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An analysis of the potential Korteweg-DeVries equation through regular symmetries and topological manifolds

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Abstract

In this contribution, we determine the exact solutions of the potential Korteweg-de Vries equation, a partial differential equation (PDE). The procedure followed involves first transforming this equation into an ordinary differential equation (ODE), using Sophus Lie' symmetry group theoretical methods. The resulting ODE is then resolved through a procedure developed through differentiable topological methods, a technique developed by the third author. The pure Lie approach leads to un-integrable integrals.

Keywords: Potential Korteweg -de Vries equation, Partial Differential Equations, Ordinary Differential Equations, Lie Symmetry Group Theoretical Methods, Differentiable Topological Manifolds.

1 INTRODUCTION

The potential Korteweg -de Vries (p-KdV) equation is obtained from the pioneer model KdV equation, see [1] and [2]. The papers describe the evolution of waves under the competing, but comparable effects of weak nonlinearity and weak dispersion. In the present work, the target is:

$$u_t + u_x^2 + u_{xxx} = 0. \qquad (1)$$

Tchiera et. al., present some results in [3]. Unfortunately, their approach involves too many assumptions, resulting in compromised deductions. In their study, a parameter λ, arises. They then proceed by determining the solutions for the case of $\lambda > 0$, and then $\lambda = 0$; finally concluding with $\lambda < 0$. This is a wide spread practise. The third author demonstrated in [4] that this partitioning of solutions tend to lead to faulty conclusions.

Here, we do our best to avoid following this trend. As stated in the Abstract, we first transform equation (1) into an ODE using Sophus Lie's symmetry group theoretical methods [5]. This we do in the next section, Section 2. Lie's theory, made famous through the paper [6], and also [5]. It was revived in the 1950s by likes of Ovsiaanikov. Some of his works include [7], [8], [9]. A number of scholars have since followed. These include Ibragimov [10], [11], and Leach [12] amongst others and Mahomed [13].

The technique we use to determine exact solutions to the ODE is built on differentiable topological manifolds. We follow it as presented [14]. It is precursor is in [15]. We lay the foundation of this idea in Section 3.

Section 4 is dedicated solely to the application of the technique discussed in Section 3.

2 PURE LIE SYMMETRY ANALYSIS

The process of transforming a PDE, in this case (1), into an ODE through Lie group theoretical methods, involves first determining an appropriate symmetry generator, leading to what is called the determining equation, which in turn separates into several simple linear equations known as monomials. Solving the monomials lead to defining equations, subsequently to the much sought after symmetries, and finally resulting into invariants, from which an ODE or more can be determined.

2.1 The infinitesimal generators and prolongations

The symmetry generator for the p-KDV equation is given by:

$$\boldsymbol{X} = \boldsymbol{\tau}\frac{\partial}{\partial t} + \boldsymbol{\xi}\frac{\partial}{\partial x} + \boldsymbol{\eta}\frac{\partial}{\partial u}. \qquad (2)$$

The first prolongation is then given by

$$\boldsymbol{X}^{[1]} = \boldsymbol{X} + \boldsymbol{\eta}^{t}\frac{\partial}{\partial u_t} + \boldsymbol{\eta}^{x}\frac{\partial}{\partial u_x} \qquad (3)$$

and the third prolongation is given by

$$\boldsymbol{X}^{[3]} = \boldsymbol{X}^{[1]} + \boldsymbol{\eta}^{xxx}\frac{\partial}{\partial u_{xxx}}. \qquad (4)$$

To find the prolongations of the transformations we use the total derivative operator given as:

$$\boldsymbol{D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots} \tag{5}$$

$$\boldsymbol{D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{xt} \frac{\partial}{\partial u_x} + \cdots} \tag{6}$$

The prolongations of the generator which we use are as follows:

$$\boldsymbol{\eta^x = \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t} , \tag{7}$$

$$\boldsymbol{\eta^t = \eta_t + (\eta_u - \tau_t)u_t - \xi_t u_x - \tau_u u_x^2 - \xi_u u_x u_t} , \tag{8}$$

$$\boldsymbol{\eta^{xxx} = \eta_{xxx} + (3\eta_{xxu} - \xi_{xxx})u_x + (3\eta_{xuu} - 3\xi_{xxu})u_x^2 + (\eta_{uuu} - 3\xi_{xuu})u_x^3 - \xi_{uuu}u_x^4 + \tau_{xxx}u_t - 3\tau_{xxu}u_x u_t - 3\,\tau_{xuu}u_x^2 u_t - \tau_{uuu}u_x^3 u_t - 3\tau_{xx}u_{xt} + (3\eta_{xu} - \xi_{xx})u_{xx} + (3\eta_{uu} - 9\xi_{xu})u_x u_{xx} - 6\xi_{uu}u_x^2 u_{xx} - 6\tau_{xu}u_x u_{xt} - 3\tau_{uu}u_x^2 u_{xt} - 3\tau_{xu}u_t u_{xx} - 3\xi_u u_{xx}^2 - 3\tau_u u_{xt}u_{xx} - 3\tau_{uu}u_x u_{xx}u_t + (\eta_u - 3\xi_x)u_{xxx} - 4\xi_u u_x u_{xxx} - 3\tau_x u_{xxt} - 3\tau_x u_{xxt} - 3\tau u_x u_{xxt} - \tau_u u_t u_{xxx}} . \tag{9}$$

