

INTEGRAL EXPRESSIONS FOR DERIVATIONS OF MULTIARRANGEMENTS

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ABSTRACT. The construction of an explicit basis for a free multiarrangement is not easy in general. Inspired by the integral expressions for quasi-invariants of quantum Calogero-Moser systems, we present integral expressions for specific bases of certain multiarrangements. Our construction covers the cases of three lines in dimension 2 (previously examined by Wakamiko) and free multiarrangements associated with complex reflection groups (Hoge, Mano, Röhrle, Stump). Furthermore, we propose a conjectural basis for the module of logarithmic vector fields of the extended Catalan arrangement of type B_2 .

1. INTRODUCTION

Let V be an ℓ -dimensional linear space over \mathbb{C} . Let $\{x_1, \dots, x_\ell\}$ be a basis of V^* and let $S = S(V^*) = \mathbb{C}[x_1, \dots, x_\ell]$ be the polynomial ring. Define $\text{Der}_S = \bigoplus_{i=1}^{\ell} S\partial_i$ as the module of polynomial vector fields (\mathbb{C} -linear derivations of S), where $\partial_i = \frac{\partial}{\partial x_i}$.

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement of hyperplanes. The pair (\mathcal{A}, m) consisting of \mathcal{A} and a map $m : \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0}$ is called a multiarrangement. For each $H \in \mathcal{A}$, choose a linear form $\alpha_H \in V^*$ such that $H = \text{Ker}(\alpha_H)$. The polynomial

$$Q(\mathcal{A}, m) = \prod_{H \in \mathcal{A}} \alpha_H^{m(H)}$$

is the defining equation of the multiarrangement (\mathcal{A}, m) .

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For a multiarrangement (\mathcal{A}, m) , define the module of multiderivations $D(\mathcal{A}, m)$ by

$$D(\mathcal{A}, m) := \{\delta \in \text{Der}_S \mid \delta\alpha_H \in (\alpha_H^{m(H)}), \text{ for any } H \in \mathcal{A}\}. \quad (1.1)$$

The multiarrangement (\mathcal{A}, m) is said to be free with exponents (e_1, \dots, e_ℓ) if $D(\mathcal{A}, m)$ is a free S -module with a basis $\{\delta_1, \dots, \delta_\ell\} \subset D(\mathcal{A}, m)$ of the form $\delta_i = \sum_{j=1}^{\ell} f_{ij} \partial_j$, where each non-zero f_{ij} is a homogeneous polynomial of $\deg f_{ij} = e_i$. The module $D(\mathcal{A}, m)$ was first introduced by K. Saito in [13] for $m \equiv 1$ and by Ziegler [26] for general m . $D(\mathcal{A}, 1)$ is simply denoted by $D(\mathcal{A})$. We also define $\deg \delta_i = e_i$.

We will frequently use the following criterion for the freeness of $D(\mathcal{A}, m)$.

Proposition 1.1. (*Saito-Ziegler criterion for freeness*) *Let (\mathcal{A}, m) be a multiarrangement. Suppose that there exist homogeneous vector fields $\delta_1, \dots, \delta_\ell \in D(\mathcal{A}, m)$ such that*

- $\delta_1, \dots, \delta_\ell$ are linearly independent over S , and
- $\sum_{i=1}^{\ell} \deg \delta_i = \sum_{H \in \mathcal{A}} m(H)$.

Then $D(\mathcal{A}, m)$ is a free S -module with a basis $\{\delta_1, \dots, \delta_\ell\}$.

The module $D(\mathcal{A})$ plays a crucial role in the study of hyperplane arrangements [12]. The algebraic structures of these modules are thought to reflect the combinatorial structures of \mathcal{A} . For example, when $D(\mathcal{A})$ is a free module, Tearo's factorization theorem [19] asserts that the characteristic polynomial of \mathcal{A} factors into the product of linear terms over \mathbb{Z} . The module $D(\mathcal{A}, m)$ of multiderivations is also important for characterizing the freeness of $D(\mathcal{A})$ [24, 25].

Multiderivations for Coxeter arrangements have various relations with other research topics, e.g., flat structures [14, 15, 16], Frobenius manifolds [7]. In recent years, the connection between such multiderivations and quantum integrable systems has become apparent [1], which is linked to significant advancements in [3, 10]. Indeed, it has been observed that (the invariant part of) the module $D(\mathcal{A}, m)$ is identified with a space of quasi-invariants for the quantum generalized Calogero-Moser system [6, 8]. The relationship between these two fields is expected to enhance our understanding of both research topics.

Despite numerous studies, it is still difficult to construct an explicit basis for the module $D(\mathcal{A}, m)$ of multiderivations, even in simple cases, e.g., 2-dimensional arrangements [21, 23].

The purpose of this paper is to investigate integral expressions for the basis of $D(\mathcal{A}, m)$, inspired by the integral expressions of quasi-invariants [4, 6, 8, 9]. The paper is organized as follows. In §2, we examine the basis for the multiarrangement on three lines, specifically that of the form

$x_1^p x_2^q (x_1 - x_2)^r$. The freeness of $D(\mathcal{A}, m)$ and the exponents are already known [26, 23], as well as an existing basis [22]. However, in this work, we present a novel and convenient expression for the basis using integrals. Additionally, we discuss a higher dimensional analogue.

In §3, we discuss multiarrangements associated with complex reflection groups. In dimension ≥ 3 , the freeness of $D(\mathcal{A}, m)$ depends on the multiplicity m . In §3.1, we recall two families of free multiarrangements (\mathcal{A}, μ) and (\mathcal{A}, μ') , discovered by Hoge, Mano, Röhrle, and Stump [10]. Subsequently, in §3.2 and §3.3, we provide explicit integral expressions for the basis. In §3.4, we consider special cases, including constant multiplicities for type B_ℓ and D_ℓ arrangements.

In §4, we discuss the relationship between integral expressions and K. Saito's primitive derivation. Saito first introduced the primitive derivation and the Hodge filtration on the module of logarithmic derivations in the context of singularity theory [14]. Later, Saito described these structures in terms of invariant theory for Coxeter arrangements [15, 16]. Recently, these structures have been generalized to well-generated complex reflection groups [10] based on the study of the Okubo system of linear differential equations [11]. We observe that our integral expressions are consistent with the primitive derivation and the Hodge filtration.

In §5, we propose a conjectural basis for the extended Catalan arrangements of type B_2 .

2. INTEGRAL EXPRESSIONS FOR CERTAIN MULTIARRANGEMENTS

2.1. Three lines. In this section, we consider the multiarrangement (\mathcal{A}, m) defined by $x_1^p x_2^q (x_1 - x_2)^r$ in \mathbb{R}^2 (Figure 1). Without loss of generality, we

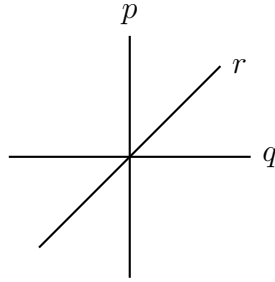


FIGURE 1. The multiarrangement $x_1^p x_2^q (x_1 - x_2)^r$

may assume

$$\max\{p, q\} \leq r. \quad (2.1)$$

If $r \geq p + q - 1$, then it is easily seen that the vector fields

$$x_1^p x_2^q (\partial_1 + \partial_2), \text{ and} \\ \left(\sum_{i=p}^r \binom{r}{i} (-x_1)^i x_2^{r-i} \right) \partial_1 - \left(\sum_{i=0}^{p-1} \binom{r}{i} (-x_1)^i x_2^{r-i} \right) \partial_2 \quad (2.2)$$

form a basis of $D(\mathcal{A}, m)$. When $r < p + q - 1$, the construction of the basis is more involved. Wakamiko [21, Theorem 1.5] constructed a basis using the generalized binomial coefficients. Here, we construct a basis with an integral expression.

