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Generalized topology and the family of monotonic maps $\Gamma(X)$

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Abstract

In this paper, interesting properties of the generalized topological spaces, generated by the monotonic maps $\sigma = (cl_\delta \circ int_\delta), \alpha = (int_\delta \circ cl_\delta \circ int_\delta), \pi = (int_\delta \circ cl_\delta)$ and $\beta = (cl_\delta \circ int_\delta \circ cl_\delta)$, for any generalized topological space (X, g_δ) are deduced and analyzed. Special subfamilies of the family of monotonic maps $\Gamma(X)$ are studied and interesting results regarding generalized topologies are obtained.

Keywords: Family of monotonic maps $\Gamma(X)$, Császár generalized topological space, Interesting monotonic maps $\{int_\delta, cl_\delta, \sigma, \alpha, \pi, \beta\}$

Mathematics Subject Classification: Primary 54C50, Secondary 54A05

Introduction

In [4], Á. Császár introduced the generalized topological spaces. He showed that each monotonic map $\delta : P(X) \to P(X)$ ($\delta(A) \subset \delta(B)$, for each $A \subset B$) defines a generalized topology g_δ on X, containing all the subsets O, that satisfy $\delta(O) \supset O$. The family of all monotonic maps δ is denoted by $\Gamma(X)$. Moreover, each generalized topology g on the set X defines a monotonic map δ_g, such that $\delta_g(O) \supset O$, for every $O \in g$.

To learn about the studies of the $\gamma-$generalized topological spaces (X, g_γ), see the references [4–8]. Moreover, to learn about the studies of the generalized continuity of functions on the $\gamma-$generalized topological spaces (X, g_γ), which is generated by the monotonic functions $\gamma \in \{int_\delta, cl_\delta, \sigma, \alpha, \pi, \beta\}$, see the references [1–3, 9–13], where g_δ is a given generalized topology on X. In addition to that the properties of interior and closure operators are outlined in [14].

The outline of this manuscript is as follows: In the first section, some properties of the subclasses of the family of monotonic maps $\Gamma(X)$, whose elements generate the same generalized topology, are studied. Moreover, the relations between the family $\Phi \subset \Gamma(X)$ of all monotonic maps $\gamma \in \Gamma(X)$, for which there exists a function $f : X \to X$ such that $\gamma(A) = f^{-1}(A)$; $A \in P(X)$, and the generalized topologies on X are studied.

In the second section, the study of some properties and examples on the family of all monotonic maps $\Gamma(X)$ are outlined.

In the third section, the generalized topologies generated by the monotonic maps: $\sigma = (cl_\delta \circ int_\delta), \alpha = (int_\delta \circ cl_\delta \circ int_\delta), \pi = (int_\delta \circ cl_\delta), \beta = (cl_\delta \circ int_\delta \circ cl_\delta)$ are studied.

In the fourth section, some interesting relations between the elements of the subfamily $\{int_\delta, cl_\delta, \sigma, \alpha, \pi, \beta\} \subset \Gamma(X)$, for any $\delta-$Csázsár generalized topological space (X, g_δ), are found.

Study of special classes of the family of all monotonic maps $\Gamma(X)$

Equivalence classes on the family of all monotonic maps $\Gamma(X)$

Definition 1 Consider the following binary operations on the family $\Gamma(X)$, where $\gamma, \delta \in \Gamma(X)$ and A is a subset of X:

1 $(\gamma \circ \delta)(A) = \gamma(\delta(A))$.
2 $(\gamma \cap \delta)(A) = \gamma(A) \cap \delta(A)$.
3 $(\gamma \cup \delta)(A) = \gamma(A) \cup \delta(A)$.

It is clear that for every $\gamma, \delta \in \Gamma(X)$, the maps $\gamma \circ \delta, \gamma \cap \delta, \gamma \cup \delta$ are monotonic maps and are elements of $\Gamma(X)$.

Definition 2 (*Equivalent relation on the family* $\Gamma(X)$) The maps $\gamma, \delta \in \Gamma(X)$ are called equivalent maps, if the family of all $\gamma-$open sets is identical with the family of all $\delta-$open sets $(g_\delta = g_\gamma)$, and we write $\gamma \approx \delta$.

Each equivalence class of this relation is characterized by its family of open sets,which forms a generalized topology. The equivalence class, which contains the map δ, will be denoted by $\Gamma_\delta(X)$ or simply Γ_δ.

Theorem 3 *The interior operator int_δ of the generalized topology g_δ is the smallest element of the class Γ_δ. Moreover, for every $A \subset X$:*

$$int_\delta(A) = \bigcap_{\gamma \in \Gamma_\delta} \gamma(A)$$

Proof The proof is obtained through the following steps:

1 Let $I_\delta(A) = \bigcap_{\gamma \in \Gamma_\delta} \gamma(A)$, for every $A \subset X$.
It is clear that $\gamma(A) \supset I_\delta(A) = \bigcap_{\gamma \in \Gamma_\delta} \gamma(A)$, for every $\gamma \in \Gamma_\delta$ and $A \subset X$. Moreover, $I_\delta \in \Gamma(X)$.
2 Let A be an open set in the class Γ_δ, then for every $\gamma \in \Gamma_\delta$, we have $A \subset \gamma(A)$ and $A \subset I_\delta(A)$. Therefore, every open set in Γ_δ is an open set relative to I_δ.

Now, if C is an open set relative to I_δ, i.e. $C \subset I_\delta(C) = \bigcap_{\gamma\in\Gamma_\delta} \gamma(C)$. It follows that $C \subset \gamma(C)$; for each $\gamma \in \Gamma_\delta$. Therefore, C is $\gamma-$open set for all $\gamma \in \Gamma_\delta$ and the family of open sets in Γ_δ is identical with the family of open sets of I_δ and so $I_\delta \in \Gamma_\delta$. Hence I_δ is the smallest element in Γ_δ.