2.2 Solving the determining equation

The determining equation follows from the invariant condition

$$\boldsymbol{X^{[3]} F \,\big|_{F=0} = 0}, \tag{10}$$

with $F = u_t + u_x^2 + u_{xxx}$. Equation (10) in expanded form assumes the form

$$\boldsymbol{\eta^t + 2u_x\eta^x + \eta^{xxx} = 0}. \tag{11}$$

After substituting (7), (8), and (9) into (7), (8) and (9) in (11), we obtain the determining equation:

$$\boldsymbol{[\eta_t + (\eta_u - \tau_t)u_t - \xi_t u_x - \tau_u u_x^2 - \xi_u u_x u_t] + 2[\eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t] + [\eta_{xxx} + (3\eta_{xxu} - \xi_{xxx})u_x + (3\eta_{xuu} - 3\xi_{xxu})u_x^2 + (\eta_{uuu} - 3\xi_{xuu})u_x^3 - \xi_{uuu}u_x^4 + \tau_{xxx}u_t - 3\tau_{xxu}u_x u_t - 3\,\tau_{xuu}u_x^2 u_t - \tau_{uuu}u_x^3 u_t - 3\tau_{xx}u_{xt} + (3\eta_{xu} - \xi_{xx})u_{xx} + (3\eta_{uu} - 9\xi_{xu})u_x u_{xx} - 6\xi_{uu}u_x^2 u_{xx} - 6\tau_{xu}u_x u_{xt} - 3\tau_{uu}u_x^2 u_{xt} - 3\tau_{xu}u_t u_{xx} - 3\xi_u u_{xx}^2 - 3\tau_u u_{xt}u_{xx} - 3\tau_{uu}u_x u_{xx}u_t + (\eta_u - 3\xi_x)u_{xxx} - 4\xi_u u_x u_{xxx} - 3\tau_x u_{xxt} - 3\tau_x u_{xxt} - 3\tau u_x u_{xxt} - \tau_u u_t u_{xxx}] = 0} . \tag{12}$$

2.2.1 The monomials

Substituting $u_{xxx} = -(u_t + u_x^2) + u_{xxx}$ into (12) and separating the result it in terms of the factors constituted by u_x, u_t , u_{xx} of and u_{xxt} leads to monomials:

$$\boldsymbol{u_{xxt}: \tau_x = 0} \;, \tag{13}$$

$$\boldsymbol{u_x u_{xxt}: \tau_u = 0} \;, \tag{14}$$

$$\boldsymbol{u_x u_t: \xi_u = 0} \;, \tag{15}$$

$$\boldsymbol{u_x u_{xx}: \eta_{uu} = 0} \;, \tag{16}$$

$$\boldsymbol{u_{xx}: \eta_{xu} - \xi_{xx} = 0} \;, \tag{17}$$

$$\boldsymbol{u_t: 3\xi_x - \tau_t = 0} \;, \tag{18}$$

$$\boldsymbol{u_x: 2\eta_x - \xi_t + 3\eta_{xxu} - \xi_{xxx} = 0}, \tag{19}$$

$$\boldsymbol{u_x^2: \eta_u + \xi\_x = 0} \;, \tag{20}$$

$$\boldsymbol{u^0: \eta_t + \eta_{xxx} = 0}. \tag{21}$$

We began with equations (13) and (14), I solving them. We notice that the two lead to

$$\boldsymbol{\tau = a(t)} \;. \tag{22}$$

From equation (22), we have

$$\boldsymbol{\xi = b(x,t)} \;. \tag{23}$$

Equation (16) gives

$$\boldsymbol{\eta = c(x,t)u + d(x,t)} \;. \tag{24}$$

We substitute (22) into (18) and have

$$\boldsymbol{3\xi_x - a'(t) = 0} \;, \tag{25}$$

$$\boldsymbol{\xi = \frac{1}{3}\, a'(t)x + e(t)} \;. \tag{26}$$

This we follow with the substitution of (25) and (12) into (17) to yield

$$\boldsymbol{c_x(x,t) - 0 = 0}, \tag{27}$$

which gives $c(x,t) = f(t)$.

Using (26) we can rewrite (24) as

$$\boldsymbol{\eta = f(t)u + d(x,t)}. \tag{28}$$

Substituting (27) into (21) gives

$$\boldsymbol{f'(t)u + d_t(x,t) + d_{xxx}(x,t) = 0} \;. \tag{29}$$

We now consider the coefficients of various powers of u in equation (29), and set them to equal 0. This gives

$$\boldsymbol{u^0}: \boldsymbol{d_t + d_{xxx} = 0}\ , \qquad (30)$$

$$\boldsymbol{u^1}: \boldsymbol{f'(t) = 0}\ . \qquad (31)$$

To proceed we introduce arbitrary constants A_i, with $i = 1, 2, 3, \ldots$.

From (31), we have

$$\boldsymbol{f = A_1}\ . \qquad (32)$$

From (27), and using (32), we have

$$\boldsymbol{f = c = A_1}. \qquad (33)$$

Substituting (26) and (28) into (20), and applying (33), we get $A_1 + \frac{1}{3}a'(t) = 0$, so that

$$\boldsymbol{a = -3A_1 t + A_2}\ . \qquad (34)$$

We substitute (26) and (28) in (19), we arrive at:

$$\boldsymbol{2d_x - \frac{1}{3}\, a''x - e' = 0} \qquad (35)$$

Differentiating (35) with respect to t:

$$\boldsymbol{a'' = 0}. \qquad (36)$$

We substitute (36) into (35):

$$\boldsymbol{2d_x - e' = 0}. \qquad (37)$$

Now differentiate (37) with respect to x:

