The Ripple Effect in a Supply Chain: A Sudden Demand Increase and with a Sinusoidal Component

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This paper studies the backward travel of a disturbance ripple up the chain to the upstream units subsequent to a disturbance at the downstream end, on which topic there have been a lesser number of studies. The nature of the problem leads quite naturally to the use of dynamic analysis; and hence we examine the ripple effect in the chain through the study of the dynamic behavior of a three-stage chain. We derive the dynamic behavior of each of the stages under the most common and frequently encountered dynamic replenishment control schemes under such conditions. Concomitantly we derive sufficient conditions on the control parameter settings for the arrest and prevention of the back propagation of the disturbance upstream. A notable feature of these dynamic controls is that they can be made pro-active as is shown in the paper to effectively decouple upstream stages from the downstream ones. The paper delineates the roles of each of the control parameters and presents the planner with effective ways and means to arrest and prevent the backward travel of the ripple to upstream units. The paper also makes a case for effective information sharing between the stages in order to achieve this objective.

Keywords: Supply Chain Ripple Effect, Dynamic Modelling, Dynamic Replenishment Controls, Sudden Increase in Demand

1. Introduction

Two fundamental strategies for supply chains (SCs) are those of an 'efficient' chain with focus on cost minimization but with a somewhat higher response time, versus a 'responsive' chain with focus on response time, albeit with a marginally higher cost (Chopra and Meindl 2016). While an efficient strategy is more suited to a relatively predictable environment with low uncertainty, the responsive strategy is more useful in an environment of higher uncertainty. A vital requirement of a responsive chain is that it should be able to respond quickly to any/unanticipated changes in demand. However, when the demand increase is *sudden and sustained*, the system can go into stock-outs and/or back-order positions, which can be very deleterious to its performance. Simultaneously, the effect of the disturbance propagates backwards along the chain to the upstream units sequentially, thereby causing a ripple effect along and up the chain. Thus the inventories and flows all along the entire chain and upstream units can begin to fluctuate, leading to unsteady operation and delayed and/or insufficient replenishments. And one of the underlying factors impacting this ripple effect phenomenon in a major way is the *dynamics* of the system.

Concomitantly, the determination of the inventory levels to be carried at various points in the chain to maintain uninterrupted material flow is also governed by the dynamics of the system. In such situations, the use of *dynamic* replenishment control schemes could be an effective way to steady the system, smooth out fluctuations, and consequently arrest and prevent the back propagation of the disturbance up the chain to upstream units. This would not only damp out and minimize the ripple effect along the chain, but also enhance its responsiveness performance. Though such sudden increases in demand can be expected to be encountered in warehouses/SCs, the use of *dynamic controls* and their impact on the ripple effect phenomenon has not been found to have been investigated in specific terms and in detail in the literature.

Hence there is a felt need to be able to study, understand, and *predict* the dynamic behavior of a SC and the ripple effect under such conditions. This would then enable the design of better controls to minimize the ripple effect and enhance performance. In this paper, we address this need, and examine the performance of the most common and frequently encountered dynamic replenishment policies under such conditions. Simultaneously we also attempt to derive sufficient conditions to be satisfied by the control parameters, to arrest and stamp out the ripple in the chain. Thus, the important research questions that this paper seeks to answer are the following:

1) Can we model and *predict the behavior of the disturbance ripple* created by a sudden demand disturbance?
2) Can we *arrest and prevent the backward travel of the disturbance ripple* up the chain to the upstream units?
3) And if so, can we *derive sufficient conditions on the system parameters* which will achieve this?

And hence the key contributions of this paper would be the prediction and arrest of the backward travel of a disturbance ripple upstream through dynamic modeling on the one hand, and the determination of sufficient conditions on the system control parameters to achieve this phenomenon on the other.

To this end, section 2 briefly reviews the relevant literature while section 3 defines the dynamic modeling framework for the study. Section 4 examines the various types of control schemes, while section 5 derives the system response of the various stages of the chain in detail for a sudden demand increase (step disturbance). Section 6 then examines the behavior under an additional seasonal (sinusoidal) component in the disturbance and highlights the ripple effect along the chain. Section 7 then presents numerical examples illustrating the ripple effect, while Section 8 delineates the salient observations and summarizes the results. Section 9 then presents the practical significance and managerial implications of the study. Finally, Section 10 concludes the paper with a mention of the salient contributions, limitations of the study, and scope for further work.

2. Literature Review

2.1 Supply Chain Dynamics and the Ripple Effect

The literature on supply chain dynamics is both rich and vast, commencing with the application of control theory in production-control (Simon, 1952) and the use of system dynamic methodologies (Forrester 1958, 1961) subsequently. The survey papers of Axsater (1985), and Ortega and Lin (2004) comprehensively capture the subsequent use of these methods in production-inventory systems, while Simchi-Levi et al (2004a, 2004b) take up the mathematical modeling aspects. Several modeling aspects are also covered in the recent books by Marquez (2010), Dolgui and Proth (2010), and Ivanov and Sokolov (2010), and the edited book by Ivanov et. al. (2019). Recently there has been a very large amount of work on SC disruption and its mitigation (spearheaded by the numerous recent works of Ivanov, Dolgui, and Sokolov) which is too vast to describe here. We cite herein only a few of them which focus directly on the ripple effect.

Dolgui et. al. (2019) studies the cause-and-effect relationships between the ripple effect and the bullwhip effect on each other and shows through a simulation-based study that the ripple effect can be a driver of the bullwhip effect through backlog accumulation and proposes a contingent inventory control policy through *information sharing and coordination* to mitigate both these effects (thereby corroborating the results presented in our paper subsequently). Most recently, Ivanov (2019) proposes a method to mitigate 'disruption tails' (accumulated backlogs and delayed deliveries in the post-disruption period) through not only the contingent recovery policies during disruption but also a special 'revival policy' to restore and stabilize the ordering policies and performance post disruption and has also suggested the inclusion of the revival policy in the resilience framework.

Further motivation for our study stems for the works of Ivanov et. al. (2014) which looks at SC disruptions from a multi-disciplinary perspective and concludes that *quantitative analysis of the ripple effect could be phenomena that could consolidate SC disruption management*, thereby pointing to a direction for further research, Ivanov et. al. (2017) which presents a literature review on SC disruption recovery relating the existing quantitative methods to empirical research, and identifying future research needs and directions in the SC risk management domain including *response stabilization post deviations and disruptions, along with recovery and time aspects*, and Dolgui et. al. (2018) which presents an analysis of the literature on the ripple effect, classifying them into mathematical optimization, simulation, control theory, and complexity and reliability approaches and identifying some future research avenues including *study of disruption propagation and dynamic inventory policies*. Most recently Hosseini et.al. (2019) presents a comprehensive review of quantitative methods in SC resilience analysis identifying gaps in the literature and opportunities for future research, again mentioning the areas of *mathematical modeling and dynamic analysis*.

Thus, it can be seen a *mathematical study of disruption propagation and dynamic SC response stabilization using dynamic inventory control policies* have been identified by several researchers to be areas for further research in the SC and ripple effect domain. Our paper attempts to bridge this gap and takes up the study of disruption propagation in a supply chain using dynamic control analysis.

Notably, these dynamic controls can be made *pro-active* as is shown subsequently, as compared to the conventional optimal inventory controls which are static and hence more *reactive* in nature. Furthermore, and from a practical standpoint, sudden demand increases *cannot usually be anticipated in advance by the planner, or to a sufficient* *degree of accuracy*; and hence have the potential to perturb the entire chain, and hence would *need specific controls to steady the system*. And hence the apparent gap and pointers in the literature together with the practical standpoint further reinforce the need for such a study.

2.2 Dynamic Analysis

It is only recently that attention and research has started focusing on the *dynamic control* of *responsive supply chains* and *tuning of their control parameters* to improve their response times (Ortega and Lin 2004, Moudgalya 2007, Haralambos et. al. 2008, Nilakantan 2010), which would also consequently have a major impact on the damping out of the disturbance ripple. A few recent papers (Lin et. al. 2004; Nilakantan 2010) have analyzed the behavior of a single stage under sudden demand increases. Most recently, Nilakantan (2019) has analyzed SC performance under dynamic/sudden lead-time disturbances obtaining the system response under different control types for a single stage of a supply chain. Whereas our paper specifically takes up the study of the ripple effect in a multi-stage (three-stage) serial supply chain, with the focus being on the *control of the upstream travel of the ripple* up the chain.

3. The Dynamic Modeling Framework

3.1 Notation and Model Framework

We use the standard notation with deviation variables as is standard in the literature (e.g. Moudgalya 2007, Nilakantan 2019):

$x_i(k)$ is the inventory *deviation* (from its nominal value) at time k, at stage i of the chain

$q_i(k)$ is the material flow *deviation* (from its nominal value) in period (k-1, k], into stage i

$r_3(k)$ is the *deviation* (from the predicted) in demand observed at the warehouse in (k-1, k]

with: i = 1 representing (the raw material) the upstream end of the production facility

i = 2 representing (the finished goods) the downstream end of the production facility

i = 3 representing (the finished goods) the warehouse.

The dynamic equations of the system are then given by:

$$x_i(k+1) = x_i(k) + q_i(k+1) - q_{i+1}(k+1) \tag{3.1}$$

Where for i = 3, $q_4(k) = r_3(k)$, the demand outflow from the warehouse.

The standard initial conditions are: $\{ (x_i(k), q_i(k), r_3(k)) = (0,0,0), k \le 0 \}$. The control variables are the replenishment flows, through which control is exercised over the system.

The Demand in the downstream end normally can be split into two distinct components, viz. the Mean Demand, and a (stochastic) random component, a detailed procedure for which can be found in Gouriroux and Monfort (1990). The random component represents the residual variation which are the normally prevalent variations in demand that could be expected under normal circumstances. These normal fluctuations are captured by the variance of the stochastic term $\varepsilon(k)$

.(Gouriroux and Monfort 1990). Our interest is in studying the system behavior when the *Mean Demand suddenly increases*. And hence our study focuses on the case when the Mean Demand increases, *over and above the normal random fluctuation*s.

The demand disturbance in our model is hence represented by a Heaviside step function of magnitude b_0, and a random disturbance term superimposed on it, given by:

$$r_3(k+1) = b_0 H(k) + \varepsilon(k+1), \text{ where } H(k) = \begin{bmatrix} 1, k \geq 0 \\ 0, k < 0 \end{bmatrix} \tag{3.2}$$

The first term represents the *increase in Mean Demand* and the second, $\varepsilon(k)$, the stochastic part of the disturbance, taken to be a White Noise process, $\varepsilon(k) \approx WN(0,\sigma^2)$.

3.2 Performance Metrics for Dynamic Analysis

The response of the system to a disturbance will have two components:

1) The deterministic component: the *mean response,* which describes and characterizes the system behavior, and,
2) The stochastic component, representing the random fluctuations that could occur even after the system is brought under full control, and characterized by the inventory variance.

In our system, due to the sudden additional demand, the inventory levels can be expected to decrease and fluctuate before being restored to normalcy again, thereby causing the ripple to move backwards to the upstream units in the chain. And thus, the *dynamic* performance metrics for the *mean response* that would be of interest from the point of view of ripple travel as well as replenishment system design would be the following:

a. The permanent depletion of the inventory level if any (the "offset"), which indicates whether the system is restored to its original level or not.
b. The average inventory levels, which we would like to maintain at +ve levels for comfortable operation of the system, as measured by the center line about which fluctuations occur. This would have a direct bearing on the Stock-out Risk in the system.
c. The amplitude of fluctuations of inventory, which we would like to damp out as rapidly as possible (damping rate) to restore the system to steady operation.
d. The limiting inventory variance, which is normally taken as a measure of the robustness of the system to random disturbances, and which we require to be *bounded* for the validity and meaningful interpretation of the mean response.

The first three metrics above are for the mean response, while the last is a condition imposed by us for meaningful operation and control.

3.3 Dynamic Replenishment Controls and Initial Conditions

In most supply systems, the replenishment control action is *triggered* by the inventory and demand levels at the warehouse (Weindahl and Breithaupt 2000, Nilakantan 2019). The replenishment flow is given by a function of the latest available/observed set of inventory and demand deviations, as under: for i=1,2,3,…

$$q_i(k+1) = f\left(x_3(k-1), r_3(k-1), x_2(k-1), x_1(k-1)\right) \tag{3.3}$$

We can note that the orders placed in period k would be based on the latest fully observed inventory and demand deviations at the end of the previous period (k-1), while the consignments ordered in period k would be delivered in the next period (k+1), as is captured in the eqn. (3.3) above.

The magnitude of the sudden demand increase that the system is to be designed to handle, is a fundamental input to the design process; but this is however an environmental parameter and is not under the control of the planner. And hence, in our paper, we analyze the system behavior for a step increase in demand of any arbitrary magnitude of b_0 units, the *objective* being to design a control scheme which would automatically mitigate and minimize the ripple effect.

We focus our attention hereafter on a three-stage serial supply chain, consisting of a manufacturer/supplier at stage 1 at the upstream end, followed by an intermediate warehouse at stage 2 in the intermediate stage, and finally a retail warehouse at stage 3 at the downstream end which caters to the demand off-take of the chain.

We now derive the Initial Conditions (ICs) for the system when the disturbance is applied to the downstream end of the chain.

Since the demand increases suddenly from the first period onwards, the sudden increased demand would keep pulling down the inventory at the retail warehouse end *by the quantum of increase in demand, say* ‘ b_0 ’ units till period 2, *only after* which the system would begin to feel the effect of the additional replenishment flow due to the replenishment control action.

We first look at the retail warehouse inventory levels in stage 3 at the downstream end of the chain. Now, the first consignment arrives in the second period (period 2, since the first deviation is in period 1). And hence the inventory deviation due to the sudden demand increase of b_0 units would be given by:

$\{x_3(0) = 0, x_3(1) = -b_0, x_3(2) = -2b_0 + q_3(2), \ldots\}$.