Definition 2.1. Let $a, b, c \in \mathbb{Z}_{\geq 0}$. Define $\theta_{a,b,c} \in \text{Der}_S$ to be

$$\theta_{a,b,c} = \left(\int^{x_1} t^c (t - x_1)^b (t - x_2)^a dt \right) \partial_1 + \left(\int^{x_2} t^c (t - x_1)^b (t - x_2)^a dt \right) \partial_2,$$

where $\int^{x_i} dt$ is the linear operator defined by $\int^{x_i} t^k dt = \frac{x_i^{k+1}}{k+1}$ for $k \neq -1$. When $k \geq 0$, it is equivalent to the definite integral $\int_0^{x_i} dt$. Later we will also need the same operator for $k < -1$. In this case, $\int^{x_i} dt$ is equivalent to $\int_{-\infty}^{x_i} dt$.

The following are the basic properties of $\theta_{a,b,c}$.

Proposition 2.2. *The degree of $\theta_{a,b,c}$ is $\deg \theta_{a,b,c} = a + b + c + 1$ and $\theta_{a,b,c}$ is contained in $D(\mathcal{A}, m)$ for $Q(\mathcal{A}, m) = x_1^{b+c+1} x_2^{a+c+1} (x_1 - x_2)^{a+b+1}$.*

Proof. The claim $\deg \theta_{a,b,c} = a + b + c + 1$ is clear. Let us compute the order of vanishing of $\theta_{a,b,c}(x_1 - x_2)$ at $x_1 = x_2$. Let $t = x_1 + s$. Then

$$\begin{aligned} \theta_{a,b,c}(x_2 - x_1) &= \int_{x_1}^{x_2} t^c (t - x_1)^b (t - x_2)^a dt \\ &= \int_0^{x_2 - x_1} (x_1 + s)^c s^b (s - (x_2 - x_1))^a ds \end{aligned}$$

which is divisible by $(x_1 - x_2)^{a+b+1}$. Other assertions are proved similarly. \square

Theorem 2.3. *Let (\mathcal{A}, m) be the multiarrangement defined by $x_1^p x_2^q (x_1 - x_2)^r$ with $p, q > 0$ and $r \leq p + q - 1$.*

(1) *Suppose $p + q + r$ is odd. Let*

$$(a, b, c) = \left(\frac{-p + q + r - 1}{2}, \frac{p - q + r - 1}{2}, \frac{p + q - r - 1}{2} \right).$$

Then $\theta_{a,b,c}$ and $\theta_{a,b,c+1}$ form a basis of $D(\mathcal{A}, m)$.

(2) Suppose $p + q + r$ is even. Let

$$(a, b, c) = \left(\frac{-p + q + r}{2}, \frac{p - q + r - 2}{2}, \frac{p + q - r - 2}{2} \right),$$

and

$$(a', b', c') = \left(\frac{-p + q + r - 2}{2}, \frac{p - q + r}{2}, \frac{p + q - r - 2}{2} \right).$$

Then $x_1\theta_{a,b,c}$ and $x_2\theta_{a',b',c'}$ form a basis of $D(\mathcal{A}, m)$.

Proof. (1) It is clear that $\theta_{a,b,c}$ and $\theta_{a,b,c+1}$ are contained in $D(\mathcal{A}, m)$ by Proposition 2.2. Since $\deg \theta_{a,b,c} = \frac{p+q+r-1}{2}$ and $\deg \theta_{a,b,c+1} = \frac{p+q+r+1}{2}$ are the exponents of (\mathcal{A}, m) ([21]), it suffices to show that these two vector fields are linearly independent over S , or equivalently, the Saito determinant

$$\det \begin{pmatrix} \theta_{a,b,c}x_1 & \theta_{a,b,c}x_2 \\ \theta_{a,b,c+1}x_1 & \theta_{a,b,c+1}x_2 \end{pmatrix}$$

is nonzero. The highest degree terms with respect to the variable x_1 in each entry are

$$\begin{pmatrix} x_1^{a+b+c+1} & x_1^b x_2^{a+c+1} \\ x_1^{a+b+c+2} & x_1^b x_2^{a+c+2} \end{pmatrix},$$

up to nonzero constant factors. Then, it is clear that the highest degree term in the determinant is a non-zero multiple of $x_1^{a+2b+c+2} x_2^{a+c+1}$.

(2) It is clear that $x_1\theta_{a,b,c}$ and $x_2\theta_{a',b',c'}$ are contained in $D(\mathcal{A}, m)$ by Proposition 2.2. Since $\deg x_1\theta_{a,b,c} = \deg x_2\theta_{a',b',c'} = \frac{p+q+r}{2}$ are the exponents of (\mathcal{A}, m) ([21]), it suffices to show that these two vector fields are linearly independent over S , equivalently, the Saito determinant

$$\det \begin{pmatrix} x_1\theta_{a,b,c}x_1 & x_1\theta_{a,b,c}x_2 \\ x_2\theta_{a',b',c'}x_1 & x_2\theta_{a',b',c'}x_2 \end{pmatrix}$$

is nonzero. Note that $(a', b', c') = (a - 1, b + 1, c)$. The highest degree terms with respect to the variable x_1 in each entry are

$$\begin{pmatrix} x_1^{a+b+c+2} & x_1^{b+1} x_2^{a+c+1} \\ x_1^{a+b+c+1} x_2 & x_1^{b+1} x_2^{a+c+1} \end{pmatrix},$$

up to nonzero constant factor. Then, the highest degree term in the determinant is a non-zero multiple of $x_1^{a+2b+c+3} x_2^{a+c+1}$. \square

Example 2.4. The following are generic examples.

(1) Let (\mathcal{A}, m) be the multiarrangement defined by $x_1^{101} x_2^{115} (x_1 - x_2)^{157}$. Then the following vector fields form a basis of $D(\mathcal{A}, m)$ with exponents

(186, 187):

$$\begin{aligned} & \left(\int^{x_1} t^{29} (t-x_1)^{71} (t-x_2)^{85} dt \right) \partial_1 + \left(\int^{x_2} t^{29} (t-x_1)^{71} (t-x_2)^{85} dt \right) \partial_2, \\ & \left(\int^{x_1} t^{30} (t-x_1)^{71} (t-x_2)^{85} dt \right) \partial_1 + \left(\int^{x_2} t^{30} (t-x_1)^{71} (t-x_2)^{85} dt \right) \partial_2. \end{aligned}$$

