = (\dots, y_{-2}, y_{-1}, y_0)$ with components R the expressions (0.2.4) all converge in R and define $(\dots, z_{-2}, z_{-1}, z_0) = z =: x + y$, which is in turn simultaneously admissible with x and y. In the $\mathbb{Q}_p$-algebra case a Witt covector $x = (\dots, x_{-2}, x_{-1}, x_0)$ has *ghost components* $(\dots, x^{(-2)}, x^{(-1)}, x^{(0)})$ defined by

$$x^{(i)} = x_i + p^{-1}x_{i-1}^p + p^{-2}x_{i-2}^{p^2} + \dots \quad , \quad i = 0, -1, -2, \dots \ . \tag{0.3.1}$$

Under very general assumptions [16, Teorema 1.11], a finite family of sequences $(x_{\alpha,-i})_{i=0,1,\dots}$, for $\alpha \in A$ in a $\mathbb{Q}_p$-Fréchet algebra are simulaneously PD-admissible iff the same holds for the family of sequences of ghost components $(x_\alpha^{(-i)})_{i=0,1,\dots}$, for $\alpha \in A$. Under these assumptions, for simultaneously PD-admissible covectors x and y, $x + y = z$ is equivalent to

$$z^{(i)} = x^{(i)} + y^{(i)} \quad , \quad i = 0, -1, -2, \dots \ . \tag{0.3.2}$$

In the present case, which coincides with the case treated in [2], the sequences $i \mapsto x_{-i} := p^i x$ and $i \mapsto y_{-i} := p^i y$ are simultaneously PD-admissible in the standard $\mathbb{C}_p$-Fréchet algebra $\mathbb{C}_p\{x, y\}$ of entire functions on $\mathbb{C}_p^2$ [2, Lemma 1 and Lemma 3]. It follows from the relation (0.0.5) that $i \mapsto x_{-i} := p^i x$, for $i = 0, 1, 2, \dots$ is the sequence of ghost components of $x \mapsto x^{(-i)} := \Psi(p^i x)$. Therefore from [16, *loc.cit.*] we conclude that the two sequences $i \mapsto \Psi(p^i x)$ and $i \mapsto \Psi(p^i y)$ are simultaneously admissible in $\mathbb{C}_p\{x, y\}$, as well. Moreover, by [16, *loc.cit.*] and the definition of the addition law of Witt covectors with coefficients in $\mathbb{C}_p\{x, y\}$, we have

$$\begin{aligned}(\dots, \Psi(p^2(x+y)), \Psi(p(x+y)), \Psi(x+y)) = \\ (\dots, \Psi(p^2 x), \Psi(px), \Psi(x)) + (\dots, \Psi(p^2 y), \Psi(py), \Psi(y)) \ .\end{aligned} \tag{0.3.3}$$

Equivalently, Ψ satisfies the addition law [2, (11)]

$$\Psi(x + y) = \Phi(\Psi(x), \Psi(px), \dots; \Psi(y), \Psi(py), \dots) \tag{0.3.4}$$

where

$$\begin{aligned}\Phi(\Psi(x), \Psi(px), \dots; \Psi(y), \Psi(py), \dots) = \\ \lim_{i \to \infty} \varphi_i(\Psi(p^i x), \dots, \Psi(px), \Psi(x); \Psi(p^i y), \dots, \Psi(py), \Psi(y)) \ ,\end{aligned} \tag{0.3.5}$$

for the polynomials φ_i of (0.1.2) and (0.2.2). Notice that (1.12.2) may be restated to say that, for any $x \in \mathbb{Q}_p$,

$$x = (\dots, x_{-2}, x_{-1}; x_0, x_1, \dots) \ ,$$

where $x_i \in \mathbb{F}_p$ is given by (1.12.3), as a *Witt bivector* [16] with coefficients in $\mathbb{F}_p$.

0.4 Our present approach

We present here in section 2 direct elementary proofs of the main properties of Ψ, which make no use of the Barsotti-Witt algorithms of [16]. Actually, following a suggestion of M. Candilera, we consider rather than (0.0.5), the more general functional relation for $\Psi = \Psi_q$, $q = p^f$,

$$\sum_{j=0}^{\infty} p^{-j} \Psi(p^j T)^{q^j} = T \; . \tag{0.3.6}$$

The result, at no extra work, will then be that (0.3.6) admits a unique solution $\Psi_q(T) \in T + T^2\mathbb{Z}[[T]]$. The series $\Psi_q(T)$ represents a p-adically entire function such that $\Psi_q(\mathbb{Q}_q) \subset \mathbb{Z}_q$. In section 3 we describe in the same elementary style the Newton and valuation polygons of the entire function Ψ_q, and obtain new estimates on the growth of $|\Psi_q(x)|$ as $|x| \to \infty$, which will be crucial for the sequel [3]. From these estimates we also deduce, modulo the self-contained technical Proposition 3.10 whose proof appears in [3], the location of the zeros of Ψ_p (Theorem 3.12). Namely, any ball of radius 1, $a + \mathbb{Z}_p \in \mathbb{Q}_p/\mathbb{Z}_p$, contains precisely one (simple) zero of Ψ_p.

We present in Appendix C below some numerical calculations due to M. Candilera, which exhibit the first coefficients of Ψ_p, for small values of p. These calculations have been useful to us and we believe they may be quite convincing for the reader.

The function $\Psi_q : \mathbb{A}^1_{\mathbb{Q}_p} \to \mathbb{A}^1_{\mathbb{Q}_p}$ is a quasi-finite covering of the Berkovich affine line over $\mathbb{Q}_p$ by itself. We do not know whether the previous covering is Galois.

0.5 Convergence of Fourier-type expansions

Section 1.1 describes some Hopf algebras whose existence follows from the addition properties of Ψ_p. The next section 1.2 suggests an interpretation of the functions $\Psi_p(x/\lambda)$, for $\lambda \in \mathbb{Q}_p^\times$, as p-adic analogs of $\exp(\frac{2\pi i}{\lambda} z)$, for $\lambda \in \mathbb{R}^\times$. We are naturally lead to the question of which functions can be expressed as uniform limits on $\mathbb{Q}_p$ of the previous functions. By analogy to the classical case, we call these functions uniformly almost periodic on $\mathbb{Q}_p$ and denote by $AP_{\mathbb{Q}_p}$ the corresponding closed subalgebra of the Banach algebra $\mathscr{C}^{\text{bd}}_{\text{unif}}(\mathbb{Q}_p, \mathbb{Q}_p)$ of bounded uniformly continuous functions $\mathbb{Q}_p \to \mathbb{Q}_p$. Although we do not have an intrinsic characterization of these functions, we can show that they may be seen as germs of holomorphic functions on a neighborhood of $\mathbb{Q}_p$. We point out that colimits for topological algebras are not in general supported by set-theoretic inductive limits (see Remark 5.8 below). Therefore, our Uniform Approximation Theorem 1.25 does not state that any uniformly almost periodic function on $\mathbb{Q}_p$ necessarily extends to an analytic function on a p-adic strip around $\mathbb{Q}_p$. On the other hand, $AP_{\mathbb{Q}_p}$ is dense in the Fréchet algebra $\mathscr{C}(\mathbb{Q}_p, \mathbb{Q}_p)$ of continuous functions $\mathbb{Q}_p \to \mathbb{Q}_p$, equipped with the topology of uniform convergence on compact open subsets of $\mathbb{Q}_p$. The proofs of these facts are detailed in sections 4 and 5. We spend some time in section 4 to explain in categorical terms (clearly stated in Appendix A) the natural limit/colimit/tensor product formulas which justify the linear topologies of the previous function algebras. For example, $\mathscr{C}(\mathbb{Q}_p, \mathbb{Z}_p)$ (but not $\mathscr{C}_{\text{unif}}(\mathbb{Q}_p, \mathbb{Z}_p)$) is a Hopf algebra related to the constant p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$ over $\mathbb{Z}_p$ and to its "universal covering" $\mathbb{Q}_p$. A more complete discussion of these topological algebras and of their duality relation with the affine algebra of the universal covering of the p-divisible torus, interpreted as an algebra of measures, will appear in [5].

In section 5 we prove the facts announced in section 1.2, namely Theorem 1.15, Theorem 1.17, Proposition 1.18, Proposition 1.19, Proposition 1.21, Proposition 1.22, and Theorem 1.25.

0.6 Aknowledgments

It is a pleasure to acknowledge that the proofs in sections 2 and 3 of this text are based on a discussion with Philippe Robba which took place in April 1980. I am strongly indebted to him for this and for his friendship.
I thank my colleague Giuseppe De Marco for his patient explanations on classical Fourier theory.
I thank the MPIM of Bonn for hospitality during March 2018, when this article was completed.

1 Rings of p-adic analytic functions

1.1 Entire functions bounded on p-adic strips

(See Appendix A for notation of topological algebra and non-archimedean functional analysis.) We describe here the Hopf algebra object $\mathbb{Q}_p\{x\}$ in the category of Fréchet $\mathbb{Q}_p$-algebras equipped with the completed projective = inductive tensor product $\widehat{\otimes}_{\pi,\mathbb{Q}_p} = \widehat{\otimes}_{\iota,\mathbb{Q}_p}$, which consists of the global sections of the $\mathbb{Q}_p$-analytic group $\mathbb{G}_a$. We also consider boundedness conditions for the functions in $\mathbb{Q}_p\{x\}$ on suitable neighborhoods of $\mathbb{Q}_p$ in the Berkovich affine line $\mathbb{A}^1_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$.

Our notation for coproduct (resp. counit, resp. inversion) of a Hopf algebra object A in a symmetric monoidal category with monoidal product $\otimes$ and unit object I is usually $\mathbb{P} = \mathbb{P}_A : A \to A \otimes A$ (resp. $\varepsilon = \varepsilon_A : A \to I$, resp. $\rho = \rho_A : A \to A$).

Definition 1.1. *For any closed subfield K of $\mathbb{C}_p$, we denote by $K\{x\} = K\{x_1, \dots, x_n\}$ the ring of entire functions on the K-analytic affine space $(\mathbb{A}^n_K, \mathcal{O}_K)$. The* standard Fréchet topology *on the K-algebra $K\{x\}$ is induced by the family $\{w_r\}_{r\in\mathbb{Z}}$ of valuations*

$$w_r(f) := \inf_{x\in(p^{-r}\mathbb{C}_p^\circ)^n} v(f(x)) \ ,$$

for any $f \in K\{x\}$.

Remark 1.2. More generally, for bounded functions $f : X \to (S, |\ |)$, where X is a set and $(S, |\ |)$ is a Banach ring in multiplicative notation, $||f||_X = \sup_{x\in X} |f(x)|$ will denote the supnorm on X.

Definition 1.3. *For any $\rho > 0$ and any finite extension $K/\mathbb{Q}_p$, the* p-adic n-strip of width ρ around K^n *is the analytic domain which is the union $\Sigma_\rho(K) = \Sigma^{(n)}_\rho(K)$ of all affinoid n-polydiscs of radius ρ centered at K-rational points. We denote by*

$$Res_\rho : \mathbb{C}_p\{x\} \longrightarrow \mathcal{O}(\Sigma_\rho) \ , \ f \longmapsto f_{|\Sigma_\rho}$$

the restriction map. Clearly, the map Res_ρ is an injection. We let $\mathcal{O}^{\rm bd}_K(\Sigma_\rho(K))$ (resp. $\mathcal{O}^\circ_K(\Sigma_\rho(K))$) denote the subring of $\mathcal{O}_K(\Sigma_\rho(K))$, consisting of functions bounded (resp. bounded by 1) on $\Sigma_\rho(K)$. We denote by $||\ ||_{K,\rho}$ the supnorm on $\Sigma_\rho(K)$. The Banach algebra structure on $\mathcal{O}^{\rm bd}_K(\Sigma_\rho(K))$ (resp. $\mathcal{O}^\circ_K(\Sigma_\rho(K))$) induced by the norm $||\ ||_{K,\rho}$ will be called K-uniform. *The Fréchet structure of $\mathcal{O}_K(\Sigma_\rho(K))$ (resp. $\mathcal{O}^\circ_K(\Sigma_\rho(K))$) induced by the family of seminorms of Definition 1.1 will be called* standard. *We set in particular $\Sigma_\rho = \Sigma^{(1)}_\rho(\mathbb{Q}_p)$ but will keep the notation $||\ ||_{\mathbb{Q}_p,\rho}$. We also denote by $\mathscr{H}^{(n),\rm bd}_K(\rho)$ (resp. $\mathscr{H}^{(n),\circ}_K(\rho)$) the subring of $K\{x\}$ of functions which are bounded (resp. bounded by 1) on $\Sigma_\rho(K)$. We set*

$$\mathscr{H}^{(n),\rm bd}_K := \bigcap_\rho \mathscr{H}^{(n),\rm bd}_K(\rho) \ .$$

For any $\rho > 0$ and any $f \in \mathscr{H}_K^{(n),\mathrm{bd}}(\rho)$ we introduce one further valuation

$$w_{K,\infty}(f) := \inf_{x \in K^n} v(f(x)) . \tag{1.3.1}$$

For $n = 1$ and $K = \mathbb{Q}_p$, we shorten $\mathscr{H}_K^{(n),\mathrm{bd}}(\rho)$ (resp. $\mathscr{H}_K^{(n),\circ}(\rho)$, resp. $\mathscr{H}_K^{(n),\mathrm{bd}}$, resp. $w_{K,\infty}$, resp. K-uniform) to $\mathscr{H}^{\mathrm{bd}}(\rho)$ (resp. $\mathscr{H}^{\circ}(\rho)$, resp. $\mathscr{H}^{\mathrm{bd}}$, resp. w_∞, resp. uniform).

Remark 1.4. It is not a priori clear that $\mathscr{H}^{\mathrm{bd}}$ contains non-constant functions. We will prove below (Theorem 3.11) that $\Psi(x) \in \mathscr{H}^{\mathrm{bd}}$.

Remark 1.5. For any n and any $\rho > 0$, $\mathscr{H}_K^{(n),\circ}(\rho)$ is a closed K°-subalgebra of $K\{x_1, \dots, x_n\}$; the induced Fréchet K°-algebra structure on $\mathscr{H}_K^{(n),\circ}(\rho)$ will be called *standard*. It follows from formula 0.0.5 below that, by contrast, $\mathscr{H}_K^{(n),\mathrm{bd}}(\rho) = \mathscr{H}_K^{(n),\circ}(\rho)[1/p]$ is dense in $K\{x_1, \dots, x_n\}$.

Remark 1.6. The Fréchet structure on $\mathcal{O}_K(\Sigma_\rho)$ which we call "standard" is the one of analytic geometry: it coincides with the topology of uniform convergence on rigid discs of radius ρ. Similarly for $\mathcal{O}_K^\circ(\Sigma_\rho(K))$. The standard Fréchet algebra $K\{x\}$ identifies with

$$K\{x\} = \varprojlim_{\rho \to +\infty} (\mathcal{O}_K(\Sigma_\rho), \text{standard}) \tag{1.6.1}$$

Definition 1.7. *The* strip topology *on $\mathscr{H}_K^{(n),\mathrm{bd}}$ is the projective limit topology of the uniform topologies of Definition 1.3. So,*

$$(\mathscr{H}_K^{(n),\mathrm{bd}}, \text{strip}) = \varprojlim_{\rho \to +\infty} (\mathcal{O}_K^{\mathrm{bd}}(\Sigma_\rho(K)), ||\ ||_{K,\rho}) , \tag{1.7.1}$$

is a K-Fréchet space.

Remark 1.8. We have a dense embedding $\mathscr{H}_K^{(n),\mathrm{bd}} \subset K\{x_1, \dots, x_n\}$. The strip topology on $\mathscr{H}_K^{(n),\mathrm{bd}}$, for which this algebra is complete, is finer than its (non complete) standard topology.

The next lemma shows that, for any non archimedean field K and $\mathbb{G}_a = \mathbb{G}_{a,K}$,

$$\mathcal{O}(\mathbb{G}_a \times \mathbb{G}_a) = \mathcal{O}(\mathbb{G}_a) \widehat{\otimes}_{\pi,K} \mathcal{O}(\mathbb{G}_a)$$

so that $\mathcal{O}(\mathbb{G}_a)$ is a Hopf algebra object in the category of Fréchet K-algebras.

Lemma 1.9. *There are natural identifications*

$$K\{x_1, \dots, x_n\} \widehat{\otimes}_{\pi,K} K\{y_1, \dots, y_m\} = K\{x_1, \dots, x_n, y_1, \dots, y_m\} . \tag{1.9.1}$$

sending $x_i \otimes 1 \mapsto x_i$ and $1 \otimes y_j \mapsto y_j$, for $i = 1, \dots, n$, $j = 1, \dots, m$.

Proof. For any $s \in \mathbb{Z}$ the map of the statement produces isomorphisms of K-Tate algebras [9, §6.1.1, Cor. 8]
(1.9.2)

$$K\langle p^{-s}x_1, \dots, p^{-s}x_n\rangle \widehat{\otimes}_{\pi,K} K\langle p^{-s}y_1, \dots, p^{-s}y_m\rangle = K\langle p^{-s}x_1, \dots, p^{-s}x_n, p^{-s}y_1, \dots, p^{-s}y_m\rangle .$$

We now apply Proposition 6.6. □

Corollary 1.10. *Let K be a finite extension of $\mathbb{Q}_p$ and $\rho > 0$. The identifications* (1.9.1) *induce identifications*

$$(\mathscr{H}_K^{(n),\circ}(\rho), \text{standard}) \widehat{\otimes}^u_{K^\circ} (\mathscr{H}_K^{(m),\circ}(\rho), \text{standard}) \xrightarrow{\sim} (\mathscr{H}_K^{(m+n),\circ}(\rho), \text{standard}) . \tag{1.10.1}$$

Similarly for $(\mathcal{O}_K^\circ(\Sigma_\rho(K)), \text{standard})$.

Corollary 1.11. *The map* $\mathbb{P} : x_i \mapsto x_i \widehat{\otimes} 1 + 1 \widehat{\otimes} x_i$ *makes* $K\{x_1, \dots, x_n\}$ *into a Hopf algebra object in the category of Fréchet* K*-algebras. The restriction of* $\mathbb{P}$ *to* $\mathscr{H}_K^{(n),\circ}(\rho)$ *induces a map*

$$\mathbb{P} : (\mathscr{H}_K^{(n),\circ}(\rho), \text{standard}) \longrightarrow (\mathscr{H}_K^{(n),\circ}(\rho), \text{standard}) \widehat{\otimes}^u_{K^\circ} (\mathscr{H}_K^{(n),\circ}(\rho), \text{standard})$$

which makes $(\mathscr{H}_K^{(n),\circ}(\rho), \text{standard})$ *a Hopf algebra object in the category of Fréchet* K°*-algebras. Similarly for* $(\mathcal{O}_K^\circ(\Sigma_\rho(K)), \text{standard})$*.*

1.2 p-adic almost periodic functions

We sketch here the the main ideas and results on p-adic almost periodic functions. Proofs are given in section 5 below. We freely use in this introduction the (quite self-explanatory) notation of section 4 for continuous, uniformly continuous, bounded rings of p-adic functions $\mathbb{Q}_p \to \mathbb{Q}_p$ and their topologies.

The following elementary lemma shows that a naive p-adic analog of real Bohr's uniformly almost periodic functions (see Definition 7.2 in Appendix B), where "an interval of length ℓ_ε in $\mathbb{R}$" is taken to mean a coset $a + p^h\mathbb{Z}_p$, for $a \in \mathbb{Q}_p$ and $p^{-h} = \ell_\varepsilon$, does not lead to a meaningful definition.

Lemma 1.12. *A continuous function* $f : \mathbb{Q}_p \to \mathbb{Q}_p$ *which has the property that for any* $\varepsilon > 0$*, there exists* $h \in \mathbb{Z}$ *such that any coset* $a + p^h\mathbb{Z}_p$ *in* $\mathbb{Q}_p/p^h\mathbb{Z}_p$ *contains an element* t_a *such that*

$$|f(x + t_a) - f(x)| < \varepsilon \;\; \forall\, x \in \mathbb{Q}_p \,, \tag{1.12.1}$$

is constant.

Proof. In fact, from condition (1.12.1), for any $a \in \mathbb{Q}_p$, it follows by iteration that t_a may be replaced by any $t \in \mathbb{Z}t_a$. By continuity, we may replace t_a by any $t \in \mathbb{Z}_p t_a$. For $a \notin p^h\mathbb{Z}_p$, $\mathbb{Z}_p t_a = \mathbb{Z}_p a$. So, if we pick $a = p^{-N}$, for $N >> 0$, (1.12.1) implies that the variation of $f(x)$ in $p^{-N}\mathbb{Z}_p$ is less than ε. So, the variation of $f(x)$ in $\mathbb{Q}_p$ is less than ε for any $\varepsilon > 0$, hence f is constant. □

We resort to an *ad hoc* definition. For $x \in \mathbb{Q}_p$, let us consider the classical Witt vector expression

$$x = \sum_{i >> -\infty}^{\infty} [x_i]\, p^i \in \mathrm{W}(\mathbb{F}_p)[1/p] = \mathbb{Q}_p \,, \tag{1.12.2}$$

where $[t]$, for $t \in \mathbb{F}_p$, is the Teichmüller representative of t in $\mathrm{W}(\mathbb{F}_p) = \mathbb{Z}_p$. Notice that, for any $i \in \mathbb{Z}$, the function

$$x_i : \mathbb{Q}_p \longrightarrow \mathbb{F}_p \;\;,\;\; x \longmapsto x_i \tag{1.12.3}$$

factors through a function, still denoted by x_i,

$$x_i : \mathbb{Q}_p/p^{i+1}\mathbb{Z}_p \longrightarrow \mathbb{F}_p \;\;,\;\; h \longmapsto h_i \,. \tag{1.12.4}$$

We regard the function in (1.12.4) as an $\mathbb{F}_p$-valued *periodic function of period* p^{i+1} on $\mathbb{Q}_p$. In the following, for any $i \in \mathbb{Z}$ and any $\lambda \in \mathbb{Q}_p^\times$, we denote by "$[(\lambda x)_i]$" the uniformly continuous function $\mathbb{Q}_p \to \mathbb{Z}_p$, $x \mapsto [(\lambda x)_i]$. We observe that

$$[(\lambda p^j x)_i] = [(\lambda x)_{i-j}]$$

for any $i, j \in \mathbb{Z}$ and $\lambda \in \mathbb{Q}_p^\times$.