$$\boldsymbol{d_{xx} = 0}\ . \qquad (38)$$

We substitute (38) into (30) to get

$$\boldsymbol{d_t = 0}\ . \qquad (39)$$

This yields

$$\boldsymbol{d = h(x)}. \qquad (40)$$

When we differentiate (40) twice and substitute the result into (38), to get

$$\boldsymbol{d_{xx} = h_{xx} = 0}. \qquad (41)$$

From (41) we have

$$\boldsymbol{h = A_3 x + A_4} \tag{42}$$

Differentiating (42) with respect to x, and substituting the result into (37), we get $e' = 2A_3$, which solves into

$$\boldsymbol{e = A_3 t + A_5} \ . \tag{43}$$

We have now fully determined the infinitesimals. That is, the defining equations

$$\boldsymbol{\tau = -3tA_1 + A_2} , \tag{44}$$

$$\boldsymbol{\xi = -A_1 x + A_3 t + A_5} \ , \tag{45}$$

$$\boldsymbol{\eta = A_1 u + A_3 x + A_4} \ . \tag{46}$$

2.2.2 The Symmetries

To get the symmetries, we substitute the defining equations (44), (45) and (46) into the generator (2). The subsequent results are

$$\boldsymbol{X_1 = -3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}} , \tag{47}$$

$$\boldsymbol{X_2 = \frac{\partial}{\partial t}} , \tag{48}$$

$$\boldsymbol{X_3 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}} , \tag{49}$$

$$\boldsymbol{X_4 = \frac{\partial}{\partial u}} , \tag{50}$$

$$\boldsymbol{X_5 = \frac{\partial}{\partial x}} . \tag{51}$$

2.3. Invariant Solutions

The symmetries lead to invariants. The tool to use is the characteristic equation

$$\boldsymbol{\frac{dx}{\xi(t,x,u)} = \frac{dt}{\tau(t,x,u)} = \frac{du}{\eta(t,x,u)}} . \tag{52}$$

2.3.1 The case for X_1

The characteristic equations related to X_1 are

$$\boldsymbol{\frac{dx}{-x} = \frac{dt}{-3t} = \frac{du}{u}} . \tag{53}$$

From (53) we determine our first invariant as follows:

$$\frac{dx}{-x} = \frac{dt}{-3t}, \quad (54)$$

$$\frac{dx}{x} = \frac{dt}{3t}, \quad (55)$$

$$\ln x = \frac{1}{3}\ln t + \ln K_1 , \quad (56)$$

$$K_1 = xt^{-\frac{1}{3}} , \quad (57)$$

$$r = xt^{-\frac{1}{3}} . \quad (58)$$

We now use (53) to determine the second invariant:

$$\frac{dt}{-3t} = \frac{du}{u}, \quad (59)$$

$$-\frac{1}{3}\ln t + \ln K_2 = \ln u , \quad (60)$$

$$K_2 = ut^{-\frac{1}{3}} , \quad (61)$$

$$v = ut^{\frac{1}{3}} . \quad (62)$$

We use (62) to define u in terms of v:

$$u = vt^{-\frac{1}{3}} . \quad (63)$$

From our first invariant we can define v as

$$v = F(r) . \quad (64)$$

This allows us to rewrite u as

$$u = t^{-\frac{1}{3}}F(xt^{-\frac{1}{3}}) . \quad (65)$$

From (65), we determine u_t, u_x and u_{xxx}:

$$u_x = t^{-\frac{2}{3}} v_r , \quad (66)$$

$$u_{xx} = t^{-1} v_{rr} , \quad (67)$$

$$u_{xxx} = t^{-\frac{4}{3}} v_{rrr} , \quad (68)$$

$$u_t = -\frac{1}{3} t^{-\frac{4}{3}} (v + v_r) . \quad (69)$$

We substitute (66) to (69) into (1), and arrive at:

$$-\frac{1}{3} t^{-\frac{4}{3}} (v + v_r) + \left(t^{-\frac{2}{3}} v_r\right)^2 + t^{-\frac{4}{3}} v_{rrr} = 0 . \quad (70)$$

From (70) we arrive at the ODE

$$-\frac{1}{3} (v + v_r) + v_r^2 + v_{rrr} = 0. \quad (71)$$

3. THE DIFFERENTIABLE TOPOLOGICAL MANIFOLDS

In this section, we lay a foundation and elaborate on the approach used to find the solution of ODE Equation (71), which is borrowed from the method variation of parameters, that is mostly used to solve second-order non-homogeneous linear ordinary differential equations.

$$\ell \frac{d^2y}{dx^2} + m \frac{dy}{dx} + ny = \mathcal{J}(x) \quad (72)$$

Where ℓ, m and n are constants.

3.1 The variation of parameters method

We first consider the homogeneous form of (72)

$$\mathcal{J}(x) = 0, \quad (73)$$

so that we get

$$\ell \frac{d^2y}{dx^2} + m \frac{dy}{dx} + ny = 0 \quad (74)$$

The complementary solution to (74) is given as

$$y_C = C_1 y_1 + C_2 y_2 \quad (75)$$

Where C_1 and C_2 are the parameters that need to be varied. At times, we will have to let

$$\varsigma_i = C_i, \quad where \quad i = 1, 2. \quad (76)$$

Then we get this particular solution

$$\boldsymbol{y_a = \varsigma_1 y_1 + \varsigma_2 y_2} \qquad (77)$$

We therefore have the general solution given as

$$\boldsymbol{y = y_C + y_a}, \qquad (78)$$

We will make use of these assumptions going forward. But the assumption that gave rise to (74) will be interpreted as describing points within quotient spaces. And the second assumption that lead to (76) , relates the space to the entire differentiable topological manifold.

3.2 Differentiable Topological Manifold

We begin with the Haupsdorff topological space $\mathcal{S} = (\mathcal{X}, J_{\mathcal{X}})$, defined as a set $\mathcal{X}$ with a topology $J_{\mathcal{X}}$. And so, for it to be a differentiable topological manifold or simply a differentiable manifold, we need an atlas D. Thereafter we have $K\mathcal{S} = (\mathcal{X}, J_{\mathcal{X}}, D)$.