(2) Let (\mathcal{A}, m) be the multiarrangement defined by $x_1^{100} x_2^{115} (x_1 - x_2)^{157}$. Then the following vector fields form a basis of $D(\mathcal{A}, m)$ with exponents (186, 186):

$$\begin{aligned} & x_1 \left\{ \left(\int^{x_1} t^{28} (t-x_1)^{70} (t-x_2)^{86} dt \right) \partial_1 + \left(\int^{x_2} t^{28} (t-x_1)^{70} (t-x_2)^{86} dt \right) \partial_2 \right\}, \\ & x_2 \left\{ \left(\int^{x_1} t^{28} (t-x_1)^{71} (t-x_2)^{85} dt \right) \partial_1 + \left(\int^{x_2} t^{28} (t-x_1)^{71} (t-x_2)^{85} dt \right) \partial_2 \right\}. \end{aligned}$$

2.2. A higher-dimensional analogue. Let $\ell \geq 2$. Let $a_1, a_2, \dots, a_\ell, b \in \mathbb{Z}_{\geq 0}$. We define the vector field $\theta_{a_1, \dots, a_\ell, b}$ as

$$\theta_{a_1, \dots, a_\ell, b} := \sum_{i=1}^{\ell} \left(\int^{x_i} t^b (t-x_1)^{a_1} \dots (t-x_\ell)^{a_\ell} dt \right) \partial_i. \quad (2.3)$$

Note that $\deg \theta_{a_1, \dots, a_\ell, b} = \sum_{i=1}^{\ell} a_i + b + 1$. By a similar computation to the proof of Proposition 2.2, we can easily verify the following.

Proposition 2.5. *Let (\mathcal{A}, m) be the multiarrangement defined by*

$$Q(\mathcal{A}, m) = \prod_{i=1}^{\ell} x_i^{a_i+b+1} \cdot \prod_{1 \leq i < j \leq \ell} (x_i - x_j)^{a_i+a_j+1}. \quad (2.4)$$

Then the vector field $\theta_{a_1, \dots, a_\ell, b}$ is contained in $D(\mathcal{A}, m)$.

The vector fields $\theta_{a_1, \dots, a_\ell, b}$ are useful for constructing elements in the logarithmic vector fields. The following statement gives one of the basic properties.

Lemma 2.6. *Let $b_1, \dots, b_\ell \in \mathbb{Z}_{\geq 0}$ with $b_1 < b_2 < \dots < b_\ell$. Then the ℓ vector fields $\theta_{a_1, \dots, a_\ell, b_1}, \theta_{a_1, \dots, a_\ell, b_2}, \dots, \theta_{a_1, \dots, a_\ell, b_\ell}$ are linearly independent over S .*

Proof. It suffices to show that the determinant of the matrix $M = (\theta_{a_1, \dots, a_\ell, b_i}(x_j))_{i,j}$ is nonzero. We consider the lexicographic order among monomials. Namely, $x_1^{p_1} x_2^{p_2} \dots x_\ell^{p_\ell} > x_1^{q_1} x_2^{q_2} \dots x_\ell^{q_\ell}$ if and only if $p_1 = q_1, p_2 = q_2, \dots, p_i = q_i$ and $p_{i+1} > q_{i+1}$ for some i . The nonzero leading monomial of each entry

of the Saito matrix M is, up to a constant factor,

$$\begin{pmatrix} x_1^{a_1+\dots+a_\ell+b_1+1} & x_1^{a_1} x_2^{a_2+\dots+a_\ell+b_1+1} & \dots & x_1^{a_1} \dots x_{\ell-1}^{a_{\ell-1}} x_\ell^{a_\ell+b_1+1} \\ x_1^{a_1+\dots+a_\ell+b_2+1} & x_1^{a_1} x_2^{a_2+\dots+a_\ell+b_2+1} & \dots & x_1^{a_1} \dots x_{\ell-1}^{a_{\ell-1}} x_\ell^{a_\ell+b_2+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{a_1+\dots+a_\ell+b_\ell+1} & x_1^{a_1} x_2^{a_2+\dots+a_\ell+b_\ell+1} & \dots & x_1^{a_1} \dots x_{\ell-1}^{a_{\ell-1}} x_\ell^{a_\ell+b_\ell+1} \end{pmatrix}.$$

The product of anti-diagonal entries attains the unique maximum monomial

$$x_1^{\ell a_1+a_2+\dots+a_\ell+b_\ell+1} x_2^{(\ell-1)a_2+a_3+\dots+a_\ell+b_{\ell-1}+1} \dots x_\ell^{a_\ell+b_1+1}.$$

Hence $\det M \neq 0$. \square

We can prove the following.

Proposition 2.7. *Let (\mathcal{A}, m) be the multiarrangement defined by $Q(\mathcal{A}, m)$ given by (2.4). Then the vector fields $\theta_{a_1, \dots, a_\ell, b}, \theta_{a_1, \dots, a_\ell, b+1}, \dots, \theta_{a_1, \dots, a_\ell, b+\ell-1}$ give a free basis for (\mathcal{A}, m) .*

Proof. By Proposition 2.5 and Lemma 2.6, these vector fields are contained in $D(\mathcal{A}, m)$ and are linearly independent over S . Since the sum of the degrees of these vector fields is equal to $\deg Q(\mathcal{A}, m) = \ell(\sum_{i=1}^{\ell} a_i + b) + \frac{\ell(\ell+1)}{2}$, by Proposition 1.1, they form a basis of $D(\mathcal{A}, m)$. \square

As another consequence we get a free basis of a multi-braid arrangement whose freeness is known from [2].

Proposition 2.8. *Let (\mathcal{A}, m) be the multiarrangement defined by*

$$Q(\mathcal{A}, m) = \prod_{1 \leq i < j \leq \ell} (x_i - x_j)^{a_i+a_j+1}.$$

Then the vector fields $\theta_{a_1, \dots, a_\ell, b}$ for $b = 0, \dots, \ell - 2$ together with $E = \sum_{i=1}^{\ell} \partial_i$ give a free basis for (\mathcal{A}, m) .

Proof. By Proposition 2.5 the vector fields belong to $D(\mathcal{A}, m)$. Similarly to Lemma 2.6 they are linearly independent over S . We also have $\deg E = 0$ and

$$\sum_{b=0}^{\ell-2} \deg \theta_{a_1, \dots, a_\ell, b} = \frac{\ell(\ell-1)}{2} + (\ell-1) \sum_{i=1}^{\ell} a_i = \deg Q(\mathcal{A}, m).$$

The statement follows by Proposition 1.1. \square

Remark 2.9. Multiarrangements of Propositions 2.7 and 2.8 are connected directly as follows. Consider the ring homomorphism φ from $S = \mathbb{C}[x_1, \dots, x_\ell]$

to $T = \mathbb{C}[y_1, \dots, y_\ell, y_{\ell+1}]$ defined by $\varphi(x_i) = y_i - y_{\ell+1}$ ($i = 1, \dots, \ell$). Then we have

$$\varphi \left(\prod_{i=1}^{\ell} x_i^{a_i + a_{\ell+1} + 1} \cdot \prod_{1 \leq i < j \leq \ell} (x_i - x_j)^{a_i + a_j + 1} \right) = \prod_{1 \leq i < j \leq \ell+1} (y_i - y_j)^{a_i + a_j + 1}.$$

The map φ induces a linear map $f : \text{Spec } T \simeq \mathbb{C}^{\ell+1} \longrightarrow \text{Spec } S \simeq \mathbb{C}^{\ell}$ with $\text{Ker } f = \mathbb{C} \cdot (\mathbf{e}_1 + \dots + \mathbf{e}_{\ell+1})$, where vectors \mathbf{e}_i form the standard basis of $\mathbb{C}^{\ell+1}$. Thus the multiarrangement of Proposition 2.8 in $\mathbb{C}^{\ell+1}$ with $a_{\ell+1} = b$ is obtained from that of Proposition 2.7 in \mathbb{C}^{ℓ} by taking direct product with one-dimensional space.