Now the replenishment flows begin to be felt by the system from period 2 onwards, and hence the ICs of the system are simply: $\{x_3(0) = 0, x_3(1) = -b_0\}$.

For numerical computations we take the mean demand disturbance as a unit step increase i.e. ($b_0 = 1$), giving the ICs as: $\{x_3(0) = 0, x_3(1) = -1\}$. (3.4)

The ICs for stages 2 and 1 can be similarly derived and are presented subsequently.

Next, the types of replenishment control triggers frequently used are given below.

The most common and most frequently encountered are the Inventory-triggered Proportional controls abbreviated as P(I) controls, where “P” stands for “Proportional”, and the “I” within the parentheses denotes “Inventory-triggered”. Likewise, “ID” within parentheses would denote “Inventory and Demand-triggered”, wherein the replenishment action is triggered both by the inventory deviations as well as demand deviations, thereby making the control more proactive. (Whereas the inventory-triggered controls are reactive in nature). We can refer to them as “P of I” and “P of ID” to distinguish them from the PI/PID (Proportional-Integral/Proportional-Integral-Derivative) controls.

4. Types of Control Schemes

4.1 Cascade Control Schemes – Inventory-Triggered (P(I) Controls)

In these schemes the control actions in the chain are initiated serially up the chain, from the downstream end (demand end) to the upstream end (supply end). These are essentially inventory-triggered schemes, and replenishment action at each stage is based on the inventory deviation at *that stage only*. Also, these use the 'proportional' type of control, wherein the replenishment flow is 'proportional' to the latest observed inventory deviation at that stage. The replenishment control flows are given in each stage by: $q_i(k+1) = K_i^i x_i(k-1)$ for i = 1, 2, 3, where the K_i^i s are the proportionality constants in the control. These K_i^i are also called as the "cascade" parameters. These controls can be found in SCs in which each stage acts independently of the others with no coordination mechanism between the stages.

In this scheme the demand disturbance causes a dip in the inventory level at the retail warehouse, which triggers increased replenishment flow to stage 3 from stage 2, which in turn reduces the inventory level in stage 2, and hence triggers increased replenishment flow to stage 2 from stage 1 at the supply end, and so on. Thus the demand disturbance has a cascading effect up the chain to the upstream units serially, causing the ripple/fluctuations to travel up the chain.

The significant point to be noted is that the cascade controls are *reactive* in nature. Replenishment action is initiated only when the inventory deviation at *that* stage is non-zero, or the inventory level at *that* stage is disturbed. The schematic diagram is shown in Figure 1a**.**

Figure 1: *Types of Control Schemes*

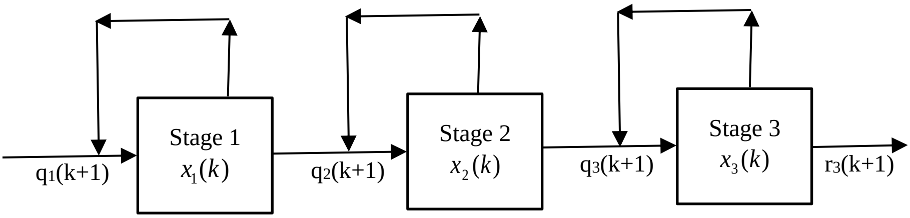

Figure 1a: *Cascade Control Scheme: The Forward Arrows show Material Flows, while Backward Arrows show the Information Flow*

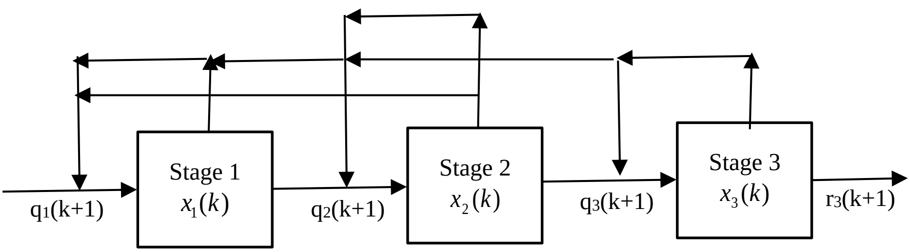

Figure 1b: *All Forward Inventory-Triggered Control Scheme: The Forward Arrows show Material Flows, while Backward Arrows show Information Flow*

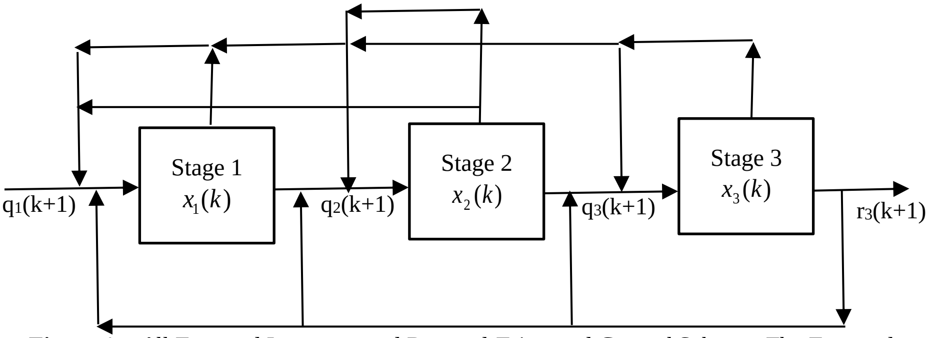

Figure 1c: *All Forward Inventory and Demand-Triggered Control Scheme: The Forward Arrows show Material Flows, while Backward Arrows show the information Flow*

4.2 Control Triggered by all Downstream/Forward Inventories (P(In) Controls)

In this type of control scheme, the replenishment is triggered by an inventory deviation in any of the downstream stages and is hence more forward-looking/proactive than the previous cascade control scheme. The control does not wait for the disturbance to propagate backwards to that particular stage before initiating control action, but rather initiates action no sooner than a deviation in any of the downstream inventory levels occurs. The replenishment controls in this scheme are given by:

$q_1(k+1) = K_1^1 x_1(k-1) + K_2^1 x_2(k-1) + K_3^1 x_3(k-1)$ for stage 1 at the upstream end

$q_2(k+1) = K_2^2 x_2(k-1) + K_3^2 x_3(k-1)$ for stage 2 in the intermediate stage

$q_3(k+1) = K_3^1 x_3(k-1)$ for stage 3 at the downstream end

Thus, replenishment action for stages 1 and 2 are triggered by an inventory deviation in any of the forward/downstream units, while the control for stage 3 is of the cascade type. The parameters $\{K_1^1, K_2^2, K_3^3\}$ are the 'cascade' parameters, while $\{\{K_2^1, K_3^1, K_3^2\}$ are the 'forward' parameters in the scheme. These controls are denoted as P(In) controls, since there are multiple inventory triggers (i.e. n>1), with 'In' denoting multiple inventory triggers. The schematic diagram is shown in Figure 1b**.**

4.3 Forward Inventory and Demand-Triggered Proportional Schemes (P(InD) Controls)

These are even more proactive with control action being initiated no sooner than a deviation is observed either in any of the downstream inventory levels or the demand level at the downstream end and is hence the most proactive among the controls seen hitherto. The control flows are given by:

$q_1(k+1) = K_1^1 x_1(k-1) + K_2^1 x_2(k-1) + K_3^1 x_3(k-1) + K_1^0 r_3(k-1)$ for stage 1 at upstream end

$q_2(k+1) = K_2^2 x_2(k-1) + K_3^2 x_3(k-1) + K_2^0 r_3(k-1)$ for stage 2 in the intermediate stage

$q_3(k+1) = K_3^1 x_3(k-1) + K_3^0 r_3(k-1)$ for stage 3 at the downstream end

Where $r_3(k)$ is the demand deviation in period k, and the K_i^0 s are the proportionality constants for the demand triggers in each stage?

This type of control can be expected to perform better than the previous types since it the most proactive among them. These controls are denoted as P(InD) controls, indicating that there are multiple inventory triggers and a single demand trigger. The schematic diagram of the P(InD) control is given in Figure1c.

Analysis shows that this last type of control, i.e., the P(InD) control is the most effective, and hence in the succeeding sections we derive the system response for this last type of control and analyze its performance in minimizing the ripple effect. We also simultaneously derive the most appropriate values for the control parameters i.e., the K values, and the sufficient conditions to be satisfied by them for control of ripple travel.

5. Ripple Effect Performance Analysis of the P(InD) Control for Step Disturbance

The vector equation of the system under P(InD) Control is given by Eqn. (5.1) below:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} \equiv \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} + \begin{bmatrix} K_1^1 & K_2^1 - K_2^2 & K_3^1 - K_3^2 \\ 0 & K_2^2 & K_3^2 - K_3^3 \\ 0 & 0 & K_3^3 \end{bmatrix} \begin{pmatrix} x_1(k-1) \\ x_2(k-1) \\ x_3(k-1) \end{pmatrix} + \begin{pmatrix} K_1^0 \\ K_2^0 \\ K_3^0 \end{pmatrix} r_3(k-1) + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} r_3(k+1)$$

Or in canonical form by Eq. (5.2) below:

$$\begin{pmatrix} x_1(k+2) \\ x_2(k+2) \\ x_3(k+2) \end{pmatrix} \equiv \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} + \begin{bmatrix} K_1^1 & K_2^1 - K_2^2 & K_3^1 - K_3^2 \\ 0 & K_2^2 & K_3^2 - K_3^3 \\ 0 & 0 & K_3^3 \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} + \begin{pmatrix} K_1^0 \\ K_2^0 \\ K_3^0 \end{pmatrix} r_3(k) + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} r_3(k+2)$$

We study first the downstream end and then work our way backwards along the upstream units serially to study the ripple travel.

5.1 Stage 3 at the Downstream End

5.1.1 The Mean Response

For the retail warehouse at the downstream end, the system equation for the mean response in canonical form is:

$$x_3(k+2) - x_3(k+1) - K_3^3 x_3(k) \equiv K_3^0 r_3(k) - r_3(k+1) \qquad (5.3)$$

For a demand disturbance of magnitude b_0 units, the equation above becomes:

$x_3(k+2) - x_3(k+1) - K_3^3 x_3(k) \equiv b_0(K_3^0 - 1)$ valid in $k \geq 1$, with the ICs:

$$\{x_3(1) = -b_0, x_3(2) = -2b_0\} \qquad (5.4)$$

Introducing the Forward-shift Operator E defined by:

$$Ex(k) \equiv x(k+1) \qquad (5.5)$$

the system equation can be written in its Operator form as a Linear Difference Equation (LDE) of order 2 as:

$[E^2 - E - K_3^3]x_3(k) \equiv b_0(K_3^0 - 1)$ valid in $k \geq 0$, with ICs as above (5.6)

Or equivalently,

$[E^2 - E - K_3^3]x_3(k)/b_0 \equiv K_3^0 - 1$ valid in $k \geq 0$, with ICs: $\{x_3(1) = -1, x_3(2) = -2\}$

(5.7)

Eqn. (5.7) clearly shows that the solution and the system deviations will be *proportional to the magnitude of the disturbance*. However, the *nature* or *characteristics* of the response would remain the same, but however will be magnified by the magnitude of the demand disturbance.

And hence it suffices to analyze the behavior of the system for a unit step disturbance. We hence take the demand disturbance as a unit step increase in period 1, i.e. $r_3(k+1) \equiv 1, k \geq 0$, and the system equation as eqn. (5.7) with $b_0 = 1$. Elementary analysis yields the stability conditions as under:

{ $K_3^3 \geq 0$ & $K_3^3 < -1$} : Unstable Response

(with Non-oscillatory Unstable response for $K_3^3 \geq 0$, and Unstable Oscillations for $K_3^3 < -1$)

$-1/4 < K_3^3 < 0$: Real Roots, Stable Non-oscillatory response

$K_3^3 = -1/4$: Repeated Real Roots, Non-oscillatory Stable Response

$-1 < K_3^3 < -1/4$: Complex Roots, Stable Damped Oscillations

$K_3^3 = -1$:Un-damped Just Stable Response, Constant Amplitude Oscillations (The Marginal Stability Case)

We choose the region of stable operation for study, and in particular the region: $-1 < K_3^3 < -1/4$ from the practitioners' point of view, since during discussions the practitioners were found to favor K_3^3 values of higher magnitude so as to make good the inventory shortfall quickly. (The magnitude of K_3^3 is the fraction of inventory deviation ordered for replenishment, and hence a value of K_3^3 close to – 1 is naturally preferred by practitioners). The solution of the LDE is obtained as below (Kelley and Peterson 2001):

Firstly, the roots of the LHS Operator are complex being given by $\rho_3 e^{\pm j\theta_3}$, (where $j = \sqrt{-1}$)

with $\rho_3 = \sqrt{|K_3^3|}$ and $\tan\theta_3 = \sqrt{4|K_3^3| - 1}$, and hence the solution is given by:

$$x_3(k) \equiv C_0 + (\sqrt{|K_3^3|})^k (A_3 Cosk\theta_3 + B_3 Sink\theta_3) \tag{5.8}$$

Now plugging in the ICs yields the solution as:

$$x_3(k) \equiv \frac{K_3^0 - 1}{|K_3^3|} + (\sqrt{|K_3^3|})^k (A_3 Cosk\theta_3 + B_3 Sink\theta_3) \tag{5.9}$$

Where $A_3 = \frac{1}{|K_3^3|}$, $B_3 = \frac{2(K_3^0 - K_3^3 - 1/2)}{(K_3^3)\sqrt{4|K_3^3| - 1}} = \frac{2(K_3^0 + |K_3^3| - 1/2)}{(K_3^3)\sqrt{4|K_3^3| - 1}}$, (5.10)

The solution can also be written as:

$$x_3(k) \equiv \frac{K_3^0 - 1}{|K_3^3|} + (\sqrt{|K_3^3|})^k \{2\frac{\sqrt{((|K_3^3| + K_3^0)^2 - K_3^0)}}{|K_3^3|\sqrt{4|K_3^3| - 1}} Cosk(\theta_3 + \phi_3)) \tag{5.11a}$$

Where $\tan\phi_3 = -\frac{1 + 2K_3^3 - 2K_3^0}{\sqrt{4|K_3^3| - 1}}$. The response shows the following characteristics:

The Damping Rate is $O(|K_3^3|)^k$ which decreases with increase of $|K_3^3|$ while the Offset value is $\dfrac{K_3^0 - 1}{|K_3^3|}$ which decreases with increase of $|K_3^3|$. The offset can be made zero by setting $K_3^0 = 1$; for any other value of K_3^0, there is a trade-off between the Damping Rate and Offset value. Under normal circumstances we would like the inventories and flows in the system to be restored to their original operating levels as quickly as possible. Hence the recommended values for the control parameters would be:

a) $K_3^0 = 1$ reducing the offset to zero, and thereby restoring the inventory to its original level, (5.11b)

b) $K_3^3 \gtrapprox -1$, i.e., a value close to -1 and $|K_3^3|$< 1, say of 0.8 or 0.7 say. This would provide a damping rate of between $\sqrt{0.8}$ and $\sqrt{0.7}$, i.e. between 0.894 and 0.837. (5.11c)

These equations above describe the *mean* response or the *mean* inventory levels.