We then consider points $r \in U_r$ and $s \in U_s$ such that the sets U_r and U_s are elements of the same manifold that is considered. We now can build the sub-topologies $(U_r, J_{\mathcal{X}}|_{U_r})$ and $(U_s, J_{\mathcal{X}}|_{U_s})$. If a mapping T_a, exists, and maps the space $(U_r, J_{\mathcal{X}}|_{U_r})$ into the Euclidean space $(\mathbb{R}^N, \mathcal{S}_{\mathbb{R}^N}|_{T_r}(U_r)$. Also, T_s maps $(U_s, J_{\mathcal{X}}|_{U_s})$ into the Euclidian space $(\mathbb{R}^N, \mathcal{S}_{\mathbb{R}^N}|_{T_s}(U_s)$.

If these mapping are homomorphisms, then the set

$$\boldsymbol{K} = \{(\boldsymbol{U_r}, \boldsymbol{T_r}), (\boldsymbol{U_s}, \boldsymbol{T_s})\}, \qquad (79)$$

is called an atlas, where T_r, T_s are called coordinates.

We now concentrate on one of the cherts mapping equivalence classes

$$\boldsymbol{K} = \{([\boldsymbol{U_r}], [\boldsymbol{T_r}]), (\boldsymbol{U_s}, \boldsymbol{T_s})\}. \qquad (80)$$

Similarly, for mapping manifolds in derivatives of T, we get atlases,

$$\text{Ꝁ}^{(i)} = \left\{\left([\boldsymbol{U_r}], \left[\mathbf{T}^{(i)}_{(r)}\right]\right), \left(\boldsymbol{U_s}, \mathbf{T}^{(i)}_{(s)}\right)\right\}. \qquad (81)$$

3.2.1 Transition mapping

The mapping from $\left(\mathbb{R}^N, J_{\mathbb{R}}|_{\mathrm{T}([U_r])}\right)$ to $\left(\mathbb{R}^N, J_{\mathbb{R}}|_{\mathrm{T}([U_s])}\right)$, having stepped down from $\mathbb{R}^N$ to $\mathbb{R}$, is given by

$$\boldsymbol{T_r}\left(\boldsymbol{T_s^{-1}}\left(\boldsymbol{T_s}([\boldsymbol{U_r}])\right)\right), \qquad (82)$$

is called a transition mapping, and its inverse

$$\boldsymbol{T}_{s}\left(\boldsymbol{T}_{r}^{-1}\left(\boldsymbol{T}_{r}([\boldsymbol{U}_{s}])\right)\right). \qquad (83)$$

Our main focus is in the cases where $[U_r]$ and $[U_s]$ overlap such that we have point w in the neighbourhood of both r and q such that:

$$\left[\boldsymbol{T}[\boldsymbol{w}]\right] = \boldsymbol{T}(\boldsymbol{w}). \qquad (84)$$

The transmission mapping in derivative spaces, leads to

$$\frac{\boldsymbol{d^n}[\boldsymbol{T}[\boldsymbol{w}]]}{\boldsymbol{dw^n}} = \frac{\boldsymbol{d^n T}(\boldsymbol{w})}{\boldsymbol{dw^n}}, \qquad (85)$$

for $\text{n} = 1, 2, 3, \ldots$

3.2.2 Tangent Spaces

The tangent spaces are useful in finding a function $\mathcal{J}$, that projects the results into metric space. A vector can be represented as a tangent space, as

$$\boldsymbol{PQ} = \left\{\boldsymbol{V}_{\boldsymbol{\Upsilon},\boldsymbol{Q}} \,\middle|\, \boldsymbol{\Upsilon} : \mathbb{R} \rightarrow \boldsymbol{\mathcal{X}}\right\}, \qquad (86)$$

such that

$$\boldsymbol{V}_{\boldsymbol{\Upsilon},\boldsymbol{Q}}\boldsymbol{\mathcal{J}} = (\boldsymbol{\mathcal{J}} \circ \boldsymbol{\Upsilon}^{-1})[\boldsymbol{\Upsilon}(\boldsymbol{\tau_0})], \qquad (87)$$

where $\mathcal{J} \epsilon C^{\infty}(\mathcal{X}), V_{\Upsilon,Q}: C^{\infty}(M) \rightarrow \mathbb{R}, \Upsilon(\tau_0) = Q$.

Now PQ is a tangent space which has the basis vectors $\{\partial \mathcal{X}^i\}$. Thus represented by

$$\boldsymbol{\mathcal{X}} = \boldsymbol{\xi^i} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial \mathcal{X}^i}} |_{\boldsymbol{Q}}, \qquad (88)$$

where $\mathcal{X} \epsilon P_r\mathcal{X} = P_r M$.

3.2.3 Cotangent Spaces

A tangent space is a vector space, and where there exist a vector space there should also be a co-vector space, and hence the cotangent space. It is the set of all maps in the tangent space to $\mathbb{R}$. That is,

$$\boldsymbol{\omega} : \boldsymbol{P_r \mathcal{X}} \rightarrow \mathbb{R}, \qquad (89)$$

where ω is an element of the cotangent space. A cotangent space is

$$\boldsymbol{PQ}^* = \{(\boldsymbol{d\mathcal{J}})_{\boldsymbol{r}} | \boldsymbol{\mathcal{J}} \in \boldsymbol{C}^{\infty}(\boldsymbol{\mathcal{X}})\}, \qquad (90)$$

and is a vector space with the dual of PQ.