Definition 1.13. *We define the* $\mathbb{Q}_p$*-algebra* $AP_{\mathbb{Q}_p}$ *(resp. the* $\mathbb{Z}_p$*-algebra* $AP_{\mathbb{Z}_p}$*) of (resp.* integral*) uniformly almost periodic (*u.a.p. *for short) functions* $\mathbb{Q}_p \to \mathbb{Q}_p$ *(resp.* $\mathbb{Q}_p \to \mathbb{Z}_p$*) as the closure of*

$$\mathbb{Q}_p[[(\lambda x)_i] \,|\, i \in \mathbb{Z}\,,\ \lambda \in \mathbb{Z}_p^\times] \ \ \textit{(resp. of } \mathbb{Z}_p[[(\lambda x)_i] \,|\, i \in \mathbb{Z}\,,\ \lambda \in \mathbb{Z}_p^\times]\ \textit{)}$$

in the $\mathbb{Q}_p$*-Banach algebra* $\mathscr{C}^{\mathrm{bd}}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{Q}_p)$ *(resp. in the* $\mathbb{Z}_p$*-Banach ring* $\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{Z}_p)$*), equipped with the induced valuation* w_∞*.*

Remark 1.14. This remark is made to partially justify Definition 1.13. For any $N \in \mathbb{Z}$ we denote by $S_N : \mathbb{Q}_p \to p^N\mathbb{Z}_p$ the function *N-th order fractional part*, namely

$$x = \sum_{i>>-\infty}^{\infty} [x_i]\, p^i \longmapsto S_N(x) = \sum_{i=N}^{\infty} [x_i]\, p^i\ . \tag{1.14.1}$$

It is clear that, for any N and $\lambda \in \mathbb{Q}_p^\times$, $x \mapsto S_N(\lambda x)$ is a bounded uniformly continuous function. The function S_3, certainly not periodic, is a p-adic analog of the function

$$\mathbb{R} \to [0,1)\ \ ,\ \ ....1234.56789.... \mapsto 0.789...$$

which is genuinely periodic of period 0.01.

We will prove the following partial analog to Bohr's "Approximation Theorem" (Theorem 7.3 in Appendix B), where in fact the functions $\cos(\frac{2\pi}{\lambda}x)$ and $\sin(\frac{2\pi}{\lambda}x)$, for $\lambda \in \mathbb{R}^\times$ are replaced by the functions $\Psi(\lambda x)$, for $\lambda \in \mathbb{Q}_p^\times$.

Theorem 1.15. $(AP_{\mathbb{Q}_p}, w_\infty)$ *(resp.* $(AP_{\mathbb{Z}_p}, w_\infty)$*) is the completion of the valued ring*

$$(\mathbb{Q}_p[\Psi(\lambda x) \,|\, \lambda \in \mathbb{Q}_p^\times], w_\infty)\ \ \textit{(resp. } (\mathbb{Z}_p[\Psi(\lambda x) \,|\, \lambda \in \mathbb{Q}_p^\times], w_\infty)\,\textit{)}\ .$$

Definition 1.16. *For any* $\lambda \in \mathbb{Q}_p^\times$*, the Fréchet* $\mathbb{Z}_p$*-algebra* $\mathscr{E}^\circ_\lambda$ *(resp.* $\mathscr{T}^\circ_\lambda$*) is the closure of*

$$\mathbb{Z}_p[\Psi(\lambda^{-1}p^j x) \,|\, j = 0, 1, \dots] \tag{1.16.1}$$

in $\mathbb{Q}_p\{x\}$ *(resp. in* $\mathcal{O}(\Sigma_{|\lambda|})$*) with the standard topology. We then set* $\mathscr{E}^{\mathrm{bd}}_\lambda := \mathscr{E}^\circ_\lambda[1/p]$ *(resp.* $\mathscr{T}^{\mathrm{bd}}_\lambda := \mathscr{T}^\circ_\lambda[1/p]$*).*

Finally, we define the Fréchet $\mathbb{Z}_p$*-algebra* $\mathscr{E}^\circ$ *as the closure of* $\mathbb{Z}_p[\Psi(\lambda^{-1}p^j x) \,|\, j = 0, 1, \dots]$ *in* $\mathbb{Q}_p\{x\}$*, and set* $\mathscr{E}^{\mathrm{bd}} := \mathscr{E}^\circ[1/p]$*.*

Theorem 1.17. (Approximation Theorem on compacts) *The completion of the multivalued ring*

$$(\mathscr{E}^{\mathrm{bd}}, \{||\ ||_{p^r\mathbb{Z}_p}\}_{r\in\mathbb{Z}})\ \ \textit{(resp. } (\mathscr{E}^\circ, \{||\ ||_{p^r\mathbb{Z}_p}\}_{r\in\mathbb{Z}})\,\textit{)}$$

is the Fréchet $\mathbb{Q}_p$*-algebra (resp.* $\mathbb{Z}_p$*-algebra)* $\mathscr{C}(\mathbb{Q}_p, \mathbb{Q}_p)$ *(resp.* $\mathscr{C}(\mathbb{Q}_p, \mathbb{Z}_p)$*).*

The following proposition follows from the estimates of Proposition 0.1 (see Corollary 3.4 or Theorem 3.11 for the proof) together with the fact that the conditions listed below are closed for the standard Fréchet structure. The proof of the latter fact is given in Lemma 5.6.

Proposition 1.18. *For any* $f \in \mathscr{E}^\circ_\lambda$ *(resp.* $f \in \mathscr{T}^\circ_\lambda$*) we have*

1. *f is bounded by 1 on the p-adic strip* $\Sigma_{|\lambda|}$*;*
2. $f(\mathbb{Q}_p) \subset \mathbb{Z}_p$*;*
3. *For any* $r \in \mathbb{Z}$*,* $a, j \in \mathbb{Z}_{\geq 0}$*, the function* $g(x) := f(p^{-r}x)^{p^a}$ *satisfies*

$$g(x + p^{r+j}\lambda\mathbb{C}_p^\circ) \subset g(x) + p^{a+j}\mathbb{C}_p^\circ\ \ ,\ \ \forall\, x \in \mathbb{Q}_p\ .$$

Proposition 1.19. *For any* $\lambda \in \mathbb{Q}_p^\times$*,* $(\mathscr{E}_\lambda^\circ, \mathrm{standard})$ *(resp.* $(\mathscr{T}_\lambda^\circ, \mathrm{standard})$*) is a Hopf algebra object in the monoidal category* $(\mathcal{CLM}^u_{\mathbb{Z}_p}, \widehat{\otimes}^u_{\mathbb{Z}_p})$ *for the coproduct* $\mathbb{P}$ *and coidentity* ε *given by*

$$\mathbb{P}(\Psi(\lambda^{-1}p^j x)) \mapsto \Psi(\lambda^{-1}p^j x \widehat{\otimes} 1 + 1 \widehat{\otimes} \lambda^{-1}p^j x) \ , \ \varepsilon(\Psi(\lambda^{-1}p^j x)) = 0 \ , \tag{1.19.1}$$

for $j = 0, 1, \dots$ *. This structure only depends upon* $|\lambda|$*.*

Definition 1.20. *We define* $\mathbb{E}_\lambda$ *in* (0.0.1) *(resp.* $\mathbb{T}_\lambda$*) as the abelian group functor on* $\mathcal{ACLM}^u_{\mathbb{Z}_p}$*, represented by the Hopf algebras* $(\mathscr{E}_\lambda^\circ, \mathrm{standard})$ *(resp. by* $(\mathscr{T}_\lambda^\circ, \mathrm{standard})$*).*

A partial p-adic analog of Féjer's Theorem, or, more precisely, of Theorem 7.1 in Appendix B, is then

Proposition 1.21. *For any* $\lambda \in \mathbb{Q}_p^\times$*, the completion of the valued ring*

$$(\mathbb{Z}_p[\Psi(\lambda^{-1}p^j x) \,|\, j = 0, 1, \dots], w_\infty)$$

coincides with its closure in $\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{Z}_p) = \mathrm{W}(\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{F}_p))$ *equipped with the* p*-adic topology, and identifies with* $\mathrm{W}(\mathbb{F}_p[[(\lambda^{-1}x)_{-j}] \,|\, j = 0, 1, \dots])$ *also equipped with the* p*-adic topology.*

For the standard topology we have

Proposition 1.22. *For any* $\lambda \in \mathbb{Q}_p^\times$*, the completion of the valued ring* $(\mathscr{E}_\lambda^\circ, w_\infty)$ *(resp.* $(\mathscr{T}_\lambda^\circ, w_\infty)$*) coincides with its closure in* $\mathscr{C}(\mathbb{Q}_p, \mathbb{Z}_p) = \mathrm{W}(\mathscr{C}(\mathbb{Q}_p, \mathbb{F}_p))$ *equipped with the product topology of the prodiscrete topologies on the components* (4.18.4)*, and identifies with* $\mathrm{W}(\mathbb{F}_p(v(\lambda), \infty))$ *(see Proposition 4.16 below for notation) also equipped with the product topology of the prodiscrete topologies on the components.*

The ring $\mathscr{E}_\lambda^\circ \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ *(resp.* $\mathscr{T}_\lambda^\circ \otimes_{\mathbb{Z}_p} \mathbb{F}_p$*) equipped with the quotient topology coincides with* $\mathbb{F}_p(v(\lambda), \infty) = \mathscr{C}(\mathbb{Q}_p/\lambda p \mathbb{Z}_p, \mathbb{F}_p)$*.*

We now introduce our p-adic analog of the sheaf $\mathcal{APH}_{\mathbb{C}}$ of almost periodic analytic functions (see subsection 7.2 in Appendix B).

Definition 1.23.

1. *For any* $\rho > 0$*, we define the algebra of (resp.* integral*)* almost periodic p-adic analytic functions on the strip Σ_ρ *as the closure* $\mathcal{APH}_{\mathbb{Q}_p}(\Sigma_\rho)$ *(resp.* $\mathcal{APH}_{\mathbb{Z}_p}(\Sigma_\rho)$*) of* $\mathbb{Q}_p[\Psi(\lambda x) \,|\, \lambda \in \mathbb{Q}_p^\times]$ *(resp.* $\mathbb{Z}_p[\Psi(\lambda x) \,|\, \lambda \in \mathbb{Q}_p^\times]$*) in* $(\mathcal{O}^{\mathrm{bd}}(\Sigma_\rho), \mathrm{uniform})$*, with the induced Banach ring structure.*

2. *The algebra of* germs at 0 of almost periodic p-adic analytic functions *is the locally convex inductive limit*

$$(\mathcal{APH}_{0,\mathbb{Q}_p}, \mathrm{strip}) := \varinjlim_{\rho \to 0} \mathcal{APH}_{\mathbb{Q}_p}(\Sigma_\rho) \ . \tag{1.23.1}$$

3. *The algebra of* germs at 0 of integral almost periodic p-adic analytic functions *is*

$$(\mathcal{APH}_{0,\mathbb{Z}_p}, \mathrm{strip}) := \varinjlim_{\rho \to 0}{}^{u} \mathcal{APH}_{\mathbb{Z}_p}(\Sigma_\rho) \ . \tag{1.23.2}$$

4. *The algebra of* almost periodic p-adic entire functions *is*

$$(APH_{\mathbb{Q}_p}, \mathrm{strip}) := \varprojlim_{\rho \to +\infty} \mathcal{APH}_{\mathbb{Q}_p}(\Sigma_\rho) \ . \tag{1.23.3}$$

5. *The algebra of* integral almost periodic p-adic entire functions *is the closure* $(APH_{\mathbb{Z}_p}, \text{strip})$ *of* $\mathbb{Z}_p[\Psi(\lambda x) \,|\, \lambda \in \mathbb{Q}_p^\times]$ *in* $(APH_{\mathbb{Q}_p}, \text{strip})$ *equipped with the induced Fréchet* $\mathbb{Z}_p$*-algebra structure.*

6. *The Fréchet* $\mathbb{Z}_p$*-algebra* $\mathscr{E}^\circ$ *is a Hopf algebra object in the category* $\mathcal{CLM}^u_{\mathbb{Z}_p}$ *for the laws* (1.19.1)*. The corresponding group functor*

$$\mathbb{E} : \mathcal{ACLM}^u_{\mathbb{Z}_p} \longrightarrow \mathcal{A}b \tag{1.23.4}$$

will be called the universal covering *of* $\mathbb{E}_\lambda$*, for any* $\lambda \in \mathbb{Q}_p^\times$*.*

Remark 1.24. The special fiber of $\mathbb{E}$ is the constant group

$$\mathbb{Q}_p = \varprojlim_{|\lambda| \to 0} \mathbb{Q}_p/\lambda\mathbb{Z}_p$$

over $\mathbb{F}_p$. On the other hand, equation 0.0.5 shows that $\mathscr{E}^\circ[1/p]$ is dense in $\mathbb{Q}_p\{x\}$, so that the generic fiber of $\mathbb{E}$ is $\mathbb{G}_{a,\mathbb{Q}_p}$.

Our Definition 1.23 is designed as to make the analog of Theorem 7.5 in Appendix B a true statement. In the p-adic case, we actually get the following more precise statement.

Theorem 1.25. (Uniform Approximation Theorem) *The natural* $\mathcal{CLM}^u_{\mathbb{Q}_p}$*-morphism (resp.* $\mathcal{CLM}^u_{\mathbb{Z}_p}$*-morphism)*

$$(\mathcal{APH}_{0,\mathbb{Q}_p}, \text{strip}) \longrightarrow (AP_{\mathbb{Q}_p}, w_\infty)$$

(resp.

$$(\mathcal{APH}_{0,\mathbb{Z}_p}, \text{strip}) \longrightarrow (AP_{\mathbb{Z}_p}, w_\infty)\ )\ ,$$

is an isomorphism.

The similarity with classical Fourier expansions will be made more stringent in [3], where the classical Mahler binomial expansions of continuous functions $\mathbb{Z}_p \to \mathbb{Z}_p$ is generalized to an expansion of any uniformly continuous functions $\mathbb{Q}_p \to \mathbb{Q}_p$ as a series with countably many terms of entire functions of exponential type. Such a p-adic Fourier theory on $\mathbb{Q}_p$ presents the same power and limitations as the classical Fourier theory on $\mathbb{R}$. Functions in $AP_{\mathbb{Q}_p}$ play the role of Bohr's uniformly almost periodic functions and a variation of the Bochner-Fejér approximation theorem [7, I.9] holds. On the other hand, a Fourier series $\mathscr{F}(f)$ (with countably many terms) does exist for a much more general class of functions $f : \mathbb{Q}_p \to \mathbb{Q}_p$ and the classical question as to what extent the series $\mathscr{F}(f)$ approximates f makes perfect sense, precisely as in classical Harmonic Analysis.

We ask whether the classical Bohr compactification of $\mathbb{Q}_p$ has a p-adic analytic description, as it has one in terms of classical (*i.e.* complex-valued) harmonic theory on the locally compact group $(\mathbb{Q}_p, +)$.

We expect that a completely analogous theory should exist for any finite extension $K/\mathbb{Q}_p$. To develop it properly it will be necessary to extend Barsotti covector's construction to ramified Witt vectors modeled on K and to relate this construction to Lubin-Tate groups over K° [19].

2 Elementary proofs of the main properties of Ψ

We prove here the basic properties of the function Ψ. In contrast to [2], the proofs are here completely self-contained.

Proposition 2.1. *The equation* (0.3.6) *has a unique solution in* $\Psi = \Psi_q \in T + T^2\mathbb{Z}[[T]]$ *.*

Proof. We endow $\mathbb{Z}[[T]]$ of the T-adic topology. It is clear that, for any $\varphi \in T\mathbb{Z}[[T]]$, the series $\sum_{j=1}^{\infty} p^{-j}\varphi(p^jT)^{q^j}$ converges in $T\mathbb{Z}[[T]]$. Moreover, the map

$$\mathscr{L} : \varphi \longmapsto T - \sum_{j=1}^{\infty} p^{-j}\varphi(p^jT)^{q^j} \ , \tag{2.1.1}$$

is a contraction of the complete metric space $T+T^2\mathbb{Z}[[T]]$. In fact, let $\varepsilon(T) \in T^r\mathbb{Z}[[T]]$, with $r \geq 3$. For any $\varphi \in T + T^2\mathbb{Z}[[T]]$ we see that

$$\mathscr{L}(\varphi+\varepsilon) - \mathscr{L}(\varphi) \in T^{r(q-1)+q}\mathbb{Z}[[T]] \ .$$

Since $r(q-1)+q > r$ this shows that $\mathscr{L}$ is a contraction. So, this map has a unique fixed point which is $\Psi_q(T)$. □

The following proposition, due to M. Candilera, provides an alternative proof of Proposition 2.1 and finer information on $\Psi_q(T)$.

Proposition 2.2. (M. Candilera) *The functional equation for the unknow function* u

$$1 = \sum_{j=0}^{\infty} p^{j(q^j-1)} T^{\frac{q^j-1}{q-1}} u(p^{j(q-1)}T)^{q^j} \tag{2.2.1}$$

admits a unique solution $u(T) = u_q(T) \in 1 + T\mathbb{Z}[[T]]$. *We have*

$$\Psi_q(T) = Tu_q(T^{q-1}) \ . \tag{2.2.2}$$

Proof. In this case we consider the T-adic metric space $1 + T\mathbb{Z}[[T]]$ and the map

$$\begin{aligned} \mathscr{M} : 1 + T\mathbb{Z}[[T]] &\longrightarrow 1 + T\mathbb{Z}[[T]] \\ \varphi &\longmapsto 1 - \sum_{j=1}^{\infty} p^{j(q^j-1)} T^{\frac{q^j-1}{q-1}} \varphi(p^{j(q-1)}T)^{q^j} \ . \end{aligned} \tag{2.2.3}$$

We endow $\mathbb{Z}[[T]]$ of the T-adic topology. It is clear that, for any $\varphi \in T\mathbb{Z}[[T]]$, the series $\sum_{j=1}^{\infty} p^{-j}\varphi(p^jT)^{q^j}$ converges in $T\mathbb{Z}[[T]]$. If $\varepsilon(T) \in T^r\mathbb{Z}[[T]]$, with $r \geq 2$. For any $\varphi \in 1 + T\mathbb{Z}[[T]]$ we see that

$$\mathscr{M}(\varphi+\varepsilon) - \mathscr{M}(\varphi) \in T^{r+1}\mathbb{Z}[[T]] \ .$$

So, the map $\mathscr{M}$ is a contraction and its unique fixed point has the properties stated for the series u in the statement. □

Proposition 2.3. *The series* $\Psi(T) = \Psi_q(T)$ *is entire.*

Proof. Since $\Psi \in T + T^2\mathbb{Z}[[T]] \subset T\mathbb{Z}[[T]]$, we deduce that Ψ converges for $v_p(T) > 0$. Since the coefficient of T in $\Psi(T)$ is 1, whenever $v_p(T) > 0$ we have $v_p(\Psi(T)) = v_p(T)$.

Suppose Ψ converges for $v_p(T) > \rho$, for $\rho \leq 0$. Then, for $j \geq 1$, $\Psi(p^jT)^{q^j}$ converges for $v_p(T) > \rho - 1$. Moreover, if $j > -\rho + 1$ and $v_p(T) > \rho - 1$, we have

$$v_p(p^{-j}\Psi(p^jT)^{q^j}) = -j + q^j(v_p(p^jT)) > -j + q^j(j + \rho - 1) \ , \tag{2.3.1}$$

and this last term $\to +\infty$, as $j \to +\infty$.

This shows that the series $T - \sum_{j=1}^{\infty} p^{-j}\Psi(p^jT)^{q^j}$ converges uniformly for $v_p(T) > \rho - 1$, so that its sum, which is Ψ, is analytic for $v_p(T) > \rho - 1$. It follows immediately from this that Ψ is an entire function. □

Remark 2.4. We have proven that, for any $j = 0, 1, \ldots$ and for $v_p(T) > -j$,

$$v_p(p^{-j}\Psi(p^jT)^{q^j}) = -j + q^j(j + v_p(T)) \ . \tag{2.4.1}$$

In particular, for any $a \in \mathbb{Z}_q$, ($\Psi(a) \in \mathbb{Z}_q$ and) $\Psi_q(a) \equiv a$, modulo $p\mathbb{Z}_q$.