5.1.2 The Stochastic Component of the Response

For the stochastic component of the response, the system stochastic difference equation (SDE) is as under: $[E^2 - E - K_3^3]x_3(k)^{stoc} / b_0 \equiv K_3^0\varepsilon(k) - \varepsilon(k+2)$ valid in $k \geq 1$, or in equivalent form using the Lag Operator L (defined by $Lx(k) \equiv x(k-1)$) as:

$$[1 - L - K_3^3L^2]x_3(k)^{stoc} / b_0 \equiv K_3^0\varepsilon(k-2) - \varepsilon(k) \qquad (5.12)$$

Since unity is not a root of the LHS Operator (for the chosen values of K_3^3), the solution admits an infinite Moving Average representation (Gouriroux and Monfort 1990) given by: $x_3(k)^{stoc} = \sum_{l=0}^{\infty} \beta_l \varepsilon(k-l)$, (where ϵ(k) is a White Noise process, i.e. $\varepsilon(k) \approx WN(0,\sigma^2)$) satisfying the Operator Equation:

$$[1 - L - K_3^3L^2](\sum_{l=0}^{\infty} \beta_l \varepsilon(k-l)) / b_0 \equiv K_3^0\varepsilon(k-2) - \varepsilon(k), \qquad (5.13)$$

Since the system starts at k = 0, the limiting inventory variance (for control parameter values: $K_3^3 = -0.81, K_3^0 = 1$) is obtained as: $\lim_{k\to\infty} \mathrm{var}(x_3(k)) = 21.6\sigma^2$. The detailed solution is given in Appendix – 1.

Henceforth we focus our attention on the Mean Response and impose the condition of bounded-ness of the Limiting Inventory Variance.

5.1.3 The Just Stable Case (Marginal Stability)

For the case when $K_3^3 = -1$, i.e. $|K_3^3|$ = 1, which is the maximum permitted magnitude for stability, and $K_3^0 = 1$, the response is obtained as:

$$x_3(k) \equiv K_3^0 - 1 + 2Cos((k+1)\pi/3) \qquad (5.14a),$$

which shows un-damped oscillations of amplitude 2 units (or twice the magnitude of the demand disturbance) about a center-line of zero, and the response can be seen to be ‘just stable’.

5.1.4 For other values of K_3^0 with $K_3^3 = -1$

In this case $|K_3^3| = 1$, and $\tan\theta_3 = \sqrt{3}$, yielding $\theta_3 = \pi/3$, and hence the solution is given by: $x_3(k) \equiv K_3^0 - 1 + (\frac{2}{\sqrt{3}}\sqrt{(1 + K_3^0 + (K_3^0)^2})Cos((k+1)\pi/3)$ (5.14b),

which shows un-damped constant-amplitude fluctuations, with the amplitude given by:

$(\frac{2}{\sqrt{3}}\sqrt{(1 + K_3^0 + (K_3^0)^2})$, while the offset is as before given by: $K_3^0 - 1$.

The question of interest in such a case would be the trough value or the lowest dip in inventory levels, which is given by: $K_3^0 - 1 - (\frac{2}{\sqrt{3}}\sqrt{(1 + K_3^0 + (K_3^0)^2})$. The value of K_3^0 that maximizes this value can be obtained by elementary methods as: $K_3^0 = 1$, yielding a trough value of -2, and constant-amplitude un-damped oscillations of amplitude 2 units, about a center-line of zero, which is the case of marginal stability above.
We next take up the intermediate stage below.

5.2 The Intermediate Stage – Stage 2

5.2.1 The System Response

The system equation for stage 2 as given in eqn. (5.2) in canonical form is as under:

$$x_2(k+2) - x_2(k+1) - K_2^2 x_2(k) \equiv (K_3^2 - K_3^3)x_3(k) + K_2^0 r_3(k) \qquad (5.15)$$

Now since $r_3(k) \equiv 1$ for $k \geq 1$, the LDE for stage 2 in Operator form becomes:

$$[E^2 - E - K_2^2]x_2(k) \equiv (K_3^2 - K_3^3)x_3(k) + K_2^0 \text{ valid in } k \geq 1 \qquad (5.16)$$

with ICs (obtained from eqn. (5.15) above) as: $\{x_2(1) = 0, x_2(2) = 0\}$.

Now plugging in the solution of stage 3 into eqn. (5.15) yields: (5.17)

$$[E^2 - E - K_2^2]x_2(k) \equiv (K_3^2 - K_3^3)\{\frac{K_3^0 - 1}{|K_3^3|} + (\sqrt{|K_3^3|})^k\{2\frac{\sqrt{((|K_3^3| + K_3^0)^2 - K_3^0)}}{|K_3^3|\sqrt{4|K_3^3| - 1}}Cosk(\theta_3 + \phi_3)\}\} + K_2^0 \quad (5.17)$$

And we can readily see the backward propagation of the demand disturbance to stage 2 through the effect of the $x_3(k)$ term in eqns. (5.16) and (5.17).

The salient observation is that choosing $K_3^2 = K_3^3$, makes the coefficient of the $x_3(k)$ term in the equations (5.16) and (5.17) above zero, thereby eliminating the backward effect of stage 3 on stage 2, i.e. thereby effectively *decoupling stage 2 from stage 3,* and *arresting the backward travel of the ripple* to stage 2. However, *exact* realization of this condition may not always be possible in practice due to imprecise control on the replenishment flows, and hence we need to make allowance for the fact that this *decoupling condition* of $K_3^2 = K_3^3$ may not always hold. Hence, we further analyze this system under the condition: $K_3^2 \neq K_3^3$, with the ripple moving backwards to stage 2. Next, we look at the choice of K_2^2, the cascade parameter for stage 2.

5.2.2 Choice of K_2^2, the cascade parameter for stage 2:

5.2.2.1 Stability: The stability of the response of stage 2 is again dependent on the LHS Operator and is again determined by the values of K_2^2 the cascade parameter; and elementary analysis again leads to the same stability conditions as for K_3^3 in stage 1 earlier. Thus, it is evident that the cascade parameter determines the stability of the response (at every stage), and the stable range of values are the same for both (all) stages.

This is essentially because, at every stage it is only the cascade parameter K_j^j at each stage that is part of the LHS Operator of the system LDE which determines stability, while it is the other forward-inventory-trigger parameters that play a role in arresting the backward travel of the ripple through the forward inventory terms.

5.2.2.2 Damping: From the above discussion it would hence be tempting to choose the same values for both stages i.e., set $K_2^2 = K_3^3$, so that we have similar/maximum damping in both stages. We next examine this condition i.e., $K_2^2 = K_3^3 = K$ (with $-1 < K < -1/4$):

Firstly, we can note that the term $(\sqrt{|K_3^3|})^k\{2\frac{\sqrt{((|K_3^3| + K_3^0)^2 - K_3^0)}}{|K_3^3|\sqrt{4|K_3^3| - 1}} Cosk(\theta_3 + \phi_3)$ in the solution for $x_3(k)$ satisfies the Homogeneous LDE: $[E^2 - E - K_3^3]x_3(k) \equiv 0$, and hence is annihilated by the LHS Operator as given above. Hence the annihilator for the RHS terms in eqn. (5.16) and (5.17) above is precisely the Operator: $[E-1][E^2 - E - K]$ (Kelley and Peterson 2001). And hence the LDE for stage 2 in its homogeneous form is given by:

$$[E-1][E^2 - E - K]^2 x_2(k) \equiv 0 \qquad (5.18)$$

Which yields the form of the solution as: (in the range: { $-1 < K < -1/4$ })

$$x_2(k) \equiv C_0 + (\sqrt{|K|})^k\{(A_0 + A_1 k)Cosk\theta + (B_0 + B_1 k)Sink\theta\} \qquad (5.19)$$

where, $\tan\theta = \sqrt{4|K| - 1}$ as before.

The solution shows a damping rate of $(\sqrt{|K|})^k$ as for stage 3 earlier, which can be controlled by us by choice of the cascade parameters: $K_2^2 = K_3^3 = K$, and an offset of: $C_0 = \frac{(K_3^2 - K)(K_3^0 - 1)}{|K|} + K_2^0$ which can then be precisely controlled by us by choice of K_2^0, the demand-trigger parameter in stage 2. Hence, we can obtain similar and equal damping in both stages even though the ripple travels back to stage 2 in the chain (due to imprecise control on the flows: the condition: $K_3^2 \neq K_3^3$) .

The values of the constants { $\{A_0, A_1, B_0, B_1\}$ can be obtained from the ICs.

5.2.3 The Just Stable Case with $K_2^2 = K_3^3 = K = -1$ (The case with Resonance)

We next look at the special case of: $K_2^2 = K_3^3 = K = -1$, which is the 'just stable' condition, and $K_3^2 \neq K_3^3 = -1$ (and hence the ripple travels backwards due to imprecise control on flows).

For the case: $K_2^2 = K_3^3 = K = -1$ and $K_3^2 \neq K_3^3 = -1$, the constants in the solution are obtained as:

$C_0 = K_2^0, A_0 = -2(K_3^2+1), A_1 = \frac{2}{3}(K_3^2+1), B_0 = -\frac{2K_2^0}{\sqrt{3}} - \frac{8(K_3^2=1)}{3\sqrt{3}}, B_1 = \frac{4(K_3^2+1)}{\sqrt{3}}$, and hence

$$x_2(k) \equiv K_2^0 + \{-2(K_3^2+1) + \frac{2(K_3^2+1)}{3}k\}Cos(k\pi/3) + \{-\frac{2K_2^0}{\sqrt{3}} - \frac{8(K_3^2=1)}{3\sqrt{3}} + \frac{4(K_3^2+1)}{\sqrt{3}}k\}Sin(k\pi/3) \quad (5.20)$$

which clearly shows *resonance* with *un-damped oscillations of increasing magnitude.* And hence the particular choice of $K_2^2 = K_3^3 = -1$ can lead to resonance and instability in the upstream unit. For other values of $-1 < K_2^2 = K_3^3 < -1/4$ in the stable range, there is no resonance. And hence it would be *safer to have both* K_2^2 *and* K_3^3 *sufficiently far removed from the value of – 1.*

5.2.4 The Most General Case

We finally look at the most general case with { $K_3^2 \neq K_3^3, K_2^2 \neq K_3^3$ and $-1 < K_3^3, K_2^2 < -1/4$ (stable region)}: With { $K_3^2 \neq K_3^3, K_2^2 \neq K_3^3$ and $-1 < K_3^3, K_2^2 < -1/4$ (stable region)}, the equivalent homogenous LDE is:

$$[E-1][E^2 - E - K_3^3][E^2 - E - K_2^2]x_2(k) \equiv 0 \text{ with the ICs as before} \quad (5.21)$$

And hence by the ‘Annihilator Method’ the solution is obtained as:

$$x_2(k) \equiv C_0 + \rho_3^k(ACosk\theta_3 + B\sin k\theta_3) + \rho_2^k(A_2Cosk\theta_2 + B_2Sink\theta_2) \quad (5.22)$$

where,

$\rho_3 = \sqrt{|K_3^3|}$, $\tan\theta_3 = \sqrt{4|K_3^3|-1}$, $\rho_2 = \sqrt{|K_2^2|}$, $\tan\theta_2 = \sqrt{4|K_2^2|-1}$,

$C_0 = \{(K_3^2 - K_3^3)(K_3^0 - 1) + K_3^0|K_3^3|\}/(|K_3^3||K_2^2|$

$A_2 = (b_1\rho_2 Sin2\theta_3 - b_2 Sin\theta_2)/(\rho_2^2 Sin\theta_2)$, $B_2 = -(b_1\rho_2 Cos2\theta_2 - b_2 Cos\theta_2)/(\rho_2^2 Sin\theta_2)$

$b_1 = -C_0 - \rho_3(ACos\theta_3 + BSin\theta_3)$, $b_2 = -C_0 - \rho_3^2(ACos2\theta_3 + BSin2\theta_3)$

$A = (K_3^2 - K_3^3)(aA_3 - bB_3)/(a^2+b^2)$, $B = (K_3^2 - K_3^3)(bA_3 + aB_3)/(a^2+b^2)$

$a = \rho_3^2 Cos2\theta_3 - \rho_3 Cos\theta_3 - K_2^2$, $b = \rho_3^2 Sin2\theta_3 - \rho_3 Sin\theta_3$

$A_3 = \frac{1}{|K_3^3|}$, $B_3 = \frac{-2(K_3^0 - K_3^3 - 1/2)}{|K_3^3|\sqrt{4|K_3^3|-1}}$

5.2.5 An Illustration for stage 2

The Case with $\{K_3^3 = 1/2 = K_2^2, K_3^0 = 1$ and $-1 = K_3^2 \neq K_3^3, K_2^0 = K_2^0 = arbitrary\}$:

For this case, firstly the LDE for stage 3 can be solved to yield the solution:

$$x_3(k) \equiv -(\frac{1}{\sqrt{2}})^k(2\sqrt{5})\{2Cos(k\pi/4) + 4Sin(k\pi/4)\} \quad (5.23)$$

Which shows damped oscillations with a damping rate of $O(0.707)^k)$, and offset value of zero.

