Hindawi
Journal of Complex Analysis
Volume 2018, Article ID 5915864, 7 pages
https://doi.org/10.1155/2018/5915864

Research Article

Certain Integral Operator Related to the Hurwitz–Lerch Zeta Function

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Received 16 August 2017; Accepted 6 March 2018; Published 8 April 2018

Academic Editor: Pranay Goswami

The aim of the present paper is to investigate several third-order differential subordinations, differential superordination properties, and sandwich-type theorems of an integral operator $\mathscr{W}_{s,b}f(z)$ involving the Hurwitz–Lerch Zeta function. We make some applications of the operator $\mathscr{W}_{s,b}f(z)$ for meromorphic functions.

1. Introduction

Denote by $\mathscr{H}(\mathbb{U})$ the class of functions analytic in the unite disk

$$\mathbb{U} = \{z : z \in \mathbb{C},\ |z| < 1\} \tag{1}$$

of the form

$$\mathscr{H}[a,n] = \left\{f : f \in \mathscr{H}(\mathbb{U}),\ f(z) = a + \sum_{k=n}^{\infty} a_k z^k\right\} \quad (a \in \mathbb{C};\ n \in \mathbb{N} = \{1,2,\ldots\}) \tag{2}$$

and let $\mathscr{H} = \mathscr{H}[1,1]$.

For two functions $f(z)$ and $g(z)$ to be analytic in $\mathbb{U}$, $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$ and written by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}), \tag{3}$$

if there exists a Schwarz function $\omega(z)$, which is analytic in $\mathbb{U}$, with

$$\omega(0) = 0, \quad |\omega(z)| < 1, \tag{4}$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}). \tag{5}$$

It is generally known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{6}$$

Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{7}$$

Denote by Q the set of functions $q(z)$ that are analytic and univalent on $\overline{\mathbb{U}} \setminus \mathscr{E}(q)$, where

$$\mathscr{E}(q) = \left\{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty\right\} \tag{8}$$

are such that $\min|q'(\zeta)| = \varepsilon > 0$ for $\zeta \in \partial\mathbb{U} \setminus \mathscr{E}(q)$. Furthermore, let

$$Q(a) = \{q(z) \in Q : q(0) = a\}, \quad Q_1 = Q(1). \tag{9}$$

Denote by $\mathscr{A}^*$ the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{10}$$

which are *analytic* in the *punctured* unit disk

$$\mathbb{U}^* = \{z \in \mathbb{C},\ 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}. \tag{11}$$

We recall the general Hurwitz–Lerch Zeta function $\Phi(z, s, a)$ (see, e.g., [1, p. 121] and [2, p. 194]) defined by

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s} \quad \left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z| < 1;\ \Re(s) > 1 \text{ when } |z| = 1\right), \tag{12}$$

where

$$\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} = \{0, -1, -2, \ldots\}. \tag{13}$$

In recent years, the general Hurwitz–Lerch Zeta function $\Phi(z, s, a)$ was investigated by many researchers. A huge amount of interesting properties and consequences can be found in, for example, Choi and Srivastava [3], Garg et al. [4], Lin and Srivastava [5], and Srivastava et al. [6].

In 2007, by involving the general Hurwitz–Lerch Zeta function $\Phi(z, s, a)$, Srivastava and Attiya [7] (also see [8–11]) introduced the integral operator

$$\mathcal{J}_{s,b} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^s c_k z^k \quad \left(b \in \mathbb{C} \setminus \mathbb{Z}^-;\ s \in \mathbb{C};\ z \in \mathbb{U}\right). \tag{14}$$

Analogous to abovementioned operator $\mathcal{J}_{s,b} f$, Wang and Shi [12] introduced a new integral operator

$$\mathscr{W}_{s,b} : \Sigma \longrightarrow \Sigma \tag{15}$$

defined by

$$\mathscr{W}_{s,b} f(z) := \Theta_{s,b}(z) * f(z) \quad \left(b \in \mathbb{C} \setminus \{\mathbb{Z}_0^- \cup \{1\}\};\ s \in \mathbb{C};\ f \in \Sigma;\ z \in \mathbb{U}^*\right), \tag{16}$$

where

$$\Theta_{s,b}(z) := (b-1)^s \left[\Phi(z, s, b) - b^{-s} + \frac{1}{z(b-1)^s}\right] \quad \left(z \in \mathbb{U}^*\right), \tag{17}$$

and "$*$" denotes the Hadamard product.