The basis of a cotangent space needs

$$(\boldsymbol{d\omega^j})_{\boldsymbol{r}}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}}\right)|_{\boldsymbol{Q}} = \boldsymbol{\sigma_i^j}, \tag{91}$$

so that:

$$(\boldsymbol{\mathcal{P}Q}^*) = \{\frac{\boldsymbol{\partial}}{\boldsymbol{\partial w^i}}\}|_{\boldsymbol{r}} \ . \tag{92}$$

And thereafter we can write ω, as an element of of TQ^* as

$$\boldsymbol{\omega} = \boldsymbol{\omega_i}(\boldsymbol{dw^i})|_{\boldsymbol{r}} \ . \tag{93}$$

3.3 Quotient Spaces

The general ordinary differential equation

$$\boldsymbol{\mathcal{J}}(\boldsymbol{w}, \boldsymbol{T}, \boldsymbol{T}', \boldsymbol{T}'', \boldsymbol{T^3}, \dots) \tag{94}$$

with

$$\boldsymbol{T}: \boldsymbol{\mathcal{X}} \to \boldsymbol{Y} \ , \tag{95}$$

a set

$$\boldsymbol{l} = \{\boldsymbol{w_0}, \boldsymbol{w_1}, \boldsymbol{w_2}, \dots\} \subset \boldsymbol{\mathcal{X}}, \tag{96}$$

such that

$$\boldsymbol{w_i} = \boldsymbol{Q}(\boldsymbol{w_j}) = \boldsymbol{w_j} + \boldsymbol{2\pi t_a} \ , \tag{97}$$

with t_a being an equivalence class, which is an integer. Which leads to a quotient space, $\mathbb{R}/\sim$, that is a set of all equivalent classes given by

$$\mathbb{R}/\sim = \{[\boldsymbol{w_0}], [\ \boldsymbol{w_1}], [\boldsymbol{w_2}], \dots\}. \tag{98}$$

Equation (98) generates a differentiable topological Space. In this state, the image of topological space is an equivalence class:

$$\boldsymbol{l} = \{[\boldsymbol{T}(\boldsymbol{w_0})], [\boldsymbol{T}(\boldsymbol{w_1})], [\boldsymbol{T}(\boldsymbol{w_2})], \dots\} \ , \tag{99}$$

which is a homomorphism that extends to the derivative spaces P, which is equal to:

$$\{[\boldsymbol{T}^{(i)}(\boldsymbol{w_0})], [\boldsymbol{T}^{(i)}(\boldsymbol{w_1})], [\boldsymbol{T}^{(i)}(\boldsymbol{w_2})], \dots\} \ , \tag{100}$$

for $i = 1, 2, 3, \dots$

4 THE SOLUTION THROUGH X_1

We now apply the method in section 3 to the ODE which was obtained with the use of the invariant solutions. We differentiate equation (71) to get:

$$v^{(4)}(y) + 2v'(y)v''(y) + \frac{1}{3}(-yv''(y) - 2v'(y)) = 0. \quad (101)$$

From equation (101), we set $v^{(4)} = 0$ and $v'' = 0$, we get:

$$-\frac{2}{3}v'(y) = 0. \quad (102)$$

We then differentiate, setting $v'' = \frac{a\sin(i\omega y+\phi)}{i\omega}$ and $\phi = 0$, we get

$$v'' = \frac{a\sin(i\omega y)}{i\omega}. \quad (103)$$

From equation (103) above, we can determine:

$$v^{(3)} = a\,cos(i\omega y), \quad (104)$$

$$v^{(4)} = -ai\omega\,sin(i\omega y), \quad (105)$$

$$v' = -\frac{a\,cos(i\omega y)}{(i\omega)^2} + b_1, \quad (106)$$

$$v = -\frac{a\,sin(i\omega y)}{(i\omega)^3} + 2yb_1 + b_2. \quad (107)$$

Then the equivalence classes, setting sin $(i\omega y) = 0$ and cos $(i\omega\, y) = 1$, are:

$$[v^{(4)}] = 0, \quad (108)$$

$$[v^{(3)}] = a, \quad (109)$$

$$[v''] = 0, \quad (110)$$

$$[v'] = -\frac{a}{(i\omega)^2} + b_1. \quad (111)$$

And the finally, we have

$$[v] = 2yb_1 + b_2. \quad (112)$$

The equivalence classes are then satisfied.

We now have

$$b_1 = -\frac{a}{\omega^2}\,. \quad .(113)$$

We then integrate the original ODE, and get

$$a\,cos(i\omega y) + \left(-\frac{a}{\omega^2} - \frac{a\,cos(i\omega y)}{(i\omega)^2}\right)^2 + \frac{1}{3}\left(\frac{2ay}{\omega^2} - y\left(-\frac{a}{\omega^2} - \frac{a\,cos(i\omega y)}{(i\omega)^2}\right) + \frac{a\,cos(i\omega y)}{(i\omega)^3} - b_2\right) = 0. \quad (114)$$

Setting $\cos(i\omega y) = 1,\ \sin(i\omega y) = 0$ and expanding yields

$$\frac{3a^2y}{2\omega^4} + \frac{ay^2}{2\omega^2} - \frac{yb_2}{3} + c_1 = 0. \quad (115)$$

Solving (115) gives

$$b_2 = \frac{3(a^2y + ay^2\omega^2 + 2\omega^2c_1)}{2y\omega^4}. \quad (116)$$

We then integrate for the second time, using the same settings we set in the first integration, and solve:

$$-\frac{15a^2}{8\omega^6} - \frac{ay}{3\omega^4} + \frac{3a^2y^2}{4\omega^4} + \frac{a}{\omega^2} + \frac{ay^3}{6\omega^2} + yc_1 + b_1 + c_2 = 0, \quad (117)$$

gives $b_1 = \frac{y(3a^2y + ay^2\omega^2 + 2\omega^4c_1)}{4\omega^4}$.