While the LDE for stage 2 is obtained as:

$[E^2 - E + 1/2]x_2(k) \equiv -(1/2)x_3(k) + K_2^0$ valid in $k \geq 1$ with ICs: $x_2(1) = 0 = x_2(2)\}$ (5.24)

which yields the solution of the form

$$x_2(k) \equiv (\frac{1}{\sqrt{2}})^k \{(A_0 + A_1 k)Cos(k\pi/4) + (B_0 + B_1 k)Sin(k\pi/4)\} \tag{5.25}$$

Which, upon plugging in the ICs yields the solution as:

$$x_2(k) \equiv 2K_2^0 + (\frac{1}{\sqrt{2}})^k \{(6\sqrt{5}k - 4\sqrt{5})Cos(k\pi/4) + +(2\sqrt{5}k - 4\sqrt{5} - 4K_2^0)Sin(k\pi/4)\} \tag{5.26}$$

which clearly indicates the absence of Resonance. The damping rate is $O(0.707)^k)$, and the offset value is $2K_2^0$ which can be set by us as desired.

5.2.6 Summary of Results for stages 2 and 3

We present the salient findings below:

1) Stability in each stage j is determined by the cascade parameter K_j^j value (LHS Operator in the system LDE) j, while the backward travel of the ripple is prevented by the forward inventory-trigger parameter K_j^{j-1} in each upstream stage j (K_3^2 in stage 2). The offset value and center-line of oscillations is determined by the Demand-trigger parameter K_j^0 in each stage.

2) Setting $K_3^2 = K_3^3$ arrests and prevents back-propagation of the disturbance ripple from stage 3 back to the upstream stage 2, and hence *effectively decouples* stage 2 from the downstream stage 3

3a) Setting the cascade parameter to $K_j^j = -1$ in *either stage 1 or stage 2* would lead to un-damped oscillations of constant magnitude of 2 units in the respective stage; The center-line of oscillations can be controlled by choice of the forward demand-trigger parameter K_j^0 in each stage.

3b) Setting $K_3^2 = K_3^3 = -1$, i.e. setting both cascade parameters *simultaneously* to – 1 would lead to Resonance in the upstream unit stage 2 (Oscillations with increasing magnitude); and hence to avoid resonance both the cascade parameters should be set at values sufficiently far removed from – 1 (the *Resonance prevention condition*).

4) Setting both cascade parameters equal i.e. setting $K_2^2 = K_3^3 \neq -1$ and sufficiently far removed from – 1 causes no resonance in the upstream stage 2.

5) For the stochastic component of the response, the limiting variance can be obtained by the same method (as was for stage 3) for any chosen set of values of the control parameters.

5.3 The Upstream End Stage 1

5.3.1 The System Response

The LDE for stage 1 is given by:

$$x_1(k+2) - x_1(k+1) - K_1^1 x_1(k) \equiv (K_2^1 - K_2^2)x_2(k) + (K_3^1 - K_3^2)x_3(k) + K_1^0 r_3(k) \tag{5.27}$$

Or in Operator form:

$$[E^2 - E - K_1^1]x_1(k) \equiv (K_2^1 - K_2^2)x_2(k) + (K_3^1 - K_3^2)x_3(k) + K_1^0 r_3(k)$$

Valid in $k \geq 1$, with ICs: $\{x_1(1) = 0 = x_1(2)\}$ (5.28)

Similar analysis as above yields the following features of the response:

1) Stability is again controlled by the cascade parameter K_1^1 in the LHS Operator, while the forward inventory-trigger parameters $\{K_2^1, K_3^1\}$ play a role in the arrest and prevention of the backward travel of the ripple to stage 1. The demand-trigger parameter K_1^0 controls the offset value and the center-line of oscillations.

2) Setting $K_2^1 = K_2^2$ arrests and prevents the backward travel of the ripple from stage 2 back to the upstream stage 1, and setting $K_3^1 = K_3^2$ arrests and prevents the backward travel of the disturbance in stage 3 back to upstream end stage 1 (*decoupling conditions*). Setting $|K_1^1| < 1$ (as before) ensures stability in stage 1(stability condition)

3) For $K_1^1 = K_2^2 = -1$ or $K_1^1 = K_3^3 = -1$ or both, induces resonance in stage 1. However, setting $K_1^1 = K_2^2 = K_3^3 = K \neq -1$ i.e., setting all three cascade parameters equal but far removed from -1 does not induce resonance (*resonance prevention condition*) (5.28a)

The damping rate is given by $O(\sqrt{|K|})^k)$ and can be controlled by us.

The full solution for this case ($K_1^1 = K_2^2 = K_3^3 = K \neq -1$) is given by:

$$\begin{aligned} x_1(k) \equiv E_0 &+ (\sqrt{|K|})^k \{(C_0 + C_1 k + C_2 k^2) Cosk\theta + (D_0 + D_1 k + D_2 k^2) Sink\theta\} \\ &+ (K_2^1 - K_2^2)(\sqrt{|K|})^k \{(A_0 + A_1 k) Cosk\theta + (B_0 + B_1 k) Sink\theta \\ &+ (K_3^1 - K_3^2)(\sqrt{|K|})^k \{A_3 Cosk\theta + B_3 Sink\theta \end{aligned} \quad (5.29)$$

Where $\tan\theta_j = \sqrt{4|K_j^j| - 1} = \tan\theta = \sqrt{4|K| - 1}$ for j = 1,2,3 and the constant term E_0 (which is the offset in stable systems) is a function of $(K_3^3, K_3^0; K_2^2, K_3^2, K_2^0; K_1^1, K_2^1, K_3^1, K_1^0\}$ and can be set as desired (by choice of K_1^0) either to control offset or the center-line of oscillations, as the case maybe. The constants can be determined from the initial conditions.

5.3.2 The Most General Case

For the most general stable case:

with $\{K_1^1 \neq K_2^2 \neq K_3^3$, $|K_j^j| < 1, \forall j = 1,2,3$, $K_2^1 \neq K_2^2, K_3^1 \neq K_3^2$, $K_3^2 \neq K_3^3\}$:

The solution obtained under such conditions is of the form given by:

$$\begin{aligned} x_1(k) \equiv E_0 &+ (\sqrt{|K_1^1|})^k \{A_1 Cosk\theta_1 + B_1 Sink\theta_1\} \\ &+ (K_2^1 - K_2^2)(\sqrt{|K_2^2|})^k \{(A_2 Cosk\theta_2 + B_2 Sink\theta_2 \\ &+ (K_3^1 - K_3^2)(\sqrt{|K_3^3|})^k \{A_3 Cosk\theta_3 + B_3 Sink\theta_3 \end{aligned} \quad (5.29)$$

Where $\tan\theta_j = \sqrt{4|K_j^j| - 1}$ for j = 1,2,3 and the constant term E_0 (which is the offset in stable systems) is a function of $(K_3^3, K_3^0; K_2^2, K_3^2, K_2^0; K_1^1, K_2^1, K_3^1, K_1^0\}$ and can be set as desired either to control offset or the center-line of oscillations, as the case maybe. The constants can be determined from the initial conditions.

5.4 Salient Findings and Sufficient Conditions for Prevention of Backward Travel of Ripple

From the above results and eqn. (5.29), we can derive the following set of sufficient conditions on the control parameters:

1) *Sufficient Conditions* for the arrest and prevention of the back propagation of the disturbance to the upstream units, the *Decoupling Conditions:*

a) Condition: $K_3^2 = K_3^3$ to prevent back-propagation form stage 3 back to stage 2 (decoupling of stage 2 from stage 3) (5.30a)

b) Condition: $K_3^1 = K_3^2$ to prevent back-propagation form stage 3 back to stage 1 (decoupling of stage 1 from stage 3) (5.30b)

c) Condition: $K_2^1 = K_2^2$ to prevent back-propagation form stage 2 back to stage 1 (decoupling of stage 1 from stage 2) (5.30c)

2) Sufficient Conditions for stable Operation, the *Stability Conditions*:

a) Condition: $|K_j^j| < 1$ for j = 1, 2, 3, (5.31a)

with the concomitant Damping Rate given by

b) Damping Rate $\rho_j = \sqrt{|K_j^j|}$ for j = 1, 2, 3 (5.31b)

3) Sufficient Conditions for prevention of resonance, the *Resonance Prevention Condition*:

If $K_1^1 = K_2^2 = K_3^3 = K$ for all stages, then K should sufficiently far remove from – 1(i.e., $K << -1, |K| < 1$) to ensure that Resonance does not occur in the upstream stages. (5.32)

4) Sufficient Conditions for offset prevention, the *Offset Prevention Conditions*:

Appropriate choice of $\{K_3^0, K_2^0, K_1^0\}$ as given in the solutions above to achieve zero offset/zero center-line. (eqns. (5.11b), (5.26), (529))

Additionally, we can make the following observations:

5) Proper and *rapid restoration of the downstream end* would automatically insulate the upstream units from demand disturbances even in cases of imprecise control flows (with the above 4 conditions not being satisfied exactly). This is because all the upstream stage controls implicitly have a component dependent on $x_3(k)$, and hence quick restoration of the downstream warehouse inventory levels would implicitly have a beneficial effect on all upstream units in the chain.

6) To achieve the sufficient conditions above, it would be very essential to ensure adequate *flow of information between the stages* as under:

a. From the downstream end stage 3 to all upstream units regarding the *demand disturbance,*

b. All stages to all their upstream stages regarding their *control parameter settings*.

This flow of such information could make a significant difference in arresting and preventing the disturbance ripple from moving backward to upstream units. And even in the absence of precise control of replenishment flows (violation of the decoupling conditions), could reduce and mitigate the magnitude/effect of the disturbance ripple to a significantly large extent.

These are illustrated with numerical examples subsequently.

6. An Additional Sinusoidal Disturbance Component

We now consider the effect of a seasonal component in the demand disturbance, represented by a constant-amplitude sinusoidal term $bSin\omega k$, where $\omega = \frac{2\pi}{T}$, where T is the time-period of the seasonal term (e.g., if k is in months, then T =12 months would represent an annual seasonal cycle).
The disturbance term is now given by: $r_3(k+1) = b_0 + bSinwk$ valid in $k \geq 0$.

6.1 The P(InD) Control

We consider first the P(InD) Control studied above. The system equation for stage 3 is: $x_3(k+1) \equiv x_3(k) + K_3^3 x_3(k-1) + K_3^0 r_3(k-1) - r_3(k+1)$ (6.1)

i.e. $x_3(k+2) \equiv x_3(k+1) + K_3^3 x_3(k) + K_3^0 r_3(k) - r_3(k+2)$ valid in $k \geq 1$ (6.2a)

or, equivalently (since the disturbance is felt by the end of period 1)

$[E^2 - E - K_3^3]x_3(k) \equiv K_3^0(1 + bSinw(k-1)) - 1 - bSinw(k+1)$ valid in $k \geq 1$ (6.2b)

With ICs: $\{x_3(1) = -b_0, x_3(2) = -2b_0 - bSinw\}$.

Since the LHS Operator is the same as in eqn. (5.5), the stability conditions also remain identical as earlier. And hence we again analyze the system behavior for the stable region: $-1 < K_3^3 < -1/4$.

Since the Annihilator for the RHS terms is $(E-1)(E-e^{jw})(E-e^{-jw})$, the system equation (6.2b) reduces to: $(E-1)(E-e^{jw})(E-e^{-jw})(E^2 - E - K_3^3)x_3(k) \equiv 0$, yielding the General Solution as:

$$x_3(k) \equiv C_0 + A_0 Coswk + B_0 Sinwk + (\rho_3^k \{A_3 Cosk\theta_3 + B_3 Sink\theta_3\} \quad (6.3a)$$

Where, as before, $\rho_3 = \sqrt{|K_3^3|}$ and $\tan\theta_3 = \sqrt{4|K_3^3| - 1}$. And plugging back the solution terms into the Original System (Non-homogeneous) Eqn. (6.2b) yields values for the constants as:

$$A_0 = \frac{-bSinw(K_3^0 + 1)a_1 - bCosw(K_3^0 - 1)b_1}{(a_1^2 + b_1^2)},$$

$$B_0 = \frac{-bSinw(K_3^0 + 1)b_1 + bCosw(K_3^0 - 1)a_1}{(a_1^2 + b_1^2)}$$

$$C_0 = \frac{(K_3^0 - 1)b_0}{|K_3^3|}, \text{ with } a_1 = Cos2w - Cosw - K_3^3,\ b_1 = Sin2w - Sinw \quad (6.3b)$$

where the undetermined (arbitrary) constants (A_3, B_3) can be determined from the ICs above. From the form of the solution, we can see that the last two terms die out to zero at the rate $O(\rho_3^k)$, but the first two terms do not. The system thus exhibits *perpetual* constant-amplitude fluctuations about a center-line of $C_0 = \frac{(K_3^0 - 1)b_0}{|K_3^3|}$ (which can be made 0/ >0 by choice of $K_3^0 = 1 / K_3^0 > 1$),

with amplitude of $\sqrt{A_0^2 + B_0^2}$, with

$$A_0^2 + B_0^2 = b^2(K_3^0+1)^2 - 4K_3^0 b^2 Cos^2 w - \frac{4b^2 SinwCosw((K_3^0)^2-1)a_1b_1}{2(1-Cosw)+K_3^3(K_3^3 12Cosw - 2Cos2w)} \quad (6.4)$$

The amplitude is a complicated function of the control parameters (K_3^3, K_3^0) and the environmental parameters (b, w). These constant amplitude perpetual oscillations are essentially as a result of, and are induced by the presence of a sinusoidal component in the demand disturbance. Thus we see that the above control is *unable to damp out the constant amplitude perpetual fluctuations induced by the sinusoidal component, and hence is unable to prevent their back-propagation*.