3 It is clear that int_δ is a monotonic map, then $int_\delta \in \Gamma(X)$,; moreover, the relation $O \subset int_\delta(O)$ is valid only for the elements of g_δ. Then, $int_\delta \in \Gamma_\delta$.

4 Since $int_\delta \in \Gamma_\delta$, then $int_\delta(B) \supset I_\delta(B)$; for every $B \subset X$.

5 We shall show that $int_\delta(B) \subset I_\delta(B)$; for every $B \subset X$.
Let $B \subset X$, then $int_\delta(B) \subset B$. Since $int_\delta \in g_\delta$ and $I_\delta \in \Gamma_\delta$, then
$int_\delta(B) \subset I_\delta(int_\delta(B)) \subset I_\delta(B)$.
From 1 up to 5, it follows that $I_\delta = int_\delta$. Therefore, g_δ is the smallest element of the class Γ_δ.

□

Definition 4 Let Γ_δ be an equivalence class for any $\delta-$generalized topology on the set X. Every $\gamma \in \Gamma_\delta$ defines the map

$$\theta_\gamma(B) = X - \gamma(X - B);\, B \subset X.$$

Definition 5 To each equivalence class Γ_δ, there exists an associated class

$$\Gamma^\delta = \{\theta_\gamma : \gamma \in \Gamma_\delta\}.$$

Theorem 6 *Let (X, g_δ) be a generalized topological space, then the following properties are satisfied*:

1 $\Gamma^\delta \subset \Gamma(X)$.

2 *For any $\delta-$closed subset B, it follows that $\theta_\gamma(B) \subset B$, for all $\gamma \in \Gamma_\delta$.*

Proof

1 Let $A \subset B$, then $X - A \supset X - B$. Therefore, $\gamma(X - A) \supset \gamma(X - B)$, for every $\gamma \in \Gamma_\delta$, then

$$\theta_\gamma(A) = X - \gamma(X - A) \subset X - \gamma(X - B) = \theta_\gamma(B).$$

Which means that θ_γ is a monotonic map and $\theta_\gamma \in \Gamma(X)$. Consequently, $\Gamma^\delta \subset \Gamma(X)$.

2 Let $A = X - B$ be an open set in g_δ, then $A = X - B \subset \gamma(A) = \gamma(X - B)$, for all $\gamma \in \Gamma_\delta$. Consequently, $B \supset X - \gamma(X - B) = \theta_\gamma(B)$.

□

Theorem 7 *The closure operator cl_δ of the generalized topology g_δ is the largest element of the class Γ^δ. Moreover, for every $B \subset X$:*

$$cl_\delta(B) = \theta_{int_\delta}(B) = \bigcup_{\gamma\in\Gamma_\delta} \theta_\gamma(B).$$

Proof The proof is obtained through the following two steps:

1. Since the interior monotonic map int_δ, where $int_\delta(C) = \bigcup_{C \supset A \in g_\delta} A$ defines the map θ_{int_δ}, where $\theta_{int_\delta}(B) = X - int_\delta(X - B)$, then $\theta_{int_\delta} \in \Gamma^\delta$. Consequently,

$$\theta_{int_\delta}(B) = X - int_\delta(X - B) = X - \bigcap_{\gamma \in \Gamma_\delta} \gamma(X - B) = \bigcup_{\gamma \in \Gamma_\delta} (X - \gamma(X - B)) = \bigcup_{\gamma \in \Gamma_\delta} \theta_\gamma(B).$$

which means that the monotonic map θ_{int_δ} is the largest monotonic map in the associated class Γ^δ.

2. Let $B \subset X$, then

$$\theta_{int_\delta}(B) = X - int_\delta(X - B) = X - \bigcup_{(X-B) \supset A \in g_\delta} A = \bigcap_{A \in g_\delta, B \subset X - A} (X - A) = \bigcap_{(X-D) \in g_\delta, B \subset D} D.$$

Since

$$cl_\delta(B) = \bigcap_{(X-D) \in g_\delta, B \subset D} D,$$

then $\theta_{int_\delta}(B) = cl_\delta(B)$, for any $B \subset X$, which implies that $\theta_{int_\delta} = cl_\delta$.

□

The subfamily $\Phi \subset \Gamma(X)$ corresponding to the family of functions X^X

Let X^X be the family of all functions $f : X \to X$. Then, for every $f \in X^X$, there exists $\gamma_f \in \Gamma(X)$, which is defined as follows: for each $A \in P(X)$

$$\gamma_f : P(X) \to P(X); \gamma_f(A) = f^{-1}(A).$$

Definition 8 The map $\Psi : X^X \to \Gamma(X)$ is defined by $\Psi(f) = \gamma_f$.

The map Ψ is an injective map: Let $\Psi(f) = \Psi(g)$ and $x \in X$. If $f(x) = y$, then $x \in \gamma_f(\{y\}) = \gamma_g(\{y\})$. It follows that $g(x) = y = f(x)$. But $x \in X$ is an arbitrary element, then $f = g$.

Definition 9 The subfamily Φ of the family of monotonic maps $\Gamma(X)$ is defined as:

$$\Phi = \left\{\gamma_f \in \Gamma(X) : f \in X^X\right\} = \Psi(X^X) \subset \Gamma(X).$$

The subfamily $\Phi \subset \Gamma(X)$ has a close relationship to the family of continuous functions on the topological spaces on X.