From (10), (12), (16), and (17), we easily find that

$$\mathscr{W}_{s,b} f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{b-1}{b+k}\right)^s a_k z^k. \tag{18}$$

It is true that $b \in \mathbb{C} \setminus \{\mathbb{Z}^- \cup \{1\}\}$, the integral operator $\mathscr{W}_{s,b}$ defined as

$$\mathscr{W}_{s,0} f(z) := \lim_{b \to 0} \{\mathscr{W}_{s,b} f(z)\}. \tag{19}$$

We can deduce that

$$\mathscr{W}_{0,b} f(z) = f(z), \tag{20}$$

$$\mathscr{W}_{-1,0} f(z) = -z f'(z), \tag{21}$$

$$\mathscr{W}_{-1,-1} f(z) = \frac{f(z) - z f'(z)}{2}, \tag{22}$$

$$\mathscr{W}_{s,2} f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{k+2}\right)^s a_k z^k, \tag{23}$$

$$\mathscr{W}_{1,b+1} f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{b}{k+b+1}\right) a_k z^k = \frac{b}{z^{b+1}} \int_0^z t^b f(t)\, dt \quad (b > 0), \tag{24}$$

$$\mathscr{W}_{\alpha,\beta+1} f(z) = \frac{\beta^\alpha}{\Gamma(s) z^{\beta+1}} \int_0^z t^b \left(\log \frac{z}{t}\right)^{s-1} f(t)\, dt \quad (\alpha > 0;\ \beta > 0). \tag{25}$$

We also see that

$$\mathscr{W}_{1,\gamma} f(z) = \frac{\gamma - 1}{z^\gamma} \int_0^z t^{\gamma-1} f(t)\, dt \quad (\Re(\gamma) > 1). \tag{26}$$

Furthermore, by (18), we observe that

$$\mathscr{W}_{s+1,b} f(z) = \frac{b-1}{z^b} \int_0^z t^{b-1} \mathscr{W}_{s,b} f(z)\, dt \quad (\Re(b) > 1). \tag{27}$$

Operator (23) was introduced and studied by Alhindi and Darus [13]; operators (24) and (25) were introduced by Lashin [14].

The main purpose of this paper is to derive some third-order differential subordination, differential superordination properties, and sandwich-type theorems of the integral operator $\mathscr{W}_{s,b} f(z)$.

2. Preliminary Results

We will investigate our main results by using following definitions and lemmas.

Definition 1 (see [15, p. 440, Definition 1]). Suppose that $\Psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$, $q(z)$, and $h(z)$ are univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the third-order differential subordination

$$\psi\left(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z\right) \prec h(z), \tag{28}$$

then $p(z)$ is called a solution of the differential subordination. $q(z)$ is called a dominant of the solutions of the differential subordination or more simply a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (28). A dominant $\tilde{q}(z)$ that satisfies

$$\tilde{q}(z) \prec q(z), \tag{29}$$

for all dominants of (28), is called the best dominant of (28).

As the second-order differential superordinations were introduced and investigated by Miller and Mocanu [16], Tang et al. [17] introduced the following third-order differential superordinations.

Definition 2 (see [17, p. 3, Definition 5]). Suppose that $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and the function $h(z)$ is analytic in $\mathbb{U}$. If the functions $p(z)$ and

$$\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) \tag{30}$$

are univalent in $\mathbb{U}$ and satisfy the third-order differential superordination

$$h(z) \prec \psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right), \tag{31}$$

then $p(z)$ is called a solution of the differential superordination. An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination or more simply a subordinant if $q(z) \prec p(z)$ satisfies (31) for $p(z)$ satisfying (31). A univalent subordinant $\tilde{q}(z)$ that satisfies

$$q(z) \prec \tilde{q}(z) \tag{32}$$

for all superordinants $q(z)$ of (31) is said to be the best superordinant.