Of course, proper choice of the forward-inventory trigger parameters (to satisfy the *decoupling conditions:* $K_3^2 = K_3^3, K_3^1 = K_3^2, K_2^1 = K_2^2$) could mitigate this to a significant extent. However, the system would be *more prone to transmit the disturbance* and permit the backward travel of the ripple to upstream units in case of imprecise control of flows.

Thus, in summary, we can conclude that while the P(InD) control is able to arrest and prevent the back propagation of the step component of the demand disturbance to a substantially large extent (by proper choice of the control parameters), it however, is less prone to do so in the case of the sinusoidal (seasonal) component.

We hence look for a modification of the control with a view to enable it to arrest the backward travel of the sinusoidal component also.

6.2 Modified P(InD) Controls to Arrest Backward Travel of the Sinusoidal Component

We have seen that the solution (inventory level) in stage 3 replicates the constant-amplitude perpetual fluctuations of the demand disturbance and is essentially due to the control flows being proportional to the inventory levels, which in turn mirrors the demand disturbance.

Two possible methods which could mitigate this phenomenon would be:

a) use of additional (multiple) inventory triggers as in the case of Moving Average (MA) controls

b) use of additional (multiple) demand triggers.

We take up these two modifications next below.

6.2.1 Moving Average Controls – MA(InD) Control

In these controls the distinguishing feature is that instead of having only a single cascade parameter K_j^j in each stage, there are several of them, thereby constituting a Moving Average of the latest few fully observed inventory deviations as indicated below. The Moving Average component of the flows are thus set as:

$$K_j^j x_j(k-1) + M_j^1 x_j(k-2) + M_j^2 x_j(k-3) + ... + M_j^{r-1} x_j(k-r)$$

Where 'r' is the order of the MA. The full control flow would be as under (in the case of MA(InD) control): (6.5)

$$q_j(k+1) \equiv K_j^j x_j(k-1) + M_j^1 x_j(k-2) + M_j^2 x_j(k-3) + ... + M_j^{r-1} x_j(k-r) + K_j^0 r_3(k+1)$$

The system equation for stage 3 is thus given in Canonical Operator form by: (6.6)

$$[E^{r+1} - E^r - K_j^j E^{r-1} - M_j^1 E^{r-2} - ..... - M_j^{r-1}]x_3(k) \equiv K_3^0 r_3(k-1+r) - r_3(k+1+r), \quad k \geq 0.$$

The choice of the multiple cascade (MA) parameters in the LHS Operator can be done so as to achieve the maximum damping which is the most important criterion for us, and yields (Nilakantan 2010):

$$M_j^l = (-1)^{l+1}\binom{r+1}{l+1}\left(\frac{1}{r+1}\right)^{l+1}, \text{ for } l = 1,2,\dots\dots, r-1 \tag{6.7}$$

With this choice the LHS Operator can be put in the form $[E-1/r+1)]^{r+1}$, and hence the system equation becomes:

$$[E-1/(r+1)]^{r+1}x_3(k) \equiv K_3^0 r_3(k-1+r) - r_3(k+1+r)\ , \quad k \geq 0 \tag{6.8}$$

Which, substituting for the demand yields:

$$[E-1/(r+1)]^{r+1}x_3(k) \equiv K_3^0(b_0 + bSinw(k+r-2)) - (b_0 + bSinw(k+r)), k \geq 0 \tag{6.9}$$

Which simplifies to:

$$[E-1/(r+1)]^{r+1}x_3(k) \equiv (K_3^0 - 1)b_0 + b(K_3^0 Cosw(r-2) - Coswr)Sinwk + \\ + b(K_3^0 Sinw(r-2) - Sinwr)Coswk \tag{6.10}$$

With ICs: $\{x_3(0), x_3(1), x_3(2), \dots\dots, x_3(r)\}$ which can be obtained from the system equation (6.6).

As before using the Annihilator Operator $(E-1)(E-e^{jw})(E-e^{-jw})$, the system reduces to its homogeneous form: $(E-1)(E-e^{jw})(E-e^{-jw})[E-1/(r+1)]^{r+1}x_3(k) \equiv 0$ (6.11)

The solution has the form:

$$x_3(k) \equiv D + (C_0 + C_1 k + C_2 k^2 + C_3 k^3 + \dots + C_r k^r)(1/(r+1))^k + ACoswk + BSinwk \tag{6.12}$$

the constants in which can be obtained by plugging back the solution into the original system equation (6.10) and the ICs. The first term is the offset, while the second is the effect of the step disturbance component and which is rapidly damped out. The third set of sinusoidal terms is induced by the sinusoidal component in the demand disturbance and are not damped out.

And hence we can see that the control will damp out the step component very rapidly at the rate $O(1/(r+1))^k)$ which can be very fast even for very moderate values of r. For r = 3 say, the damping rate is $O((1/4)^k)$ which is very rapid. But the control will be *unable to damp out the demand induced constant-amplitude perpetual oscillations*.

Essentially the above results show that the MA control parameters act mainly on the step component, but do not have much effect on the sinusoidal component of the disturbance.

We next look at the other option of using multiple demand triggers.

6.2.2 Multiple Demand Triggers – the P(InDm) Controls

6.2.2.1 Sufficient Conditions for Damping of Sinusoidal Components

We first look at Stage 3 at the downstream end. We propose a control flow of the form:

$$q_3(k+1) \equiv K_3^3 x_3(k-1) + K_3^0 r_3(k-1) + L_3^1 r_3(k-2) + L_3^2 r_3(k-3) \tag{6.13}$$

Where the last two terms on the RHS in eqn. (6.13) above are the two additional demand triggers.

The logic is to be able to annihilate the step term as well as the Sinwk and Coswk terms of the demand on the RHS, for which we need at least two additional demand triggers (one for the Sinwk terms and one for the Coswk terms) thus yielding a P(InD3) control. Thus the system equation is given by:

$$x_3(k+1) \equiv x_3(k) + K_3^3 x_3(k-1) + K_3^0 r_3(k-1) + L_3^1 r_3(k-2) + L_3^2 r_3(k-3) - r_3(k+1) \quad (6.14a)$$

Or equivalently:

$$x(k+1) - x(k) - K_3^3 x(k-1) \equiv K_3^0 r(k-1) + L_3^1 r(k-2) + L_3^2 r(k-3) - r_3(k+1) \quad (6.14b)$$

We look at only the demand trigger terms on the RHS of eqn. (6.14b) above and impose the condition that the entire RHS be annihilated to become *identically zero* by suitable choice of the demand trigger parameters (K_3^0, L_3^1, L_3^2). Thus, we have the equation:

$$K_3^0 r(k+3) + L_3^1 r(k+2) + L_3^2 r(k+1) - r_3(k+5) \equiv 0 \quad (6.15)$$

Or equivalently, (6.16)

$$K_3^0(b_0 + bSinw(k+2)) + L_3^1(b_0 + bSinw(k+1)) + L_3^2(b_0 + bSinwk) - (b_0 + bSinw(k+4)) \equiv 0$$

Which (equating the coefficients of the constant term, Sinkw and Coskw individually) yields the system of equations in the control parameters as under:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & bCosw & bCos2w \\ 0 & bSinw & bSin2w \end{bmatrix} \begin{pmatrix} L_3^2 \\ L_3^1 \\ K_3^0 \end{pmatrix} = \begin{pmatrix} 1 \\ Cos4w \\ Sin4w \end{pmatrix} \quad (6.17)$$

(The first equation yields: $L_3^2 + L_3^1 + K_3^0 = 1$) which yields the values of the control parameters as:

$$\begin{pmatrix} L_3^2 \\ L_3^1 \\ K_3^0 \end{pmatrix} = \begin{pmatrix} b^2 Sinw + bSin2w - bSin3w \\ -2bSin2w + Sin4w \\ bSinw + bSin3w - Sin4w \end{pmatrix} / (b^2 Sinw - bSin2w + bSinw) \quad (6.18)$$

This is the '*Annihilation*' *condition*. This choice of parameters will completely annihilate the RHS terms in eqn. (6.14b) for $k \geq 5$, and hence the solution of the LDE for $k \geq 5$ will be that of the homogeneous equation, i.e., $[E^2 - E - K_3^3]x_3(k) \equiv 0$ valid in $k \geq 5$, with ICs: $\{x_3(5), x_3(6)\}$.

The solution is hence obtained as (in the region: $-1 < K_3^3 < -1/4$) as:

$$x_3(k) \equiv \rho^k \{A_3 Cosk\theta_3 + B_3 Sink\theta_3\} \quad \text{where} \quad \rho = \sqrt{|K_3^3|} < 1, \quad \tan\theta_3 = \sqrt{4|K_3^3| - 1}$$

(6.19)

which decays to zero at the rate of $O(\rho^k)$, and thus *restores the system to its original state*.

The period $k < 5$ is the *transient phase* during which time the *full* action of the control *in its entirety* is *not* felt by the system due to the lagged variables in the control scheme. The system begins to feel the full action of the control in its entirety only after $k \geq 5$, subsequent to which the control damps out the fluctuations and restores the system to its original state.

In such controls it would be advantageous to keep $|K_3^3|$ as low as possible ($|K_3^3|$ just greater than ¼) so as to achieve a high damping rate of approximately $O((1/2)^k)$, which can be quite fast in restoring the system to its original state.

The values of (A_3, B_3) can be obtained by plugging in the ICs which are obtained as:

$$\{ x_3(5) = -5b_0 - b(Sinw + Sin2w + Sin3w + Sin4w) + b_0(-K_3^3 + 3K_3^0 + 2L_3^1 + L_3^2) + $$
$$+ bK_3^0(Sinw + Sin2w) + bL_3^1 Sinw + K_3^3 x_3(3)$$
$$x_3(6) = -6b_0 - b(Sinw + Sin2w + Sin3w + Sin4w + Sin5w) + b_0(-K_3^3 + 4K_3^0 + 3L_3^1 + 2L_3^2) +$$
$$+ bK_3^0(Sinw + Sin2w + Sin3w) + bL_3^1(Sinw + Sin2w) + bL_3^2 Sinw + K_3^3(x_3(3) + x_3(4)) \}$$

where,

$$x_3(3) = -3b_0 - b(Sinw + Sin2w) + b_0(-K_3^3 + K_3^0)$$
$$x_3(4) = -4b_0 - b(Sinw + Sin2w + Sin3w) + b_0(-K_3^3 + 2K_3^0 + L_3^1) + bK_3^0 Sinw \qquad (6.20)$$

Hence this type of control can achieve the arrest and prevention of the backward travel of the ripple to a very great extent. This would have very positive spin-offs for the upstream units also even in case of imprecise flow control (the *decoupling conditions*: $K_3^2 = K_3^3$ for stage 2, and the conditions: $K_2^1 = K_2^2$ and $K_3^1 = K_3^2$ for stage 1 not being satisfied) since the inventory levels in the downstream units themselves would be restored to their original levels rapidly.

And from a similar analysis for the upstream units, *resonance during the transient phase would not occur in the upstream units even if t*he cascade parameters K_j^j s are set so as to have $\tan^{-1}(\sqrt{4|K_j^j|-1} = \theta_j = w$, the frequency of the sinusoidal component of the demand disturbance, *provided* $w \neq \pi/3$. In case $w = \pi/3$, then to avoid resonance in the transient phase in the upstream units, it would have to be ensured that none of the cascade parameters K_j^j s are set as to have $\tan^{-1}(\sqrt{4|K_j^j|-1} = \theta_j = w = \pi/3$, (this would essentially imply that none of the cascade parameters are set to -1, i.e. $K_j^j \neq -1$ for any j = 1, 2, 3).

However, it can also be noted that in the case of $w = \pi/3$, even setting the cascade parameters such that $\tan^{-1}(\sqrt{4|K_j^j|-1} = \theta_j = w = \pi/3$, resonance would occur only in the transient phase ($k < 5$), after which the control would damp out the disturbance completely and restore the system to its original state. And thenceforth the fluctuations would be damped out at the rate $O(\rho_j)^k = O\left(\sqrt{|K_j^j|}\right)^k$, which can be set by us as desired.

6.2.2.2 Sufficient Conditions for Prevention of the Backward Travel of Ripple

From the preceding discussions we can see that the *sufficient conditions for damping out the sinusoidal component in the demand disturbance* in stage 3 are those given in eqn. (6.18) above.

And the same conditions would hold in the upstream units also (i.e we can set $K_j^0 = K_3^0, L_j^1 = L_3^1, L_j^2 = L_3^2$ for j = 1, 2) in the upstream units also.

These can be used in conjunction with the sufficient conditions in section 5 earlier for the cascade and forward-inventory parameters (the stability conditions, decoupling conditions, resonance prevention, and offset-prevention conditions).

Such a choice would insulate the upstream units from demand disturbances to a very large extent.

Thus, we can see that the P(InD3) control is able to quickly arrest and prevent the backward travel of both the components (the step as well as the sinusoidal component) of the ripple upstream, thereby providing a degree of insulation to the upstream units from market-linked/demand disturbances at the downstream end. Also, the scheme is *more robust to imprecise flow control and non-satisfaction of the decoupling conditions.*

6.2.3 Multiple Inventory as well as Multiple Demand Triggers: MA(InDm) Controls

We could also envisage a control with both multiple inventories as well as multiple demand triggers with these multiple inventory triggers being of the MA type, and in addition to the forward-inventory triggers. For example, in stage 1 at the upstream end the control would be of the form as under:

$$q_1(k+1) \equiv K_1^1 x_1(k-1) + M_1^2 x_1(k-2) + ... + M_1^r x_1(k-r) + K_2^1 x_2(k-1) + K_3^1 x_3(k-1) \\ + K_1^0 r_3(k-1) + L_1^1 r(k-2) + L_1^2 r(k-3) \qquad (6.21)$$

Where the first r terms on the RHS of eqn. (6.21) above (i.e. $K_1^2, M_1^2, M_1^3, ..., M_1^r$) constitute the MA (cascade) part of the control, while the next two terms (K_2^1, K_3^1) constitute the forward-inventory triggers, and the last three terms (K_1^0, L_1^1, L_1^2) constitute the demand-triggered part.