Lemma 10 *The monotonic map $\gamma \in \Gamma(X)$ is an element of Φ, if and only if it satisfies the following conditions*:

(*a*) $\gamma(\{y_1\}) \cap \gamma(\{y_2\}) = \emptyset$, *for all* $y_1, y_2 \in X$ *and* $y_1 \neq y_2$.
(*b*) $\bigcup_{y \in X} \gamma(\{y\}) = X$.

(*c*) *$\gamma(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \gamma(A_i)$ and $\gamma(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} \gamma(A_i)$, where I is an arbitrary indexed set.*

Then, $\gamma = f^{-1}$, where f is a function from X to itself, where $f(x) = y$; if $x \in \gamma(\{y\})$.

Proof Let $\gamma \in \Phi$, then there exists $f \in X^X$ and $\gamma = \gamma_f$.

(*a*) Let $y_1, y_2 \in X$ such that $y_1 \neq y_2$, then

$$\gamma(\{y_1\}) \cap \gamma(\{y_2\}) = \gamma_f(\{y_1\}) \cap \gamma_f(\{y_2\}) = f^{-1}(y_1 \cap y_2) = f^{-1}(\emptyset) = \emptyset.$$

(*b*) $\bigcup_{y\in X} \gamma(\{y\}) = \bigcup_{y\in X} \gamma_f(\{y\}) = \bigcup_{y\in X} f^{-1}(\{y\}) = f^{-1}(\bigcup_{y\in X}\{y\}) = f^{-1}(X) = X.$

(*c*)

$$\gamma\left(\bigcup_{i\in I} A_i\right) = \gamma_f\left(\bigcup_{i\in I} A_i\right) = f^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f^{-1}(A_i) = \bigcup_{i\in I} \gamma_f(A_i) = \bigcup_{i\in I} \gamma(A_i).$$

Moreover,

$$\gamma\left(\bigcap_{i\in I} A_i\right) = \gamma_f\left(\bigcap_{i\in I} A_i\right) = f^{-1}\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f^{-1}(A_i) = \bigcap_{i\in I} \gamma_f(A_i) = \bigcap_{i\in I} \gamma(A_i).$$

□

Lemma 11 *The subfamily $\Phi \subset \Gamma(X)$ is closed relative to the composition binary operation.*

Proof Let $\gamma_f, \gamma_g \in \Phi$. Since $\gamma_f(A) = f^{-1}(A)$ and $\gamma_g(B) = g^{-1}(B)$, for all $A, B \subset X$, then

$$(\gamma_f \circ \gamma_g)(A) = (f^{-1} \circ g^{-1})(A) = (g \circ f)^{-1}(A) = \gamma_{g\circ f}(A).$$

□

For each $G \subset P(X)$, define the family $\Omega_G \subset P(\Phi)$ as:

$$\Omega_G = \{\mathcal{H} \subset \Phi : h(G) \subset G \quad \forall \quad h \in \mathcal{H}\}$$

Definition 12 The subfamily $G \subset P(X)$ is called invariant relative to $\mathcal{H}_G$ if $\mathcal{H}_G$ is the maximal element of the family Ω_G with respect to the inclusion relation.

Remark 13 The family $\mathcal{H}_G$ is not empty for every $G \subset P(X)$, since the identity function $id_X(A) = A$, for all $A \in P(X)$ belongs to every $\mathcal{H}_G$.

For each $G \subset P(X)$, define the subfamily $\mathbf{F}_G$ of the family X^X as:

$$\mathbf{F}_G = \{f \in X^X \quad : \quad \gamma_f \in \mathcal{H}_G\}$$

Theorem 14 *If $G \subset P(X)$ is a generalized topology on X, then the family $\mathbf{F}_G$ is the family of generalized continuous functions on the topological space (X, G).*

Proof The proof is clear, since: If $f \in \mathbf{F}_G$ and $A \in G$, then $\gamma_f \in \mathcal{H}_G$, which implies that $\gamma_f(A) = f^{-1}(A) \in G$. Therefore, $f : (X, G) \to (X, G)$ is a generalized continuous function. □

Lemma 15 *All the elements $h \in \Phi$ satisfy the relations*:

$$h\left(\bigcup_{i\in K} A_i\right) = \bigcup_{i\in K} h(A_i), \quad h\left(\bigcap_{i\in K} A_i\right) = \bigcap_{i\in K} h(A_i),$$

for any arbitrary family $\{A_i \subset X : i \in K\} \subset P(X)$, where K is an arbitrary index set.

Proof The proof is straightforward, since for any function $f \in X^X$,

$$f^{-1}\left(\bigcup_{i\in K} A_i\right) = \bigcup_{i\in K} f^{-1}(A_i), \quad f^{-1}\left(\bigcap_{i\in K} A_i\right) = \bigcap_{i\in K} f^{-1}(A_i).$$

□

Theorem 16 *If G and G_0 are subsets of P(X) and $G \subset G_0$, then $\mathcal{H}_G \subset \mathcal{H}_{G_0}$, if each element of G_0 can be written as arbitrary unions of finite (arbitrary) intersections of elements of G.*

Proof Let G, G_0 be subsets of P(X), where $G \subset G_0$. Let G, G_0 be invariant relative to $\mathcal{H}_G$ and $\mathcal{H}_{G_0}$ respectively. Then, $h(G) \subset G$; $h \in \mathcal{H}_G$. Let $g \in G_0$, then from the assumption, g can be written in the form: $g = \bigcup_{i\in I_0} \bigcap_{j_i\in K_i} A_{j_i}$; where $A_{j_i} \subset G$, for all i, j. Consequently, if $h \in \mathcal{H}_G$, then from Lemma (1.15), it follows that

$$h(g) = h\left(\bigcup_{i\in I_0} \bigcap_{j_i\in K_i} A_{j_i}\right) = \bigcup_{i\in I_0} \bigcap_{j_i\in K_i} h(A_{j_i}) = \bigcup_{i\in I_0} \bigcap_{j_i\in K_i} A^*_{j_i} \subset G_0,$$

since $A^*_{j_i} = h(A_{j_i}) \in G$. Then, $h \in \mathcal{H}_{G_0}$, and so $\mathcal{H}_G \subset \mathcal{H}_{G_0}$. □