Proposition 2.5. *For any $a \in \mathbb{Q}_q$, $\Psi_q(a) \in \mathbb{Z}_q$.*

Proof. Let $a \in \mathbb{Z}_q$. We define by induction the sequence $\{a_i\}_{i=0,1,\ldots}$:

$$a_0 = a \ \ , \ \ a_i = \sum_{j=0}^{i-1} p^{j-i}(a_j^{q^{i-j-1}} - a_j^{q^{i-j}}) \ . \tag{2.5.1}$$

Since, for any $a, b \in \mathbb{Z}_q$, if $a \equiv b \mod p$, then $a^{q^n} \equiv b^{q^n} \mod pq^n$, hence modulo p^{n+1}, while $a \equiv a^q \mod p$, we see that $a_i \in \mathbb{Z}_q$, for any i.

We then see by induction that, for any i,

$$a_i = p^{-i}(a - \sum_{j=0}^{i-1} p^j a_j^{q^{i-j}}) \ \text{ or, equivalently, } \ a = \sum_{j=0}^{i} p^j a_j^{q^{i-j}} \ . \tag{2.5.2}$$

Explicitly, if we substitute in the formula which defines a_i, namely

$$p^i a_i = \sum_{j=0}^{i-1} p^j a_j^{q^{i-j-1}} - \sum_{j=0}^{i-1} p^j a_j^{q^{i-j}}$$

the $(i-1)$-st step of the induction, namely, $a = \sum_{j=0}^{i-1} p^j a_j^{q^{i-j-1}}$, we get

$$p^i a_i = a - \sum_{j=0}^{i-1} p^j a_j^{q^{i-j}} \ ,$$

which is precisely the i-th inductive step.

From the functional equation (0.3.6) and from Remark 2.4 we have, for $a \in \mathbb{Z}_q$ and $i = 0, 1, 2, \ldots$,

$$\Psi(p^{-i}a) \equiv p^{-i}a - \sum_{\ell=1}^{i} p^{-\ell}\Psi(p^\ell p^{-i}a)^{q^\ell} = p^{-i}(a - \sum_{j=0}^{i-1} p^j \Psi(p^{-j}a)^{q^{i-j}}) \mod p\mathbb{Z}_q \ . \tag{2.5.3}$$

Notice that $\Psi(a) \in \mathbb{Z}_q$ and that, modulo $p\mathbb{Z}_q$, $\Psi_q(a) \equiv a = a_0$, defined as in (2.5.1). We now show by induction on i that for $a_1, \ldots, a_i, \ldots$ defined as in (2.5.1),

$$\Psi(p^{-i}a) \equiv a_i \mod p\mathbb{Z}_q \ , \tag{2.5.4}$$

which proves the statement. In fact, assume $\Psi(p^{-j}a) \equiv a_j \mod p\mathbb{Z}_q$, for $j = 0, 1, \ldots, i-1$, and plug this information in (2.5.3). We get

$$\Psi(p^{-i}a) \equiv p^{-i}a - \sum_{\ell=1}^{i} p^{-\ell} a_{i-\ell}^{q^\ell} = p^{-i}(a - \sum_{j=0}^{i-1} p^j a_j^{q^{i-j}}) = a_i \mod p\mathbb{Z}_q \ , \tag{2.5.5}$$

which is the i-th inductive step. □

Remark 2.6. Notice that from (2.5.3) it follows that, for any $a \in p^{-n}\mathbb{Z}_q$,

$$a \equiv \sum_{\ell=0}^{n} p^{-\ell} \Psi_q(p^\ell a)^{q^\ell} \mod p\mathbb{Z}_q \ .$$

The formula can be more precise using the functional equation (0.3.6) and Remark 2.4. We get, for any $a \in \mathbb{Q}_q$,

$$a \equiv \sum_{\ell=0}^{-v_p(a)+i} p^{-\ell} \Psi_q(p^\ell a)^{q^\ell} \mod p^{i+1}\mathbb{Z}_q \ , \ \forall\, i \in \mathbb{Z}_{\geq 0} \ . \tag{2.6.1}$$

that is

$$a \equiv \sum_{\ell=0}^{i} p^{-\ell} \Psi_q(p^\ell a)^{q^\ell} \mod p^{i+v_p(a)+1}\mathbb{Z}_q \ , \ \forall\, i \in \mathbb{Z}_{\geq -v_p(a)} \ . \tag{2.6.2}$$

We generalize (1.12.2) as

Corollary 2.7. *For any* $a \in \mathbb{Q}_q$, *let*

$$a_i := \Psi_q(p^{-i}a) \mod p\mathbb{Z}_q \ \in \mathbb{F}_q \ .$$

We have

$$a = \sum_{i>>-\infty}^{\infty} [a_i]\, p^i \in \mathrm{W}(\mathbb{F}_q)[1/p] = \mathbb{Q}_q \ . \tag{2.7.1}$$

Proof. Assume first that $a \in \mathbb{Z}_q$. In this case (2.6.2) implies

$$a \equiv \sum_{\ell=0}^{i} p^{-\ell} \Psi_q(p^\ell a)^{q^\ell} \mod p^{i+1}\mathbb{Z}_q \ , \ \forall\, i \in \mathbb{Z}_{\geq 0} \ . \tag{2.7.2}$$

So, the statement follows from the following

Lemma 2.8. *Let* $i \mapsto b_i$ *and* $i \mapsto c_i$, *for* $i = 0, 1, \dots,$ *be two sequences in* $\mathbb{Z}_q$ *such that*

$$\sum_{j=0}^{i} p^j b_j^{q^{i-j}} \equiv \sum_{j=0}^{i} p^j c_j^{q^{i-j}} \mod p^{i+1}\mathbb{Z}_q \ , \ \forall\, i \in \mathbb{Z}_{\geq 0} \ .$$

Then

$$b_i \equiv c_i \mod p\mathbb{Z}_q \ , \ \forall\, i \in \mathbb{Z}_{\geq 0} \ .$$

Proof. Immediate by induction on i. □

In the general case, assume $a \in p^{-n}\mathbb{Z}_q$. Then

$$p^n a = \sum_{i=0}^{\infty} [\Psi_q(p^{n-i}a) \bmod p\mathbb{Z}_q]\, p^i \in \mathrm{W}(\mathbb{F}_q) \ . \tag{2.8.1}$$

hence

$$a = \sum_{i=0}^{\infty} [\Psi_q(p^{n-i}a) \bmod p\mathbb{Z}_q]\, p^{i-n} \in p^{-n}\mathrm{W}(\mathbb{F}_q) \ . \tag{2.8.2}$$

□

From the previous corollary, it follows that $a \in \mathbb{Q}_q$ has the following expression as a Witt bivector with coefficients in $\mathbb{F}_q$

$$a = (\dots, a_{-i}^{(q/p)^i}, \dots, a_{-2}^{(q/p)^2}, a_{-1}^{q/p}; a_0, a_1^p, a_1^{p^2}, \dots) . \tag{2.8.3}$$

which obviously equals $(\dots, a_{-i}, \dots, a_{-2}, a_{-1}; a_0, a_1, a_1, \dots)$, if $q = p$.

Remark 2.9. We have tried to provide a simple addition formula for Ψ_q of the form (0.3.4), in terms of the same power-series Φ. We could not get one, nor were we able to establish the relation between Ψ_q and Ψ_p, for $q = p^f$ and $f > 1$. On the other hand it is clear that Barsotti's construction of Witt bivectors, based on classicals Witt vectors, extends to the L-Witt vectors of [19, Chap. 1], where $L/\mathbb{Q}_p$ denotes any fixed finite extension. In our case, we would only need the construction of *loc.cit.* in the case of the field $L = \mathbb{Q}_q$. We believe that the inductive limit of $\mathbb{Z}_q$-groups $\mathrm{W}_{\mathbb{Q}_q,n} \to \mathrm{W}_{\mathbb{Q}_q,n+1}$ under Verschiebung

$$V : (x_{-n}, \dots, x_{-1}, x_0) \to (0, x_{-n}, \dots, x_{-1}, x_0)$$

is a $\mathbb{Z}_q$-formal groups whose addition law is expressed by a power-series Φ_q analog to Barsotti's Φ. We believe that equation (0.3.4) still holds true for Ψ_q if we replace Φ by Φ_q. We also believe that a generalized Ψ exists for any finite extension $L/\mathbb{Q}_p$, with analogous properties.

3 Valuation and Newton polygons of Ψ_q

This section is dedicated to establishing the growth behavior of $|\Psi_q(x)|$ as $|x| \to \infty$. These results will be essential to get the delicate estimates of [3].

3.1 Valuation polygon of Ψ_q

We recall from [15] that the valuation polygon of a Laurent series $f = \sum_{i \in \mathbb{Z}} a_i T^i$ with coefficients $a_i \in \mathbb{C}_p$, converging in an annulus $A := \alpha \le v_p(T) \le \beta$, is the graph $\mathrm{Val}(f)$ of the function $\mu \mapsto v(f, \mu) := \inf_i (v_p(a_i) + i\mu)$, which is in fact finite along the segment $\alpha \le \mu \le \beta$. The function $\mu \mapsto v(f, \mu)$ is continuous, piecewise affine, and concave on $[\alpha, \beta]$. For any $\mu \in [\alpha, \beta]$, we have $v(f, \mu) = \inf\{v_p(\Psi(x)) \,|\, v_p(x) = \mu\}$. In the case of Ψ, $A = \mathbb{C}_p$ and the segment $[\alpha, \beta]$ is the entire μ-line. For the convenience of the reader we have recalled below the relation between the valuation polygon and the Newton polygon of f.

We prove

Theorem 3.1. *The valuation polygon of Ψ_q goes through the origin, has slope 1 for $\mu > -1$, and slope q^j, for $-j-1 < \mu < -j$, $j = 1, 2, \dots$ (see Figure 1).*

Proof. We recall that if both f and g converge in the annulus $A := \alpha \le v_p(T) \le \beta$, then, for any $\mu \in [\alpha, \beta]$, $v(f + g, \mu) \ge \inf(v(f, \mu), v(g, \mu))$, and that equality holds at μ if $v(f, \mu) \ne v(g, \mu)$. Moreover, for any $n \in \mathbb{N}$, $v(f^n, \mu) = n\, v(f, \mu)$.

In the polygon in Figure 1, for $j = 1, 2, \dots$, the side of projection $[-j, 1-j]$ on the μ-axis is the graph of the function

$$\sigma_j(\mu) := q^{j-1}(\mu + j - 1) - q^{j-2} - \dots - q - 1 . \tag{3.1.1}$$

Notice that

$$\sigma_{j+1}(\mu) = -1 + q\, \sigma_j(\mu + 1) , \tag{3.1.2}$$

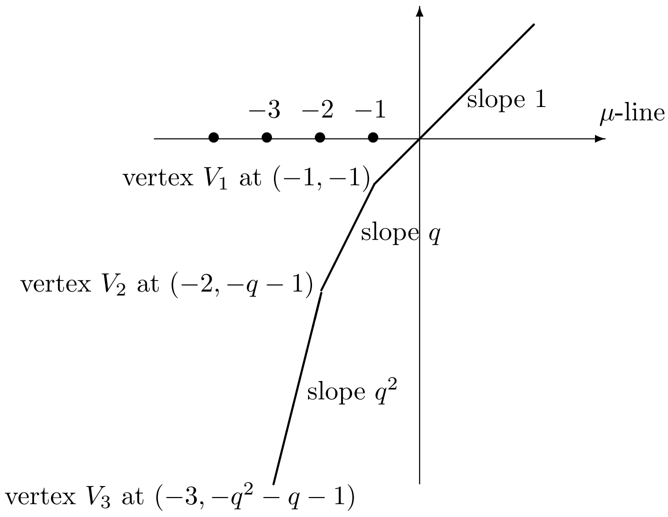

Figure 1: The valuation polygon of Ψ_q.

and therefore

$$\sigma_{j+i}(\mu) = -1 - q - \cdots - q^{i-1} + q^i \, \sigma_j(\mu + i) \ , \tag{3.1.3}$$

for any $i = 0, 1, 2, \ldots$.

Since $\Psi \in T\mathbb{Z}[[T]]$ and since the coefficient of T is 1, we have $v(\Psi, \mu) = \mu$, for $\mu \geq 0$. For $0 > \mu > -j$, $j \geq 1$, we have

$$\begin{aligned} v(p^{-j}\Psi(p^jT)^{q^j}, \mu) = -j + v(\Psi(p^jT)^{q^j}, \mu) = -j + q^j v(\Psi(p^jT), \mu) = \\ -j + q^j v(\Psi(S), j + \mu) = -j + (j+\mu)q^j > \mu = v(T, \mu) \ , \end{aligned} \tag{3.1.4}$$

where we have used the variable $S = p^jT$.

Remark 3.2. For $\mu = -j$ we get equality in the previous formula.

Let us set, for $j = 0, 1, 2, \ldots$,

$$\ell_j(\mu) = -j + (j+\mu)q^j \ ,$$

so that (3.1.4) becomes

$$v(p^{-j}\Psi(p^jT)^{q^j}, \mu) = \ell_j(\mu) > \ell_0(\mu) = \mu = v(T, \mu) \ , \tag{3.2.1}$$

for $0 > \mu > -j$, $j \geq 1$, with equality holding if $\mu = -j$. Notice that

$$\ell_0(\mu) = \mu = \sigma_1(\mu) \ .$$

Because of (3.2.1) and (0.0.5), and by continuity of $\mu \mapsto v(\Psi, \mu)$, we have

$$v(\Psi, \mu) = v(T, \mu) = \mu = \sigma_1(\mu) \ , \ \text{for } \mu \geq -1 \ . \tag{3.2.2}$$

We now reason by induction on $n = 1, 2, \ldots$. We assume that, for any $j = 1, 2, \ldots, n$ the side of projection $[-j, 1-j]$ on the μ-axis of the valuation polygon of Ψ is the graph of $\sigma_j(\mu)$. This at least was proven for $n = 1$. We consider the various terms in the functional equation

$$\Psi = T - p^{-1}\Psi(pT)^q - p^{-2}\Psi(p^2T)^{q^2} - \sum_{j=3}^{\infty} p^{-j}\Psi(p^jT)^{q^j} \ .$$

We assume $n > 1$. For $j = 1, 2, \dots, n$, and $-n-1 < \mu < -n$, we have

$$v(p^{-j}\Psi(p^jT)^{q^j}, \mu) = -j + v(\Psi(p^jT)^{q^j}, \mu) = -j + q^j v(\Psi(p^jT), \mu) = -j + q^j v(\Psi(S), j+\mu) = -j + q^j \sigma_{n-j+1}(\mu + j) \ , \tag{3.2.3}$$

since $j - n - 1 < j + \mu < j - n$, and therefore the inductive assumption gives $v(\Psi, j+\mu) = \sigma_{n-j+1}(\mu + j)$ in that interval. For $j > n$, and $-n-1 < \mu$, we have instead, from (3.2.1), $v(p^{-j}\Psi(p^jT)^{q^j}, \mu) = \ell_j(\mu)$.

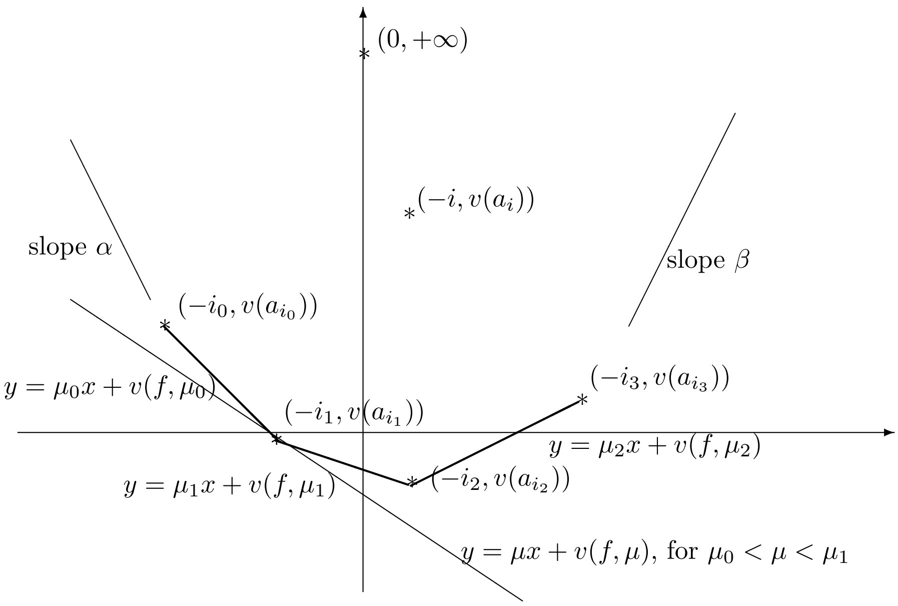

Figure 2: The Newton polygon $\mathrm{Nw}(f)$ of f.

Lemma 3.3. *Let $n > 1$.*

1. *For $j = 1, 2, \dots, n$ and for any $\mu \in \mathbb{R}$,*

$$\sigma_{n+1}(\mu) < -j + q^j \sigma_{n-j+1}(\mu + j) \ . \tag{3.3.1}$$

2. *For $j > n$ and $\mu > -n-1$, we have*

$$\sigma_{n+1}(\mu) < \ell_j(\mu) \ . \tag{3.3.2}$$

3. *For $-n-1 < \mu < -n$,*

$$\sigma_{n+1}(\mu) < \mu \ . \tag{3.3.3}$$

Proof. Assertion (3.3.1) is clear, since the two affine functions $\mu \mapsto \sigma_{n+1}(\mu)$ and $\mu \mapsto -j + q^j \sigma_{n-j+1}(\mu + j)$, have the same slope q^n, while their values at $\mu = -n$ are $-q^{n-1} - q^{n-2} - \cdots - q - 1$ and $-j - q^{n-1} - q^{n-2} - \cdots - q^{j+1} - q^j$, respectively. Notice that

$$-j-q^{n-1}-q^{n-2}-\cdots-q^{j+1}-q^j = (q^j-j)-q^{n-1}-q^{n-2}-\cdots-p-1 > -q^{n-1}-q^{n-2}-\cdots-q-1 ,$$

so that the conclusion follows.

We examine assertion (3.3.2), namely that, for $j > n$ and $\mu > -n-1$, we have

$$q^n(\mu + n) - q^{n-1} - q^{n-2} - \cdots - q - 1 < -j + (j + \mu)q^j .$$

The previous inequality translates into

$$q^n(\mu + n) - q^{n-1} - q^{n-2} - \cdots - q - 1 < -j + (j - n)q^j + (n + \mu)q^{n+(j-n)} ,$$

that is

$$q^{n-1} + q^{n-2} + \cdots + q + 1 - j + (j - n)q^j + (n + \mu)q^n(q^{j-n} - 1) > 0 ,$$

for $\mu > -n-1$. Since the l.h.s. is an increasing function of μ, it suffices to show that the inequality hold for $\mu = -n-1$, that is to prove that

$$q^{n-1} + q^{n-2} + \cdots + q + 1 - j + (j - n)q^j - q^n(q^{j-n} - 1) > 0 , \tag{3.3.4}$$

for any $j > n > 1$. We rewrite the l.h.s. of (3.3.4) as

$$\begin{aligned} &q^{n-1} + q^{n-2} + \cdots + q + 1 - n + (n - j) + (j - n)q^j - q^j + q^n = \\ &(q^{n-1} + q^{n-2} + \cdots + q + 1 - n) + (q^j - 1)(j - n) + (q^n - q^j) , \end{aligned} \tag{3.3.5}$$

where the four terms in round brackets on the r.h.s. are each, obviously, positive numbers. The conclusion follows.

We finally show (3.3.3) , namely that for $-n-1 < \mu < -n$,

$$q^n(\mu + n) - q^{n-1} - q^{n-2} - \cdots - p - 1 < \mu .$$

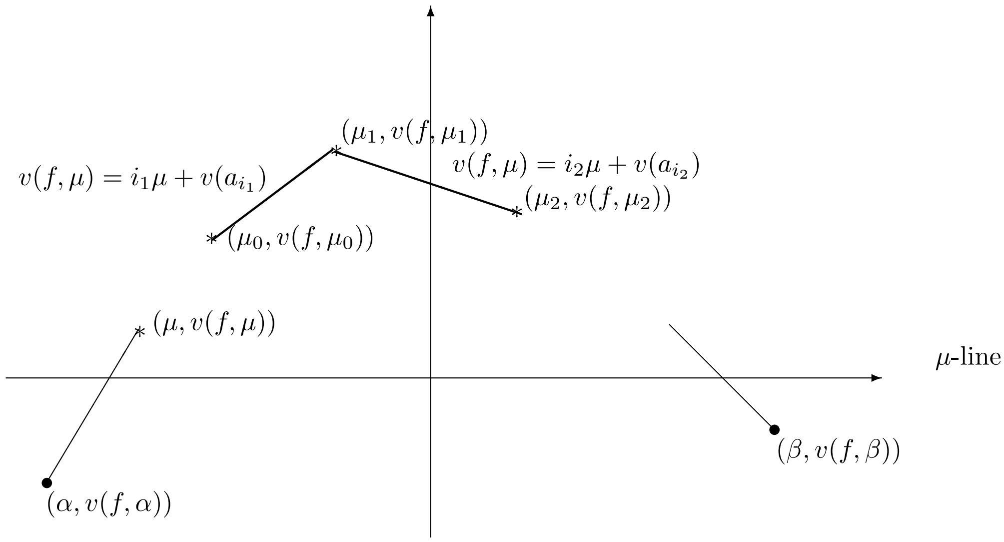

Figure 3: The valuation polygon $\mathrm{Val}(f)$.