3. INTEGRAL EXPRESSIONS FOR REFLECTION MULTIARRANGEMENTS

3.1. Free reflection multiarrangements for monomial groups. Let $\ell \geq 2$ and $r \geq 2$. Let $V = \bigoplus_{i=1}^{\ell} \mathbb{C} \mathbf{e}_i$ be an ℓ -dimensional \mathbb{C} -linear space. Let $C_r := \langle \zeta_r \rangle = \{\zeta_r^k \mid k = 0, \dots, r-1\}$ be the multiplicative cyclic group of order r , where $\zeta_r = e^{2\pi\sqrt{-1}/r}$ is the primitive r -th root of 1. Let $[\ell] := \{1, \dots, \ell\}$. Let $\varepsilon : [\ell] \longrightarrow C_r$ be a map and $\sigma \in \mathfrak{S}_\ell$ be a permutation of $[\ell]$. Then, define the linear map $\varphi_{\sigma, \varepsilon} : V \longmapsto V$ as

$$\varphi_{\sigma, \varepsilon}(\mathbf{e}_i) = \varepsilon(i) \cdot \mathbf{e}_{\sigma(i)}. \quad (3.1)$$

The linear maps $\varphi_{\sigma, \varepsilon}$ ($\sigma \in \mathfrak{S}_\ell$ and $\varepsilon : [\ell] \longrightarrow C_r$) form a subgroup of $GL(V)$, denoted by $G(r, 1, \ell)$ and called the full monomial group. The order of the group is $|G(r, 1, \ell)| = r^\ell \cdot \ell!$.

Now let p be a divisor of r . The monomial group $G(r, p, \ell)$ is the collection of $\varphi_{\sigma, \varepsilon}$ such that $\prod_{i=1}^{\ell} \varepsilon(i)^{r/p} = 1$. In other words, the product $\prod_{i=1}^{\ell} \varepsilon(i)$ is contained in $\langle \zeta_r^p \rangle = C_{r/p}$. Clearly, $|G(r, p, \ell)| = r^\ell \cdot \ell! / p$.

The hyperplane defined by $x_i - \zeta_r^k x_j = 0$ ($i \neq j, k = 0, \dots, r-1$) is a reflecting hyperplane of $G(r, p, \ell)$. Furthermore, if $r > p$, the coordinate hyperplane $x_i = 0$ is also a reflecting hyperplane. The defining equation $Q(\mathcal{A})$ of the arrangement \mathcal{A} of reflecting hyperplanes is as follows.

$$Q(\mathcal{A}) = \begin{cases} x_1 x_2 \dots x_\ell \cdot \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r), & \text{if } r > p, \\ \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r), & \text{if } r = p. \end{cases} \quad (3.2)$$

Let $u \geq 1$, and $m_i \geq 0$ ($i = 1, \dots, \ell$). We are considering the free multiarrangement (\mathcal{A}, μ) of the form

$$Q(\mathcal{A}, \mu) = \prod_{i=1}^{\ell} x_i^{m_i} \cdot \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r)^u. \quad (3.3)$$

Hoge, Mano, Röhrle, and Stump [10] proved that the following multiarrangements are free.

Theorem 3.1. ([10, Theorem 4.1]) *Let $m \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq -m-1}$ and \overline{m}_i ($i = 1, \dots, \ell$) be integers such that $0 \leq \overline{m}_i \leq r - 1$ and*

$$m_i := r(m + k) + 1 + \overline{m}_i \geq 0,$$

for $i = 1, \dots, \ell$. Note that the quotient $q := \lfloor \frac{m_i - 1}{r} \rfloor = m + k$ does not depend on i . Set $a := (\ell - 1)r$, $m' := \sum_{i=1}^{\ell} m_i$, and $c := ma + qr + 1$.

(1) (The case $u = 2m + 1$ is odd.) Let (\mathcal{A}, μ) be the multiarrangement defined by

$$Q(\mathcal{A}, \mu) = \prod_{i=1}^{\ell} x_i^{m_i} \cdot \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r)^{2m+1}. \quad (3.4)$$

Then (\mathcal{A}, μ) is free with exponents

$$\begin{aligned} \exp(\mathcal{A}, \mu) &= (c + m' - \ell(qr + 1), c + r, c + 2r, \dots, c + (\ell - 1)r) \\ &= (mrl + rk + 1 + \sum \overline{m}_i, \\ &\quad r(k + 1) + mrl + 1, r(k + 2) + mrl + 1, \dots, r(k + \ell - 1) + mrl + 1). \end{aligned} \quad (3.5)$$

(2) (The case $u = 2m$ is even.) Let (\mathcal{A}, μ') be the multiarrangement defined by

$$Q(\mathcal{A}, \mu') = \prod_{i=1}^{\ell} x_i^{m_i} \cdot \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r)^{2m}. \quad (3.6)$$

Then (\mathcal{A}, μ') is free with exponents

$$\exp(\mathcal{A}, \mu') = (ma + m_1, \dots, ma + m_{\ell}). \quad (3.7)$$

Remark 3.2. If $m_i = 0$ then $k = -m - 1$ and $\overline{m}_i = r - 1$. If $m_i = r(m + k) + 1$ then we have $k \geq -m$ and $\overline{m}_i = 0$.

3.2. Integral expression for (\mathcal{A}, μ) . In this section, we provide integral expressions for the basis of free multiarrangements in Theorem 3.1 (1). Let $\lambda(t) = \prod_{i=1}^{\ell} (t^r - x_i^r)$. For $m \in \mathbb{Z}_{\geq 0}$ and $u \in \mathbb{Z}$, define the vector field η_u^m as

$$\eta_u^m = \sum_{i=1}^{\ell} \left(\int^{x_i} t^{ru} \lambda(t)^m dt \right) \partial_i. \quad (3.8)$$

Note that η_u^m may have poles in general. However, it does not have poles for the following cases.

Proposition 3.3. *Let $\ell, m, k, r, \overline{m}_i$ be as above.*

- (1) If $u \geq -m$, then η_u^m is a polynomial vector field.
(2) The vector field $\left(\prod_{i=1}^{\ell} x_i^{\overline{m}_i}\right) \eta_k^m$ is a polynomial vector field.

Proof. (1) We will see $\eta_u^m x_1 = \int^{x_1} t^{ru} \lambda(t)^m dt$ is a polynomial. Note that $\lambda(t)$ can be expressed as

$$\lambda(t) = A \cdot t^r + B \cdot x_1^r,$$

where A and B are polynomials in t, x_i . Therefore, $t^{ru} \cdot \lambda(t)^m$ has the form

$$t^{ru} \cdot \lambda(t)^m = \sum_{i+j=m} A_i \cdot x_1^{ri} \cdot t^{r(j+u)},$$

where A_i is a polynomial. The degree of x_1 in $\int^{x_1} t^{ru} \lambda(t)^m dt$ is at least $ri + r(j+u) + 1 = r(m+u) + 1 \geq 1$ (recall $u \geq -m$). Therefore, it is a polynomial.