Integrating for the third time, with the same settings made previously, we have:

$$\frac{2a}{3\omega^6} + \frac{a^2y^3}{4\omega^4} + \frac{ay^4}{24\omega^2} + \frac{y^2c_1}{2} - B_2 + yc_2 + c_3 = 0, \quad (118)$$

gives $B_2 = \frac{y^2(3a^2y + ay^2\omega^2 + 2\omega^4c_1)}{12\omega^4}$.

We can therefore determine a as

$$a = \frac{8\omega^6(y^2c_1 + 3yc_2 + 3c_3)}{y^4\omega^6 - 16}\,. \quad (119)$$

And we now determine ω. Collecting the numerator:

$$\omega^2\,(256\,c_1\,y^3 + 768\,c_2\,y^2 + 768\,c_3y) + \omega^4(-32\,c_1y^5 + 192\,c_3y^3 - 768\,c_1\,y^2 - 2304\,c_2\,y - 2304c_3) + \omega^8\,(-c_1y^9 - 6c_2\,y^8 - 12c_3y^7 + 48\,c_1\,y^6 + 144\,c_2\,y^5 + 144\,c_3\,y^4) + \omega^6\,(-16\,c_1\,y^7 - 48\,c_2\,y^6 - 48\,c_3\,y^5 - 720\,c_1^2\,y^4 - \quad (120)$$

$4320\, c_1\, c_2\, y^3 - 6480\, c_2^2\, y^2 -$
$4320\, c_1\, c_3\, y^2 - 12960\, c_2\, c_3\, y -$
$6480\, c_3^2) + 768\, c_1\, y + 1536\, c_2 = 0.$

Tis equation solves into eight solutions. All cannot be displayed her. We thus present only solution. It solves into

$$\omega^2 = -\sqrt{f_1 - \frac{f_2}{f_3} + \frac{f_4}{f_5} - \frac{1}{2}\sqrt{\frac{f_6}{f_7} - \frac{f_8}{f_9} - \frac{f_{10}}{f_{11}}}}\ , \quad (121)$$

where

$$f_1 = -4\frac{C_1 y^3}{g_1} - 12\, C_2 \frac{y^2}{g_1} - 12\, C_3 \frac{y}{g_1} - 180\frac{C_1^2}{g_1} - 1080\frac{C_1 C_2}{g_1} - 1620\frac{C_2^2}{y^2 g_1} - 1080\frac{C_1 C_3}{y^2 g_1} - \frac{3240 C_2 C_3}{y^3 g_1} - 1620\frac{C_3^2}{y^4 g_1} \quad (122)$$

$$g_1 = C_1\, y^5 + 6\, C_2\, y^4 + 12\, C\_3\, y\text{^}3 - 48\, C\_1\, y\text{^}2 - 144\, C_2\, y - 144\, C_3 \quad (123)$$

$$f_2 = \frac{-32\, C_1 y^5 - 6 C_3 y^3 + 72\, C_3}{y^4 g_1} + \frac{64\,(g_3)^2}{y^8 g_1^2} + \frac{32\, g_4}{3 g_1^2} + \sqrt{(1024\, g_4} - 9216\,(C_1\, y + 2\, C_2)\,(C_1\, y^9 + 6\, C_2\, y^8 + 12\, C_3 y^7 - 48\, C_1\, y^6 - 144\, C_2\, y^5 - 144\, C_3\, y^4)^3 + 12288\,(C_1\, y^3 + 3\, C_2\, y^2 + 3\, C_3 y)(C_1\, y^7 + 3\, C_2\, y^6 + 3\, C_3\, y^5 + 45\, C_1^2\, y^4 + 270\, C_1\, C_2\, y^3 + 405\, C_2^2\, y^2 + 270\, C_1\, C_3\, y^2 + 810\, C_2\, C_3\, y + 405\, C_3^2), \quad (124)$$

$$g_3 = C_1\, y^7 + 3\, C_2\, y^6 + 3\, C_3\, y^5 + 45\, C_1^2\, y^4 + 270\, C_1\, C_2\, y^3 + 405\, C_2^2\, y^2 + 270\, C_1\, C_3\, y^2 + 810\, C_2\, C_3\, y + 405\, C_3^2\ , \quad (125)$$

$$g_4 = C_1 y^5 - 6\, C_3\, y^3 + 24\, C_1\, y^2 + 72\, C_2\, y + 72\, C_3\,, \quad (126)$$

$$f_3 = 3y^4(C_1 y^5 + 6\, C_2 y^4 + 12\, C_3 y^3 - 48\, C_1 y^2 - 144 C_2\, y - 144\, C_3)(65536\,(C_1\, y^5 - 6\, C_3\, y^3 + 24\, C_1\, y^2 + 72\, C_2\, y + 72\, C_3)^3 + 1769472\,(C_1\, y^3 + 3\, C_2\, y^2 + 3\, C_3\, y)^2(g_5) + 1769472\,(C_1 y + 2\, C_2)(C_1 y^5 - 6\, C_3 y^3 + 24\, C_1 y^2 + 72\, C_2 y + 72\, C_3)g_5 + 1179648(C_1\, y^3 + 3\, C_2\, y^2 + 3\, C_3\, y)(C_1\, y^5 - 6\, C_3\, y^3 + 24\, C_1\, y^2 + 72\, C_2\, y + 72\, C_3)(C_1\, y^7 + 3\, C_2\, y^6 + 3\, C_3\, y^5 + 45\, C_1^2\, y^4 + 270\, C_1\, C_2\, y^3 + 405\, C_2^2\, y^2 + 270\, C_1\, C_3\, y^2 + 810\, C_2\, C_3\, y + 405\, C_3^2) + (65536\,(C_1\, y^5 - 6\, C_3\, y^3 + 24\, C_1\, y^2 + 72\, C_2\, y + 72\, C_3)^3 + 1769472\,(C_1 y^3 + 3\, C_2\, y^2 + 3\, C_3\, y)^2 g_5 + 1769472\,(C_1 y + 2\, C_2)(C_1 y^5 - 6\, C_3 y^3 + 24\, C_1 y^2 + 72\, C_2 y + 72\, C_3)g_5, \quad (127)$$