It suffices to compare the values at $\mu = -n-1$ and at $\mu = -n$. We get

$$-q^n - q^{n-1} - q^{n-2} - \cdots - q - 1 < -n - 1 ,$$

and

$$-q^{n-1} - q^{n-2} - \cdots - q - 1 < -n ,$$

respectively, both obviously true. □

The previous calculation shows that the side of projection $[-n-1, -n]$ on the μ-axis of the valuation polygon of Ψ is the graph of $\sigma_{n+1}(\mu)$. We have then crossed the inductive step Case $n \Rightarrow$ Case $n+1$, and Theorem 3.1 is proven. □

Corollary 3.4. *Proposition 0.1 holds true.*

Proof. We have seen that $v_p(\Psi_p(x)) = v_p(x)$ if $v_p(x) > 0$. Then Proposition 0.1 follows from (0.3.4) and Lemma 0.2. □

Corollary 3.5. *For any $i = 1, 2, \ldots,$ and $v_p(x) \geq -i$ (resp. $v_p(x) > -i$), we have $v_p(\Psi_q(x)) \geq -\frac{q^i-1}{q-1}$ (resp. $v_p(\Psi_q(x)) > -\frac{q^i-1}{q-1}$). If $v_p(x) > -1$, we have $v_p(\Psi_q(x)) = v_p(x)$.*

Proof. The last part of the statement is a general fact for automorphisms of an open k-analytic disk D with one k-rational fixed point $a \in D(k)$ (the disk $v_p(x) > -1$ and $x(a) = 0$, in the present case) [6, Lemma 6.4.4]. □

3.2 Newton polygon of Ψ_q

We now recall that to a Laurent series $f = \sum_{i \in \mathbb{Z}} a_i T^i$ with coefficients $a_i \in \mathbb{C}_p$, converging in an annulus $A := \alpha \leq v_p(T) \leq \beta$, one associates two, dually related, polygons. The valuation polygon $\mu \mapsto v(f, \mu)$, was recalled before. The *Newton polygon* $\mathrm{Nw}(f)$ of f is the convex closure in the standard affine plane $\mathbb{R}^2$ of the points $(-i, v(a_i))$ and $(0, +\infty)$. If $a_i = 0$, then $v(a_i)$ is understood as $= +\infty$. We define $s \mapsto \mathrm{Nw}(f, s)$ to be the function whose graph is the lower-boundary of $\mathrm{Nw}(f)$. The main property of $\mathrm{Nw}(f)$ is that the length of the projection on the X-axis of the side of slope σ is the number of zeros of f of valuation $= \sigma$. The formula

$$v(f, \mu) = \inf_{i \in \mathbb{Z}} i\,\mu + v(a_i)$$

indicates (*cf.* [15]) that the relation between $\mathrm{Nw}(f)$ and $\mathrm{Val}(f)$ "almost" coincides with the duality formally described in the following lemma.

Lemma 3.6. (Duality of polygons) *In the projective plane $\mathbb{P}^2$, with affine coordinates (X, Y), we consider the polarity with respect to the parabola $X^2 = -2Y$*

$$\mathbb{P}^2 \to (\mathbb{P}^2)^* \to \mathbb{P}^2 ,$$

$$\text{point } (\sigma, \tau) \longmapsto \text{ line } (Y = -\sigma X - \tau) \longmapsto \text{ point } (\sigma, \tau) .$$

Assume the graph Γ of a continuous convex piecewise affine function has consecutive vertices

$$\ldots, (-i_0, \varphi_0), (-i_1, \varphi_1), (-i_2, \varphi_2), (-i_3, \varphi_3), \ldots$$

joined by the lines

$$\ldots, Y = \sigma_1 X + \tau_1 , Y = \sigma_2 X + \tau_2 , Y = \sigma_3 X + \tau_3 , \ldots .$$

Then, the lines joining the points

$$\ldots, (-\sigma_1, -\tau_1), (-\sigma_2, -\tau_2), (-\sigma_3, -\tau_3), \ldots$$

are

$$\dots\,,\, Y = i_1 X - \varphi_1\,,\, Y = i_2 X - \varphi_2\,,\, \dots\,,$$

and the polarity transforms these back into

$$\dots\,,\, (-i_1, \varphi_1)\,,\, (-i_2, \varphi_2)\,,\, \dots\,.$$

We say that the graph Γ^* *joining the vertices* (σ_i, τ_i), $(\sigma_{i+1}, \tau_{i+1})$ *by a straight segment is the* dual graph *of* Γ. *It is clear that the relation is reciprocal, that is* $(\Gamma^*)^* = \Gamma$ *and that* Γ^* *is a continuous concave piecewise affine function.*

Proof. It is the magic of polarities. □

The precise relation between $\mathrm{Nw}(f)$ and $\mathrm{Val}(f)$ is

Proposition 3.7.

$$\mathrm{Val}(f) = (-\mathrm{Nw}(f))^*$$

where $-\mathrm{Nw}(f)$ *is the polygon obtained from* $\mathrm{Nw}(f)$ *by the transformation* $(X, Y) \mapsto (X, -Y)$.

Proof. The most convincing proof follows from comparing Lemma 3.6 with Figures 2 and 3. □

We now apply the previous considerations to the two polygons associated to the function Ψ_q.

Corollary 3.8. *The Newton polygon* $\mathrm{Nw}(\Psi_q)$ *has vertices at the points*

$$V_i := (-q^i, i\,q^i - \frac{q^i - 1}{q - 1}) = (-q^i, i\,q^i - q^{i-1} - \dots - q - 1)\,.$$

The equation of the side joining the vertices V_i *and* V_{i-1} *is*

$$Y = -iX - \frac{q^i - 1}{q - 1}\,;$$

its projection on the X*-axis is the segment* $[-q^i, -q^{i-1}]$. *So,* $\mathrm{Nw}(\Psi)$ *has the form described in Figure 4.*

Corollary 3.9. *For any* $i = 0, 1, \dots$, *the map* $\Psi = \Psi_q$ *induces coverings of degree* q^i,

(3.9.1) $$\Psi : \{\, x \in \mathbb{C}_p \,|\, v_p(x) > -i - 1 \,\} \longrightarrow \{\, x \in \mathbb{C}_p \,|\, v_p(x) > -\frac{q^{i+1} - 1}{q - 1} \,\}\,,$$

(in particular, an isomorphism

(3.9.2) $$\Psi : \{\, x \in \mathbb{C}_p \,|\, v_p(x) > -1 \,\} \xrightarrow{\ \sim\ } \{\, x \in \mathbb{C}_p \,|\, v_p(x) > -1 \,\}\,,$$

for $i = 0$*), finite maps of degree* q^i

(3.9.3)
$$\Psi : \{\, x \in \mathbb{C}_p \,|\, -(i+1) < v_p(x) < -i \,\} \longrightarrow \{\, x \in \mathbb{C}_p \,|\, -\frac{q^{i+1} - 1}{q - 1} < v_p(x) < -\frac{q^i - 1}{q - 1} \,\}\,,$$

and finite maps of degree $q^{i+1} - q^i$

(3.9.4) $$\Psi : \{\, x \in \mathbb{C}_p \,|\, v_p(x) = -i - 1 \,\} \longrightarrow \{\, x \in \mathbb{C}_p \,|\, -\frac{q^{i+1} - 1}{q - 1} \le v_p(x) \,\}\,.$$

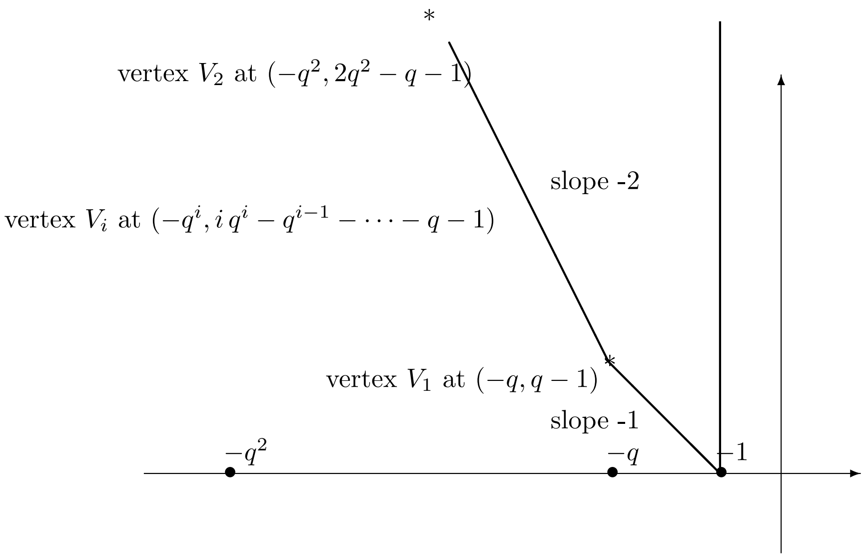

Figure 4: The Newton polygon $\mathrm{Nw}(\Psi_q)$ of Ψ_q.

Proof. The shape of the Newton polygon of Ψ indicates that, for any $a \in \mathbb{C}_p$, with $v_p(a) > -1$, the side of slope $= v_p(a)$ of the Newton polygon of $\Psi - a$ has projection of length 1 on the X-axis. So, $\Psi : \{\, x \in \mathbb{C}_p \,|\, v_p(x) > -1 \,\} \to \{\, x \in \mathbb{C}_p \,|\, v_p(x) > -1 \,\}$ is bijective, hence biholomorphic. Now we recall from Corollary 3.5 that for any given $i \geq 1$,

$$\Psi(\{\, x \in \mathbb{C}_p \,|\, v_p(x) > -i-1 \,\}) \subset \{\, x \in \mathbb{C}_p \,|\, v_p(x) > -\frac{q^{i+1}-1}{q-1} \,\} \,. \tag{3.9.5}$$

So, let a be such that $-\frac{q^{i+1}-1}{q-1} < v_p(a) \leq -\frac{q^i-1}{q-1}$, say $v_p(a) = -\frac{q^i-1}{q-1} - \varepsilon$, with $\varepsilon \in [0, q^i)$. Then, the Newton polygon of $\Psi - a$ has a single side of slope $> -i-1$, which has precisely slope $= -\varepsilon\, q^{-i} - i$ and has projection of length q^i on the X-axis. So, the equation $\Psi(x) = a$ has precisely q^i solutions x in the annulus $-i-1 < v_p(x) \leq -i$. If, for the same i, $-\frac{q^i-1}{q-1} < v_p(a) \leq -\frac{q^{i-1}-1}{q-1}$, the Newton polygon of $\Psi - a$ has a side of slope $-i$, whose projection on the X-axis has length $q^i - q^{i-1}$, and a side of slope σ, $1 - i \geq \sigma > -i$, whose projection on the X-axis has length q^{i-1}. So again $\Psi^{-1}(a)$ consists of q^i distinct points. We go on, for a in an annulus of the form $-\frac{q^{i-j}-1}{q-1} < v_p(a) \leq -\frac{q^{i-j-1}-1}{q-1}$, up to $j = i-2$, *i.e.* to $-\frac{q^2-1}{q-1} < v_p(a) \leq -1$. In that case, the Newton polygon of $\Psi - a$ has a side of slope $-i$ of projection $q^i - q^{i-1}$, a side of slope $1-i$ of projection $q^{i-1} - q^{i-1}$,..., a side of slope $j - i$ of projection $q^{i-j} - q^{i-j-1}$ on the X-axis, ..., up to a side of slope -1 of projection $q-1$ on the X-axis. Finally, for $v_p(a) > -1$, there is still exactly one solution of $\Psi(x) = a$, with $v_p(x) > -1$. This means that Ψ induces a (ramified) covering of degree q^i in (3.9.1). $\square$

3.3 The addition law of Ψ

We now extend the estimates of Corollary 3.9 to translates $\Psi(a + x)$ of Ψ, for $a \in \mathbb{Q}_p$. Although we expect that the same discussion carries over to $\Psi_q(a + x)$, where $a \in \mathbb{Q}_q$, we assume for simplicity that $q = p$ in the rest of this subsection.

Proposition 3.10. *Let $m \in \mathbb{Z}_{>0}$ and let $M(x_{-m} \ldots, x_{-1}, x_0)$ be a monomial in $\mathbb{Z}_p[x_{-m} \ldots, x_{-1}, x_0]$ divisible by x_{-m} and of pure weight 1, where x_i weighs p^i, for any i. Set*

$$M(x) := M(\Psi(p^m x), \ldots, \Psi(x)) \,.$$

Then, for any $r = 1, 2, \dots,$

$$
\begin{gathered}
w_r(M(x)) \geq \\
m+1-(p-1)r(m-r+1)+(p-1)\left(\binom{m+1}{2}-\binom{r}{2}\right)-\frac{p^{r+1}-1}{p-1} > \\
m+1+(p-1)\frac{(m-r)^2+(m-r)}{2}-\frac{p^{r+1}-1}{p-1}\ \left(> -\frac{p^{r+1}-1}{p-1}\right),
\end{gathered}
\tag{3.10.1}
$$

while, for $r = 0, -1, -2, \dots,$ *we get*

$$
w_r(M(x)) \geq m - r - (p-1)mr + (p-1)\binom{m+1}{2}\ \left(\geq p\,(1-r)\right). \tag{3.10.2}
$$

Proof. This follows from the estimates of Corollary 3.9 via a totally self-contained, but lengthy, computation on isobaric polynomials of Witt-type. We refer to the upcoming paper [3] for the proof of a more general statement. □

We apply Proposition 3.10 to the study of the addition law of Ψ. From (0.3.4) and (0.3.5), we deduce, taking into account Proposition 2.5, that, for any $c \in \mathbb{Q}_p$

$$
\begin{gathered}
\Psi(x+c) = \\
\lim_{i\to\infty} \varphi_i(\Psi(p^i x), \dots, \Psi(px), \Psi(x); \Psi(p^i c), \dots, \Psi(pc), \Psi(c)),
\end{gathered}
\tag{3.10.3}
$$

where

$$
\begin{gathered}
\varphi_m(\Psi(p^m x), \dots, \Psi(px), \Psi(x); \Psi(p^m c), \dots, \Psi(pc), \Psi(c)) - \\
\varphi_{m-1}(\Psi(p^{m-1} x), \dots, \Psi(px), \Psi(x); \Psi(p^{m-1} c), \dots, \Psi(pc), \Psi(c))
\end{gathered}
\tag{3.10.4}
$$

is a sum of monomials $M(x)$ as in Proposition 3.10.

Theorem 3.11.

1. *The function* Ψ *is bounded and uniformly continuous on any* p*-adic strip around* $\mathbb{Q}_p$*. In particular,*
$$\Psi(x) \in \mathscr{H}^{\mathrm{bd}}\,.$$
2. *For any* $j = 0, 1, \dots$ *and* $x \in \mathbb{Q}_p$*,*
$$\Psi_p(x + p^j \mathbb{C}_p^\circ) \subset \Psi_p(x) + p^j \mathbb{C}_p^\circ\,. \tag{3.11.1}$$

Proof. For the first part of the statement, we observe that Proposition 3.10 shows that, for any fixed $r \geq 0$, the sequence

$$
i \longmapsto \varphi_i(\Psi(p^i x), \dots, \Psi(px), \Psi(x); \Psi(p^i c), \dots, \Psi(pc), \Psi(c))
$$

converges in the w_r-valuation. This means that for any $\rho > 0$, the previous sequence is a sequence of entire functions bounded on the p-adic strip Σ_ρ around $\mathbb{Q}_p$, which converges to $\Psi(x+c)$ uniformly on Σ_ρ.

The second part of the statement was already proved in Corollary 3.4. It also follows from the estimates of Proposition 3.10 when $r \leq 0$. □

3.4 The zeros of Ψ

The following theorem is formulated in a way to make sense for $q =$ any power of p. We expect that it is true in that generality. However, for the time being, we can only prove it for $q = p$.

Theorem 3.12. *In this statement, let $q = p$.*

1. *For any $n = 1, 2, \ldots$, the map Ψ_q has $q_n := q^n - q^{n-1}$ simple zeros of valuation $-n$ in $\mathbb{Q}_q$. More precisely, for any system of representatives $a_1, \ldots, a_{q_n} \in \mathbb{Z}_q$ of $(\mathbb{Z}_q/p^n\mathbb{Z}_q)^\times = \mathrm{W}_{n-1}(\mathbb{F}_q)^\times$, and any $j = 1, \ldots, q_n$, the open disc $D(a_j p^{-n}, p^-)$ contains a unique zero $z_j^{(n)} \in \mathbb{Q}_q$ of Ψ_q. Then $z_1^{(n)}, \ldots, z_{q_n}^{(n)}$ are all the zeros of Ψ_q of valuation $-n$.*

2. *For $n = 1, 2, \ldots$ let $z_1^{(n)}, \ldots, z_{q_n}^{(n)}$ be the zeros of Ψ_q of valuation $-n$. We set*

$$\psi_n(x) = \prod_{j=1}^{q_n} (1 - \frac{x}{z_j^{(n)}}) \in 1 + p^n x \mathbb{Z}_q[x] \ .$$

   Then

$$\Psi_q(x) = x \prod_{n=1}^{\infty} \psi_n(x) \tag{3.12.1}$$

   is the canonical convergent infinite Schnirelmann product expression [15, (4.13)] of $\Psi_q(x)$ in the ring $\mathbb{Q}_p\{x\}$.

3. *The inverse function $\beta(T) = \beta_q(T)$ of $\Psi_q(T)$ (*i.e.* the power series such that, in $T\mathbb{Z}[[T]]$, $\Psi_q(\beta_q(T)) = T = \beta_q(\Psi_q(T))$) belongs to $T + T^2\mathbb{Z}[[T]]$. Its disc of convergence is exactly $v_p(T) > -1$.*

Proof. We now prove the first statement in Theorem 3.12. We recall that here $q = p$, so that $a_1, \ldots, a_{p_n} \in \mathbb{Z}_p$, with $p_n = p^n - p^{n-1}$, are a system of representatives of $(\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times$.

Lemma 3.13. *For any $m, n \in \mathbb{Z}_{>0}$, with $m \leq n$, and any $j = 1, \ldots, p_n$, the value of Ψ at the maximal point $\xi_{a_j p^{-n}, p^m}$ (of Berkovich type 2) of the rigid disc $D(a_j p^{-n}, (p^m)^+)$, that is $-\log|\Psi(\xi_{a_j p^{-n}, p^m})| = w_m(\Psi(a_j p^{-n} + x))$, is $-\frac{p^m - 1}{p-1} < 0$.*

Proof. The proof follows from the addition law (3.10.3) in which

$$\Psi(a_j p^{-n}), \ldots, \Psi(a_j p^{i-n}) \in \mathbb{Z}_p$$

so that, for any $i = 0, 1, 2, \ldots$,

$$\varphi_i(\Psi(p^i x), \ldots, \Psi(px), \Psi(x); \Psi(a_j p^{i-n}), \ldots, \Psi(a_j p^{1-n}), \Psi(a_j p^{-n}))$$

is a sum of a dominant (at $\xi_{a_j p^{-n}, p^m}$) term

$$\Psi(x) + \Psi(a_j p^{-n})$$

and of terms $M(x)$ described, for $m = n - i$, in Proposition 3.10. □

From the harmonicity of the function $|\Psi(x)$ at the point $\xi_{a_j p^{-n}, p^m}$, the estimate of Lemma 3.13, and the fact that $\Psi(a_j p^{-n}) \in \mathbb{Z}_p$, we deduce that each of the $p_n p^{m-n}$ open discs of radius p^m, centered at points of $p^{-n}\mathbb{Z}_p \setminus p^{1-n}\mathbb{Z}_p$ contains at least one zero of Ψ_q in $\mathbb{Q}_q$. For $m = 1$, this proves the first part of the statement.

For the second part of the statement we refer to [15, §4]. The fact that every $\psi_n(x) \in \mathbb{Q}_p[x]$ is $-n$-extremal follows from the fact that its zeros are all of exact valuation $-n$ [15, (2.7')].

The fact that β_p belongs to $T+T^2\mathbb{Z}[[T]]$ is obvious. The convergence of β_q for $v_p(T) > -1$ follows from (3.9.2). The fact that it cannot converge in a bigger disk is a consequence of the fact that Ψ_q has $q-1$ zeros of valuation -1. □

Corollary 3.14. *All zeros of Ψ_q are simple and are contained in $\mathbb{Q}_q$. Each ball $a + \mathbb{Z}_q \in \mathbb{Q}_q/\mathbb{Z}_q$ contains a single zero of Ψ_q.*

Remark 3.15. We believe that Theorem 3.12 holds, with essentially the same proof, for any power q of p. See Remark 2.9.

4 Rings of continuous functions on $\mathbb{Q}_p$

The point of this section is that of establishing the categorical limit/colimit formulas for the linear topologies of rings of p-adic functions on $\mathbb{Q}_p$. For topological algebra notions, we take the viewpoint and use the definitions explained in Appendix A (see also [4]).

We consider here a linearly topologized separated and complete ring k, whose family of open ideals we denote by $\mathcal{P}(k)$. In practice $k = \mathbb{Z}_p$ or $= \mathbb{F}_p$, or $= \mathbb{Z}_p/p^r\mathbb{Z}_p$, for any $r \in \mathbb{Z}_{\geq 1}$. More generally, A will be a complete and separated topological ring equipped with a $\mathbb{Z}$-linear topology, defined by a family of open additive subgroups of A. In particular we have in mind $A =$ a fixed finite extension K of $\mathbb{Q}_p$, whose topology is K°-linear but not K-linear. Again, a possible k would be K° or any $K^\circ/(\pi_K)^r$, for a parameter $\pi = \pi_K$ of K, and r as before.