Lemma 3 (see [18, p. 132], [19, p. 190]). *Suppose that q is univalent in the open unit disk $\mathbb{U}$ and θ and ϕ are analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $\phi(\omega) \neq 0$ when $\omega \in q(\mathbb{U})$. Set $\Phi(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + \Phi(z)$. Suppose that*

(1) *Φ is star-like in $\mathbb{U}$;*

(2) $\Re(zh'(z)/\Phi(z)) > 0$.

If $p \in \mathcal{H}[q(0), n]$ for some $n \in \mathbb{N}$ with $p(\mathbb{U}) \subset \mathbb{D}$ and

$$\begin{aligned}&\theta(p(z)) + zp'(z)\phi(p(z)) \\ &\quad \prec \theta(q(z)) + zq'(z)\phi(q(z)),\end{aligned} \tag{33}$$

then $p \prec q$ and q is the best dominant.

Lemma 4 (see [20, p. 332]). *Suppose that q is univalent in the open unit disk $\mathbb{U}$ and θ and ϕ are analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$. Set $\Phi(z) = zq'(z)\phi(q(z))$. Suppose that*

(1) *Φ is star-like in $\mathbb{U}$;*

(2) $\Re(\theta'(q(z))/\phi(q(z))) > 0$.

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\mathbb{U}) \subseteq \mathbb{D}$, $\theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in $\mathbb{U}$, and

$$\begin{aligned}&\theta(q(z)) + zq'(z)\phi(q(z)) \\ &\quad \prec \theta(p(z)) + zp'(z)\phi(p(z)),\end{aligned} \tag{34}$$

then $q \prec p$ and q is the best dominant.

Lemma 5 (see [16, p. 822]). *Suppose that q is univalent complex in the open unit disk $\mathbb{U}$ and $\gamma \in \mathbb{C}$, with $\Re(\gamma) > 0$. If $p \in \mathcal{H}[q(0), 1] \cap Q$, $p(z) + \gamma zp'(z)$ is univalent in $\mathbb{U}$, and*

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z) \quad (z \in \mathbb{U}), \tag{35}$$

then $q \prec p$ and q is the best dominant.

3. Main Results

In this section, we state several third-order differential subordination and differential superordination results associated with the operator $\mathscr{W}_{s,b}f(z)$.

Theorem 6. *Suppose that the function $q \in \mathscr{A}^*$ is nonzero univalent in $\mathbb{U}$ with $q(0) = 1$ and*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{36}$$

Let $0 \leq \rho \leq 1$ and $\eta \in \mathbb{C}$. If $f \in \mathcal{H}[0, p]$ satisfies

$$\left[(1-\rho) z\mathscr{W}_{s,b}f(z) + \rho z\mathscr{W}_{s+1,b}f(z)\right] \neq 0 \quad (z \in \mathbb{U}), \tag{37}$$

$$\begin{aligned}&\eta\left[\frac{(1-\rho) z\left(\mathscr{W}_{s,b}f(z)\right)' + \rho z\left(\mathscr{W}_{s+1,b}f(z)\right)'}{(1-\rho)\mathscr{W}_{s,b}f(z) + \rho\mathscr{W}_{s+1,b}f(z)} - 1\right] \\ &\quad \prec \frac{zq'(z)}{q(z)},\end{aligned} \tag{38}$$

then

$$\left[(1-\rho) z\mathscr{W}_{s,b}f(z) + \rho z\mathscr{W}_{s+1,b}f(z)\right]^{\eta} \prec q(z) \tag{39}$$

and q is the best dominant in (39). When $\eta = 0$ the left hand side expressions in (39) are interpreted as 1.