$$g_5 = C_1\, y^9 + 6\, C_2\, y^8 + 12\, C_3\, y^7 - 48\, C_1\, y^6 - 144\, C_2\, y^5 - 144\, C_3\, y^4\,, \quad (128)$$

$$f_4 = \frac{1}{2}\sqrt{}(-\frac{g_8\{128 g_7\}}{g_9} - \frac{\{32\,(C_1 y^5 - 6\, C_3 y^3 + 24\, C_1 y^2 + 72\, C_2 y + 72\, C_3)\}}{\{3\, g_6\}} - \left(\sqrt{2}\right)^3(1024\,(C_1 y^5 - 6\, C_3 y^3 + 24\, C_1 y^2 + 72\, C_2 y + 72\, C_3)^2 - 9216(yC_1 + 2C_2)g_5 + 12288(y^3 C_1 + 3y^2 C_2 + 3yC_3)(y^7 C_1 + 45y^4 C_1^2 + 3y^6 C_2 + 270y^3 C_1 C_2 + 405y^2 C_2^2 + 3y^5 C_3 + 270y^2 C_1 C_3 + 810yC_2C_3 + 405C_3^2), \quad (129)$$

$$g_6 = (C_1y^9 + 6\,C_2y^8 + 12\,C_3y^7 - 48\,C_1y^6 - 144\,C_2y^5 - 144\,C_3y^4), \tag{130}$$

$$g_7 = (C_1y^7 + 3\,C_2y^6 + 3\,C_3y^5 + 45\,C_1^2y^4 + 270\,C_1C_2y^3 + 405\,C_2^2y^2 + 270\,C_1C_3y^2 + 810\,C_2C_3y + 405\,C_3^2)^2 \ , \tag{131}$$

$$f_5 = (3y^4g_1)(65536\,k_1^3 + 1769472(y^3C_1 + 3y^2C_2 + 3yC_3)^2g_5 + 1769472(yC_1 + 2C_2)(24y^2C_1 + y^5C_1 + 72yC_2 + 72C_3 - 6y^3C_3)g_6 + 1179648(y^3C_1 + 3y^2C_2 + 3yC_3)k_1k_2 - 5308416(yC_1 + 2C_2)k_2 - (65536k_1 1769472(y^3C_1 + 3y^2C_2 + 3yC_3)^2g_5 + 1179648(y^3C_1 + 3y^2C_2 + 3yC_3)k_1k_2, \tag{132}$$

$$k_1 = 24y^2C_1 + y^5C_1 + 72yC_2 + 72C_3 - 6y^3C_3 \tag{130}$$

$$k_2 = y^7C_1 + 45y^4C_1^2 + 3y^6C_2 + 270y^3C_1C_2 + 405y^2C_2^2 + 3y^5C_3 + 270y^2C_1C_3 + 810yC_2C_3 + 405C_3^2\ , \tag{131}$$

$$f_7 = \left(3 * 2^{1/3}y^4(g_5)\right) - \left(\frac{2048\left(y^2C_1+3yC_2+3C_3\right)}{y^3(g_5)}\right) + \frac{2048k_1k_2}{y^8(g_5)^2} - \frac{4096(k_2)^3}{y^{12}(g_5)^3}, \tag{132}$$

$$f_8 = \left(2^{1/3}\left(1024(k_1)^2 - 9216(yC_1 + 2C_2)(g_5) + 12288(y^3C_1 + 3y^2C_2 + 3yC_3)(\mathrm{k}_2)\right)\right), \tag{133}$$

$$f_9 = 3y^4(g_5)(65536(k_1)^3 + 1769472(y^3C_1 + 3y^2C_2 + 3yC_3)^2(g_5) + 1769472(yC_1 + 2C_2)(g_5k_1) + 1179648(y^3C_1 + 3y^2C_2 + 3yC_3)(k_1k_2) - 5308416(yC_1 + 2C_2)(\mathrm{k}_2)^2 + \sqrt{}(-4(1024(k_1)^2 - 9216(yC_1 + 2C_2)(g_5) + 12288(y^3C_1 + 3y^2C_2 + 3yC_3)(\mathrm{k}_2))^3 + (65536(k_1)^3 + 1769472(y^3C_1 + 3y^2C_2 + 3yC_3)^2(g_5) + 1769472(yC_1 + 2C_2)(g_5k_1) + 1179648(y^3C_1 + 3y^2C_2 + 3yC_3)(k_1k_2) - 5308416(yC_1 + 2C_2)(\mathrm{k}_2)^2)^2))^{1/3}\ , \tag{134}$$