(2) If $k \geq -m$, the assertion follows from (1). Now we consider the case $k = -m - 1$. In this case, $m_i = 0$ and $\overline{m}_i = r - 1$ for all $i = 1, \dots, \ell$. The degree of x_1 in $\prod_{i=1}^{\ell} x_i^{\overline{m}_i} \cdot \eta_k^m x_1$ is at least

$$\overline{m}_1 + r(m+k) + 1 = 0.$$

□

Theorem 3.4. *Let $\ell, m, k, r, \overline{m}_i$ be as above. Then the vector fields*

$$\left(\prod_{i=1}^{\ell} x_i^{\overline{m}_i}\right) \eta_k^m, \quad \eta_{k+1}^m, \quad \eta_{k+2}^m, \quad \dots, \quad \eta_{k+\ell-1}^m \quad (3.9)$$

form a basis of $D(\mathcal{A}, \mu)$.

Proof. By Proposition 3.3, these are polynomial vector fields. It is easily seen that

$$\eta_u^m x_j = \int^{x_j} t^{ru} \lambda(t)^m dt$$

is divisible by $x_j^{ru+rm+1}$. When $u \geq k+1$, we have

$$ru + rm + 1 \geq r(m+k+1) + 1 > r(m+k) + \overline{m}_j + 1 = m_j.$$

Similarly, $\left(\prod_{i=1}^{\ell} x_i^{\overline{m}_i}\right) \eta_k^m x_j$ is also divisible by $x_j^{\overline{m}_j+r k+rm+1} = x_j^{m_j}$.

Next we compute the multiplicity of $\eta_u^m(x_i - \zeta x_j)$ for $\zeta \in C_r$. Since the integrand is invariant under the transformation $t = \zeta^{-1}t'$, we have

$$\begin{aligned} \eta_u^m(x_i - \zeta x_j) &= \int^{x_i} t^{ru} \lambda(t)^m dt - \zeta \int^{x_j} t^{ru} \lambda(t)^m dt \\ &= \int^{x_i} t^{ru} \lambda(t)^m dt - \zeta \int^{\zeta x_j} (t')^{ru} \lambda(t')^m \zeta^{-1} dt' \quad (3.10) \\ &= \int_{\zeta x_j}^{x_i} t^{ru} \lambda(t)^m dt. \end{aligned}$$

Since $\lambda(t)^m$ is divisible by $(t - x_i)^m(t - \zeta x_j)^m$, and $\eta_u^m(x_i - \zeta x_j)$ is a polynomial, it follows that $\eta_u^m(x_i - \zeta x_j)$ is divisible by $(x_i - \zeta x_j)^{2m+1}$.

We can prove the linear independence of $\eta_k^m, \eta_{k+1}^m, \dots, \eta_{k+\ell-1}^m$ in a similar way to Lemma 2.6. It is also straightforward that degrees of these vector fields are exactly equal to the exponents $\exp(\mathcal{A}, \mu)$ in Theorem 3.1 (1). Hence, by Saito-Ziegler criterion, the vector fields (3.9) form a free basis of $D(\mathcal{A}, \mu)$. \square

3.3. Integral expression for (\mathcal{A}, μ') . In this section, we provide integral expressions for the basis of free multiarrangements in Theorem 3.1 (2). Let

$$\lambda_i(t) = \frac{\lambda(t)}{t^r - x_i^r}.$$

For $m \in \mathbb{Z}_{>0}$ and $1 \leq i \leq \ell$, define the vector field σ_i^m as

$$\sigma_i^m = \sum_{j=1}^{\ell} \left(\int^{x_j} t^{r(k+1)} \lambda(t)^{m-1} \lambda_i(t) dt \right) \partial_j. \quad (3.11)$$

Theorem 3.5. *Let ℓ, m, k, r, \bar{m}_i be as above. Then the vector fields*

$$x_1^{\bar{m}_1} \sigma_1^m, x_2^{\bar{m}_2} \sigma_2^m, \dots, x_\ell^{\bar{m}_\ell} \sigma_\ell^m \quad (3.12)$$

form a basis of $D(\mathcal{A}, \mu')$.

Proof. First we prove that $x_i^{\bar{m}_i} \sigma_i^m$ is a polynomial vector field and $x_i^{\bar{m}_i} \sigma_i^m x_j$ is divisible by $x_j^{m_j}$. The exponent of x_n in each monomial of

$$x_i^{\bar{m}_i} \sigma_i^m x_j = x_i^{\bar{m}_i} \int^{x_j} t^{r(k+1)} \lambda(t)^{m-1} \lambda_i(t) dt$$

is clearly nonnegative if $n \neq j$. Now we consider the exponent of x_j . We note that $x_i^{\bar{m}_i} \sigma_i^m x_j$ is divisible by $x_j^{r(k+1)+mr+1}$ if $i \neq j$, and it is divisible by $x_j^{\bar{m}_i+r(k+1)+r(m-1)+1}$ if $i = j$. In each case, the exponent is at least $m_i \geq 0$.

Next we can prove that $x_i^{\bar{m}_i} \sigma_i^m(x_j - \zeta x_n)$, ($\zeta \in C_r$) is divisible by $(x_j - \zeta x_n)^{2m}$ in a similar way to Theorem 3.4. We can also easily check that the degrees of these vector fields are

$$\deg x_i^{\bar{m}_i} = \bar{m}_i + r(k+1) + \ell r(m-1) + r(\ell-1) + 1 = m_i + ma,$$

which are equal to the exponents in Theorem 3.1 (2).

It remains to prove that $\sigma_1^m, \dots, \sigma_\ell^m$ are linearly independent over S . Let $M = (\sigma_i^m x_j)_{i,j}$. To prove that $\det M \neq 0$, we look at the leading term of each entry with respect to the lexicographic order. To look at the leading term, we need

$$\int^x t^{rn} (t^r - x^r)^m dt \neq 0, \quad (3.13)$$

($r \geq 2$, $m \geq 0$ and $n \in \mathbb{Z}$), which is a nonzero multiple of $x^{r(n+m)+1}$. Actually, (3.13) is equivalent to $\int_1^{\zeta^r} t^{rn} (t^r - 1)^m dt \neq 0$, which can be proved by using integration by parts and the induction on m . Then the nonzero leading monomials of entries of M are (up to a constant factors) as follows:

$$\begin{pmatrix} x_1^{rk+\ell rm+1} & x_1^{r(m-1)} x_2^{r(k+1)+rm(\ell-1)+1} & x_1^{r(m-1)} x_2^{rm} x_3^{r(k+1)+rm(\ell-2)+1} & \dots \\ x_1^{rk+\ell rm+1} & x_1^{rm} x_2^{rk+rm(\ell-1)+1} & x_1^{rm} x_2^{r(m-1)} x_3^{r(k+1)+rm(\ell-2)+1} & \dots \\ x_1^{rk+\ell rm+1} & x_1^{rm} x_2^{rk+rm(\ell-1)+1} & x_1^{rm} x_2^{rm} x_3^{rk+rm(\ell-2)+1} & \dots \\ x_1^{rk+\ell rm+1} & x_1^{rm} x_2^{rk+rm(\ell-1)+1} & x_1^{rm} x_2^{rm} x_3^{rk+rm(\ell-2)+1} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In j -th column, the monomial increases strictly up to j -th place, and it is equal to $x_1^{rm} \dots x_{j-1}^{rm} x_j^{rk+rm(\ell-j+1)+1}$ at the j -th place and below. In the determinant of M , the product of diagonal entries produces the unique nonzero leading monomial of $\det M$. Therefore $\det M \neq 0$. \square