Proof. Suppose that

$$p(z) := \left[(1-\rho) z\mathscr{W}_{s,b}f(z) + \rho z\mathscr{W}_{s+1,b}f(z)\right]^{\eta}. \tag{40}$$

Then p is analytic in $\mathbb{U}$. Logarithmically differentiating both sides of (40) with respect to z, we have

$$\begin{aligned}&\frac{zp'(z)}{p(z)} \\ &\quad = \eta\left[\frac{(1-\rho) z\left(\mathscr{W}_{s,b}f(z)\right)' + \rho z\left(\mathscr{W}_{s+1,b}f(z)\right)'}{(1-\rho)\mathscr{W}_{s,b}f(z) + \rho\mathscr{W}_{s+1,b}f(z)} \right. \\ &\quad \left. - 1\right].\end{aligned} \tag{41}$$

To apply Lemma 3, we set

$$\begin{aligned}&\theta(\omega) := 1, \\ &\phi(\omega) := \frac{1}{\omega} \\ &\qquad\qquad (\omega \in \mathbb{C} \setminus \{0\}), \\ &\Phi(z) = zq'(z)\phi(q(z)) = \frac{zq'(z)}{q(z)} \quad (z \in \mathbb{U}), \\ &h(z) = \theta(q(z)) + \Phi(z) = 1 + \frac{zq'(z)}{q(z)}.\end{aligned} \tag{42}$$

By means of (36) we see that $\Phi(z)$ is univalent star-like in $\mathbb{U}$. Since $h(z) = 1 + \Phi(z)$, we furthermore get that

$$\mathfrak{R}\left(\frac{zh'(z)}{\Phi(z)}\right) > 0. \tag{43}$$

By a routine calculation using (40) and (41) we find that

$$\theta(p(z)) + zp'(z)\phi(p(z)) = 1 + \eta\left[\frac{(1-\rho)z(\mathscr{W}_{s,b}f(z))' + \rho z(\mathscr{W}_{s+1,b}f(z))'}{(1-\rho)\mathscr{W}_{s,b}f(z) + \rho\mathscr{W}_{s+1,b}f(z)} - 1\right]. \tag{44}$$

Therefore, hypothesis (38) is equivalently written as

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec 1 + \frac{zq'(z)}{q(z)} = \theta(q(z)) + zq'(z)\phi(q(z)). \tag{45}$$

We know that condition (33) is also satisfied. From an application of Lemma 3, we have

$$p(z) \prec q(z). \tag{46}$$

Thus, we get the assertions in (39). Thus, the proof of Theorem 6 is completed. □

Theorem 7. *Suppose that the function $q \in \mathscr{A}^*$ is a univalent mapping of $\mathbb{U}$ into the right half plane with $q(0) = 1$ and*

$$\mathfrak{R}\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{47}$$

Let $0 \le \rho \le 1$ and $\eta \in \mathbb{C}$, $f \in \mathscr{H}[0, p]$ satisfy

$$[(1-\rho)z\mathscr{W}_{s,b}f(z) + \rho z\mathscr{W}_{s+1,b}f(z)] \neq 0 \quad (z \in \mathbb{U}). \tag{48}$$

If

$$\Delta(z) \prec q(z) + \frac{zq'(z)}{q(z)}, \tag{49}$$

where

$$\Delta(z) = [(1-\rho)z\mathscr{W}_{s,b}f(z) + \rho z\mathscr{W}_{s+1,b}f(z)]^{\eta} + \eta\left[\frac{(1-\rho)z(\mathscr{W}_{s,b}f(z))' + \rho z(\mathscr{W}_{s+1,b}f(z))'}{(1-\rho)\mathscr{W}_{s,b}f(z) + \rho\mathscr{W}_{s+1,b}f(z)} - 1\right], \tag{50}$$

then

$$[(1-\rho)z\mathscr{W}_{s,b}f(z) + \rho z\mathscr{W}_{s+1,b}f(z)]^{\eta} \prec q(z) \tag{51}$$

and q is the best dominant in (51). When $\eta = 0$, the left hand side expression of (51) is interpreted as 1.