Corollary 17 *If G is a subset of P(X), then $\mathcal{H}_G \subset \mathcal{H}_{\tau(G)}$, where $\tau(G)$ is the (generalized topology) topology on the set X, generated by G as a (generalized base) sub-base. Since the elements of $\tau(G)$ are obtained from the elements of G, using (arbitrary unions) arbitrary unions and arbitrary finite intersections.*

The following example shows that in general, if G, G_0 are subsets of P(X), where $G \subset G_0$. Then it is not necessary that $\mathcal{H}_G \subset \mathcal{H}_{G_0}$.

Example 18 Let $X = \{a, b, c\}$, $G_1 = \{\{a\}\}$ and $G_2 = \{\{a\}, \{b\}\}$. Consider the function $f : X \to X$, where $f(a) = a, f(b) = c$ and $f(c) = b$. Then, $\gamma_f(G_1) = f^{-1}(G_1) = \{\{a\}\} = G_1$, which implies that $\gamma_f \in \mathcal{H}_{G_1}$. But $\gamma_f(G_2) = f^{-1}(G_2) = \{\{a\}, \{c\}\} \not\subset G_2$, which implies that $\gamma_f \notin \mathcal{H}_{G_2}$. Therefore, $\mathcal{H}_{G_1} \not\subset \mathcal{H}_{G_2}$.

It is clear that the element $\{b\} \in G_2$ can't be obtained from the elements of G_1, using the union and intersection operations. This justifies why $\mathcal{H}_{G_1}$ is not contained in $\mathcal{H}_{G_2}$, although $G_1 \subset G_2$.

The following example shows that in general, if $G \subset P(X)$. Then $\mathcal{H}_G \neq \mathcal{H}_{\tau(G)}$, where $\tau(G)$ is the generalized topology generated by G.

Example 19 Let $X = \{a, b, c\}$. Choose $G = \{\{a\}, \{b\}\}$. Then, $\tau(G) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Consider the function $g : X \to X$, where $g(a) = b, g(b) = b$ and $g(c) = c$. Then, the action of γ_g is defined as follows: $\gamma_g(G) = g^{-1}(G) = \{\emptyset, \{a, b\}\} \not\subset G$, then $\gamma_g \notin \mathcal{H}_G$, $\gamma_g(\tau(G)) = g^{-1}(\tau(G)) = \{\emptyset, \{a, b\}\} \subset \tau(G)$, then $\gamma_g \in \mathcal{H}_{\tau(G)}$. Therefore, $\mathcal{H}_G \neq \mathcal{H}_{\tau(G))}$.

Theorem 20 *Let $G_1, G_2 \subset P(X)$, then $\mathcal{H}_{G_1} \cap \mathcal{H}_{G_2} \subset \mathcal{H}_{G_1 \cap G_2}$.*

Proof Let $\gamma \in \mathcal{H}_{G_1} \cap \mathcal{H}_{G_2}$, then $\gamma \in \mathcal{H}_{G_i}, i \in \{1, 2\}$. It follows that $\gamma(G_i) \subset G_i, i \in \{1, 2\}$. Consequently, $\gamma(G_1 \cap G_2) = \gamma(G_1) \cap \gamma(G_2) \subset G_1 \cap G_2$. It follows that $\gamma \in \mathcal{H}_{G_1 \cap G_2}$. Therefore, $\mathcal{H}_{G_1} \cap \mathcal{H}_{G_2} \subset \mathcal{H}_{G_1 \cap G_2}$. □

Example (1.17) shows that the inverse statement of Theorem (1.19) is not valid. Since $G_1 \cap G_2 = G_1$ and $\mathcal{H}_{G_1 \cap G_2} = \mathcal{H}_{G_1} \not\subset \mathcal{H}_{G_2}$.

Remark 21 If $\mathcal{H} \subset \Gamma(X)$ and $\gamma \in \Gamma(X)$, then $\gamma \circ \mathcal{H}$ and $\mathcal{H} \circ \gamma$ can be defined as:

$$\gamma \circ \mathcal{H} = \{\gamma \circ h : h \in \mathcal{H}\}, \quad \mathcal{H} \circ \gamma = \{h \circ \gamma : h \in \mathcal{H}\} \subset \Gamma(X).$$

Definition 22 Let $G \subset P(X)$, then the ordered pair $\prec G, \mathcal{H}_G \succ$ is called an invariant system.

Theorem 23 *If $\prec G, \mathcal{H}_G \succ$ is an invariant system and $f \in X^X$ is a one-to-one correspondence, then $\prec f(G), \gamma_{f^{-1}} \circ \mathcal{H}_G \circ \gamma_f \succ$ is an invariant system and $\mathcal{H}_{f(G)} = \gamma_{f^{-1}} \circ \mathcal{H}_G \circ \gamma_f$.*

Proof Since $\prec G, \mathcal{H}_G \succ$ is an invariant system, then $h(G) \subset G; h \in \mathcal{H}_G$. Consequently, we have: $(\gamma_{f^{-1}} \circ h \circ \gamma_f)(f(G)) = \gamma_{f^{-1}}(h(\gamma_f(f(G)))) = f(h(f^{-1}(f(G)))) \subset f(h(G)) \subset f(G)$. Therefore, $\gamma_{f^{-1}} \circ h \circ \gamma_f \in \mathcal{H}_{f(G)}$, for all $h \in \mathcal{H}_G$. Hence, $\gamma_{f^{-1}} \circ \mathcal{H}_G \circ \gamma_f \subset \mathcal{H}_{f(G)}$. Now, we show that $\gamma_{f^{-1}} \circ \mathcal{H}_G \circ \gamma_f$ is a maximal element of the family

$$\Omega_{f(G)} = \{\mathcal{H} \subset \Phi : h(f(G)) \subset f(G) \quad \forall \quad h \in \mathcal{H}\}.$$