3.4. Special cases. Let us list a few examples illustrating Theorems 3.4 and 3.5.

Example 3.6. Consider the case of Theorem 3.4 when $k = 0$ and $\overline{m}_i = 0$. Then the defining equation of the multiarrangement is

$$\prod_{i=1}^{\ell} x_i^{rm+1} \cdot \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r)^{2m+1}.$$

The vector fields

$$\eta_u^m = \sum_{i=1}^{\ell} \left(\int^{x_i} t^{ru} \prod_{j=1}^{\ell} (t^r - x_j^r)^m dt \right) \partial_i, \quad (3.14)$$

$0 \leq u \leq \ell - 1$, form a basis.

Example 3.7. Consider the case of Theorem 3.4 when $k = -m - 1$ and $\overline{m}_i = r - 1$. Then the defining equation of the multiarrangement is

$$\prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r)^{2m+1}.$$

The vector fields

$$(x_1 \dots x_\ell)^{r-1} \eta_{-m-1}^m = (x_1 \dots x_\ell)^{r-1} \sum_{i=1}^{\ell} \left(\int^{x_i} t^{-r(m+1)} \prod_{j=1}^{\ell} (t^r - x_j^r)^m dt \right) \partial_i,$$

and

$$\eta_u^m = \sum_{i=1}^{\ell} \left(\int^{x_i} t^{ru} \prod_{j=1}^{\ell} (t^r - x_j^r)^m dt \right) \partial_i, \quad (-m \leq u \leq -m + \ell - 2),$$

form a basis.

Example 3.8. (Coxeter arrangement of type B_ℓ) Let $r = 2$ and $m \geq 0$.

(1) Let $m_i = 2m + 1$, hence, $k = 0$, $\overline{m}_i = 0$ ($i = 1, \dots, \ell$). Then the defining equation (3.4) takes the form

$$\prod_{i=1}^{\ell} x_i^{2m+1} \cdot \prod_{1 \leq i < j \leq \ell} (x_i^2 - x_j^2)^{2m+1}.$$

The vector fields $\eta_0^m, \eta_1^m, \dots, \eta_{\ell-1}^m$, or, more explicitly,

$$\sum_{i=1}^{\ell} \left(\int_0^{x_i} t^{2u} \prod_{j=1}^{\ell} (t^2 - x_j^2)^m dt \right) \partial_i,$$

($u = 0, 1, \dots, \ell - 1$) form a basis of $D(B_\ell, 2m + 1)$.

(2) Let $m_i = 2m$, hence, $k = -1$, $\overline{m}_i = 1$ ($i = 1, \dots, \ell$). Then the defining equation (3.6) takes the form

$$\prod_{i=1}^{\ell} x_i^{2m} \cdot \prod_{1 \leq i < j \leq \ell} (x_i^2 - x_j^2)^{2m}.$$

The vector fields $x_1 \sigma_1^m, x_2 \sigma_2^m, \dots, x_\ell \sigma_\ell^m$, or, more explicitly,

$$x_i \cdot \sum_{j=1}^{\ell} \left(\int_0^{x_j} \frac{1}{t^2 - x_i^2} \prod_{k=1}^{\ell} (t^2 - x_k^2)^m dt \right) \partial_j,$$

($i = 1, \dots, \ell$) form a basis of $D(B_\ell, 2m)$.

Example 3.9. (Coxeter arrangement of type D_ℓ) Let $r = 2$ and $m \geq 0$.

(1) Let $m_i = 0$, hence, $k = -m - 1$, $\overline{m}_i = 1$ ($i = 1, \dots, \ell$). Then by Theorem 3.4, the vector fields

$$\begin{aligned} & \prod_{i=1}^{\ell} x_i \cdot \sum_{j=1}^{\ell} \left(\int^{x_j} t^{-2(m+1)} \prod_{k=1}^{\ell} (t^2 - x_k^2)^m dt \right) \partial_j, \\ & \sum_{j=1}^{\ell} \left(\int^{x_j} t^{-2u} \prod_{k=1}^{\ell} (t^2 - x_k^2)^m dt \right) \partial_j, \end{aligned}$$

$(u = m, m-1, \dots, m-\ell+2)$ form a basis of the multiarrangement defined by $\prod_{1 \leq i < j \leq \ell} (x_i^2 - x_j^2)^{2m+1}$.

(2) Again let $m_i = 0$, hence, $k = -m - 1$, $\overline{m}_i = 1$ ($i = 1, \dots, \ell$). Then by Theorem 3.5, the vector fields

$$x_i \cdot \sum_{j=1}^{\ell} \left(\int^{x_j} \frac{t^{-2m}}{t^2 - x_i^2} \prod_{k=1}^{\ell} (t^2 - x_k^2)^m dt \right) \partial_j$$

($i = 1, \dots, \ell$), form a basis of the multiarrangement defined by $\prod_{1 \leq i < j \leq \ell} (x_i^2 - x_j^2)^{2m}$.

Remark 3.10. By the results of [1], Examples 3.6 – 3.9 and, more generally, Theorems 3.4–3.5 also lead to integral formulas of certain quasi-invariants associated with the monomial groups $G(r, p, \ell)$ with $p = 1, r$.

4. PRIMITIVE DERIVATION FOR WELL-GENERATED MONOMIAL GROUPS

4.1. The invariant rings of monomial groups. Let

$$e_i(x_1, \dots, x_\ell) = \sum_{1 \leq s_1 < \dots < s_i \leq \ell} x_{s_1} \dots x_{s_i}$$

be the elementary symmetric polynomial of degree i . The basic invariants, which are the generators of the invariant ring $\mathbb{C}[x_1, \dots, x_\ell]^{G(r, p, \ell)}$, can be explicitly described using e_i . Namely, let

$$\begin{cases} P_i & := (-1)^i e_i(x_1^r, \dots, x_\ell^r), \text{ for } 1 \leq i \leq \ell - 1 \\ P_\ell & := (x_1 x_2 \dots x_\ell)^{r/p}. \end{cases}$$

Note that $P_\ell^p = e_\ell(x_1^r, \dots, x_\ell^r)$. In particular, if $p = 1$, we have $P_\ell = e_\ell(x_1^r, \dots, x_\ell^r)$. We also have

$$\lambda(t) = t^{r\ell} + P_1(x)t^{r(\ell-1)} + \dots + P_{\ell-1}(x)t^r + (-1)^\ell P_\ell(x)^p. \quad (4.1)$$

We formally have

$$\frac{\partial \lambda(t)}{\partial P_i} \doteq \begin{cases} t^{r(\ell-i)}, & \text{if } 1 \leq i \leq \ell - 1 \\ P_\ell(x)^{p-1}, & \text{if } i = \ell. \end{cases}$$

Recall that ([5]) P_1, \dots, P_ℓ are algebraically independent, and

$$\mathbb{C}[x_1, \dots, x_\ell]^{G(r, p, \ell)} = \mathbb{C}[P_1, \dots, P_\ell].$$

The Jacobian of the basic invariants is

$$\det \left(\frac{\partial P_i}{\partial x_j} \right) \doteq (x_1 x_2 \dots x_\ell)^{\frac{r}{p}-1} \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r). \quad (4.2)$$

Later, the basic invariant of the highest degree plays an important role.