We will express our statements for an abelian topological group G, which is separated and complete in the $\mathbb{Z}$-linear topology defined by a countable family of profinite subgroups G_r, with $G_r \supset G_{r+1}$, for any $r \in \mathbb{Z}$. So,

$$G = \varprojlim_{r \to +\infty} G/G_r = \varinjlim_{r \to -\infty} G_r \ ,$$

where G/G_r is discrete, G_r is compact, and limits and colimits are taken in the category of topological abelian groups separated and complete in a $\mathbb{Z}$-linear topology. We denote by $\pi_r : G \to G/G_r$ the canonical projection. Then, G is canonically a uniform space in which a function $f : G \to A$ is uniformly continuous iff, for any open subgroup $J \subset A$, the induced function $G \to A/J$ factors via a π_r, for some $r = r(J)$. A subset of G of the form $\pi_r^{-1}(\{h\}) = g + G_r$, for $g \in G$ and $h = \pi_r(g)$ is sometimes called the *ball of radius* G_r and *center* g. In particular, G is a locally compact, paracompact, 0-dimensional topological space. A general discussion of the duality between k-valued functions and measures on such a space, will appear in [5]. In practice here $G = \mathbb{Q}_p$ or $\mathbb{Q}_p/p^r\mathbb{Z}_p$ or $p^r\mathbb{Z}_p$, with the obvious uniform and topological structure.

Definition 4.1. *Let G and A be as before. We define $\mathscr{C}(G, A)$ (resp. $\mathscr{C}^{\rm bd}_{\rm unif}(G, A)$) as the A-algebra of continuous (resp. bounded and uniformly continuous) functions $f : G \to A$. We equip $\mathscr{C}(G, A)$ (resp. $\mathscr{C}^{\rm bd}_{\rm unif}(G, A)$) with the topology of uniform convergence on compact subsets of X (resp. on X). For any $r \in \mathbb{Z}$ and $g \in G$, we denote by χ_{g+G_r} is the characteristic function of $g + G_r \in G/G_r$. If G is discrete and $h \in G$, by $e_h : G \to k$ we denote the function such that $e_h(h) = 1$, while $e_h(x) = 0$ for any $x \neq h$ in G.*

Remark 4.2. It is clear that if $A = k$ is a linearly topologized ring any subset of k and therefore any function $f : G \to k$, is bounded. So, we write $\mathscr{C}_{\rm unif}(G, k)$ instead of $\mathscr{C}^{\rm bd}_{\rm unif}(G, k)$ in this case. If G is discrete, any function $G \to k$ is (uniformly) continuous; still, the bijective map $\mathscr{C}_{\rm unif}(G, k) \to \mathscr{C}(G, k)$ is not an isomorphism in general, so we do keep the difference in notation. If G is compact, any continuous function $G \to k$ is uniformly continuous, and $\mathscr{C}_{\rm unif}(G, k) \to \mathscr{C}(G, k)$ is an isomorphism, so there is no need to make any distinction.

Lemma 4.3. *Notation as above, but assume G is discrete (so that the G_r's are finite). Then $\mathscr{C}(G,k)$ (resp. $\mathscr{C}_{\mathrm{unif}}(G,k)$) is the k-module of functions $f : G \to k$ endowed with the topology of simple (resp. of uniform) convergence on G. So*

$$\mathscr{C}(G,k) = \varprojlim_{r \to -\infty} \mathscr{C}(G_r,k) = \prod_h k\, e_h \ \ , \ \ h \in G \ .$$

Similarly,

$$\mathscr{C}_{\mathrm{unif}}(G,k) = \varprojlim_{I \in \mathcal{P}(k)} \prod_{h \in G}^{\square,u} (k/I)\, e_h = \prod_{h \in G}^{\square,u} k\, e_h \ ,$$

where $\prod_{h \in G}^{\square,u} (k/I)\, e_h$ carries the discrete topology.

Proof. Clear from the definitions. □

The next lemma is a simplified abstract form, in the framework of linearly topologized rings and modules, of the classical decomposition of a continuous function as a sum of characteristic functions of balls (see for example Colmez [11, §1.3.1]).

Lemma 4.4. *Notation as above but assume G is compact (so that the G/G_r's are finite). Then*

$$\mathscr{C}(G,k) = \mathscr{C}_{\mathrm{unif}}(G,k) = \varinjlim_{r \to +\infty}{}^{u} \mathscr{C}(G/G_r,k) = \varinjlim_{r \to +\infty}{}^{u} \bigoplus_{g+G_r \in G/G_r} k\chi_{g+G_r} \ .$$

For any r, the canonical morphism $\mathscr{C}(G/G_r,k) \to \mathscr{C}(G,k)$ is injective.

Proof. This is also clear from the definitions. □

Remark 4.5. We observe that the inductive limit appearing in the formula hides the complication of formulas of the type

$$\chi_{g+G_r} = \sum_i \chi_{g_i+G_{r+1}} \ \text{ if } \ g + G_r = \dot{\bigcup_i} \, g_i + G_{r+1}$$

which we do not need to make explicit for the present use (see [5] for a detailed discussion).

Proposition 4.6. *Notation as above, with G general. Then in the category $\mathcal{CLM}_k^u$ we have :*

1. $$\mathscr{C}(G,k) = \varprojlim_{r \to -\infty} \mathscr{C}(G_r,k) \ \textit{ for the restrictions } \ \mathscr{C}(G_r,k) \to \mathscr{C}(G_{r+1},k) \ .$$

   In particular, for any fixed $r \in \mathbb{Z}$,

   $$\mathscr{C}(G,k) = \prod_{g+G_r \in G/G_r} \mathscr{C}(g+G_r,k) \ .$$

2. $$\mathscr{C}_{\mathrm{unif}}(G,k) = \varinjlim_{r \to +\infty}{}^{u} \mathscr{C}_{\mathrm{unif}}(G/G_r,k)$$

   for the embeddings

   $$\mathscr{C}_{\mathrm{unif}}(G/G_r,k) \hookrightarrow \mathscr{C}_{\mathrm{unif}}(G/G_{r+1},k)$$

3. *The natural morphism*

   $$\mathscr{C}_{\mathrm{unif}}(G,k) \longrightarrow \mathscr{C}(G,k)$$

   is injective and has dense image.

Proof. The first two parts follow from the universal properties of limits and colimits. The morphism in part 3 comes from the injective morphisms, for $r \in \mathbb{Z}$,

$$\mathscr{C}_{\mathrm{unif}}(G/G_r, k) \longrightarrow \mathscr{C}(G, k)$$

and the universal property of colimits. The inductive limit of these morphisms in the category $\mathcal{CLM}_k^u$ is a completion of the inductive limit taken in the category $\mathcal{M}od_k$ of k-modules equipped with the k-linear inductive limit topology. Since the latter is separated and since the axiom AB5 holds for the abelian category $\mathcal{M}od_k$, we deduce that the morphism in part 3 is injective. The morphism has dense image because, for any $r \in \mathbb{Z}$ and for any $s \in \mathbb{Z}_{\geq 0}$, the composed morphism

$$\mathscr{C}_{\mathrm{unif}}(G_r/G_{r+s}, k) \longrightarrow \mathscr{C}_{\mathrm{unif}}(G/G_{r+s}, k) \longrightarrow \mathscr{C}(G, k) \longrightarrow \mathscr{C}(G_r, k)$$

is the canonical map of Lemma 4.4

$$\mathscr{C}_{\mathrm{unif}}(G_r/G_{r+s}, k) \longrightarrow \mathscr{C}(G_r, k)$$

for the compact group G_r and its subgroup G_{r+s}. The fact that the set theoretic union $\bigcup_{s\geq 0} \mathscr{C}_{\mathrm{unif}}(G_r/G_{r+s}, k)$ is dense in $\mathscr{C}(G_r, k)$ is built-in in the definition of $\varinjlim^u$. □

Proposition 4.7. *Let $(H, \{H_r\}_r)$ be a locally compact group with the same properties as $(G, \{G_r\}_r)$ above, so that $(G \times H, \{G_r \times H_r\}_r)$ also has the same properties. Then we have a natural identification in $\mathcal{CLM}_k^u$*

$$\mathscr{C}(G, k)\widehat{\otimes}_k^u \mathscr{C}(H, k) \xrightarrow{\ \sim\ } \mathscr{C}(G \times H, k) \tag{4.7.1}$$

and a continuous strictly closed embedding

$$\mathscr{C}_{\mathrm{unif}}(G, k)\widehat{\otimes}_k^u \mathscr{C}_{\mathrm{unif}}(H, k) \longrightarrow \mathscr{C}_{\mathrm{unif}}(G \times H, k)\ . \tag{4.7.2}$$

Proof. We prove (4.7.1) first. By the first part of point 1 in Proposition 4.6 and the fact that $\widehat{\otimes}_k^u$ commutes with projective limits, we are reduced to the case of G and H compact. We are then in the situation of Lemma 4.4 for both G and H (in particular, the G/G_r's and the H/H_r's are finite). We need to prove

$$\varinjlim_{r\to+\infty}{}^u \bigoplus_{(g,h)+(G_r\times H_r)} k\chi_{(g,h)+G_r\times H_r} = \varinjlim_{r\to+\infty}{}^u \bigoplus_{g+G_r} k\chi_{g+G_r} \ \widehat{\otimes}_k^u \ \varinjlim_{r\to+\infty}{}^u \bigoplus_{h+H_r} k\chi_{h+H_r}\ .$$

Let M (resp. N) be the l.h.s. (resp. the r.h.s.) in the previous equation. Then

$$M = \varprojlim_{I\in\mathcal{P}(k)} M/\overline{IM}\ \ ,\ \ N = \varprojlim_{I\in\mathcal{P}(k)} N/\overline{IN}\ .$$

We show that $M/\overline{IM} \xrightarrow{\ \sim\ } N/\overline{IN}$, for any $I \in \mathcal{P}(k)$. Now,

$$M/\overline{IM} = \varinjlim_{r} \bigoplus_{(g,h)+(G_r\times H_r)} (k/I)\chi_{(g,h)+G_r\times H_r}\ .$$

Let

$$P := \varinjlim_{r}{}^u \bigoplus_{g+G_r} k\chi_{g+G_r}\ ,\ Q := \varinjlim_{r}{}^u \bigoplus_{h+H_r} k\chi_{h+H_r}\ .$$

Then

$$N/\overline{IN} = P/\overline{IP} \otimes_{k/I} Q/\overline{IQ} = \varinjlim_{r} \bigoplus_{g+G_r} (k/I)\chi_{g+G_r} \ \otimes_{k/I}\ \varinjlim_{r} \bigoplus_{h+H_r} (k/I)\chi_{h+H_r} = M/\overline{IM}\ .$$

This concludes the proof of (4.7.1).

We now pass to (4.7.2). We use formula 2 of Proposition 4.6, to replace the map in the statement by

$$\varinjlim_{r}{}^{u}\, \mathscr{C}_{\mathrm{unif}}(G/G_r,k)\widehat{\otimes}^{u}_{k} \varinjlim_{r\to+\infty}{}^{u}\ \mathscr{C}_{\mathrm{unif}}(H/H_r,k) \longrightarrow \varinjlim_{r}{}^{u}\, \mathscr{C}_{\mathrm{unif}}((G\times H)/(G_r\times H_r),k)\ .$$

By Lemma 4.3 this reduces to considering

$$\varinjlim_{r}{}^{u}\ \varprojlim_{I\in\mathcal{P}(k)}\ \prod_{g\in G}^{\square,u}(k/I)\, e_{g+G_r}\widehat{\otimes}^{u}_{k}\ \varinjlim_{r\to+\infty}{}^{u}\ \varprojlim_{I\in\mathcal{P}(k)}\ \prod_{h\in H}^{\square,u}(k/I)\, e_{h+H_r} \longrightarrow$$

$$\varinjlim_{r}{}^{u}\ \varprojlim_{I\in\mathcal{P}(k)}\ \prod_{(g,h)}^{\square,u}(k/I)\, e_{(g+G_r,h+H_r)}$$

As before, let M (resp. N) be the l.h.s. (resp. the r.h.s.) in the previous equation. Then

$$M = \varprojlim_{I\in\mathcal{P}(k)}\ M/\overline{IM}\ \ ,\ \ N = \varprojlim_{I\in\mathcal{P}(k)}\ N/\overline{IN}\ .$$

We show that $M/\overline{IM} \hookrightarrow N/\overline{IN}$ in an embedding with the relative topology, for any $I \in \mathcal{P}(k)$. Now,

$$\begin{aligned} M/\overline{IM} = & \varinjlim_{r}{}^{u} \prod_{g}(k/I)\, e_{g+G_r}\widehat{\otimes}^{u}_{k/I} \varinjlim_{r}{}^{u}\prod_{h}(k/I)\, e_{h+H_r} = \\ & \varinjlim_{r}{}^{u}(\prod_{g}(k/I)\, e_{g+G_r}\widehat{\otimes}^{u}_{k/I}\prod_{h}(k/I)\, e_{h+H_r})\ , \end{aligned}$$

and

$$N/\overline{IN} = \varinjlim_{r}{}^{u}\ \prod_{(g,h)}\ (k/I)\, e_{(g+G_r,h+H_r)}$$

where $\varinjlim^{u}$ and $\widehat{\otimes}^{u}_{k/I}$ are taken in the category $\mathcal{CLM}^{u}_{k/I}$. So, our statement is reduced to the fact that, for k, G, and H, discrete, if k^G (resp. k^H, resp. $k^{G\times H}$) indicates the k-algebra of functions $G \to k$ (resp. $H \to k$, resp. $G\times H \to k$) with the discrete topology, we have an inclusion

$$k^G \otimes_k k^H \hookrightarrow k^{G\times H}\ .$$

□

We are especially interested in

Corollary 4.8.

1. *For any* $r\in\mathbb{Z}$*,*

   (4.8.1) $$\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p,\mathbb{Z}_p) = \varprojlim_{s\to+\infty}\ \mathscr{C}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p,\mathbb{Z}_p/p^s\mathbb{Z}_p)$$

   where $\mathscr{C}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p,\mathbb{Z}_p/p^s\mathbb{Z}_p)$ *is equipped with the discrete topology. It is the* $\mathbb{Z}_p$*-algebra of all maps* $\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p \to \mathbb{Z}_p$ *equipped with the* p*-adic topology;*

2. *For any* $r\in\mathbb{Z}$*,*

   (4.8.2) $$\mathscr{C}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p,\mathbb{Z}_p) = \varprojlim_{s,t\to+\infty}\ \mathscr{C}(p^{-t}\mathbb{Z}_p/p^{r+1}\mathbb{Z}_p,\mathbb{Z}_p/p^s\mathbb{Z}_p)$$

   where $\mathscr{C}(p^{-t}\mathbb{Z}_p/p^{r+1}\mathbb{Z}_p,\mathbb{Z}_p/p^s\mathbb{Z}_p)$ *is equipped with the discrete topology. It is the* $\mathbb{Z}_p$*-Hopf algebra of all maps* $\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p \to \mathbb{Z}_p$ *equipped with the topology of simple convergence on* $\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p$ *for the* p*-adic topology of* $\mathbb{Z}_p$*;*

3.

$$\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p,\mathbb{Z}_p) = \varinjlim_{r\to+\infty}{}^{u}\ \mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p,\mathbb{Z}_p)\ ; \tag{4.8.3}$$

4.

$$\mathscr{C}(\mathbb{Q}_p,\mathbb{Z}_p) = \varprojlim_{s\to+\infty}\ \mathscr{C}(\mathbb{Q}_p,\mathbb{Z}_p/p^s\mathbb{Z}_p)\ . \tag{4.8.4}$$

Remark 4.9. Formula 4.7.1 shows that $\mathscr{C}(\mathbb{Q}_p,\mathbb{Z}_p)$ is a Hopf algebra object in $\mathcal{CLM}^u_{\mathbb{Z}_p}$.

Remark 4.10. We point out a tautological, but useful, formula which holds in $\mathscr{C}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p,\mathbb{F}_p)$. For any $h \in \mathbb{Q}_p/p^{r+1}\mathbb{Z}_p$, let e_h denote as before the function $\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p \to \mathbb{F}_p$ such that $e_h(h) = 1$ while $e_h(x) = 0$, if $x \in \mathbb{Q}_p/p^{r+1}$, $x \neq h$. For any $i \leq r$, the function

$$x_i : \mathbb{Q}_p/p^{r+1}\mathbb{Z}_p \longrightarrow \mathbb{F}_p\ ,$$

was introduced in (1.12.4). We then have

$$x_i = \sum_{h\in\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p} h_i e_h\ , \tag{4.10.1}$$

where $h_i = x_i(h)$.

Lemma 4.11. *Let G and K be as above but assume G is discrete. Then in the category $\mathcal{CLC}_K$*

1.

$$\mathscr{C}(G,K) = \prod_{g\in G} K e_g$$

is a Fréchet K-algebra.

2.

$$\mathscr{C}^{\mathrm{bd}}_{\mathrm{unif}}(G,K) = \ell_\infty(G,K)$$

is the Banach K-algebra of bounded sequences $(a_g)_{g\in G}$ of elements of K, equipped with the componentwise sum and product and with the supnorm.

Proof. Obvious from the definitions. □

Lemma 4.12. *Let G and K be as above, but assume G is compact. Then in the category $\mathcal{CLC}_K$*

$$\mathscr{C}(G,K) = \mathscr{C}^{\mathrm{bd}}_{\mathrm{unif}}(G,K) = \ell^0_\infty(G,K)$$

is the Banach K-algebra of sequences $(a_g)_{g\in G}$, with $a_g \in K$, such that $a_g \to 0$ along the filter of cofinite subsets of G, equipped with componentwise sum and product and with the supnorm.

Proof. This is a straightforward generalization of the classical wavelet decomposition. See [11, Prop. 1.16]. □

Proposition 4.13. *Let G and K be as in all this section. Then in the category $\mathcal{CLC}_K$ we have :*

1.

$$\mathscr{C}(G,K) = \varprojlim_{r\to-\infty}\ \mathscr{C}(G_r,K)\ \text{ for the restrictions } \mathscr{C}(G_r,K)\to\mathscr{C}(G_{r+1},K)\ .$$

In particular, $\mathscr{C}(G,K)$ is a Fréchet K-algebra.

2.

$$\mathscr{C}^{\mathrm{bd}}_{\mathrm{unif}}(G,K) = \varinjlim_{r\to+\infty} \mathscr{C}^{\mathrm{bd}}_{\mathrm{unif}}(G/G_r,K)$$

for the embeddings

$$\mathscr{C}^{\mathrm{bd}}_{\mathrm{unif}}(G/G_r,K) \hookrightarrow \mathscr{C}^{\mathrm{bd}}_{\mathrm{unif}}(G/G_{r+1},K)\ ,$$

where the inductive limit of Banach K-algebras is strict. In particular, $\mathscr{C}^{\mathrm{bd}}_{\mathrm{unif}}(G,K)$ is a complete bornological K-algebra.

3. The natural morphism

$$\mathscr{C}^{\mathrm{bd}}_{\mathrm{unif}}(G,K) \longrightarrow \mathscr{C}(G,K)$$

is injective and has dense image.

Proof. It is clear. For the notion of a bornological topological K-vector space we refer to [18, §6]; the fact that the notion is stable by strict inductive limits is Example 3 on page 39 of *loc.cit.*. The statement on completeness is proved in [18, Lemma 7.9]. □

Proposition 4.14. *Let G and H be locally compact groups as in Proposition 4.7. Then*

$$\mathscr{C}(G,K)\widehat{\otimes}_{\pi,K}\mathscr{C}(H,K) \xrightarrow{\sim} \mathscr{C}(G\times H,K)\ , \tag{4.14.1}$$

while the canonical map

$$\mathscr{C}^{\mathrm{bd}}_{\mathrm{unif}}(G,K)\widehat{\otimes}_{\pi,K}\mathscr{C}^{\mathrm{bd}}_{\mathrm{unif}}(H,K) \longrightarrow \mathscr{C}^{\mathrm{bd}}_{\mathrm{unif}}(G\times H,K) \tag{4.14.2}$$

is a strictly closed embedding of complete bornological algebras.

Proof. In the case of G and H compact this is detailed in the Example after Prop. 17.10 of [18]. In the general case (4.14.1) follows by taking projective limits. The statement for $\mathscr{C}^{\mathrm{bd}}_{\mathrm{unif}}(G,K)$ reduces instead to (4.7.2). □

We point out that $(\mathcal{CLC}_K, \widehat{\otimes}_{\pi,K})$ is a K-linear symmetric monoidal category. From Remarks 4.7 and 4.14, we conclude

Proposition 4.15. *Let G be as in Definition 4.1, and let A be either k or K, as before, and $\mathscr{C}(G,A)$ be as in* loc.cit.*. We regard $(\mathcal{CLM}^u_k, \widehat{\otimes}_k)$ and $(\mathcal{CLC}_K, \widehat{\otimes}_{\pi,K})$ as symmetric monoidal categories. The coproduct, counit, and inversion*

$$\mathbb{P}(f)(x,y) = f(x+y)\ \ ,\ \ \varepsilon(f) = f(0_G)\ \ ,\ \ \rho(f)(x) = f(-x)\ ,$$

for any $f \in \mathscr{C}(G,A)$ and any $x,y \in G$, define a structure of topological A-Hopf algebra on $\mathscr{C}(G,A)$, in the sense of the previous monoidal categories.