Let $\gamma_{f^{-1}} \circ \mathcal{H}_G \circ \gamma_f \subset \mathcal{H}$, where $\mathcal{H} \in \Omega_{f(G)}$, and let $h \in \mathcal{H}$, then $h(f(G)) = h(\gamma_{f^{-1}}(G)) \subset f(G)$. It follows that $\gamma_f(h(\gamma_{f^{-1}}(G))) \subset \gamma_f(\gamma_{f^{-1}}(G)) = G$. Hence $\gamma_f \circ h \circ \gamma_{f^{-1}} \in \mathcal{H}_G$, and so

$$\gamma_{f^{-1}} \circ \gamma_f \circ h \circ \gamma_{f^{-1}} \circ \gamma_f = h \in \gamma_{f^{-1}} \circ \mathcal{H}_G \circ \gamma_f.$$

Therefore, $\mathcal{H} = \gamma_{f^{-1}} \circ \mathcal{H}_G \circ \gamma_f$, and $\gamma_{f^{-1}} \circ \mathcal{H}_G \circ \gamma_f$ is a maximal element of the family $\Omega_{f(G)}$. Consequently, $\gamma_{f^{-1}} \circ \mathcal{H}_G \circ \gamma_f = \mathcal{H}_{f(G)}$, which implies that $\prec f(G), \gamma_{f^{-1}} \circ \mathcal{H}_G \circ \gamma_f \succ$ is an invariant system. □

Corollary 24

(1) *If $f \in X^X$ is a one-to-one correspondence function, then $f^{-1} \in X^X$ is a one-to-one correspondence. Using Theorem* (2.7), *it follows that*

$$\prec f^{-1}(G), \gamma_f \circ \mathcal{H}_G \circ \gamma_{f^{-1}} \succ$$

is an invariant system.

(2) *Each one-to-one correspondence function f and invariant system $\prec G, \mathcal{H}_G \succ$ define a sequence of invariant systems*:

$$\{\prec f^n(G), \gamma_{f^{-n}} \circ \mathcal{H}_G \circ \gamma_{f^n} \succ \quad : \quad n \in N\}.$$

Remark 25 From the study of the invariant systems $\prec G, \mathcal{H}_G \succ$, it is shown that there exists a one-to-one correspondence between the family of G—continuous functions and the family of $f(G)$—continuous functions: $h \leftrightarrow \gamma_{f^{-1}} \circ h \circ \gamma_f$; $h \in \mathcal{H}_G$, where $f \in X^X$ is any one-to-one correspondence function.

Study of some properties and examples on $\Gamma(X)$

Definition 26 [7]. Let (X, g) be a generalized topological space, then $\Sigma \subset g$ is called a base for g if: every $A \in g$ can be constructed as a union of some members of Σ.

Moreover, any subfamily $\Sigma \subset P(X)$ generates the unique generalized topology g on X, where

$$g = G(\Sigma) = \left\{ A \subset X : \exists \quad \Sigma_\circ \subset \Sigma, \quad A = \bigcup_{B_i \in \Sigma_\circ} B_i \right\},$$

and g is the smallest generalized topology on X, containing Σ.

Theorem 27 *Let $\gamma_1 \in \Gamma_{\gamma_1}$ and $\gamma_2 \in \Gamma_{\gamma_2}$. Then:*

1 $g_{\gamma_1\circ\gamma_2} \supset g_{\gamma_1} \cap g_{\gamma_2}$, *and if* $\gamma_1, \gamma_2 \in \Gamma_\delta$, *then* $g_{\gamma_1\circ\gamma_2} \supset g_\delta$, *but, in some cases the equality holds.*
2 $g_{\gamma_1\cup\gamma_2} \supset G\{g_{\gamma_1}, g_{\gamma_2}\}$, *but the equality is valid for some special cases, where* $G\{g_{\gamma_1}, g_{\gamma_2}\}$ *is the generalized topology, which is generated by the family* $g_{\gamma_1} \cup g_{\gamma_2}$.
3 $g_{\gamma_1\cap\gamma_2} = g_{\gamma_1} \cap g_{\gamma_2}$. *Moreover, if* $\gamma_1, \gamma_2 \in \Gamma_\delta$, *then* $g_{\gamma_1\cap\gamma_2} \in \Gamma_\delta$.
Then, the intersection operation forms a binary operation on Γ_δ.