Proof. Suppose that the function $p(z)$ is defined by (40). If set

$$\theta(\omega) := \omega, \quad \phi(\omega) := \frac{1}{\omega} \quad (\omega \in \mathbb{C} \setminus \{0\}), \tag{52}$$

$$\Phi(z) = zq'(z)\phi(q(z)) = \frac{zq'(z)}{q(z)} \quad (z \in \mathbb{U}),$$

$$h(z) = \theta(q(z)) + \Phi(z) = q(z) + \Phi(z)$$

we easily get

$$\mathfrak{R}\left(\frac{zh'(z)}{\Phi(z)}\right) = \mathfrak{R}\left(q(z) + 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{53}$$

By virtue of (41), hypothesis (49) can be rewritten as

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)). \tag{54}$$

Therefore, by making use of Lemma 3, we derive that

$$p(z) \prec q(z) \quad (z \in \mathbb{U}). \tag{55}$$

Thus, the assertion in (49) follows. The proof of Theorem 7 is completed. □

Theorem 8. *Suppose that the function $q \in \mathscr{A}^*$ is a univalent mapping of $\mathbb{U}$ into the right half plane with $q(0) = 1$ and satisfies condition*

$$\mathfrak{R}\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{56}$$

Let $0 \le \rho \le 1$, $\eta \in \mathbb{C}$, and $f \in \mathscr{H}[0, p]$ satisfy

$$[(1-\rho)z\mathscr{W}_{s,b}f(z) + \rho z\mathscr{W}_{s+1,b}f(z)]^{\eta} \in \mathscr{H}[1, 1] \cap Q. \tag{57}$$

Let function $\Delta(z)$ be univalent in $\mathbb{U}$, where $\Delta(z)$ is defined by (50). If

$$q(z) + \frac{zq'(z)}{q(z)} \prec \Delta(z), \tag{58}$$

then

$$q(z) \prec [(1-\rho)z\mathscr{W}_{s,b}f(z) + \rho z\mathscr{W}_{s+1,b}f(z)]^{\eta} \tag{59}$$

and q is the best subordinant in (59). When $\eta = 0$, the left hand side expressions of (59) are interpreted as 1.

Proof. By putting

$$\theta(\omega) := \omega,$$
$$\phi(\omega) := \frac{1}{\omega} \quad (\omega \in \mathbb{C} \setminus \{0\}), \tag{60}$$
$$\Phi(z) = zq'(z)\phi(q(z)) = \frac{zq'(z)}{q(z)} \quad (z \in \mathbb{U}),$$

obviously, Φ is star-like in $\mathbb{U}$ and

$$\Re\left(\frac{\theta'(q(z))}{\phi(q(z))}\right) = \Re(q(z)) \quad (z \in \mathbb{U}). \tag{61}$$

Suppose that function p is defined by (40). By simple calculation, from (41), we know that

$$\theta(p(z)) + zp'(z)\phi(p(z)) = \Delta(z). \tag{62}$$

Hence, condition (58) can be equivalently written as

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)). \tag{63}$$

Therefore, by Lemma 4, we have

$$q(z) \prec p(z) \quad (z \in \mathbb{U}) \tag{64}$$

and q is the best subordinant. The proof of Theorem 8 is completed. □

Theorem 9. *Suppose that* $0 \le \rho \le 1$, $\alpha, \eta \in \mathbb{C}$, *the function* $q \in \mathscr{A}^*$ *is univalent in* $\mathbb{U}$, *and*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\{0, -\Re(\alpha)\}. \tag{65}$$

Let $f \in \mathscr{H}[0, p]$ *satisfy*

$$[(1-\rho)z\mathscr{W}_{s,b}f(z) + \rho z\mathscr{W}_{s+1,b}f(z)] \neq 0 \quad (z \in \mathbb{U}). \tag{66}$$

Denote by

$$\Xi(z) = [(1-\rho)z\mathscr{W}_{s,b}f(z) + \rho z\mathscr{W}_{s+1,b}f(z)]^{\eta} \times \left\{\alpha + \eta\left[\frac{(1-\rho)z(\mathscr{W}_{s,b}f(z))' + \rho z(\mathscr{W}_{s+1,b}f(z))'}{(1-\rho)\mathscr{W}_{s,b}f(z) + \rho\mathscr{W}_{s+1,b}f(z)} - 1\right]\right\} \quad (z \in \mathbb{U}). \tag{67}$$