The following result describes the structure of the Hopf algebras of functions

$$\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p \to \mathbb{Z}_p/p^{a+1}\mathbb{Z}_p\ ,$$

for any $r,a \in \mathbb{Z}$ and $a \geq 0$ in terms of the functions x_i

$$x_i : \mathbb{Q}_p/p^{i+1}\mathbb{Z}_p \longrightarrow \mathbb{F}_p$$

introduced in (1.12.4). See also Remark 4.10.

Proposition 4.16. *For any $i \in \mathbb{Z}$, let x_i be as in (1.12.4) and let X_i be indeterminates. For $r \in \mathbb{Z}$ and $i \in \mathbb{Z}_{\geq 0}$, let $\mathbb{F}_p(r,i)$ denote the $\mathbb{F}_p$-algebra*

$$\mathbb{F}_p[X_r, X_{r-1}, X_{r-2}, \dots, X_{r-i}]/(1 - X_r^{p-1}, 1 - X_{r-1}^{p-1}, \dots, 1 - X_{r-i}^{p-1}) \ .$$

The dimension of $\mathbb{F}_p(r,i)$ as a $\mathbb{F}_p$-vector space is $(p-1)^{i+1}$. Let $X_{r,i} := (X_{r-i}, X_{r-i+1}, \dots, X_{r-1}, X_r)$ be viewed as a Witt vector of length $i+1$ with coefficients in $\mathbb{F}_p(r,i)$. We make $\mathbb{F}_p(r,i)$ into an $\mathbb{F}_p$-Hopf algebra by setting

$$\mathbb{P}X_{r,i} = X_{r,i} \otimes_{\mathbb{F}_p} 1 + 1 \otimes_{\mathbb{F}_p} X_{r,i} \ .$$

For any $i = 0, 1, \dots,$ the map $\mathbb{F}_p$-algebra map $\mathbb{F}_p(r, i+1) \to \mathbb{F}_p(r,i)$ sending X_{r-j} to X_{r-j} if $0 \leq j \leq i$ and X_{r-i-1} to 0 is a homomorphism of $\mathbb{F}_p$-Hopf algebras. Then, in the category $\mathcal{CLM}^u_{\mathbb{F}_p}$

1. *The map*

$$(4.16.1) \qquad \begin{aligned} \mathbb{F}_p(r,i) &\longrightarrow \mathscr{C}(p^{r-i}\mathbb{Z}_p/p^{r+1}\mathbb{Z}_p, \mathbb{F}_p) \\ X_j &\longmapsto x_j \ , \ \text{for } r - i \leq j \leq r \ , \end{aligned}$$

   is an isomorphism of $\mathbb{F}_p$-Hopf algebras.

2. *the $\mathbb{F}_p$-Hopf algebra $\mathscr{C}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{F}_p)$ equals*

$$\mathbb{F}_p(r,\infty) := \varprojlim_{i \to +\infty} \mathbb{F}_p(r,i)$$

   with the prodiscrete topology;

3. *the topological $\mathbb{F}_p$-algebra $\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{F}_p)$ equals $\mathbb{F}_p(r,\infty)$ equipped with the discrete topology.*

Proof. Parts 1 and 2 are [17, Teorema 3.31]. Part 3 follows by forgetting the topology. □

Remark 4.17. Notice that the $\mathbb{F}_p$-algebras $\mathbb{F}_p(r,i)$ are perfect.

Corollary 4.18. *For $r \in \mathbb{Z}$ and $i, a \in \mathbb{Z}_{\geq 0}$*

1. *the topological $\mathbb{Z}_p/p^{a+1}\mathbb{Z}_p$-algebra $\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{Z}_p/p^{a+1}\mathbb{Z}_p)$ equals*

$$(4.18.1) \qquad \mathrm{W}_a(\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{F}_p)) = \mathrm{W}_a(\mathbb{F}_p(r,\infty))$$

   equipped with the discrete topology. Therefore,

$$(4.18.2) \qquad \mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{Z}_p) = \mathrm{W}(\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{F}_p)) = \mathrm{W}(\mathbb{F}_p(r,\infty))$$

   equipped with the p-adic topology.

2. *the $\mathbb{Z}_p/p^{a+1}\mathbb{Z}_p$-Hopf algebra $\mathscr{C}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{Z}_p/p^{a+1}\mathbb{Z}_p)$ equals*

$$(4.18.3) \qquad \mathrm{W}_a(\mathscr{C}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{F}_p)) = \mathrm{W}_a(\mathbb{F}_p(r,\infty))$$

   with the prodiscrete topology. Therefore,

$$(4.18.4) \qquad \mathscr{C}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{Z}_p) = \mathrm{W}(\mathscr{C}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{F}_p)) = \mathrm{W}(\mathbb{F}_p(r,\infty))$$

   equipped with the product topology of the prodiscrete topology of $\mathbb{F}_p(r,\infty)$ on the components.

Definition 4.19. *We set*

$$\mathscr{C} = \mathscr{C}(\mathbb{Q}_p, \mathbb{Z}_p)\ .$$

For any $r, a \in \mathbb{Z}$ with $a \geq 0$, we define a Fréchet $\mathbb{Z}_p$-subalgebra of $\mathscr{C}$

$$\mathscr{C}_{r,a} := \{f \in \mathscr{C} \mid f(x + p^{r+1}\mathbb{Z}_p) \subset f(x) + p^{a+1}\mathbb{Z}_p\ \ , \forall\, x \in \mathbb{Q}_p\ \}\ .$$

Let F be the set-theoretic map

$$\begin{aligned} F : \mathscr{C} &\longrightarrow \mathscr{C} \\ f &\longmapsto f^p \end{aligned} \tag{4.19.1}$$

Then

$$\mathscr{C}_{r,a+1} \subset \mathscr{C}_{r,a} \ \text{ and } \ \mathscr{C}_{r,a} \subset \mathscr{C}_{r+1,a} \tag{4.19.2}$$

$$p^{a+1}\mathscr{C} \subset \mathscr{C}_{r,a} \tag{4.19.3}$$

is an ideal of $\mathscr{C}_{r,a}$, and F induces a map

$$F : \mathscr{C}_{r,a} \longrightarrow \mathscr{C}_{r,a+1}\ . \tag{4.19.4}$$

There exists a canonical map

$$\begin{aligned} R_{r,a} : \mathscr{C}_{r,a} &\longrightarrow \mathscr{C}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{Z}_p/p^{a+1}\mathbb{Z}_p) \\ f &\longmapsto R_{r,a}(f) \end{aligned} \tag{4.19.5}$$

such that

$$\pi_{a+1} \circ f = R_{r,a}(f) \circ \pi_{r+1}$$

which sits in the exact sequence

$$0 \longrightarrow p^{a+1}\mathscr{C} \longrightarrow \mathscr{C}_{r,a} \xrightarrow{R_{r,a}} \mathscr{C}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{Z}_p/p^{a+1}\mathbb{Z}_p) = \mathrm{W}_a(\mathbb{F}_p(r,\infty)) \longrightarrow 0 \tag{4.19.6}$$

We conclude

Proposition 4.20. *For any $r \in \mathbb{Z}$ and any $a \in \mathbb{Z}_{\geq 1}$, the map $f \mapsto \pi_1 \circ f$ induces an isomorphism*

$$\mathscr{C}_{r,a}/p\mathscr{C}_{r,a-1} \xrightarrow{\sim} \mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{F}_p)\ . \tag{4.20.1}$$

For $a = 0$ we similarly have

$$\mathscr{C}_{r,0}/p\mathscr{C} \xrightarrow{\sim} \mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{F}_p)\ . \tag{4.20.2}$$

The inverse of the isomorphism of discrete $\mathbb{F}_p$-algebras

$$\mathscr{C}_{r,a}/p\mathscr{C}_{r,a-1} \xrightarrow{\sim} \mathscr{C}_{r,0}/p\mathscr{C} \tag{4.20.3}$$

is provided by the map

$$\begin{aligned} F^a : \mathscr{C}_{r,0}/p\mathscr{C} &\xrightarrow{\sim} \mathscr{C}_{r,a}/p\mathscr{C}_{r,a-1} \\ f &\longmapsto f^{p^a}\ . \end{aligned} \tag{4.20.4}$$

Proof. The first formula follows from (4.19.3) and (4.19.6). In fact,

$$\mathscr{C}_{r,a}/p\mathscr{C}_{r,a-1} = (\mathscr{C}_{r,a}/p^a\mathscr{C})/p(\mathscr{C}_{r,a-1}/p^{a-1}\mathscr{C}) = \mathrm{W}_a(\mathbb{F}_p(r,\infty))/p\mathrm{W}_{a-1}(\mathbb{F}_p(r,\infty)) = \mathbb{F}_p(r,\infty)\ .$$

Similarly for the other formulas. □

By iteration, we get

Corollary 4.21.

(4.21.1) $$\mathscr{C}_{r,a}/p^{a+1}\mathscr{C} \xrightarrow{\sim} \mathscr{C}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{Z}_p/p^{a+1}\mathbb{Z}_p) = \mathrm{W}_a(\mathbb{F}_p(r,\infty)) \ .$$

For any $f \in \mathscr{C}_{r,a}$ there exist $f_0, f_1, \dots, f_a \in \mathscr{C}_{r,0}$, well determined modulo $p\mathscr{C}$, such that

(4.21.2) $$f \equiv f_0^{p^a} + pf_1^{p^{a-1}} + p^2 f_2^{p^{a-2}} + \cdots + p^a f_a \mod p^{a+1}\mathscr{C} \ .$$

5 p-adically entire functions bounded on $\mathbb{Q}_p$

We prove here the statements announced in the Introduction, namely Theorem 1.15, Theorem 1.17, Proposition 1.18, Proposition 1.19, Proposition 1.21, Proposition 1.22, and Theorem 1.25. We assume $q = p$ from now on, so in particular Ψ stands for Ψ_p.

We start with the proof of Theorem 1.15.

Proof. (of Theorem 1.15) It suffices to prove the statement over $\mathbb{Z}_p$. Notice that

$$\mathbb{Z}_p[\Psi(\lambda x) \,|\, \lambda \in \mathbb{Q}_p^\times] = \mathbb{Z}_p[\Psi(\lambda p^{-i} x) \,|\, i \in \mathbb{Z} \, , \, \lambda \in \mathbb{Z}_p^\times] \ .$$

Both rings $\mathbb{Z}_p[[(\lambda x)_i] \,|\, i \in \mathbb{Z} \, , \, \lambda \in \mathbb{Z}_p^\times]$ and $\mathbb{Z}_p[\Psi(\lambda p^{-i} x) \,|\, i \in \mathbb{Z} \, , \, \lambda \in \mathbb{Z}_p^\times]$ are contained in the $\mathbb{Z}_p$-Banach ring $\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{Z}_p)$ which may be identified with $\mathrm{W}(\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{F}_p))$ equipped with the p-adic topology. Then $AP_{\mathbb{Z}_p}$ consists of $\mathrm{W}(\mathbb{F}_p[(\lambda x)_i \,|\, i \in \mathbb{Z} \, , \, \lambda \in \mathbb{Z}_p^\times])$. Notice that $\mathbb{F}_p[(\lambda x)_i \,|\, i \in \mathbb{Z} \, , \, \lambda \in \mathbb{Z}_p^\times]$ is a perfect subring of the perfect ring $\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{F}_p)$, since $(\lambda x)_i^p = (\lambda x)_i$, for any i, λ. It suffices to prove

Lemma 5.1. *For any fixed $\lambda \in \mathbb{Z}_p^\times$, the closure of $\mathbb{Z}_p[\Psi(\lambda p^i x) \,|\, i = 0, 1, 2, \dots]$ in $\mathrm{W}(\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{F}_p))$ coincides with of $\mathrm{W}(\mathbb{F}_p[(\lambda x)_i \,|\, i = 0, -1, -2, \dots])$.*

Proof. We may as well assume $\lambda = 1$ and prove

Sublemma 5.2. *The closure of $\mathbb{Z}_p[\Psi(p^i x) \,|\, i = 0, 1, 2, \dots]$ in $\mathrm{W}(\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{F}_p))$ coincides with of $\mathrm{W}(\mathbb{F}_p[x_i \,|\, i = 0, -1, -2, \dots])$.*

Proof. Let C be the closure of $\mathbb{Z}_p[\Psi(p^i x) \,|\, i = 0, 1, 2, \dots]$ in $\mathrm{W}(\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{F}_p))$. The formula

$$[x_{-i}] = \lim_{N \to \infty} \Psi(p^i x)^N$$

shows that $\mathrm{W}(\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{F}_p)) \subset C$. It will suffice to show that, as functions $\mathbb{Q}_p \to \mathbb{Z}_p$

$$\Psi(x) \in \mathrm{W}(\mathbb{F}_p[x_i \,|\, i = 0, -1, -2, \dots]) \ .$$

We write the restriction of $\Psi(x)$ to a function $\mathbb{Q}_p \to \mathbb{Z}_p$ as

$$\Psi(x) = (\Psi_0(x), \Psi_1(x), \Psi_2(x), \dots)$$

with $\Psi_i \in \mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{F}_p)$ and $\Psi_0(x) = x_0$. We have, from (0.0.5), the formula in $\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{Q}_p)$

(5.2.1) $$\Psi(x) + p^{-1}\Psi(px)^p + \cdots + p^{-i}\Psi(p^i x)^{p^i} + \cdots = x = (\dots, x_{-i}, \dots, x_{-2}, x_{-1}; x_0, *, *, \dots)$$

From (5.2.1) we deduce that, as functions in $\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p, p^{-i}\mathbb{Z}_p)$

(5.2.2)

$$\Psi(x) + p^{-1}\Psi(px)^p + p^{-2}\Psi(p^2 x)^{p^2} + \cdots + p^{-i}\Psi(p^i x)^{p^i} = (x_{-i}, \dots, x_{-2}, x_{-1}; x_0, *, *, \dots)$$

This shows, inductively on i, that

$$\Psi_i \in \mathbb{F}_p[x_j \,|\, j = 0, -1, -2, \dots, -i] \ .$$

□

□

□

Definition 5.3. *Let $r \in \mathbb{Z}$ and $a \in \mathbb{Z}_{\geq 1}$. We define $\mathscr{E}^{\circ}_{r,a}$ (resp. $\mathscr{T}^{\circ}_{r,a}$) to be the $\mathbb{Z}_p$-subalgebra of $\mathscr{E}^{\circ}_{p^r}$ (resp. of $\mathscr{T}^{\circ}_{p^r}$) (*cf.* Definition 1.16) consisting of those functions f such that*

$$f(x + p^{r+j}\mathbb{C}_p^{\circ}) \subset f(x) + p^{a+j}\mathbb{C}_p^{\circ} \ , \ \ \forall\, x \in \mathbb{Q}_p \ \text{ and } \ \forall\, j \in \mathbb{Z}_{\geq 1} \ . \tag{5.3.1}$$

Remark 5.4. For the rest of this section the statements valid for the rings $\mathscr{E}^{\circ}_{r,a} \subset \mathbb{Q}_p\{x\}$ hold equally well, and with the same proof, for the rings $\mathscr{T}^{\circ}_{r,a} \subset \mathcal{O}(\Sigma_{p^{-r}})^{\circ}$. For short, we deal with the former only.

Notice that

$$\mathscr{E}^{\circ}_{r,a+1} \subset \mathscr{E}^{\circ}_{r,a} \subset \mathscr{E}^{\circ}_{r+1,a+1} \ \text{ and } \ p\mathscr{E}^{\circ}_{r,a} \subset \mathscr{E}^{\circ}_{r,a+1} \tag{5.4.1}$$

and that we have a map F as in Definition 4.19 such that

$$F(\mathscr{E}^{\circ}_{r,a}) \subset \mathscr{E}^{\circ}_{r,a+1} \ . \tag{5.4.2}$$

Remark 5.5. We have

$$\mathscr{E}^{\circ}_{p^r} := \mathscr{E}^{\circ}_{r,0} \ .$$

We already proved (Proposition 0.1 and Corollary 3.4) that $\Psi(x) \in \mathscr{E}^{\circ}_{0,0}$. Therefore, for any $i \in \mathbb{Z}_{\geq 0}$ and $\ell = 0, 1, \ldots, p-1$, the function $\Psi(p^{i-r}x)^{\ell p^a}$ belongs to $\mathscr{E}^{\circ}_{r-i,a} \subset \mathscr{E}^{\circ}_{r,a}$.

Lemma 5.6. *If a sequence of functions $n \mapsto f_n \in \mathscr{E}^{\circ}_{r,a}$ (resp. $\in \mathscr{T}^{\circ}_{r,a}$) converges to $f \in \mathbb{C}_p\{x\}$ (resp. to $f \in \mathcal{O}(\Sigma_{p^{-r}})^{\circ}$) uniformly on bounded subsets of $\mathbb{C}_p$ (resp. of $\Sigma_{p^{-r}}$) then $f \in \mathscr{E}^{\circ}_{r,a}$ (resp. $\in \mathscr{T}^{\circ}_{r,a}$). Therefore $\mathscr{E}^{\circ}_{r,a}$ (resp. $\mathscr{T}^{\circ}_{r,a}$) is a closed $\mathbb{Z}_p$-subalgebra of $\mathbb{C}_p\{x\}$ (resp. of $(\mathcal{O}(\Sigma_{p^{-r}})^{\circ}, \text{standard})$). The induced Fréchet algebra structure on $\mathscr{E}^{\circ}_{r,a}$ (resp. on $\mathscr{T}^{\circ}_{r,a}$) will be called* standard.

Proof. We deal, to fix ideas, with the case of $\mathscr{E}^{\circ}_{r,a}$. We show that for any $c \in \mathbb{Q}_p$ and $j = 0, 1, \ldots,$

$$f(c + p^{r+j+1}\mathbb{C}_p^{\circ}) \subset f(c) + p^{a+j+1}\mathbb{C}_p^{\circ} \ .$$

By assumption, for any $s, t \in \mathbb{N}$, there exists $N = N_{s,t}$ such that if $n \geq N$, then

$$(f_n - f)(p^{-s}\mathbb{C}_p^{\circ}) \subset p^t\mathbb{C}_p^{\circ} \ .$$

So, for c and j as before, let s be such $c + p^{r+j+1}\mathbb{C}_p^{\circ} \subset p^{-s}\mathbb{C}_p^{\circ}$, and let t be $\geq j + a + 1$. Then, for any $n \geq N_{s,t}$,

$$(f_n - f)(c + p^{r+j+1}\mathbb{C}_p^{\circ}) \subset (f_n - f)(p^{-s}\mathbb{C}_p^{\circ}) \subset p^t\mathbb{C}_p^{\circ} \subset p^{j+a+1}\mathbb{C}_p^{\circ} \ .$$

Therefore $f \in \mathscr{E}^{\circ}_{r,a}$. □

Notice that Proposition 1.18 follows from Lemma 5.6, by taking $a = 0$.

Let r, a be as in Definition 5.3. Any function $f \in \mathscr{E}^{\circ}_{r,a}$ induces a continuous function $f_{|\mathbb{Q}_p} : \mathbb{Q}_p \to \mathbb{Z}_p$. The $\mathbb{Z}_p$-linear map

$$Res^{\circ} : (\mathscr{E}^{\circ}_{r,a}, \text{standard}) \longrightarrow \mathscr{C}_{r,a} \subset \mathscr{C}(\mathbb{Q}_p, \mathbb{Z}_p) \ , \ \ f \longmapsto f_{|\mathbb{Q}_p} \ , \tag{5.6.1}$$

is continuous and injective. By composition, we obtain, for any $r \in \mathbb{Z}$ and any $a, h = 0, 1, \ldots,$ a morphism

$$R_{r,a} \circ Res^{\circ} : (\mathscr{E}^{\circ}_{r,a}, \text{standard}) \longrightarrow \mathscr{C}(p^{r-h}\mathbb{Z}_p/p^{r+1}\mathbb{Z}_p, \mathbb{Z}_p/p^{a+1}\mathbb{Z}_p) = \mathrm{W}_a(\mathbb{F}_p(r,h)) \ , \tag{5.6.2}$$

where the r.h.s. is equipped with the topology of (4.18.3). The kernel of that map is the set of $g \in \mathscr{E}^{\circ}_{r,a}$ such that $-\log ||g||_{p^{h-r}} \geq a+1$. From (5.6.1) we also get maps of Fréchet $\mathbb{Z}_p$-algebras

$$(5.6.3) \qquad Res^{\circ} : (\mathscr{E}^{\circ}_{\lambda}, \{|| \ ||_{p^r\mathbb{Z}_p}\}_{r\in\mathbb{Z}})^{\wedge} \longrightarrow \mathscr{C}(\mathbb{Q}_p, \mathbb{Z}_p) \ , \ f \longmapsto f_{|\mathbb{Q}_p} \ ,$$

$$(5.6.4) \qquad Res^{\circ} : (\mathscr{T}^{\circ}_{\lambda}, \{|| \ ||_{p^r\mathbb{Z}_p}\}_{r\in\mathbb{Z}})^{\wedge} \longrightarrow \mathscr{C}(\mathbb{Q}_p, \mathbb{Z}_p) \ , \ f \longmapsto f_{|\mathbb{Q}_p} \ .$$

Lemma 5.7. *Let $r \in \mathbb{Z}$ and $a \in \mathbb{Z}_{\geq 0}$ be as before.*

1. *Any series of functions of the form*

$$(5.7.1) \qquad \sum_{\ell=0}^{p-1}\sum_{i=0}^{\infty} c_{\ell,a,i}\Psi(p^{i-r}x)^{\ell p^a} \ , \ c_{\ell,a,i} \in \mathbb{Z}_p \ ,$$

converges in the standard Fréchet topology of $\mathbb{Q}_p\{x\}$ to an element of $\mathscr{E}^{\circ}_{r,a}$ along the filter of cofinite subsets of $\{0,1,\dots,p-1\} \times \mathbb{Z}_{\geq 0}$.