Proof

1 Let $O \in g_{\gamma_1} \cap g_{\gamma_2}$, then $O \subset \gamma_1(O)$ and $O \subset \gamma_2(O)$ Consequently, $(\gamma_1 \circ \gamma_2)(O) = \gamma_1(\gamma_2(O)) \supset \gamma_1(O) \supset O$, then $O \in g_{\gamma_1\circ\gamma_2}$. Therefore, $g_{\gamma_1\circ\gamma_2} \supset g_{\gamma_1} \cap g_{\gamma_2}$.
See Example (2.1), in which, $g_\delta \neq g_{\delta^2}$ if $\delta = \gamma$ and $g_\delta = g_{\delta^2}$ if $\delta = \gamma^2$
2 Let $O \in g_{\gamma_1} \cup g_{\gamma_2}$, then $O \subset \gamma_1(O)$ or $O \subset \gamma_2(O)$. Consequently,
$(\gamma_1 \cup \gamma_2)(O) = \gamma_1(O) \cup \gamma_2(O) \supset O$, then $O \in g_{\gamma_1\cup\gamma_2}$. Therefore, $g_{\gamma_1\cup\gamma_2} \supset g_{\gamma_1} \cup g_{\gamma_2}$.
Since $G\{g_{\gamma_1}, g_{\gamma_2}\}$ is the smallest generalized topology on X, containing $g_{\gamma_1\cup\gamma_2}$, then $g_{\gamma_1\cup\gamma_2} \supset G\{g_{\gamma_1}, g_{\gamma_2}\}$.
See Example (2.2), for the equality case.
3 Let $O \in g_{\gamma_1} \cap g_{\gamma_2}$, then $O \subset \gamma_1(O)$ and $O \subset \gamma_2(O)$. Consequently,
$O \subset \gamma_1(O) \cap \gamma_2(O) = (\gamma_1 \cap \gamma_2)(O)$, then $O \in g_{\gamma_1\cap\gamma_2}$ and $g_{\gamma_1\cap\gamma_2} \supset g_{\gamma_1} \cap g_{\gamma_2}$.
Now, et $O \in g_{\gamma_1\cap\gamma_2}$, then $O \subset (\gamma_1 \cap \gamma_2)(O) = \gamma_1(O) \cap \gamma_2(O)$. Consequently, $O \subset \gamma_1(O)$ and $O \subset \gamma_2(O)$, then $O \in g_{\gamma_1} \cap g_{\gamma_2}$ and $g_{\gamma_1\cap\gamma_2} \subset g_{\gamma_1} \cap g_{\gamma_2}$.
Therefore, the intersection operation is a binary operation on Γ_δ.

□

In the following example, a map $\gamma \in \Gamma_\delta$ is constructed to have the following properties:

(i) $\gamma^2 = \gamma \circ \gamma \notin \Gamma_\gamma$, but, $\gamma^3 \in \Gamma_\gamma$.
(ii) $g_{\gamma^{(2n+1)}} = g_\gamma = G\{O_0, O_1 \cup O_2\} \in \Gamma_\gamma$; $n \in \{0, 1, 2, 3, \ldots\}$.
(iii) $g_{\gamma^{2n}} = g_{\gamma^2} = G\{O_0, O_1, O_2\} \supset g_{\gamma^{(2n+1)}} = g_\gamma$ and $g_{\gamma^2} \notin \Gamma_\gamma$; $n \in \{1, 2, 3, \ldots\}$.

Example 28 Let X be a non-empty set and O_0, O_1, O_2 be non-empty mutually disjoint subsets of X. Define the map $\gamma : P(X) \to P(X)$ as follows:

$\gamma(A) = O_0$; if $A \supset O_0$ and A does not contain O_1, O_2.
$\gamma(A) = O_2$; if $A \supset O_1$ and A does not contain O_0, O_1.
$\gamma(A) = O_1$; if $A \supset O_2$ and A does not contain O_0, O_2.
$\gamma(A) = \bigcup_{i\in\ell} \gamma(O_i)$; if $A \supset \bigcup_{i\in\ell} O_i$, where $\ell \subset \{0, 1, 2\}$.
$\gamma(A) = \emptyset$; if A does not contain O_0 or O_1 or O_2.

The map γ satisfies the following:

(i) $\gamma(O_0) = O_0$.
(ii) $\gamma(O_1 \cup O_2) = \gamma(O_1) \cup \gamma(O_2) = O_2 \cup O_1$.

(ii) $\gamma(O_0 \cup O_1 \cup O_2) = O_0 \cup O_1 \cup O_2$.
(iv) $\gamma(A) \not\supset A$; if $A \notin G\{O_0, O_1 \cup O_2\}$.

Therefore, the topology

$$g_\gamma = G\{O_0, O_1 \cup O_2\} = \{\emptyset, O_0, O_1 \cup O_2, O_0 \cup O_1 \cup O_2\}.$$

The four steps (shown above) which constructs the map γ will be denoted by the following notation:

$$\gamma : O_0 \uparrow; O_1 \to O_2 \to O_1.$$

The map $\gamma^2 \notin \Gamma_\gamma$, since: $\gamma^2(O_i) = O_i$; $i \in \{0, 1, 2\}$, and $\gamma^2 \not\supset A$, for any $A \subset X, A \notin G\{O_0, O_1, O_2\}$.

Therefore,

$$g_{\gamma^2} = G\{O_0, O_1, O_2\} = \{\emptyset, O_0, O_1, O_2, O_0 \cup O_1, O_0 \cup O_2, O_1 \cup O_2, O_0 \cup O_1 \cup O_2\}.$$

Therefore,

$$g_{\gamma^2} = G\{O_0, O_1, O_2\} = \{\emptyset, O_0, O_1, O_2, O_0 \cup O_1, O_0 \cup O_2, O_1 \cup O_2, O_0 \cup O_1 \cup O_2\}.$$

The map γ^2 can be constructed using the following symbols:

$$\gamma^2 : [O_0, O_1, O_2] \uparrow .$$

The map $\gamma^3 \in \Gamma_\gamma$, since

$\gamma^3(O_0) = O_0$, $\gamma^3(O_1) = O_2$, $\gamma^3(O_2) = O_1$, $\gamma^3(O_1 \cup O_2) = O_2 \cup O_1$, and $\gamma^3(A) \not\supset A$, for any $A \subset X$ and $A \notin G\{O_0, O_1 \cup O_2\}$.