Similarly for stage 2 the control would be as:

$$q_2(k+1) \equiv K_2^2 x_2(k-1) + M_2^2 x_2(k-2) + ... + M_2^r x_2(k-r) + K_3^2 x_3(k-1) \\ + K_2^0 r_3(k-1) + L_2^1 r_3(k-2) + L_2^2 r_3(k-3) \qquad (6.22)$$

Where the first r terms on the RHS of eqn. (6.22) above (i.e., $K_2^2, M_2^2, M_2^3, ..., M_2^r$) again constitute the MA part, the next term (K_3^2) the forward-inventory trigger, and the last three terms (K_2^0, L_2^1, L_2^2) constitute the demand-triggered part.

(Note: The principal difference between the P(InDm) and MA(InDm) controls are in the cascade parameters alone. While the P(InDm) control has only a single cascade parameter at each stage (which are the K_j^js) the MA(InDm) control would have multiple cascade parameters making up the MA part in the control ((i.e., $K_j^j, M_j^2, M_j^3, ..., M_j^r$)).

However, it can be shown (from the system responses eqns. (6.12) and (6.19)) that the additional inventory triggers in the MA(InDm) control do not provide any significant extra benefit to the control in terms of the damping rate as compared to the simpler P(InDm) control. The damping rate of $O(1/(r+1)^k)$ in the MA(InDm) control can equivalently be achieved in the P(InDm) control by choice of the cascade parameter $K_j^j = 1/(r+1)^2$.

Hence the simpler but equally effective P(InDm) control would be preferred over the more complicated MA((InDm) control. We now summarize our findings below.

6.2.4 Summary of Results for Additional Sinusoidal Component in the Demand Disturbance

The salient results obtained above are as under:

1) The P(InD) control as well as the MA(InD) control are likely to be unable to arrest the sinusoidal component and are more prone to allow the backward propagation of the sinusoidal component upstream along the chain. However, they are able to achieve high damping out of the step increase component though.
2) The P(InD3) as well as the MA(InD3) Control studied above are more likely to be able to arrest and prevent the backward travel of both components of the demand disturbance (the step increase part as well as the sinusoidal component). The damping rate can be controlled by choice of the cascade control parameters (in P(InD3) control) and the MA parameters (in MA(InD3) control).
3) The MA(InD3) control does not provide any significant advantage over the simpler P(InD3) control with regard to the Damping Rate, and hence the simpler and equally effective P(InD3) control would be preferred in practice (though the Ma(InDm) control is also equally simple to build into the control software of the chain).
4) It is the Demand-trigger parameters that are able to arrest and prevent the backward travel of the sinusoidal component of the demand disturbance, as given by the sufficient conditions in eqn. (6.18), i.e., the *Annihilation conditions*.
5) It can be shown that in all cases of stable control (except in the case of marginal stability), the limiting inventory variance is bounded. And hence *setting the cascade parameters for stability would also automatically imply bounded limiting inventory variances*.

7. Numerical Illustrations

We provide below a set of numerical illustrations for the above discussions. We initially take up a step input, again in the marginally stable case as the worst case scenario.

7.1 P(InD) Control: Unit Step Demand Disturbance in the Marginally Stable Case (Worst-case Scenario)

For the case of a unit step demand disturbance given by: $r_3(k+1) \equiv 1$ in $k \geq 1$, the control parameters are for stage 3 are taken as: ($K_3^3 = -1, K_3^0 = 1$) for the marginally stable case in stage 3, which yields the solution as:

$$x_3(k) \equiv (K_3^0 - 1) + 2Cos((k+1)\pi/3) \tag{7.1a}$$

Now setting $K_3^0 = 1$ makes the center-line zero, with the solution being given by:

$$x_3(k) \equiv 2Cos((k+1)\pi/3) \tag{7.1b}$$

The response shows un-damped oscillations of amplitude 2 about a center-line of zero (*marginal stability or just stable condition*, which presents a worst-case scenario for the upstream stages).

We now illustrate the ripple travel backwards in the chain to stage 2 by choosing values of the control parameters in stage 2 such that the *decoupling conditions are violated at stage 2*. This would essentially represent conditions of *imprecise replenishment flow control* in stage 2.

We hence set: $K_3^2 = -1/2 \neq K_3^3 = -1$(decoupling condition in stage 2 *not satisfied*) and set $K_2^2 = -1/2$ in the stable region and noting that $K_2^2 = -1/2 \neq -1 = K_3^3$ (to avoid resonance in stage 2). Thus, we allow for about 50% variation/reduction from the desired value of the forward inventory parameter in stage 2.

Now the LDE for stage 2 is given by:

$$[E^2 - E - K_2^2]x_2(k) \equiv (K_3^2 - K_3^3)x_3(k) + K_2^0 \quad \text{valid in } k \geq 1 \tag{7.2a}$$

or, $[E^2 - E + 1/2]x_2(k) \equiv Cos((k+1)\pi/3) + K_2^0$ valid in $k \geq 1$ (7.2b)

The annihilator Operator for the RHS is: $(E-1)(E-e^{j\pi/3})(E-e^{-j\pi/3})$, and hence the homogeneous form of the equation is given by:

$(E-1)(E-e^{j\pi/3})(E-e^{-j\pi/3})[E^2 - E - K_2^2]x_2(k) \equiv 0$ or equivalently (for $K_2^2 = -1/2$) by:

$$(E-1)(E-e^{j\pi/3})(E-e^{-j\pi/3})(E-e^{h\pi/4})(E-e^{-j\pi/4})x_2(k) \equiv 0 \tag{7.3a}$$

With ICs as: { $x_2(1) = 0, x_2(2) = 0$ }, which yields the solution as: (7.3b)

$$x_2(k) \equiv D_0 + (A_3 Cos(k\pi/3) + B_3 Sin(k\pi/3)) + (1/\sqrt{2})^k \{A_2 Cos(k\pi/4) + B_2 Sin(k\pi/4)\}$$

valid in $k \geq 1$. The solution exhibits un-damped oscillations of frequency $\pi/3$ induced by the inventory fluctuations of stage 3 downstream (and this is essentially because the decoupling conditions between stages 2 and 3 are not satisfied, i.e. we have *not* set $K_3^2 = K_3^3$). And hence the control transmits the un-damped oscillations of stage 3 backwards to stage 2 upstream. The other oscillation terms are damped out at the rate $O((0.707)^k)$.

The constants in the solution are obtained as: $D_0 = 2K_2^0$ which is the center-line of oscillations, $A_3 = -1/3, B_3 = 1/\sqrt{3}$, which yields the amplitude of the un-damped oscillations as $\sqrt{(A_3^2 + B_3^2)} = 2/3$. Hence choosing $K_2^0 = 1/3$ makes the center-line of oscillations as 2/3, thereby yielding a trough value of zero (and crest value of 4/3). This particular choice of the center-line is essentially to keep the inventory deviations always positive and above zero, thereby reducing stock-out risk. The other constants are then obtained as: $A_2 = 4/3, B_2 = -2/3$, and thus the full solution with this choice of parameters is as under:

$$x_2(k) \equiv 2/3 + (2/3)Cos((k-1)\pi/3)) + (1/\sqrt{2})^k \{(4/3)Cos(k\pi/4) - (2/3)Sin(k\pi/4)\}$$

$$\text{valid in } k \geq 1 \tag{7.4}$$

A salient observation is that even though the control is unable to completely arrest the backward travel of the un-damped oscillations to stage 2, it has reduced its amplitude to a substantial extent (from a value of 2 to a value of 2/3). A suitable choice of the demand-trigger parameter K_2^0 can then set the center-line as desired, based on the policy regarding stock-out risk.

We finally take up stage 1 at the upstream end:

We again set the parameters in stage 1 such that *the decoupling conditions between stages 1 and 2, and stages 1 and 3 are not satisfied* (i.e., $-1 = K_2^1 \neq K_2^2 = -1/2$ (decoupling condition between stages 1 and 2 *not satisfied*) and $-1 = K_3^1 \neq K_3^2 = -1/2$ (decoupling condition between stages 1 and 3 also *not satisfied*). We set $K_1^1 = -1/2$ in the stable region, noting that $K_1^1 = -1/2 \neq -1 = K_3^3$ to avoid resonance in stage 1. Thus, we again allow for about 50% variation/reduction in the forward inventory parameters in stage 1 from their desired values.

We will choose the demand-trigger parameter K_1^0 in stage 1 subsequently to set the center-line appropriately so as to reduce stock-out risk. Now the LDE for stage 1 is:

$$[E^2 - E - 1/2]x_1(k) \equiv -(1/2)x_2(k) - (1/2)x_3(k) + K_1^0 \quad \text{Valid in } k \geq 1 \qquad (7.5)$$

which clearly shows the effect of stages 2 and 3 on stage 1 through the first two terms on the RHS (since the decoupling conditions are not satisfied). From the solutions of $x_3(k), x_2(k)$ above, the annihilator for the RHS of the eqn. (7.5) above is: $(E-1)(E-e^{j\phi\pi/3})(E-e^{-j\pi/3})(E-\rho_2 e^{j\pi/4})(E-\rho_2 e^{-j\pi/4})$, while the LHS Operator can be put in the form: $(E-\rho_2 e^{j\pi/4})(E-\rho_2 e^{-j\pi/4})$, and hence the homogeneous form of the equation is given by:

$$(E-1)(E-e^{j\phi\pi/3})(E-e^{-j\pi/3})[(E-\rho_2 e^{j\pi/4})(E-\rho_2 e^{-j\pi/4})]^2 x_1(k) \equiv 0, \text{ valid in } k \geq 1$$

(7.6) With ICs: { $x_1(1) = 0, x_1(2) = 0$ }. The solution is thus given by:

$$x_1(k) \equiv 2(K_1^0 - 1/3) +$$
$$(-1/18)Cos(k\pi/3) + (1/6\sqrt{3})Sin(k\pi/3) + (1/\sqrt{2})^k\{(A_0 + k/3)Cos(k\pi/4) + (B_0 - k)Sin(k\pi/4)\} \qquad (7.7)$$

with the constants (A_0, B_0) determined from the two ICs as: $A_0 = -0.8686, B_0 = 1.2222$. The solution shows un-damped oscillations of amplitude $\sqrt{(-1/18)^2 + (1/6\sqrt{3})^2} = 1/9$, about a center-line of $2(K_1^0 - 1/3)$. And hence we set $K_1^0 = 7/18$, which makes the center-line 1/9 and the trough value zero, thereby keeping the inventory deviation always above zero and reducing stock-out risk. The other oscillation terms are damped out at the rate $O((0.707)^k)$. A salient observation is that though the control is not able to completely arrest the back propagation of the oscillations upstream, it is able to reduce the amplitude of oscillations to a significant extent from 2 units in stage 3 to 2/3 units in stage 2, and finally to 1/9 units in stage 1. The demand trigger parameter can be set suitably to yield a center-line and trough value as desired based on the stock-out risk policy of the chain. The responses are shown in Figure 2.

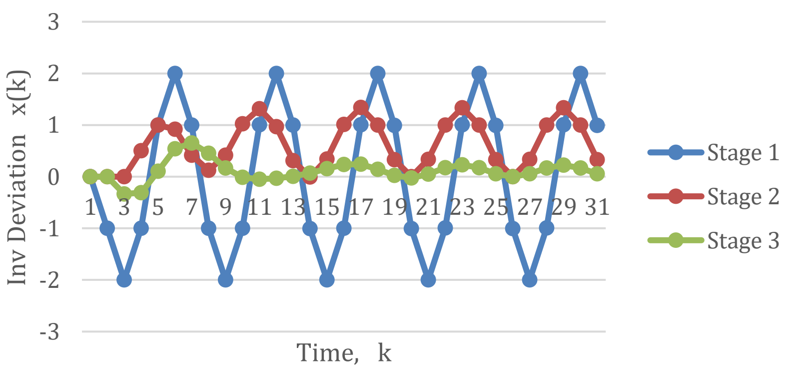

Figure 2 *Mean Response Step Disturbance (Marginal Stability in Stage 3)*

An Additional Sinusoidal Disturbance Component: P(InD3) Control
We next additionally introduce a sinusoidal component in the demand disturbance of amplitude unity and study the system response under the P(InD3) control.

Stage 3: First, for stage 3, we set the Control flows as:

$$q_3(k+1) \equiv K_3^3 x_3(k-1) + K_3^0 r_3(k-1) + L_3^1 r_3(k-2) + L_3^2 r_3(k-3) \;,$$

with $K_3^3 = -1$ for the marginally stable case in stage 3 (so as to give rise to *resonance in stage 3 in the transient phase, as the worst case scenario*). The system equation is then given by:

$$x_3(k+1) - x_3(k) - K_3^3 x_3(k-1) \equiv K_3^0 r_3(k-1) + L_3^1 r_3(k-2) + L_3^2 r_3(k-3) - r_3(k+1) \qquad (7.8)$$

The demand disturbance is now given by:

$r_3(k+1) \equiv b_0 + bSinwk \equiv 1 + Sinwk \equiv 1 + Sin(k\pi/3)$, for $k \geq 0$, where the time-period T of the seasonal component is 6 time-units ($w = 2\pi/T$) to have resonance in the transient phase in stage 3. We then set the demand-trigger parameters $\{K_3^0, L_3^1, L_3^2\}$ to annihilate the RHS demand disturbance terms completely for any arbitrary values of (b_0, b, w), which yields the system of equations as under:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & bCosw & bCos2w \\ 0 & bSinw & bSin2w \end{bmatrix} \begin{pmatrix} L_3^2 \\ L_3^1 \\ K_3^0 \end{pmatrix} = \begin{pmatrix} 1 \\ Cos4w \\ Sin4w \end{pmatrix}$$

i.e. $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1/2 & -1/2 \\ 0 & \sqrt{3}/2 & \sqrt{3}/2 \end{bmatrix} \begin{pmatrix} L_3^2 \\ L_3^1 \\ K_3^0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \\ -\sqrt{3}/2 \end{pmatrix} \qquad (7.9)$$

From which we obtain the demand-trigger parameters as: $\{K_3^0 = 2, L_3^1 = -3, L_3^2 = 2\}$. And hence we have the system equation as: $[E^2 - E + 1]x_3(k) \equiv 0$ for $k \geq 5$ with ICs as:

$\{x_3(5) = 1.5\sqrt{3} = 2.598, x(6) = \sqrt{3} = 1.732\}$ with the transient phase given by:

$\{x(0) = 0, x(1) = -1, x(2) = -2 - \sqrt{3}/2 = -2.866, x(3) = -\sqrt{3} = -1.732, x(4) = \sqrt{3}/2 = 0.866\}$

The solution is given by:

$$x_3(k) \equiv A_3 Cosk\pi/3 + B_3 Sink\pi/3 = \sqrt{3}Cosk\pi/3 - 2Sink\pi/3 \text{ valid in } k \geq 5 \qquad (7.10)$$

The response shows un-damped oscillations of amplitude: $\sqrt{A_3^2 + B_3^2} = \sqrt{7} = 2.646$ units about a center-line of zero, for a unit-amplitude sinusoidal disturbance. The high amplitude is essentially due to resonance in the transient phase (as the worst case scenario). The response is shown in Figure. 3.