If

$$\Xi(z) \prec \alpha q(z) + zq'(z), \tag{68}$$

then

$$[(1-\rho)z\mathscr{W}_{s,b}f(z) + \rho z\mathscr{W}_{s+1,b}f(z)]^{\eta} \le q(z) \tag{69}$$

and q *is the best dominant in (69). When* $\eta = 0$, *the left side hand expressions of (69) are interpreted as 1.*

Proof. Suppose that function $p(z)$ is defined by (40). Making using of (41), we have

$$zp'(z) = \eta p(z) \cdot \left[\frac{(1-\rho)z(\mathscr{W}_{s,b}f(z))' + \rho z(\mathscr{W}_{s+1,b}f(z))'}{(1-\rho)\mathscr{W}_{s,b}f(z) + \rho\mathscr{W}_{s+1,b}f(z)} - 1\right]. \tag{70}$$

Therefore, by putting

$$\theta(\omega) := \alpha\omega,$$
$$\phi(\omega) := 1 \quad (\omega \in \mathbb{C}), \tag{71}$$
$$\Phi(z) = zq'(z)\phi(q(z)) = zq'(z) \quad (z \in \mathbb{U}),$$
$$h(z) = \theta(q(z)) + \Phi(z) = \alpha q(z) + zq'(z),$$

obviously, Φ is star-like in $\mathbb{U}$ and

$$\Re\left(\frac{zh'(z)}{\Phi(z)}\right) = \Re\left(\alpha + 1 + \frac{zq''(z)}{q'(z)}\right) > 0. \tag{72}$$

Furthermore, by substituting the expression for $p(z), zp'(z)$ from (40) and (70), respectively, we get

$$\theta(p(z)) + zp'(z)\phi(p(z)) = \alpha(p(z)) + zp'(z)\phi(p(z)) = \Xi(z), \tag{73}$$

where $\Xi(z)$ is given by (67). Hypothesis (68) can be equivalently written as

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)). \tag{74}$$

From Lemma 3, we get

$$p(z) \prec q(z). \tag{75}$$

Thus, we get assertion (69) of Theorem 9. □

Theorem 10. *Suppose that* $0 \le \rho \le 1$, $\eta \in \mathbb{C}$, $\alpha \in \mathbb{C} \setminus \{0\}$, $\Re(\alpha) > 0$; *function* $q \in \mathscr{A}^*$ *is univalent in* $\mathbb{U}$ *with* $q(0) = 1$. *Let function* $f \in \mathscr{H}[0, p]$ *satisfy*

$$[(1-\rho)z\mathscr{W}_{s,b}f(z) + \rho z\mathscr{W}_{s+1,b}f(z)] \neq 0 \quad (z \in \mathbb{U}),$$
$$[(1-\rho)z\mathscr{W}_{s,b}f(z) + \rho z\mathscr{W}_{s+1,b}f(z)]^{\eta} \in \mathscr{H}[1,1] \cap Q. \tag{76}$$

If $\Xi(z)$ *defined by (67) is univalent and satisfies*

$$\alpha q(z) + zq'(z) \prec \Xi(z), \tag{77}$$

then

$$\left[(1-\rho)z\mathscr{W}_{s,b}f(z)+\rho z\mathscr{W}_{s+1,b}f(z)\right]^{\eta}\prec q(z) \tag{78}$$

and q is the best subordinant in (78). When $\eta=0$, the left hand side expressions of (78) are interpreted as 1.

Proof. Suppose that function $p(z)$ is defined by (40). From (41), we get

$$\alpha(p(z))+zp'(z)\phi(p(z))=\Xi(z). \tag{79}$$

Hypothesis (77) can be rewritten as

$$q(z)+\left(\frac{1}{\alpha}\right)zq'(z)\prec p(z)+\left(\frac{1}{\alpha}\right)zp'(z). \tag{80}$$

Then, combining Lemma 5 with $\gamma=1/\alpha$, we have (78). Theorem 10 follows immediately. □

Following that, we display some sandwich-type theorems associated with the operator $\mathscr{W}_{s,b}f(z)$.