Therefore,

$$g_{\gamma^3} = g_\gamma = G\{O_0, O_1 \cup O_2\} = \{\emptyset, O_0, O_1 \cup O_2, O_0 \cup O_1 \cup O_2\}.$$

Consequently,

$$g_{\gamma^{(2n+1)}} = g_\gamma = G\{O_0, O_1 \cup O_2\} \in \Gamma_\gamma; n \in \{0, 1, 2, 3, \ldots\}.$$

and

$$g_{\gamma^{2n}} = g_{\gamma^2} = G\{O_0, O_1, O_2\} \supset g_{\gamma^{(2n+1)}} = g_\gamma$$

and

$$g_{\gamma^2} \notin \Gamma_\gamma; n \in \{1, 2, 3, \ldots\}.$$

A general construction of Example (2.3) can be illustrated in the following theorem.

Theorem 29 *Let X be a non-empty set and $O_0, O_1, O_2, \ldots, O_n$ be non-empty mutually disjoint subsets of X. Define the map $\gamma : P(X) \to P(X)$ as follows*:

$\gamma(A) = O_0$; if $A \supset O_0$ and A does not contain any of the subsets O_i; $1 \le i \le n$.
$\gamma(A) = O_{i+1}$; if $A \supset O_i$ and A does not contain any of the subsets O_s; $s \ne i, 0 \le s \le n$, and $1 \le i \le n-1$.
$\gamma(A) = O_1$; if $A \supset O_n$ and A does not contain any subset of $O_s, 0 \le s < n$.
$\gamma(A) = \bigcup_{i\in\ell} \gamma(O_i)$; if $A \supset \bigcup_{i\in\ell} O_i$, where $\ell \subset \{0, 1, 2, 3, 4, \ldots, n\}$.
$\gamma(A) = \emptyset$; if A does not contain any set $O_i, i \in \{0, 1, 2, 3, \ldots, n\}$.

The map γ can be defined as:

$$\gamma : O_0 \uparrow; O_1 \to O_2 \to O_3 \to O_4 \to .... \to O_n \to O_1.$$

Then, for any positive integer numbers n, s, k, it follows that:

1 $$g_\gamma = G\{O_0, O_1 \cup O_2 \cup O_3 \cup \cdots \cup O_n\}.$$

2 $$g_\gamma \ne g_{\gamma^s} = G\{O_0, O_1 \cup O_{1+s} \cup O_{1+2s} \cup \cdots \cup O_{n-s+1}, \ldots, O_s \cup O_{2s} \cup O_{3s} \cup \cdots \cup O_n\},$$

whenever $1 \le s \le n-1$ and $n = ks, k > 1$.

3 $$g_\gamma \ne g_{\gamma^s} = G\{O_0, O_1, O_2, O_3, \ldots, O_n\},$$

whenever $s \ge n$ and $s = kn$.

4 $$g_\gamma = g_{\gamma^s} = G\{O_0, O_1 \cup O_2 \cup O_3 \cup \cdots \cup O_n\},$$

whenever ($s \ge n$ and $s \ne kn$) or ($s \le n-1$ and $n \ne ks, k > 1$).

Proof

1 The definition of the map γ implies that

$$\gamma(O_0) = O_0, \gamma(O_1 \cup O_2 \cup O_3 \cup \cdots \cup O_n) = O_1 \cup O_2 \cup O_3 \cup ...... \cup O_n$$

and $\gamma(A) \not\supset A$, for all $A \subset X$;

$$A \notin \{\emptyset, O_0, O_1 \cup O_2 \cup O_3 \cup ...... \cup O_n, O_0 \cup O_1 \cup O_2 \cup O_3 \cup ...... \cup O_n\}.$$

Therefore, the generalized topology generated by the monotonic map γ is

$$g_\gamma = G\{O_0, O_1 \cup O_2 \cup O_3 \cup ...... \cup O_n\}$$

or

$$g_\gamma = \{\emptyset, O_0, O_1 \cup O_2 \cup O_3 \cup ...... \cup O_n, O_0 \cup O_1 \cup O_2 \cup O_3 \cup ...... \cup O_n\}.$$

2 Let s be any positive integer such that $1 \le s \le n-1$, then

$$\gamma^s(O_i) = \begin{cases} O_0 & : \quad i = 0, \\ O_{s+i} & : \quad 1 \le i \le n-s, \\ O_{i-n+s} & : \quad n-s+1 \le i \le n. \end{cases}$$

(i) Let $n = ks, k \in \{2, 3, 4, \ldots\}$, then:

$\gamma(O_0) = O_0$.
$\gamma(O_1 \cup O_{1+s} \cup O_{1+2s} \cup ...... \cup O_{n-s+1}) = O_1 \cup O_{1+s} \cup O_{1+2s} \cup ...... \cup O_{n-s+1}$.
$\gamma(O_2 \cup O_{2+s} \cup O_{2+2s} \cup ...... \cup O_{n-s+2}) = O_2 \cup O_{2+s} \cup O_{2+2s} \cup ...... \cup O_{n-s+2}$.
$\gamma(O_3 \cup O_{3+s} \cup O_{3+2s} \cup ...... \cup O_{n-s+3}) = O_3 \cup O_{3+s} \cup O_{3+2s} \cup ...... \cup O_{n-s+3}$..
.
.
$\gamma(O_s \cup O_{2s} \cup O_{3s} \cup ...... \cup O_n) = O_s \cup O_{2s} \cup O_{3s} \cup ...... \cup O_n$.