4.2. Order of reflections.

Definition 4.1. (1) Denote by β the multiplicity assigning 1 to each coordinate hyperplane $\{x_i = 0\}$, and assigning 0 to the hyperplanes $\{x_i - \zeta x_j = 0\}$, where $\zeta \in C_r$.

(2) Denote by δ_r the multiplicity assigning 1 to each hyperplane of the form $\{x_i - \zeta x_j = 0\}$, where $\zeta \in C_r$, and assigning 0 to each coordinate hyperplane $\{x_i = 0\}$.

(3) For the group $G(r, p, \ell)$, denote by ω the multiplicity assigning the order of the reflection to each reflecting hyperplane. More explicitly, ω is expressed as follows.

$$\omega = \begin{cases} \frac{r}{p}\beta + 2\delta_r, & \text{if } p < r, \\ 2\delta_r, & \text{if } p = r. \end{cases}$$

4.3. Primitive derivation D . Let $G = G(r, 1, \ell)$ or $G(r, r, \ell)$. Denote the invariant ring by $R := S^G$. Among the vector fields $\frac{\partial}{\partial P_1}, \dots, \frac{\partial}{\partial P_\ell}$, there exists unique one with the lowest degree, which we denote by D , namely,

$$D = \begin{cases} \frac{\partial}{\partial P_\ell}, & \text{if } p = 1, \\ \frac{\partial}{\partial P_{\ell-1}}, & \text{if } p = r. \end{cases}$$

This vector field is canonically determined up to nonzero scalar multiplication, it is called the primitive derivation.

4.4. The action of the primitive derivation. We will use the notation from Example 3.6. In this section, we describe the action of primitive derivation on the module $D(\mathcal{A}, m \cdot \omega + 1)^G$.

Denote by ∇ the integrable connection with flat sections $\partial_1, \dots, \partial_\ell$. More explicitly, for polynomial vector fields δ and $\eta = \sum_{i=1}^{\ell} f_i \partial_i$, we define

$$\nabla_\delta \eta = \sum_{i=1}^{\ell} (\delta f_i) \partial_i.$$

The vector fields $\frac{\partial}{\partial P_i}$ can be considered as rational vector fields on V . For example,

$$\frac{\partial}{\partial P_\ell} = \frac{1}{Q} \det \begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_2}{\partial x_1} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_1} & \frac{\partial}{\partial x_1} \\ \frac{\partial P_1}{\partial x_2} & \frac{\partial P_2}{\partial x_2} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_2} & \frac{\partial}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial P_1}{\partial x_\ell} & \frac{\partial P_2}{\partial x_\ell} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_\ell} & \frac{\partial}{\partial x_\ell} \end{pmatrix}, \quad (4.3)$$

where $Q = \det\left(\frac{\partial P_i}{\partial x_j}\right)$ is the Jacobian as in (4.2).

The primitive derivation ∇_D acts as follows.

Proposition 4.2. ([10]) *Let G be a well-generated complex reflection group. Let T be the subring of $R = S^G$ defined as*

$$T = \{P \in S^G \mid DP = 0\}.$$

The primitive derivation ∇_D induces an isomorphism of T -modules

$$\nabla_D : D(\mathcal{A}, (m+1)\omega + 1)^G \xrightarrow{\simeq} D(\mathcal{A}, m\omega + 1)^G, \quad (4.4)$$

for $m \geq 0$. It is free S^G -module and the basis also spans $D(\mathcal{A}, m\omega + 1)$.

Therefore, the basis of $D(\mathcal{A}, m\omega + 1)$ can be described by using the inverse primitive derivation ∇_D^{-1} , which is difficult to describe in general. However, for well-generated groups $G(r, 1, \ell)$ and $G(r, r, \ell)$, we can describe ∇_D^{-1} in terms of the integral expression.

4.5. The case $G = G(r, 1, \ell)$. In this case, $D = \frac{\partial}{\partial P_\ell}$. As in Example 3.6, the vector fields

$$\eta_0^m, \eta_1^m, \dots, \eta_{\ell-1}^m \quad (4.5)$$

form a basis of $D(\mathcal{A}, m\omega + 1)$. Furthermore, since η_u^m is G -invariant, the basis (4.5) forms the S^G -basis of $D(\mathcal{A}, m\omega + 1)^G$. By the expression (4.1), we have

$$D\lambda(t)^m = (-1)^\ell m\lambda(t)^{m-1}.$$

Thus we have (see also [18, Remark 2.5])

$$\nabla_D^{-m} \eta_k^0 \doteq \eta_k^m.$$

4.6. The case $G = G(r, r, \ell)$. In this case, $D = \frac{\partial}{\partial P_{\ell-1}}$. As in Example 3.7,

$$P_\ell^{r-1} \eta_{-m-1}^m, \eta_{-m}^m, \eta_{-m+1}^m, \dots, \eta_{-m+\ell-2}^m \quad (4.6)$$

form a basis of $D(\mathcal{A}, m\omega + 1)$. Furthermore, since η_u^m is G -invariant, the basis (4.6) forms the S^G -basis of $D(\mathcal{A}, m\omega + 1)^G$. By the expression (4.1), we have

$$D\lambda(t)^m = mt^r \lambda(t)^{m-1},$$

and we also have

$$\nabla_D \eta_u^m \doteq \eta_{u+1}^{m-1}.$$

Thus we have

$$\nabla_D^{-m} P_\ell^{r-1} \eta_{-1}^0 \doteq P_\ell^{r-1} \eta_{-m-1}^m, \quad \nabla_D^{-m} \eta_u^0 \doteq \eta_{-m+u}^m, \quad (0 \leq u \leq \ell - 2).$$

5. A CONJECTURE ON B_2 -CATALAN ARRANGEMENT

In this section, we propose a conjecture regarding the basis of the Catalan arrangement of type B_2 .

Definition 5.1. Let $\ell > 0$ and $m \geq 0$. Define the extended Catalan arrangement $\text{Cat}(B_\ell, m)$ of type B_ℓ by

$$\prod_{i=1}^{\ell} \prod_{k=-m}^m (x_i - k) \cdot \prod_{1 \leq i < j \leq \ell} \prod_{k=-m}^m (x_i + x_j - k)(x_i - x_j - k).$$

Note that the extended Catalan arrangement determines a central arrangement in $\mathbb{R}^{\ell+1}$, the so-called coning [12] $c \text{Cat}(B_\ell, m)$. By taking the Ziegler restriction to the hyperplane at infinity, we obtain the multiarrangement defined by

$$\prod_{i=1}^{\ell} x_i^{2m+1} \cdot \prod_{1 \leq i < j \leq \ell} (x_i^2 - x_j^2)^{2m+1}.$$

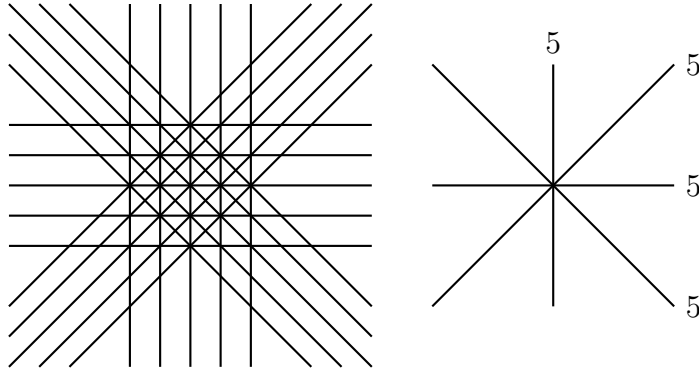


FIGURE 2. $\text{Cat}(B_2, 2)$ and associated multiarrangement

The coning $c \text{Cat}(B_\ell, m)$ is known to be free [24]. However, the explicit basis has not been constructed even for B_2 case.