Stage 2: If we try out the same form of control in stage 2, we get the system equation as:

$$[E - E - K_2^2]x_2(k) \equiv (K_3^2 - K_3^3)x_3(k) + [K_2^0 r_3(k) + L_2^1 r_3(k-1) + L_2^2 r_3(k-2)] \qquad (7.11)$$

where the terms within the square brackets constitute the demand-triggered component of the control flow in stage 2. To allow for imprecise control flows, we set $K_3^2 = -1/2 \neq -1 = K_3^3$, so as to violate the decoupling condition between stages 2 and 3 (i.e. to violate the decoupling condition: $K_3^2 = K_3^3$). We take the cascade parameter in stage 2 as: $K_2^2 = -1/2$ to avoid resonance in stage 2 since setting $K_2^2 = K_3^3 = -1$ would give rise to resonance in stage 2 even past the transient phase (as outlined in section 5 earlier). Now, attempting to annihilate the RHS terms in eqn. (7.11) above yields the trivial solution: $L_2^2 = 0$ and the conditions:

$$\begin{bmatrix} 0 & 1 & 1 \\ \sqrt{3} & 0 & -\sqrt{3}/2 \\ -2 & 1 & 1/2 \end{bmatrix} \begin{pmatrix} K_3^2 \\ K_2^0 \\ L_2^1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\sqrt{3} \\ 2 \end{pmatrix} \tag{7.12}$$

Which hence yields: $K_2^0 = L_2^1 = L_2^2 = 0$ and $K_3^2 = -1$, to satisfy the decoupling condition (i.e. $K_3^2 = K_3^3$). And hence we are led back to the P(InD) control in stage 2. (This is because stage 2 does not have the demand acting directly on it, but feels its effect only through stage 3)

Also, from the point of view of practice, it might not be desirable to have *a fixed relation* between the forward-inventory-trigger parameter K_3^2 and the demand-trigger parameters (K_2^0, L_2^1, L_2^2) as given by eqn. (7.12) due to operational uncertainties.

And also it would be better to have them *independently controllable* (for *flexibility* and *greater decoupling)*, as is in the P(InD) control.

We thus use the P(InD) control and set the control flows as:

$$q_2(k+1) \equiv K_2^2 x_2(k-1) + K_3^2 x_3(k-1) + K_2^0 r_3(k-1) \tag{7.13}$$

wherein (K_2^2, K_3^2, K_2^0) are the cascade, forward-inventory, and (single) demand trigger parameters in stage 2. The system equation is given by:

$$[E^2 - E + 1/2]x_2(k) \equiv (1/2)x_3(k) + K_2^0 r_3(k), \quad k \geq 0 \tag{7.14}$$

And hence, substituting for $x_3(k)$, we have:

$$[E^2 - E + 1/2]x_2(k) \equiv K_2^0 + (1 - K_2^0)(\sqrt{3}/2)Cos(k\pi/3) + (-1 + K_2^0/2)Sin(k\pi/3) \tag{7.15}$$

valid in $k \geq 5$ with ICs: { $x_3(5), x_3(6)$ } . The solution is obtained as:

$$x_2(k) \equiv 2K_2^0 + \sqrt{3}(K_2^0 - 1)Cosk\pi/3 + (2 - K_2^0)Sink\pi/3 + (1/\sqrt{2})^k \{A_2 Cosk\pi/4 + B_2 Sink\pi/4\} \tag{7.16}$$

which shows un-damped oscillations of amplitude $\sqrt{4(K_2^0)^2 - 10K_2^0 + 7}$ about a center-line of $2K_2^0$. The amplitude is minimum at $K_2^0 = 1.25$, and its minimum value is 0.866 about a center-line of 2.5, which shows a trough value of 1.634 and crest value of 3.366. Alternatively, setting $K_2^0 = 0.7$, yields a center-line of 1.4 and amplitude of 1.4, leading to a trough value of 0 and crest value of 2.8. However, more tractable values would be to set $K_2^0 = 1$, which yields a center-line of 2 and amplitude of unity, leading to a trough value of 1 and crest value of 3. And hence we get the solution (for $K_2^2 = -1/2, K_3^2 = -1/2 \neq -1 = K_3^3$, and $K_2^0 = 1$) and ICs{ $x_2(5) = 2.183, x_2(6) = 2.9$ } as: $x_2(k) \equiv 2 + Sink\pi/3 + (1/\sqrt{2})^k \{6.8Cosk\pi/4 - 15.2Sink\pi/4\}$ valid in $k \geq 5$, and with the transient phase induced by $x_3(k)$ given by:

$$\{ x_2(0) = 0, x_2(1) = 0, x_2(2) = 0, x_2(3) = 0.5, x_2(4) = 1.433 \} \tag{7.17}$$

Thus, we can observe that the amplitude of oscillations is unity, which is lower than that in stage 3 downstream, and hence the control is able to mitigate and reduce the magnitude of the ripple as it travels backward up the chain to stage 2. This backward travel is essentially due to the decoupling condition not being satisfied (which we have deliberately set to allow and account for imprecise flow control). Even in such cases the control is able to mitigate the oscillations to a substantial extent even in the worst-case scenario that we have taken. The response is shown in Figure 3.

Stage 1: We set the parameters as under [Note: $K_3^3 = -1, K_3^0 = 2, L_3^1 = -3, L_3^2 = 2$ in stage 3, and $K_2^2 = -1/2, K_3^2 = -1/2 \neq -1 = K_3^3, K_2^0 = 1$ in stage 2] $K_3^1 = -1 \neq -1/2 = K_3^2, K_2^1 = -1 \neq -1/2 = K_2^2, K_1^1 = -1/2$ in stage 1, to violate the decoupling conditions between stage 1 and stages 2 and 3, through the choice of the forward-inventory trigger parameters (K_3^1, K_2^1), and the cascade parameter $-1/2 = K_1^1 \neq -1$ to avoid resonance in stage 1.

The system LDE in stage 1 is given by: (7.18)

$[E^2 - E + 1/2]x_1(k) \equiv -1/2x_2(k) - 1/2x_3(k) + K_1^0 r_3(k)$ valid in $k \geq 5$, with ICs:{ $x_1(5), x_1(6)$ }, which, upon substituting the solutions for the downstream stages obtained earlier, reduces to

$$[E^2 - E + 1/2]x_1(k) \equiv (K_1^0 - 1) - (\sqrt{3}/2)(K_1^0 + 1)Cosk\pi/3 + (1/2)(K_1^0 + 1)Sink\pi/3 + (1/\sqrt{2})^k\{-3.4Cosk\pi/4 + 7.6Sink\pi/4\}$$

valid in $k \geq 5$, with ICs: { $x_1(5), x_1(6)$ } (7.19)

The solution is obtained as: $x_1(k) \equiv 2(K_1^0 - 1) + \sqrt{3}(K_1^0 + 1)Cosk\pi/3 - (K_1^0 + 1)Sink\pi/3 + (1/\sqrt{2})^k\{(A_0 - 4.2k)Cosk\pi/4 + (B_0 - 11k)Sink\pi/4\}$ valid in $k \geq 5$ (7.20)

which shows un-damped oscillations (induced by the inventory fluctuations of the downstream stages) about a center-line of $2(K_1^0 - 1)$ and amplitude of $2(K_1^0 + 1)$, which has a trough value of – 4 regardless of the value of K_1^0. The amplitude increases linearly with K_1^0, and hence to keep the amplitude low we need to have a low value of K_1^0. Setting $K_1^0 = -1$ makes the amplitude zero, but results in a severe negative offset of -4 units which would leave the inventory level in a severely depleted state, being unable to restore the system to its original state. Also having a value of $K_1^0 = -1$ would be to reduce replenishment flow when the demand increases and would be contrary to control logic and hence would not be likely to be preferred. Alternatively, setting of $K_1^0 = 0$ i.e. doing away with the demand-trigger component would yield an amplitude of 2 about a center-line of – 2 with a trough value of – 4 and crest value of zero.

The controller has some flexibility in choosing the value of K_1^0, and probably the most preferred value of K_1^0 would be unity thereby making the center-line zero and amplitude 4 units. With this choice of $K_1^0 = 1$, the other constants in the solution can be determined from the ICs:

{ $x_1(5) = 3.665, x_1(6) = 2.550$ }, and yields the full solution as:

$$x_1(k) \equiv 4Cos(k\pi/3 - \pi/6) + (1/\sqrt{2})^k\{(120 - 4.2k)Cosk\pi/4 + (73.312 - 11k)Sink\pi/4\}$$

valid in $k \geq 5$ with the transient phase given by:

$$\{ x_1(0)=0, x_1(1)=0, x_1(2)=0, x_1(3)=3/2, x\ (4)=1.933 \} \tag{7.21}$$

The response is shown in Figure. 3 and the disturbance in Figure. 4.

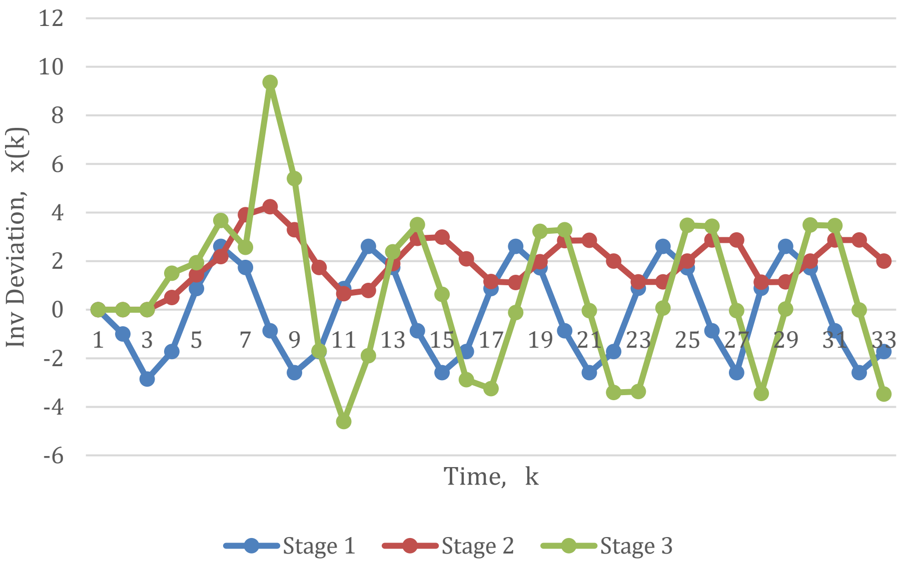

Figure 3: *Mean Response Step + Sinusoidal Disturbance*

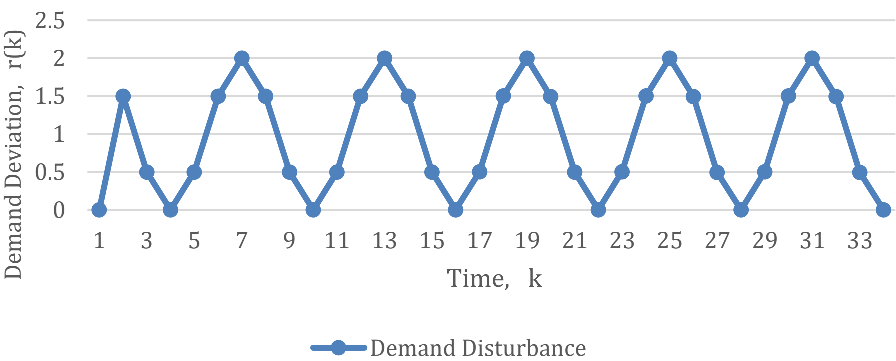

Figure 4: *Demand Disturbance = 1 + Coskπ/3*

Thus we can see that the control is unable to arrest the backward travel of the sinusoidal component of the demand disturbance and results in fluctuations of high amplitude at the upstream end. This is essentially due to the *marginal stability* deliberately set by us in the *downstream end*. It can be seen that *with imprecise flow controls (decoupling conditions not satisfied), marginal stability at the downstream end can induce and lead to marginal stability in the upstream units also* with increased/decreased magnitude, depending upon the choice of the forward-inventory trigger parameters.

However, a similar computation under conditions of *stability at the downstream end*, shows stability in the upstream stages also and restoration of the system to its original state.

And hence, the above illustrations reiterate our earlier findings of sections 5 and 6.