$$f_{10} = 65536(k_1)^3 + 1769472(y^3C_1 + 3y^2C_2 + 3yC_3)^2(g_5) + 1769472(yC_1 + 2C_2)(g_5k_1) + 1179648(y^3C_1 + 3y^2C_2 + 3yC_3)(k_1k_2) - 5308416(yC_1 + 2C_2)(k_2)^2 + \sqrt{}\left(-4\left(1024(k_1)^2 - 9216(yC_1 + 2C_2)(g_5) + 12288(y^3C_1 + 3y^2C_2 + 3yC_3)(k_2)\right)^3 + (65536(k_1)^3 + 1769472(y^3C_1 + 3y^2C_2 + 3yC_3)^2(g_5) + 1179648(y^3C_1 + 3y^2C_2 + 3yC_3)(k_1k_2) - 5308416(yC_1 + 2C_2)^2)^2)\right)^{\frac{1}{3}}\ , \tag{135}$$

$$= 3 * 2^{1/3}y^4g_1\ . \tag{136}$$

We can use the above solution of equation (120) to carry out a conclusion and visualize the solution. The solution in v can be presented as

$$v[y] = -\frac{a\text{Sin}[iy\omega]}{i^3\omega^3} + 2yb_1 + b_2. \quad (140)$$

A plot is possible when all parameters are known. We examine the case for which $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, and is plotted in Figure 1. This is the solution in u.

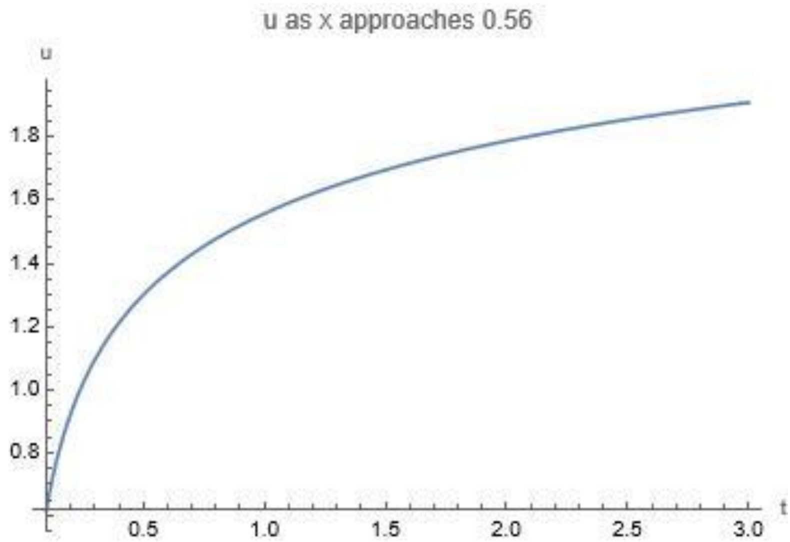

Figure 1: 2D Plot of u.

The solution illustrated in Figure 1, can also be visualized in 3D.

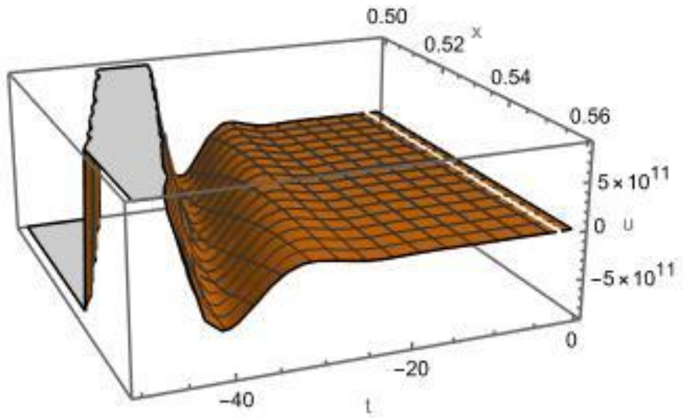

Figure 2: 3D plot of figure 1.

Figure 3 compares favorably with our results, presented in Figure 2.

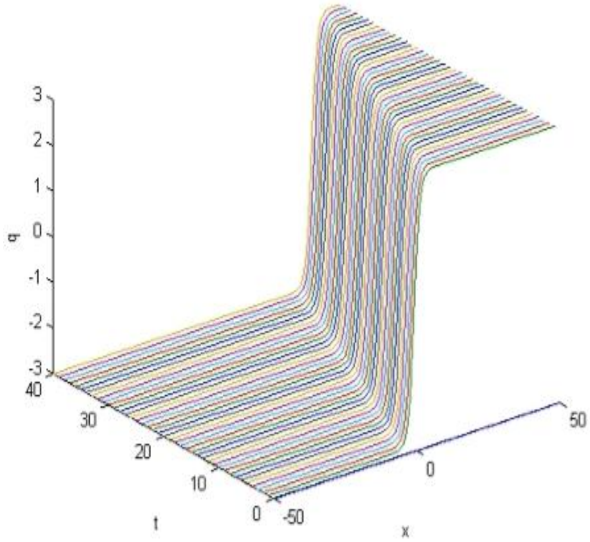

Figure 3 : Numerical solution for *q* by [9]. Biswas, Kumar, Krishnan, Ahmed, Strong, Johnson, and Yildrin have shown Figure 3 corresponds with the results that lead to Figure 2.

CONCLUSION

In the figures illustrated in the previous section, two different methods were used to arrive to a solution. Figure 3 is a small portion of Figure 2, therefore the two figures are similar. The methods in [16], although successful at arriving at a solution, is tedious. This approach, requires setting numerical results, as can be seen by the plot. In this paper we have illustrated that the method of differentiable manifolds, as introduced by [14], yields solutions that are not restricted by any boundaries or assumptions. Figure 1 illustrates that our results compare very well with the numerical results generated by a computer. The solutions obtained through coupling Lie symmetries with differentiable manifolds are superior than those previously obtained. Our solutions were not reliant on any assumptions or restrictions on parameters.

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