Let $m, i \geq 0$. Define the polynomial $f_i^m(x, y)$ to be

$$f_i^m(x, y) := \int_0^x t^{2i} (t^2 - x^2)^m (t^2 - y^2)^m dt.$$

Example 5.2. We have

$$\begin{aligned}
f_0^1(x, y) &\doteq x^5 - 5x^3y^2, \\
f_1^1(x, y) &\doteq 3x^7 - 7x^5y^2, \\
f_2^1(x, y) &\doteq 5x^9 - 9x^7y^2, \\
f_3^1(x, y) &\doteq 7x^{11} - 11x^9y^2, \\
f_0^2(x, y) &\doteq x^9 - 6x^7y^2 + 21x^5y^4, \\
f_1^2(x, y) &\doteq 5x^{11} - 22x^9y^2 + 33x^7y^4, \\
f_2^2(x, y) &\doteq 35x^{13} - 130x^{11}y^2 + 143x^9y^4, \\
f_3^2(x, y) &\doteq 21x^{15} - 70x^{13}y^2 + 65x^{11}y^4, \\
f_0^3(x, y) &\doteq 5x^{13} - 39x^{11}y^2 + 143x^9y^4 - 429x^7y^6, \\
f_1^3(x, y) &\doteq 7x^{15} - 45x^{13}y^2 + 117x^{11}y^4 - 143x^9y^6, \\
f_2^3(x, y) &\doteq 21x^{17} - 119x^{15}y^2 + 255x^{13}y^4 - 221x^{11}y^6, \\
f_3^3(x, y) &\doteq 231x^{19} - 1197x^{17}y^2 + 2261x^{15}y^4 - 1615x^{13}y^6.
\end{aligned}$$

The polynomial $f_i^m(x, y)$ has the following form,

$$f_i^m(x, y) = \sum_{k=0}^m c_k x^{4m+2i+1-2k} y^{2k}. \quad (5.1)$$

We already know that the vector field $f_i^m(x, y)\partial_x + f_i^m(y, x)\partial_y$ is contained in $D(B_2, 2m+1)$, which is equivalent to

$$f_i^m(x, y) \in (x^{2m+1}), \text{ and } f_i^m(x, y) \pm f_i^m(y, x) \in ((x \pm y)^{2m+1}).$$

In [18], for type A_ℓ , the basis of $D(c \text{Cat}(\mathcal{A}_\ell, m))$ was constructed using the discrete analogue of the integral expression. However, for type B_ℓ , the discrete integral does not work. We need to define a suitable deformation of the integral expression. In order to describe the deformation, we require deformed power of functions. For a function of one variable $f(x)$, define the falling power $f(x)^{\underline{n}}$ and the rising power $f(x)^{\overline{n}}$ as

$$\begin{aligned}
f(x)^{\underline{n}} &= f(x)f(x-1)\cdots f(x-n+1), \\
f(x)^{\overline{n}} &= f(x)f(x+1)\cdots f(x+n-1).
\end{aligned}$$

(We may also set $f(x)^{\underline{0}} = f(x)^{\overline{0}} = 1$.)

Definition 5.3. For a homogeneous polynomial of the form

$$f(x, y) = \sum_{k=0}^m c_k x^{2p+1-2k} y^{2k},$$

where $p \geq m \geq 0$ and $c_k \in \mathbb{C}$, define the deformation $\widetilde{f}(x, y)$ as

$$\widetilde{f}(x, y) = \sum_{k=0}^m c_k (x + p - k)^{2p+1-2k} (y + p - m)^k (y - p + m)^{\overline{k}}.$$

Let $g(x, y) \in \mathbb{C}[x, y]$ and $\eta = g(x, y)\partial_x + g(y, x)\partial_y$. The homogenization ([18, §4.1]) of η is contained in $D(c \text{Cat}(B_2, m))$ (which is also equivalent to $\eta \in D(\text{Cat}(B_2, m))$) if and only if the following conditions are satisfied:

$$\begin{aligned} \eta x = g(x, y) \text{ is divisible by } \prod_{k=-m}^m (x - k), \text{ and} \\ \eta(x \pm y) = g(x, y) \pm g(y, x) \text{ is divisible by } \prod_{k=-m}^m (x \pm y - k). \end{aligned} \quad (5.2)$$

Example 5.4. The following functions $\widetilde{f}_i^m(x, y)$ satisfy the conditions (5.2):

$$\begin{aligned} \widetilde{f}_0^1(x, y) &\doteq (x+2)^{\overline{5}} - 5(x+1)^{\overline{3}}(y+1)^{\overline{1}}(y-1)^{\overline{1}} \\ &= x(x^2-1)(x^2-4) - 5x(x^2-1)(y^2-1), \\ \widetilde{f}_1^1(x, y) &\doteq 3(x+3)^{\overline{7}} - 7(x+2)^{\overline{5}}(y+2)^{\overline{1}}(y-2)^{\overline{1}}, \\ &\dots \\ \widetilde{f}_3^3(x, y) &\doteq 231(x+9)^{\overline{19}} - 1197(x+8)^{\overline{17}}(y+6)^{\overline{1}}(y-6)^{\overline{1}} \\ &\quad + 2261(x+7)^{\overline{15}}(y+6)^{\overline{2}}(y-6)^{\overline{2}} - 1615(x+6)^{\overline{13}}(y+6)^{\overline{3}}(y-6)^{\overline{3}}. \end{aligned}$$

Conjecture 5.5. For any $m, i \geq 0$, $\widetilde{f}_i^m(x, y) + \widetilde{f}_i^m(y, x)$ is divisible by $\prod_{k=-m}^m (x + y - k)$.

Taking into account the fact that $\widetilde{f}_i^m(x, y)$ is divisible by $\prod_{k=-m}^m (x - k)$ (which is straightforward from (5.1)), Conjecture 5.5 implies that vector fields

$$\begin{aligned} \widetilde{\eta}_0^m &= \widetilde{f}_0^m(x, y)\partial_x + \widetilde{f}_0^m(y, x)\partial_y, \\ \widetilde{\eta}_1^m &= \widetilde{f}_1^m(x, y)\partial_x + \widetilde{f}_1^m(y, x)\partial_y \end{aligned}$$

form a basis of $D(\text{Cat}(B_2, m))$, and that the homogenization of these vector fields together with the Euler vector field form a basis of $D(c \text{Cat}(B_2, m))$.

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