8. Salient Findings and Sufficient Conditions

From the above analyses and illustrations, we can deduce as under:

8.1 Roles of the Different Parameters

The roles of the different control parameters are as under:

a) The control parameters for *Operational Stability* at each stage of the chain are the *cascade* parameters (K_1^1, K_2^2, K_3^3) in P(InDm) controls and the MA parameters. ($K_j^i s$) in MA(InDm) controls through which stability of each stage can be controlled (*stability conditions,* eqns. (5.31). Prevention of Resonance is also achieved through these parameters through adherence to the *Resonance Prevention conditions* (eqns. (5.28a), (5.32).)

b) The control parameters for achieving effective *Decoupling* of the stages are the *Forward-inventory-trigger* parameters (K_2^1, K_3^1, K_3^2), through which *Decoupling* can be achieved and controlled through adherence to *the decoupling conditions* (eqns. (5.30)).

c) The control parameters for *offset* control are the *demand-trigger* parameters at each stage (K_1^0, K_2^0, K_3^0), through which the offset at each stage can be controlled. (the *Offset control conditions* eqns. (5.11b), (5.26), (5.29)). These can be chosen based on the Stock-out Risk Policy set by the planners.

d) The control parameters for *damping out of sinusoidal disturbance components* are the *multiple demand-trigger parameters* (K_3^0, L_3^1, L_3^2) in the P(InD3) control through the *annihilation conditions* (eqn. (6.18)).

8.2 Effect of Marginal Stability at the Downstream End

a) Marginal stability at the downstream end can be very deleterious to the chain and can cause the same in the upstream units, and hence it would be *essential to ensure good stability at the downstream end, by proper choice of the cascade parameter at the downstream end to satisfy the Stability condition* ($|K_3^3| < 1$) in all Proportional controls (eqn. (5.31)).

b) However, in the event of inadvertent marginal stability at the downstream end due to unforeseen and environmental effects, it should be ensured that the *Resonance prevention conditions* are adhered to in all the units. (eqns. (5.28a), (5.32)).

c) Marginal stability at the downstream end does not cause problems if P(InD3) control is used.

8.3 Sufficient Conditions for the Arrest and Prevention of Back-propagation of the Demand Disturbance to Upstream Units

Prevention of the back propagation of the disturbance ripple can be achieved through p*roper* and *accurate* replenishment flow control in each of the upstream stages.
The sufficient conditions that would achieve it are as under:

a) Choice of *cascade* parameters with *adherence to the Stability conditions* ($1/4 \le \left|K_j^j\right| < 1$) and *Resonance Prevention conditions* in each stage ($\left|K_j^j\right| << 1$ as derived in eqns. (5.28a), (5.32))
b) Choice of the *forward inventory-trigger parameters* with *adherence to the decoupling conditions* in each stage ($K_3^2 = K_3^3, K_2^1 = K_2^2, K_3^1 = K_3^2$) as derived in section 5.3.3 (eqn. (5.30)).
c) Choice of the *demand-trigger parameters* with *adherence to the Offset prevention conditions* in each stage (as shown in sections 5.1.1, 5.1.2, and 5.1.3 to keep the trough value above zero (eqns. (5.11b), (5.26), (5.29)).
d) Choice of the multiple demand-trigger parameters to satisfy the sufficient condition (6.18) to damp out the sinusoidal components of demand disturbance (*Annihilation condition*).
e) Rapid damping out of the disturbance and rapid restoration of inventory levels in stage 3 at the downstream end would be very advantageous for the steady operation of the upstream units, and hence sufficient attention would need to be paid to it.

8.4 Information Sharing and Coordinated Action in the Stages

From the discussion above, it is clear that for the sufficient conditions to be satisfied it would be necessary to have timely information flow between the stages. Thus, for the mitigation of the ripple effect to be achieved, effective *communication and information flow between the stages* would be necessary regarding:

a) demand deviations at each stage, and,
b) replenishment flow control parameter settings at each stage

to ensure the satisfaction of the stability and resonance prevention conditions, the decoupling conditions, the offset prevention conditions, and eqn. (6.18) for damping out of sinusoidal components (annihilation conditions) at each of the stages in the SC.

9. Managerial Implications and Practical Significance

The managerial implications and practical significance can be concluded as under:

a) Operational stability in each stage of the chain can be achieved by proper monitoring of the replenishment flows based on the currently measured inventory shortfall in the respective stages of the chain. Essentially, though it would be natural to order enough to make good the current shortfall in inventory, the results show that ordering exactly or more than the current shortfall of inventory induces instability in the operation of the chain in the long run. Whereas keeping the replenishment flows just short of making good the full shortfall gives more stable and better performance in the long run. This is achieved by keeping the magnitudes of the cascade parameters just less than unity. This is the essence of the 'stability conditions'.
b) Prevention of the backward travel of the disturbance can be achieved by adherence to the decoupling conditions as given in eqn. (5,30)). In essence what these

conditions suggest is that since the replenishment flows to the downstream units are based on their own inventory shortfall and are met by the material outflow from the upstream units, any inventory fluctuations in the downstream units will also result in fluctuations in the outflows from the upstream units. This in turn will lead to inventory fluctuations in the upstream units, which in essence is the back propagation of the disturbance and the cause of the ripple effect.

Thus, the inventory fluctuations in stage 3 are directly passed on to both stage 2 and stage 1 directly; and the inventory fluctuations in stage 2 are directly passed on to stage 1. The decoupling conditions simply provide ways and means to prevent this direct transmission of the downstream fluctuations to the upstream units through choice of the forward-inventory trigger parameters.

The important and significant thing is to note that *it is possible to achieve this decoupling through proper monitoring of the replenishment flows,* which can be affected through the flow control parameters (the froward-inventory trigger parameters). This in essence, is what the ’decoupling conditions’ try to achieve.

c) To prevent and control the permanent depletion of the inventory levels (the Offset) if any, the planner needs to add a demand-triggered component in the replenishment flows. This is because the depletion of inventory levels is directly proportional to the magnitude of the demand increase. And hence making up this amount through the replenishment flows based on the extent of demand increase would be quite natural and logical. This is done through the demand-trigger parameters as discussed above. The actual choice and values of these parameters could be based on the Stock-out Risk policy set by the planners. An added advantage is that it also makes the control more proactive. And this is the essence of Offset control as described in the analyses above.

d) When a seasonal variation (sinusoidal variation) component is also present in the mean demand, then to keep the inventory levels steady, the replenishment inflows would have to also vary in a seasonal manner to maintain material flow balance in the system. Hence tying the replenishment inflows to the demand outflows would make good sense and would only be logical in this case also. This is precisely what is done and achieved through the multiple demand-trigger parameters discussed above. This again makes the control policy more proactive. The actual values of these parameters are chosen to completely nullify the seasonal variation component of the demand and hence prevent inventory fluctuations. And this is the essence of the use of multiple demand triggers in case of seasonal variations as discussed earlier.

e) In cases of Marginal Stability at the downstream end, this can essentially be interpreted as that when the inventory levels at the downstream end do not settle down to steady values but continue to fluctuate indefinitely.

From the discussions above, it can easily be seen that such a condition would be very deleterious to the chain and would cause inventory fluctuations in all the upstream units. Hence it is imperative to focus more attention on maintaining steady operation and steady inventory levels at the downstream end.

However, in extreme cases and due to unforeseen circumstances, if conditions of marginal stability prevail at the downstream end, the planner still can have recourse to the P(InD3) control as discussed above. This type of control scheme has been shown perform well even under such circumstances.

f) Thus, and as a final word, good control and steady operation of the entire chain can be achieved only through the prevention of the back propagation of the disturbance to the upstream units. This would result in steady operation all along the chain.

However, this would necessitate and require proper information sharing between the stages as discussed above.

Thus, the prevention of the backward travel of the disturbance upstream, and maintenance of steady operation all along the chain would call for *coordinated action all along the chain with proper and timely sharing of information* between the stages. In the presence of such information-sharing, the stages can accurately control their replenishment flows and hence effectively damp out the ripple and prevent its back propagation up the chain. They can thereby achieve steady operation and good control.

10. Conclusion, Limitations and Scope for Further Work

This paper has looked at the problem of the ripple effect and the backward travel of a disturbance up the chain from the downstream end to upstream units. It has examined the dynamic behavior of the various stages in the chain with the prime objective of finding ways and means to arrest the backward travel and prevent the back propagation of the disturbance up the chain. In this process, it has analyzed the functioning of the most common dynamic controls with respect to the ripple effect, and their performance in preventing the backward travel of the demand disturbance/ripple upstream along and up the chain. Concurrently, it has also derived sufficient conditions on the control parameter settings for the arrest of the ripple and prevention of its backward travel, viz. a) the *Stability conditions and Resonance-prevention conditions for the cascade parameters*, b) the *Decoupling conditions for the forward-inventory-trigger parameters*, and c) the *Offset-prevention conditions and the Annihilation conditions for the demand-trigger parameters*.

For step demand disturbances, the P(InD) control studied herein has been found to have good performance in damping out the ripple and its effect on upstream units, whereas in the presence of an additional sinusoidal (seasonal) component, the P(InD3) controls presented herein have been found to show better performance. The control parameter settings to stamp out the ripple effect have also been derived.

However, these schemes would require full information sharing among the stages and accurate replenishment flow control to be completely effective. Nevertheless, they point to a method and approach to the control of the ripple effect in supply chains.

The pathological case of marginal stability at the downstream end has also been studied and illustrated with a numerical example. And in the absence of proper replenishment flow controls, it has been shown to induce the same in the upstream units also. These illustrations emphasize the need to achieve good control in the downstream end as also the need for accurate replenishment control through timely and effective sharing of information between the stages of the chain.

A limitation of this study is that it has not considered more sophisticated forms of dynamic controls nor composite controls (for the sake of brevity), which points to an area for further work and research. Other approaches could be to study the system empirically and/or computationally to further refine the results presented herein.

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Appendix

Illustration of Limiting Inventory Variance Computation

For P(InD) Control: The Stochastic LDE is

$$[1-L-K_3^3L^2]x_3(k)^{stoc} = [1-L-K_3^3L^2]\{\textstyle\sum_{l=0}^{\infty}\beta_l\varepsilon(k-l)\} \equiv K_3^0\varepsilon(k)-\varepsilon(k-2)$$

(5.14)

Note: Since unity is not a root of the LHS Operator, $x_3(k)^{Stoc}$ admits an infinite MA representation as: $x_3(k)^{Stoc} = \sum_{l=0}^{\infty}\varepsilon(k-l)$, which gives rise to the system of equations:

LHS Opr Term	Coefft of $\varepsilon(k)$	Coefft of $\varepsilon(k-1)$	Coefft of $\varepsilon(k-2)$	Coefft of $\varepsilon(k-3)$	Coefft of $\varepsilon(k-4)$	Coefft of $\varepsilon(k-5)$	Coefft of $\varepsilon(k)$
1	β_0	β_1	β_2	β_3	β_4	β_5	β_k
- L		$-\beta_0$	$-\beta_1$	$-\beta_2$	$-\beta_3$	$-\beta_4$	$-\beta_{k-1}$
$-K_3^3L^2$			$-K_3^3\beta_0$	$-K_3^3\beta_1$	$-K_3^3\beta_2$	$-K_3^3\beta_3$	$-K_3^3\beta_{k-2}$
RHS Term	- 1	0	K_3^0	0	0	0	0

Which yields: $\beta_0 = -1 = \beta_1$ and $\beta_2 - \beta_1 - K_3^3\beta_0 = K_3^0$, thereafter the LDE: $\beta_{k+2} - \beta_{k+1} - K_3^3\beta_k \equiv 0$, valid for $k \geq 1$ with ICs:{ $\beta_0 = \beta_1 = -1, \beta_2 = \beta_1 + K_3^3\beta_0 + K_3^0$ } for control parameter values: $K_3^3 = -0.81, K_3^0 = 1$ and ICs:{ $\beta_0 = \beta_1 = -1, \beta_2 = 0.81$ }, the LDE : $\beta_{k+2} - \beta_{k+1} + 0.81\beta_k \equiv 0$ has the solution: $\beta_k \equiv (0.9)^k\{ACosk\theta + BSink\theta\}$, with $\theta = 0.982$ radians, which plugging in the ICs yields the solution as:

$\beta_k = (0.9)^k\{-2.583Cosk\theta - 0.291Sink\theta\} \equiv (0.9)^k(2.6)Cos(k\theta + \phi), \phi = 0.1122$ radians.

Hence
$$\lim_{k\to\infty} \mathrm{var}(x_3(k)) = (\sum_{k=0}^{\infty}\beta_k^2)\sigma^2 = (\beta_0^2 + \beta_1^2 + \beta_2^2 + \sum_{k=3}^{\infty}\beta_k^2)\sigma^2$$
$$\leq (1 + 1 + .81^2 + \sum_{k=3}^{\infty} 2.6(.9)^k)\sigma^2$$
$$= (2.64 + \frac{(2.6)(.729)}{(1-0.9)})\sigma^2$$
which yields: $\lim_{k\to\infty} \mathrm{var}(x_3(k)) \leq 21.6\sigma^2$, showing bounded limiting inventory variance.

About Our Author

Kannan Nilakantan holds a Bachelor's degree in Engineering from the Indian Institute of Technology, Madras, and a Ph.D. from the Indian Institute of Science, Bangalore, and a Master's in Mathematics from University of South Florida, wherein was also inducted to the Phi-Kappa-Phi Honors Society of the U.S.A. Prof. Nilakantan was with the National Institute of Industrial Engineering, Mumbai, and with the mathematical modelling group in IMT-Nagpur, India. He has published papers in international journals like *Applied Mathematical Modelling, Annals of Management Science, Engineering Intelligent Systems, IMA Journal of Management Mathematics, IJOQM, IJl of System Science, IJl of Case Studies in Mangt, IJl of Applied Mangt Science, Jl of Modelling in Mangt, IJl of Business Systems Research, IJl of Business Performance and SC Modelling, IJl of Logistics Sys Mangt, IJl of Operations Research Society (UK), IJl of SC and Operations Resilience etc.* His research interests are in the applications of Dynamic Optimization and Optimal Control Models in Industrial and Business Systems.