Theorem 11. *Suppose that functions $q_1,q_2\in\mathscr{A}^*$ are univalent mapping of $\mathbb{U}$ into the right half plane and satisfy conditions*

$$q_1(0)=q_2(0)=1,$$
$$\Re\left(1+\frac{zq_j''(z)}{q_j'(z)}-\frac{zq_j'(z)}{q_j(z)}\right)>0\quad(j=1,2;\ z\in\mathbb{U}). \tag{81}$$

Let $0\le\rho\le1$, $\alpha,\eta\in\mathbb{C}$, and $f\in\mathscr{H}[0,p]$ satisfy

$$\left[(1-\rho)z\mathscr{W}_{s,b}f(z)+\rho z\mathscr{W}_{s+1,b}f(z)\right]\neq0\quad(z\in\mathbb{U}),$$
$$\left[(1-\rho)z\mathscr{W}_{s,b}f(z)+\rho z\mathscr{W}_{s+1,b}f(z)\right]^{\eta}\in\mathscr{H}[1,1]\cap Q. \tag{82}$$

If function $\Delta(z)$ is given by (50) and satisfies

$$q_1(z)+\frac{zq_1'(z)}{q_1(z)}\prec\Delta(z)\prec q_2(z)+\frac{zq_2'(z)}{q_2(z)}, \tag{83}$$

then

$$q_1(z)\prec\left[(1-\rho)z\mathscr{W}_{s,b}f(z)+\rho z\mathscr{W}_{s+1,b}f(z)\right]^{\eta}\prec q_2(z), \tag{84}$$

where q_1 and q_2 are, respectively, the best subordinant and the best dominant in (84).

Combining Theorems 9 and 10, we get the following result.

Corollary 12. *Suppose that $0\le\rho\le1$, $\eta\in\mathbb{C}$, and $\alpha\in\mathbb{C}\setminus\{0\}$ with $\Re(\alpha)>0$. Functions q_1 and q_2 are univalent convex in $\mathbb{U}$ with $q_1(0)=q_2(0)=1$. Let $f\in\mathscr{H}[0,p]$ satisfy*

$$\left[(1-\rho)z\mathscr{W}_{s,b}f(z)+\rho z\mathscr{W}_{s+1,b}f(z)\right]\neq0\quad(z\in\mathbb{U}),$$
$$\left[(1-\rho)z\mathscr{W}_{s,b}f(z)+\rho z\mathscr{W}_{s+1,b}f(z)\right]^{\eta}\in\mathscr{H}[1,1]\cap Q. \tag{85}$$

If function $\Xi(z)$ is given by (67) and satisfies

$$q_1(z)+zq_1'(z)\prec\Xi(z)\prec\alpha q_2(z)+zq_2'(z), \tag{86}$$

then

$$q_1(z)\prec\left[(1-\rho)z\mathscr{W}_{s,b}f(z)+\rho z\mathscr{W}_{s+1,b}f(z)\right]^{\eta}\prec q_2(z), \tag{87}$$

where q_1 and q_2 are, respectively, the best subordinant and the best dominant in (87).

4. Conclusions

In the present paper, making use of the integral operator $\mathscr{W}_{s,b}f(z)$ involving the Hurwitz–Lerch Zeta function, we have derived several third-order differential subordination and differential superordination consequences of meromorphic functions in the punctured unit disk. Furthermore, the sandwich-type theorems are considered. These subordinate relationships have shown the upper and lower bounds of the operator in the punctured unit disk.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of the paper.

Acknowledgments

The present investigation was supported by the *National Natural Science Foundation* under Grant no. 11301008, the *Foundation for Excellent Youth Teachers of Colleges and Universities of Henan Province* under Grant no. 2013GGJS-146, and the *Foundation of Educational Committee of Henan Province* under Grant no. 17A110014.

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