2. *For any element $f \in \mathscr{C}_{r,a}$ and for any $s = 0,1,2,\dots$ there exist uniquely determined elements $c_{\ell,b,i} = c^p_{\ell,b,i} \in \mathbb{Z}_p$, such that for*

$$f_{r,a} := \sum_{b=0}^{a}\sum_{\ell=0}^{p-1}\sum_{i=0}^{\infty} c_{\ell,b,i}p^b\Psi(p^{i-r}x)^{\ell p^{a-b}} \in \mathscr{E}^{\circ}_{r,a} \ ,$$

where the infinite sum converges in the standard Fréchet topology of $\mathscr{E}^{\circ}_{r,a}$, we have

$$(5.7.2) \qquad -\log ||(f - f_{r,a})||_{p^{-r}\mathbb{Z}_p} \geq a+1 \ .$$

Same statement for $\mathscr{E}^{\circ}_{r,a}$ replaced by $\mathscr{T}^{\circ}_{r,a}$.

3. *For any element $f \in \mathscr{C}_{r,a}$ and any $h = 0,1,\dots,$ there exist uniquely determined elements $c_{\ell,b,i} = c^p_{\ell,b,i} \in \mathbb{Z}_p$, such that for*

$$f_{r,a,h} := \sum_{b=0}^{a}\sum_{\ell=0}^{p-1}\sum_{i=0}^{h} c_{\ell,b,i}p^b\Psi(p^{i-r}x)^{\ell p^{a-b}} \ ,$$

$$(5.7.3) \qquad -\log ||(f - f_{r,a,h})||_{p^{h-r}\mathbb{Z}_p} \geq a+1 \ .$$

4. *The map* (5.6.2) *is surjective.*

5. *The maps* (5.6.3) *and* (5.6.4) *are the isomorphisms of Theorem 1.17.*

Proof. The first part is clear. As for the second, we observe that, for any $b = 0,1,\dots,a$, the map $R_{r,a} \circ Res^{\circ}$ transforms the function $p^b\Psi(p^{i-r}x)^{\ell p^{a-b}}$, for $\ell = 0,1,\dots,p-1$, into the Witt vector

$$(0,\dots,0,w_b = x^{\ell}_{r-i},0,\dots,0) \in \mathrm{W}_a(\mathbb{F}_p(r,\infty)) \ ,$$

where x^{ℓ}_{r-i} is placed at the b-th level. Since any $y \in \mathbb{F}_p(r,\infty)$ admits a unique expression as a sum, convergent in the prodiscrete topology of $\mathbb{F}_p(r,\infty)$,

$$y = \sum_{\ell=0}^{p-1}\sum_{i=0}^{\infty} \gamma_{\ell,i}x^{\ell}_{r-i} \ , \ \gamma_{\ell,i} \in \mathbb{F}_p \ ,$$

is clear that any $w=(w_0,w_1,\dots,w_a)\in \mathrm{W}_a(\mathbb{F}_p(r,\infty))$ admits a unique expression as a sum

$$\sum_{b=0}^{a}\sum_{\ell=0}^{p-1}\sum_{i=0}^{\infty}[\gamma_{\ell,b,i}](0,\dots,0,w_b=x^{\ell}_{r-i},0,\dots,0)$$

which in turn converges in the prodiscrete topology of $\mathrm{W}_a(\mathbb{F}_p(r,\infty))$. More precisely, for any $a,h=0,1,\dots$, we can determine coefficients $c_{\ell,b,i}\in\mathbb{Z}_p$ such that

$$w-\sum_{b=0}^{a}\sum_{\ell=0}^{p-1}\sum_{i=0}^{h}c_{\ell,b,i}x^{\ell}_{r-i}$$

has zero image in $\mathrm{W}_a(\mathbb{F}_p(r,h))$. So, the function

$$f_{r,a,h}:=\sum_{b=0}^{a}\sum_{\ell=0}^{p-1}\sum_{i=0}^{h}c_{\ell,b,i}p^b\Psi(p^{i-r}x)^{\ell p^{a-b}}\ ,$$

is such that

$$\min\{v_p(f_{r,a,h}(x)-f(x))\,|\,x\in p^{r-h}\mathbb{Z}_p+p^r\mathbb{C}_p^\circ\}\geq a+1\ .$$

Finally, we already observed that the kernel of the map (5.6.2) consists of the elements $g\in\mathscr{E}^\circ_{r,a}$ such that $-\log||g||_{p^{h-r}\mathbb{Z}_p}\geq a+1$. This proves 2, 3 and 4.

As for the last part of the statement, we pick any $f\in\mathscr{C}_{\mathrm{unif}}(\mathbb{Q}_p,\mathbb{Z}_p)$ and a natural number $N=0,1,\dots$. Then there exists an $M=0,1,\dots$ and $f_M\in\mathscr{C}(\mathbb{Q}_p/p^M\mathbb{Z}_p,\mathbb{Z}_p)$ such that $w_\infty(f-f_M)\geq N$. It will suffice to determine an element $g\in\mathbb{Z}_p[\Psi(\lambda x)\,|\,\lambda\in\mathbb{Q}_p^\times]$ such that $w_\infty(g-f_M)\geq N$. We then pick $r\in\mathbb{Z}$ and $a\in\mathbb{Z}_{\geq 0}$ so that $r+1\geq M$ and $a+1\geq N$. The statement follows from the surjectivity of (5.6.2). This concludes the proof. □

As a corollary, we obtain the proof of Propositions 1.21 and 1.22. We now give the proof of Proposition 1.19.

Proof. (of Proposition 1.19) We discuss $(\mathscr{E}^\circ_\lambda,\text{standard})$ in order to fix ideas. The case of $(\mathscr{T}^\circ_\lambda,\text{standard})$ is analogous. The coproduct of $\mathscr{E}^\circ_\lambda$ originates from (0.3.4)

(5.7.4)
$$x\mapsto\Psi(x\widehat{\otimes}_{\mathbb{Z}_p}1+1\widehat{\otimes}_{\mathbb{Z}_p}x)=\Phi(\Psi(x\widehat{\otimes}_{\mathbb{Z}_p}1),\Psi(px\widehat{\otimes}_{\mathbb{Z}_p}1),\dots;\Psi(1\widehat{\otimes}_{\mathbb{Z}_p}x),\Psi(1\widehat{\otimes}_{\mathbb{Z}_p}px),\dots)=$$
$$\Phi(\Psi(x)\widehat{\otimes}_{\mathbb{Z}_p}1,\Psi(px)\widehat{\otimes}_{\mathbb{Z}_p}1,\dots;1\widehat{\otimes}_{\mathbb{Z}_p}\Psi(x),\widehat{\otimes}_{\mathbb{Z}_p}\Psi(1px),\dots)$$

and the identification (1.9.1). The fact that $\mathscr{E}^\circ_\lambda$ only depends upon $|\lambda|$ follows from the fact that, for any $f\in\mathbb{C}\{x\}$, the map $\mathbb{Q}_p\to\mathbb{C}\{x\}$, $a\mapsto f(ax)$ is continuous. For any $n\in\mathbb{Z}$, the map $n\iota:\Psi(\lambda^{-1}p^jx)\mapsto\Psi(\lambda^{-1}p^jnx)$, for any $j=0,1,\dots$, is an endomorphism of $\mathscr{E}^\circ_\lambda$. By continuity, we obtain a map $a\iota:\mathscr{E}^\circ_\lambda\to\mathscr{E}^\circ_\lambda$, for any $a\in\mathbb{Z}_p$. If $m,n\in\mathbb{Z}$ are such that $mn=1+ap^N$, for $a\in\mathbb{Z}$ and $N\in\mathbb{Z}$, $N>>0$, $\Psi(\lambda^{-1}p^jmnx)$ is close to $\Psi(\lambda^{-1}p^jx)$. Again by continuity we find that if $a\in\mathbb{Z}_p^\times$, $a\iota$ is an automorphism of $\mathscr{E}^\circ_\lambda$. □

We finally prove our Uniform Approximation Theorem 1.25.

Proof. We discuss the integral case only; the bounded case follows directly. We first observe that a $\mathcal{CLM}^u_{\mathbb{Z}_p}$-morphism

$$(\mathcal{APH}_{0,\mathbb{Z}_p},\text{strip})=\varinjlim_{\rho\to 0}{}^u(\mathcal{APH}_{\mathbb{Z}_p}(\Sigma_\rho),\text{strip})\longrightarrow(AP_{\mathbb{Z}_p},w_\infty)$$

exists because so does, for any $\rho>0$, the morphism $(\mathcal{APH}_{\mathbb{Z}_p}(\Sigma_\rho),\text{strip})\to(AP_{\mathbb{Z}_p},w_\infty)$. Moreover, that morphism is injective. An element of $\mathcal{APH}_{0,\mathbb{Z}_p}$ is represented by a sequence

$P_{\rho_n} \in \mathbb{Z}_p[\Psi(x/\lambda) \,|\, \lambda \in \mathbb{Q}_p^\times\,]$ with ρ_n decreasing to 0, such that for any $\varepsilon > 0$, there exists N_ε such that for any $m \geq n \geq N_\varepsilon$,

$$||P_{\rho_n} - P_{\rho_m}||_{\mathbb{Q}_p,\rho_m} < \varepsilon \ .$$

Let $f \in AP_{\mathbb{Z}_p}$ and let $N \in \mathbb{Z}_{>0}$. By definition of u.a.p. functions, there exists a polynomial

$$P_N := \sum_{\lambda \in \mathbb{Q}_p^\times} a_\lambda \Psi(x/\lambda)$$

where $a_\lambda \in \mathbb{Z}_p = 0$ for almost all λ, such that

$$w_\infty(f - P_N) > N \ .$$

By (3.11.1) of Theorem 3.11, for any $N > 0$ there exists $\rho_N > 0$ such that $v_p(P_N(a + x) - P_N(a)) > N$, for any $a \in \mathbb{Q}_p$ and $x \in \mathbb{C}_p$, $|x| \leq \rho_N$. We may assume that the sequence $N \to \rho_N$ decreases to 0. We deduce that for $M \geq N$

$$-\log ||P_N - P_M||_{\mathbb{Q}_p,\rho_M} > N \ .$$

So, the sequence $N \mapsto P_N$ represents a germ $P \in \mathcal{APH}_{0,\mathbb{Z}_p}$ whose restriction to $\mathbb{Q}_p$ is f. $\square$

Remark 5.8. We are not asserting here that there should be a p-adic strip Σ_ρ around $\mathbb{Q}_p$ on which f extends analytically. In fact, an inductive limit in the category $\mathcal{CLM}^u_{\mathbb{Z}_p}$ is not necessarily supported by a set-theoretic inductive limit (see section 6.1 of Appendix A below) and similarly for a locally convex inductive limit of Banach spaces.

6 Appendix A. Non archimedean topological algebra

A prime number p is fixed throughout this paper and $q = p^f$ is a power of p. So, $\mathbb{Q}_q$ will denote the unramified extension of $\mathbb{Q}_p$ of degree f, and $\mathbb{Z}_q$ will be its ring of integers. Unless otherwise specified, a *ring* is meant to be commutative with 1.

6.1 Linear topologies

Let k be a separated and complete linearly topologized ring; we will denote by $\mathcal{P}(k)$ the family of open ideals of k. We will consider the category $\mathcal{CLM}^u_k$ of separated and complete linearly topologized k-modules M such that the map multiplication by scalars

$$k \times M \longrightarrow M \ \ , \ \ (r,m) \longmapsto rm$$

is *uniformly continuous* for the product uniformity of $k \times M$. Morphisms of $\mathcal{CLM}^u_k$ are continuous k-linear maps. This is the classical category of [10, Chap. III, §2]. See [4] for more detail.

Remark 6.1. All over this paper we will assume that in a topological ring R (resp. topological R-module M), the product (resp. the scalar product) map $R \times R \to R$ (resp. $R \times M \to M$) is at least continuous for the product topology of $R \times R$ (resp. of $R \times M$); morphisms will be continuous morphisms of rings (resp. of R-modules).

By a *non-archimedean* (*n.a.*) ring R (resp. R-module M) we mean a topological ring R (resp. R-module M) equipped with a topology for which a basis of neighborhoods of 0 consists of additive subgroups and additive translations are homeomorphisms. So, any valued non-archimedean field K is a n.a. ring in the previous sense and, if K is non-trivially valued, the category $\mathcal{LC}_K$ of locally convex K-vector spaces [18] is a full subcategory of the category of n.a. K-modules. But, such a field K is never a linearly topologized ring. The ring of integers K° is indeed linearly topologized, but no non-zero object of $\mathcal{LC}_K$ is an object of $\mathcal{CLM}^u_{K^\circ}$.

Definition 6.2. *Let R be a topological ring and M be a topological R-module. A closed topological R-submodule N of M is said to be* strictly *closed if it is endowed with the subspace topology of M.*

For any object M of $\mathcal{CLM}_k^u$, $\mathcal{P}(M)$ will denote the family of open k-submodules of M. The category $\mathcal{CLM}_k^u$ admits all limits and colimits. The former are calculated in the category of k-modules but not the latter. So, a limit will be denoted by $\varprojlim$ while a colimit will carry an apex $(-)^u$ as in $\varinjlim^u$. In particular, for any family M_α, $\alpha \in A$, of objects of $\mathcal{CLM}_k^u$, the direct sum and direct product will be denoted by

$$\bigoplus_{\alpha\in A}^u M_\alpha \ , \ \prod_{\alpha\in A} M_\alpha \ ,$$

respectively. We explicitly notice that $\bigoplus_{\alpha\in A}^u M_\alpha$ is the completion of the algebraic direct sum $\bigoplus_{\alpha\in A} M_\alpha$ of the algebraic k-modules M_α's, equipped with the k-linear topology for which a fundamental system of open k-modules consists of the k-submodules

$$\bigoplus_{\alpha\in A}(U_\alpha + IM_\alpha) \text{ such that } U_\alpha \in \mathcal{P}(M_\alpha)\ \forall\alpha \ , \text{ and } I \in \mathcal{P}(k) \text{ is independent of } \alpha \ .$$

Then the k-module underlying $\bigoplus_{\alpha\in A}^u M_\alpha$ in general properly contains the algebraic direct sum $\bigoplus_{\alpha\in A} M_\alpha$. It will also be useful to introduce the *uniform box product* of the same family

$$\prod_{\alpha\in A}^{\square,u} M_\alpha \tag{6.2.1}$$

which, set-theoretically, coincides with $\prod_{\alpha\in A} M_\alpha$ but whose family of open submodules consists of all $U := \prod_{\alpha\in A} U_\alpha$, with $U_\alpha \in \mathcal{P}(M_\alpha)$, such that there exists $I_U \in \mathcal{P}(k)$ such that $I_U M_\alpha \subset U_\alpha$, for any $\alpha \in A$. The category $\mathcal{CLM}_k^u$, equipped with the tensor product $\widehat{\otimes}_k^u$ of [14, **0**.7.7] (see also [10, Chap. III, §2, Exer. 28]) is a symmetric monoidal category. The category of monoids of $\mathcal{CLM}_k^u$ is denoted by $\mathcal{ACLM}_k^u$.

For two objects M and N of $\mathcal{CLM}_k^u$, we have

$$M\widehat{\otimes}_k^u N = \varprojlim_{P\in\mathcal{P}(M),Q\in\mathcal{P}(N)} M/P \otimes_k N/Q \tag{6.2.2}$$

so that $\widehat{\otimes}_k^u$ commutes with filtered projective limits in $\mathcal{CLM}_k^u$.

6.2 Semivaluations

We describe here full subcategories of $\mathcal{CLM}_k^u$, and special base rings k, of most common use. We denote by $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$, the localization of $\mathbb{Z}$ at (p). Then $\mathbb{C}_p$ will be the completion of a fixed algebraic closure of $\mathbb{Q}_p$. On $\mathbb{C}_p$ we use the absolute value $|x| = |x|_p = p^{-v_p(x)}$, for the p-adic valuation $v = v_p$, with $v_p(p) = 1$, and $x \in \mathbb{C}_p$.

Definition 6.3. *A* semivaluation *on a ring R is a map $w : R \to \mathbb{R} \cup \{+\infty\}$ such that $w(0) = +\infty$, $w(x + y) \geq \min(w(x), w(y))$ and $w(xy) \geq w(x) + w(y)$, for any $x, y \in R$. We will say that a semivaluation is* positive *if it takes its values in $\mathbb{R}_{\geq 0} \cup \{+\infty\}$.*

Remark 6.4.
1. If $w_1, w_2, \dots, w_n$ are a finite set of semivaluations on the ring R, so is their infimum
$$w := \inf_{i=1,\dots,n} w_i \ .$$

2. The *trivial valuation* $v_0 : R \to \{0, +\infty\}$, which exists on any ring R, is (in our sense!) a positive semivaluation.

3. We will indifferently use the multiplicative notation $|x|_w = \exp(-w(x))$.

For any semivaluation w of a ring R, the family of

$$R_{w,c} := \{x \in R \,|\, w(x) \geq c\} \tag{6.4.1}$$

for $c \in \mathbb{R}$ is a fundamental set of open subgroups for a group topology of R. Moreover, $R_{w,0}$ is a subring of R and all $R_{w,c}$ are $R_{w,0}$-submodules of R. A (*multi-*) *semivalued* ring $(R, \{w_\alpha\}_{\alpha\in A})$ is a ring R equipped with a family $\{w_\alpha\}_{\alpha\in A}$ of semivaluations. A semivalued ring is endowed with the topology in which any $x \in R$ has a fundamental system of neighborhoods consisting of the subsets

$$x + \bigcap_{\alpha\in F} R_{\alpha,c_\alpha}$$

where F varies among finite subsets of A and, for any $\alpha \in F$, $c_\alpha \in \mathbb{R}$. A *Fréchet ring* (resp. *Banach ring*) is a ring R which is separated and complete in the topology induced by a countable family of semivaluations (resp. by a single semivaluation). If the semivaluations w_α are all positive, the Fréchet (resp. Banach) ring $(R, \{w_\alpha\}_{\alpha\in A})$ is linearly topologized. We will call it a *linearly topologized Fréchet* (resp. *Banach*) *ring*. When R is an algebra over a Banach ring (S, v), and the semivaluations w_α satisfy

$$w_\alpha(xy) = v(x) + w_\alpha(y) \;\; \forall\, x \in S \;,\; y \in R \;,$$

we also say that $R = (R, \{w_\alpha\}_{\alpha\in A})$ is a *Fréchet* (resp. *Banach*) *S-algebra*. In the particular case when (S, v) is a complete non-trivially valued real-valued field (K, v) a Fréchet or Banach S-algebra is a Fréchet or Banach algebra over K in the classical sense. Notice however that we allow v to be the trivial valuation of S or K. We denote by $\mathcal{CLC}_K$ the category of locally convex topological K-vector spaces of [18], where morphisms are continuous K-linear maps, which are moreover separated and complete.

We have the easy

Lemma 6.5. *Let (S, v) be a Banach ring and $(R, \{w_n\}_{n=1,2,\dots})$ be a Fréchet S-algebra. Let (R_n, w_n) be the separated completion of R in the locally convex topology induced by the semivaluation w_n. Assume $w_n(r) \geq w_m(r)$ for any $r \in R$ if $n \leq m$. Then, the identity of R extends to a morphism $R_m \to R_n$ of Banach S-algebras and R is the limit, in the category of n.a. S-algebras, of the filtered projective system $(R_n)_n$.*

In particular, a S-subalgebra T of R is dense in R if and only if it is dense in R_n, for any n.

6.3 Tensor products

Let (S, v) be a complete real-valued ring and let $R = (R, \{w_\alpha\}_{\alpha\in A})$ and $R' = (R', \{w'_\beta\}_{\beta\in B})$ be two Fréchet S-algebras. Then we define a Fréchet S-algebra $R\widehat{\otimes}_{\pi,S}R'$ as the completion of the S-algebra $R \otimes_S R'$ in the topology induced by the following semivaluations [9, 2.1.7], for any $(\alpha, \beta) \in A \times B$,

$$w_{\alpha,\beta}(g) = \sup\left(\min_{1\leq i\leq n} w_\alpha(x_i) + w'_\beta(y_i)\right) \;,$$

where the supremum runs over all possible representations

$$g = \sum_{i=1}^{n} x_i \otimes y_i \;\;,\;\; x_i \in R \;,\; y_i \in R' \;.$$

The following proposition follows immediately from Lemma 6.5.

Proposition 6.6. *Let (S,v) be a Banach ring and $(R,\{w_n\}_{n=1,2,\dots})$, $(R',\{w'_n\}_{n=1,2,\dots})$ be two Fréchet S-algebra satisfying the assumption of Lemma 6.5. Then, with the same notation, $R\widehat{\otimes}_{\pi,S}R'$ is the limit, in the category of n.a. S-algebras, of the filtered projective system of Banach S-algebras $(R_n\widehat{\otimes}_{\pi,S}R'_n)_n$.*

Notice that

1. if R and R' are Fréchet algebras over a complete real-valued field (K,v), with non-trivial valuation v, $R\widehat{\otimes}_{\pi,K}R'$ coincides with both the completed projective and the inductive tensor product of [18] (*cf.* Lemma 17.2 and Lemma 17.6 of *loc.cit.*);

2. if R and R' are linearly topologized Fréchet algebras over a linearly topologized Banach ring (S,v), $R\widehat{\otimes}_{\pi,S}R'$ coincides with $R\widehat{\otimes}^u_S R'$.

7 Appendix B. Classical theory of almost periodic functions

The main character of this paper, our function Ψ, shows many analogies with the classical holomorphic almost periodic functions of Bohr, Bochner, and Besicovitch [7]. In fact many of the subtle function theoretic difficulties which appear in the p-adic setting are also encountered in classical Harmonic Analysis. We feel that a short presentation of the basics of the classical theory might be useful. See also the survey article [12].

7.1 Fejér's Theorem

Let $(\mathscr{C}^{\rm bd}_{\rm unif}(\mathbb{R},\mathbb{R}), ||\ ||_{\mathbb{R}})$ be the Banach algebra of bounded uniformly continuous functions $\mathbb{R}\to\mathbb{R}$, equipped with the supnorm on $\mathbb{R}$. For $\lambda\in\mathbb{R}_{>0}$ let $\mathscr{P}_{\mathbb{R},\lambda}\subset\mathscr{C}^{\rm bd}_{\rm unif}(\mathbb{R},\mathbb{R})$ be the strictly closed Banach subalgebra of continuous functions periodic of period λ.

Let us recall the classical *Fejér's Theorem* [20, §13.31]. Let $f\in\mathscr{P}_{\mathbb{R},\lambda}$. The *Fourier expansion of* f is the formal trigonometric series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(\frac{2\pi n}{\lambda}z)+b_n\sin(\frac{2\pi n}{\lambda}z)\ ,$$

with

$$a_n=\frac{2}{\lambda}\int_0^{\lambda}f(t)\cos(\frac{2\pi n}{\lambda}z)dt\ \ ,\ \ b_n=\frac{2}{\lambda}\int_0^{\lambda}f(t)\sin(\frac{2\pi n}{\lambda}z)dt\ .$$

The sequence of the partial sums

$$S_N(f)=\frac{a_0}{2}+\sum_{n=1}^{N}a_n\cos(\frac{2\pi n}{\lambda}z)+b_n\sin(\frac{2\pi n}{\lambda}z)\ ,$$

does not necessarily converge to f uniformly on $\mathbb{R}$. However, the Cesaro means

$$\sigma_n=\frac{S_0+\cdots+S_{n-1}}{n}$$

converge to f uniformly on $\mathbb{R}$. In particular,

Theorem 7.1. $\mathbb{R}[\cos(\frac{2\pi}{\lambda}x),\sin(\frac{2\pi}{\lambda}x)]$ *is dense in the* $\mathbb{R}$*-Banach algebra* $(\mathscr{P}_{\mathbb{R},\lambda}, ||\ ||_{\mathbb{R}})$.

We will show below that a suitably reformulated p-adic analog of Theorem 7.1 holds true p-adically.

Definition 7.2. *(Bohr's definition of u.a.p. functions) A continuous function $f : \mathbb{R} \to \mathbb{R}$ is* uniformly almost periodic *(*u.a.p. *for short) if, for any $\varepsilon > 0$, there exists $\ell_\varepsilon > 0$ such that for any interval $I \subset \mathbb{R}$ of length ℓ_ε there exists $\tau \in I$ such that*

$$|f(x+\tau) - f(x)| < \varepsilon \ , \ \ \forall x \in \mathbb{R} \ .$$

It is easy to check that the set of uniformly almost periodic functions $\mathbb{R} \to \mathbb{R}$ is a closed subalgebra $AP_{\mathbb{R}}$ of $(\mathscr{C}^{\text{bd}}_{\text{unif}}(\mathbb{R},\mathbb{R}), || \ ||_{\mathbb{R}})$ [7, Chap. I, §1, Thms 4°,5°]. We define $AP_{\mathbb{C}} \subset (\mathscr{C}^{\text{bd}}_{\text{unif}}(\mathbb{R},\mathbb{C}), || \ ||_{\mathbb{R}})$ similarly.

The following result is Bohr's "Approximation Theorem". We refer to [7, I.5] for its proof and for a detailed description of the contributions of S. Bochner and H. Weyl.

Theorem 7.3. $(AP_{\mathbb{R}}, || \ ||_{\mathbb{R}})$ *identifies with the completion of the normed ring*

$$(\mathbb{R}[\cos(\frac{2\pi}{\lambda}x), \sin(\frac{2\pi}{\lambda}x) \,|\, \lambda \in \mathbb{R}^\times], || \ ||_{\mathbb{R}}) \ .$$

Similarly for $(AP_{\mathbb{C}}, || \ ||_{\mathbb{R}})$.

We propose p-adic analogs of those Banach algebras and of the latter theorem.

7.2 Dirichlet series

Let $\mathbb{C}\{x\}$ be the Fréchet $\mathbb{C}$-algebra of entire functions $\mathbb{C} \to \mathbb{C}$, equipped with the topology of uniform convergence on compact subsets of $\mathbb{C}$. The rotation $z \mapsto iz$ transforms trigonometric series into series of exponentials and Bohr's definition naturally propagates into the following

Definition 7.4. [7, III.2,1°]. *For any interval $(a, b) \subset \mathbb{R}$, an analytic function f on the* strip $(a, b) \times i\mathbb{R} \subset \mathbb{C}$ *is* almost periodic holomorphic *on (a, b) if, for any $\varepsilon > 0$, there exists $\ell_\varepsilon > 0$ such that for any interval $I \subset \mathbb{R}$ of length ℓ_ε there exists $\tau \in I$ such that*

$$|f(x+i\tau) - f(x)| < \varepsilon \ , \ \ \forall x \in (a,b) \times i\mathbb{R} \ .$$

We let $\mathcal{APH}_{\mathbb{C}}((a,b))$ denote the $\mathbb{C}$-algebra of almost periodic holomorphic functions on (a,b).

Notice that $\mathcal{APH}_{\mathbb{C}}((a,b))$ is a closed subalgebra of the Fréchet algebra $\mathcal{O}((a,b) \times i\mathbb{R})$; the induced Fréchet algebra structure is called *standard*. We may equip $\mathcal{APH}_{\mathbb{C}}((a,b))$ with the finer Fréchet algebra structure of uniform convergence on substrips $(a', b') \times i\mathbb{R}$, for $a < a' < b' < b$. We informally call this topology *the strip topology*.

The following Polynomial Approximation Theorem [7, III.3,3°] holds.

Theorem 7.5. *$\mathcal{APH}_{\mathbb{C}}((a,b))$ is the Fréchet completion of the $\mathbb{C}$-polynomial algebra generated by the restrictions to $(a,b) \times i\mathbb{R}$ of all continuous characters of $\mathbb{R}$, namely by the maps*

$$e_\lambda : (a,b) \times i\mathbb{R} \longrightarrow \mathbb{C} \ , \ \ z \longmapsto e^{\lambda z} \tag{7.5.1}$$

for $\lambda \in \mathbb{R}^\times$, equipped with the strip topology.

The assignment $(a,b) \mapsto \mathcal{APH}_{\mathbb{C}}((a,b))$ uniquely extends to a sheaf of Fréchet $\mathbb{C}$-algebras on $\mathbb{R}$.

Definition 7.6.

1. *We denote by $\mathcal{APH}_{0,\mathbb{C}}$ the stalk of the sheaf $\mathcal{APH}_{\mathbb{C}}$ at 0 equipped with the locally convex inductive limit topology of the system of Fréchet algebras $\mathcal{APH}_{\mathbb{C}}((-\varepsilon,\varepsilon))$ as $\varepsilon \to 0^+$.*

2. *We denote by* $APH_{\mathbb{C}} \subset \mathbb{C}\{x\}$ *the Fréchet algebra of global sections of* $\mathcal{APH}_{\mathbb{C}}$ *equipped with the strip topology.*

Notice that we have a natural injective morphism, induced by restriction of functions and the properties of the inductive limit

$$APH_{\mathbb{C}} \longrightarrow \mathcal{APH}_{0,\mathbb{C}} . \tag{7.6.1}$$

It follows from the combined theorems of approximation Theorem 7.3 and Theorem 7.5 that

Corollary 7.7. $(AP_{\mathbb{C}}, || \, ||_{\mathbb{R}})$ *identifies with the completion of the normed ring* $(\mathcal{APH}_{0,\mathbb{C}}, || \, ||_{\mathbb{R}})$.

Remark 7.8. Sections of the sheaf $\mathcal{APH}_{\mathbb{C}}$ on open subsets of $\mathbb{R}$ may be viewed as generalized Dirichlet series [7, III.3]. A p-adic analog on $\mathbb{Q}_p$ of the sheaf $\mathcal{APH}_{\mathbb{C}}$ of Dirichlet series on $\mathbb{R}$, might be useful in the theory of p-adic L-functions.

8 Appendix C: Numerical Calculations by M. Candilera

The following calculations were performed with *Mathematica*©. We computed the first coefficients of the series $\Psi_p(T) = \sum_{n=1}^{\infty} b_n T^n$, for $p = 2$, up to the term of degree 2^5, and for $p = 3$, up to degree 3^4. We also evaluated the a few coefficients of $\Psi_5(T)$ and $\Psi_7(T)$. We give here tables of the p-adic orders of the coefficients b_n for $p = 2, 3$. For those values of p, we also draw the graph of the function $n \mapsto v_p(b_n)$ and compare it with the Newton polygon of Ψ_p (flipped around the y-axis). We confirm experimentally the calculation of the corresponding valuation polygons.

8.1 Very first coefficients

1. $p = 2$

$$\Psi_2(T) = T - 2 \cdot T^2 + 2^4 \cdot T^3 - 11 \cdot 2^5 \cdot T^4 + 7 \cdot 2^{11} \cdot T^5 - 7 \cdot 37 \cdot 2^{12} \cdot T^6 + \\ 3 \cdot 751 \cdot 2^{16} \cdot T^7 - 301627 \cdot 2^{17} \cdot T^8 + 308621 \cdot 2^{26} \cdot T^9 + 2^{27} \cdot T^{10} \cdot u(T) ,$$

for a unit $u(T) \in \mathbb{Z}_{(2)}[[T]]^{\times}$.

2. $p = 3$

$$\Psi_3(T) = T - 3^2 \cdot T^3 + 3^7 \cdot T^5 - 2^2 \cdot 7 \cdot 3^{11} \cdot T^7 + \\ 2 \cdot 7 \cdot 13 \cdot 113 \cdot 3^{14} \cdot T^9 - 5 \cdot 89 \cdot 1249 \cdot 3^{22} \cdot T^{11} + 5 \cdot 117 \cdot 217667 \cdot 3^{28} \cdot T^{13} + \dots .$$

3. $p = 5$

$$\Psi_5(T) = T - 5^4 \cdot T^5 + 5^{13} \cdot T^9 - 53 \cdot 59 \cdot 5^{21} \cdot T^{13} + \\ 3 \cdot 11 \cdot 97 \cdot 1123 \cdot 1699 \cdot 5^{29} \cdot T^{17} + 5^{37} \cdot T^{21} \cdot u(T) ,$$

for a unit $u(T) \in \mathbb{Z}_{(5)}[[T]]^{\times}$.

4. $p = 7$

$$\Psi_7(T) = T - 7^6 \cdot T^7 + 7^{19} \cdot T^{13} - 2 \cdot 31 \cdot 37 \cdot 359 \cdot 7^{31} \cdot T^{19} + 7^{43} \cdot T^{25} \cdot u(T) ,$$

for a unit $u(T) \in \mathbb{Z}_{(7)}[[T]]^{\times}$.

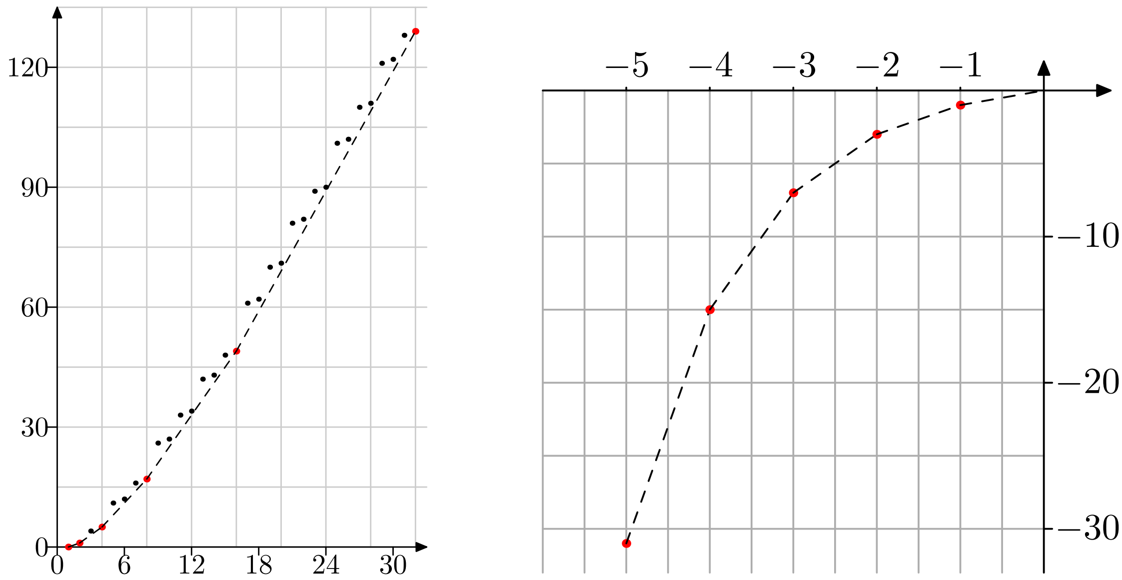

Figure 5: Newton and valuation polygons of Ψ_2.

8.2 First 24 coefficients of $\Psi_2(t)$ and 2-adic order of the 32 first

$\Psi_2(t) = \sum_{n\geq 1} b_n t^n$							
b_1	0	b_9	26	b_{17}	61	b_{25}	101
b_2	1	b_{10}	27	b_{18}	62	b_{26}	102
b_3	4	b_{11}	33	b_{19}	70	b_{27}	110
b_4	5	b_{12}	34	b_{20}	71	b_{28}	111
b_5	11	b_{13}	42	b_{21}	81	b_{29}	121
b_6	12	b_{14}	43	b_{22}	82	b_{30}	122
b_7	16	b_{15}	48	b_{23}	89	b_{31}	128
b_8	17	b_{16}	49	b_{24}	90	b_{32}	129

2-adic valuation of the coefficients of Ψ_2

$$b_1 = 1, \qquad b_2 = -2, \qquad b_3 = 16 = 2^4, \qquad b_4 = -352 = -2^5 \cdot 11$$
$$b_5 = 14336 = 2^{11} \cdot 7, \qquad b_6 = -1060864 = -2^{12} \cdot 7 \cdot 37$$
$$b_7 = 147652608 = 2^{16} \cdot 3 \cdot 751$$
$$b_8 = -39534854144 = -2^{17} \cdot 301627$$
$$b_9 = 20711204716544 = 2^{26} \cdot 308621$$
$$b_{10} = -21454855889485824 = -2^{27} \cdot 3^2 \cdot 13 \cdot 701 \cdot 1949$$
$$b_{11} = 44195700516541431808 = 2^{33} \cdot 5145056699$$
$$b_{12} = -181554407879323198423040 = -2^{34} \cdot 5 \cdot 41 \cdot 2273 \cdot 22679509$$
$$b_{13} = 1489469015852141109009448960 = 2^{42} \cdot 5 \cdot 67733208918623$$
$$b_{14} = -24421319844213105128638664146944 = -2^{43} \cdot 3^2 \cdot 8179 \cdot 37716952983613$$
$$b_{15} = 800530746908074643997623203521363968 = 2^{48} \cdot 31 \cdot 71 \cdot 1619 \cdot 826201 \cdot 966018887$$
$$b_{16} = -52473187457503996327647036404796036743168 = -2^{49} \cdot 31 \cdot 397 \cdot 13687 \cdot 2882489 \cdot 191972726039$$
$$b_{17} = 6878395240848057051122842718175351390427152384 = 2^{61} \cdot 3 \cdot 47 \cdot 59 \cdot 919 \cdot 24709 \cdot 15791216459521333$$
$$b_{18} = -1803212578568825704559863338710346864852507172012032$$
$$= 2^{62} \cdot 3^2 \cdot 19 \cdot 97 \cdot 173 \cdot 1665967 \cdot 581220517 \cdot 140723269997$$
$$b_{19} = 945424354393817092018179744741353462710753588534117924864$$
$$= 2^{70} \cdot 7^2 \cdot 23 \cdot 15973 \cdot 444853161598056649565155547941$$
$$b_{20} = -99136063278090630156034333062512951079052848307348004744986624 0$$
$$= 2^{71} \cdot 5 \cdot 167 \cdot 14503 \cdot 15445577653440901 \cdot 2244675152281633901$$
$$b_{21} = 20790458300097182146184722972326553790898170223680045176608240969973 76$$
$$= 2^{81} \cdot 109 \cdot 23549 \cdot 167442376921 \cdot 2000645152343730624200879183$$
$$b_{22} = -87201751894637405809634230575350327112612363715202067195519050312690507448 32$$
$$= 2^{82} \cdot 47 \cdot 1867 \cdot 105323 \cdot 2119591 \cdot 80618233393589 \cdot 1141865166972250409671$$
$$b_{23} = 7315023599767300841126449508348690416475855656347719558637044167637642838414458880 0$$
$$= 2^{89} \cdot 3^3 \cdot 5^2 \cdot 1750823409171113848483762658178098326058168873528317 73$$
$$b_{24} = -12272581875860699355095303554739880208834821578534282764441467365212110770018460456640839 68$$
$$= 2^{90} \cdot 54617 \cdot 76121647308197 \cdot 2384516372879688407263396723504276999519442 93$$

8.3 3-adic values of the first 81 coefficients of $\Psi_3(T)$

b_3	2	b_{23}	58	b_{43}	135	b_{63}	213
b_5	7	b_{25}	64	b_{45}	141	b_{65}	223
b_7	11	b_{27}	68	b_{47}	151	b_{67}	231
b_9	14	b_{29}	79	b_{49}	159	b_{69}	238
b_{11}	22	b_{31}	87	b_{51}	166	b_{71}	247
b_{13}	28	b_{33}	94	b_{53}	175	b_{73}	255
b_{15}	33	b_{35}	103	b_{55}	183	b_{75}	262
b_{17}	40	b_{37}	111	b_{57}	190	b_{77}	271
b_{19}	46	b_{39}	118	b_{59}	199	b_{79}	279
b_{21}	51	b_{41}	127	b_{61}	207	b_{81}	284

3-adic valuation of the coefficients of Ψ_3

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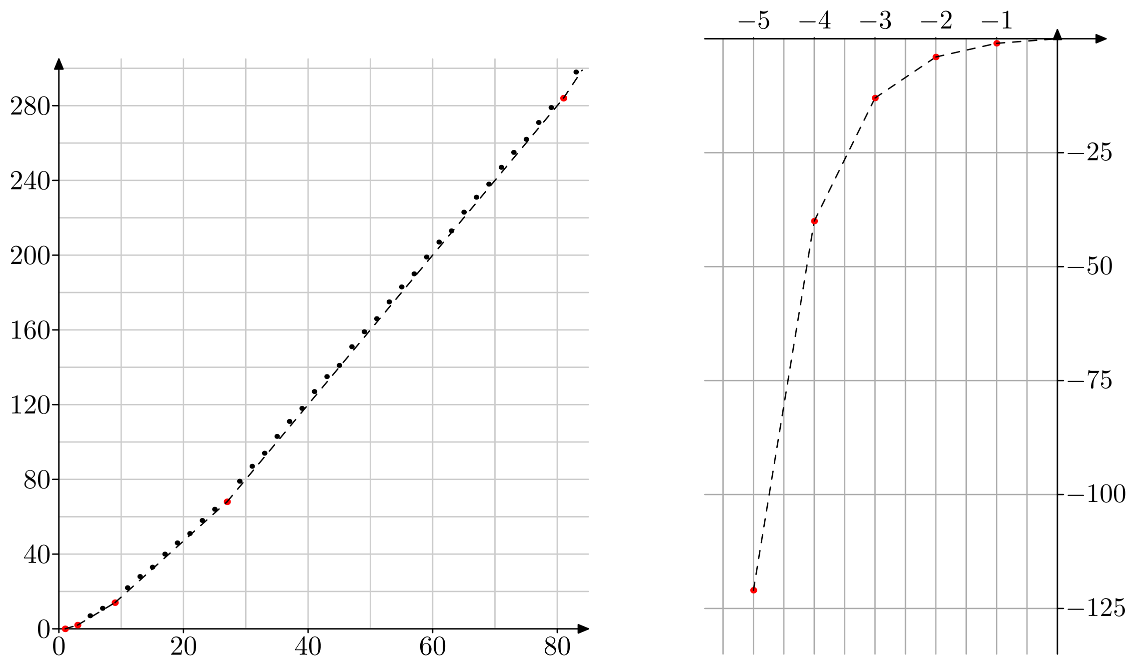

Figure 6: The Newton and valuation polygons of Ψ_3.

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