Moreover, $\gamma(A) \not\supset A$, for all $A \subset X$;

$$A \notin \{\emptyset, O_0, O_1 \cup O_{1+s} \cup O_{1+2s} \cup ...... \cup O_{n-s+1}\}$$

or

$$A \notin \{O_2 \cup O_{2+s} \cup O_{2+2s} \cup ...... \cup O_{n-s+2}, O_3 \cup O_{3+s} \cup O_{3+2s} \cup ...... \cup O_{n-s+3}\}$$

or

$$A \notin \{O_4 \cup O_{4+s} \cup O_{4+2s} \cup ...... \cup O_{n-s+4}, \ldots, O_s \cup O_{2s} \cup O_{3s} \cup ...... \cup O_n\}.$$

Therefore, γ^s can be defined as:

$\gamma^s : O_0 \uparrow; O_1 \to O_{1+s} \to O_{1+2s} \to ..... \to O_{n-s+1} \to O_1$,
$O_2 \to O_{2+s} \to O_{2+2s} \to ..... \to O_{n-s+2} \to O_2$,
$O_3 \to O_{3+s} \to O_{3+2s} \to ..... \to O_{n-s+3} \to O_3$,
.
.
$O_s \to O_{2s} \to O_{3s} \to ..... \to O_n \to O_s$.

And so, the generalized topology generated by the monotonic map γ^s is

$$g_\gamma \neq g_{\gamma^s} = G\{O_0, O_1 \cup O_{1+s} \cup O_{1+2s} \cup .... \cup O_{n-s+1}, ..., O_s \cup O_{2s} \cup O_{3s} \cup .... \cup O_n\}.$$

(ii) Let $n \neq ks, k \in \{2, 3, 4, \ldots\}$, then:

$\gamma(O_0) = O_0$.
$\gamma(O_1 \cup O_2 \cup ...... \cup O_{n-s} \cup O_{n-s+1} \cup O_{n-s+2} \cup .... \cup O_{n-1} \cup O_n)$
$= O_{1+s} \cup O_{2+s} \cup ...... \cup O_n \cup O_1 \cup O_2 .... O_{s-1} \cup O_s$
$= O_1 \cup O_2 \cup ...... \cup O_{n-s} \cup O_{n-s+1} \cup O_{n-s+2} \cup .... \cup O_{n-1} \cup O_n$.

Moreover, $\gamma(A) \not\supset A$, for all $A \subset X$;

$$A \notin \{\emptyset, O_0, O_1 \cup O_2 \cup O_3 \cup ...... \cup O_n, O_0 \cup O_1 \cup O_2 \cup O_3 \cup ...... \cup O_n\}.$$

Therefore, the generalized topology generated by the monotonic map γ^s is

$$g_{\gamma^s} = g_\gamma = G\{O_0, O_1 \cup O_2 \cup O_3 \cup ...... \cup O_n\}$$

or

$$g_{\gamma^s} = \{\emptyset, O_0, O_1 \cup O_2 \cup O_3 \cup ...... \cup O_n, O_0 \cup O_1 \cup O_2 \cup O_3 \cup ...... \cup O_n\}.$$

3 Let s be any positive integer such that $s = kn, k \in \{1, 2, 3, \ldots\}$, then

$$\gamma^s(O_i) = O_i, \quad i \in \{0, 1, 2, 3, \ldots\}.$$

Therefore, the map γ^s can be defined as:

$$\gamma^s : [O_0, O_1, O_2, \ldots, O_{n-1}, O_n] \uparrow .$$

And so the generalized topology generated by the monotonic map γ^s is

$$g_{\gamma^s} = G\{O_0, O_1, O_2, \ldots, O_{n-1}, O_n\}.$$

4 Let s be any positive integer such that

$$s \geq n, \quad s = kn + r, \quad k \in \{1, 2, 3, \ldots\}, \quad 1 \leq r \leq n-1,$$

then

$$\gamma^s(O_i) = \gamma^{nk+r} = \gamma^r(O_i) = O_{i+r}, \quad 1 \leq i \leq n, \quad 1 \leq r \leq n-1.$$

Moreover, this case is the case [2].

□

Corollary 30 *Using Theorem* (2.4), *the following results can be obtained easily*:

1 *If n is a prime number, then there exists only two generalized topologies on X, which can be constructed as follows*:

(i) $g_\gamma \neq g_{\gamma^s} = G\{O_0, O_1, O_2, \ldots, O_{n-1}, O_n\}$, whenever s is divisible by n.
(ii) $g_{\gamma^s} = g_\gamma = G\{O_0, O_1 \cup O_2 \cup O_3 \cup ...... \cup O_n\}$, whenever s is not divisible by n.

2 *If s is a prime number and $n > s$, then there exist only two generalized topologies on X, which can be constructed as follows*:

(i($g_\gamma \neq g_{\gamma_s} = G\{O_0, O_1 \cup O_{1+s} \cup O_{1+2s} \cup .... \cup O_{n-s+1}, \ldots, O_s \cup O_{2s} \cup O_{3s} \cup .... \cup O_n\}$, whenever n is divisible by s.
(ii) $g_{\gamma^s} = g_\gamma = G\{O_0, O_1 \cup O_2 \cup O_3 \cup ...... \cup O_n\}$, whenever n is not divisible by s.

3 *If s, n are prime numbers, and $n \neq s$, then there exists only one generalized topology on X, which can be constructed as follows*:

$$g_{\gamma^s} = g_\gamma = G\{O_0, O_1 \cup O_2 \cup O_3 \cup ...... \cup O_n\}.$$

The following example shows that in general $g_{\gamma_1 \cup \gamma_2} \neq G\{\gamma_1, \gamma_2\}$, for some $\gamma_1 \in \Gamma_{\gamma_1}$ and $\gamma_2 \in \Gamma_{\gamma_2}$. Moreover, the equality will be valid for some special cases.

Example 31 Let X be non-empty set and O_1, O_2, O_3, O_4 be non-empty mutually disjoint subsets of X. Suppose that O_1, O_2 are disjoint, $O_2 = O_3 \cup O_4$ and $O_3 \cap O_4 = \emptyset$. Define the three monotonic maps $\gamma_1, \gamma_2, \gamma_3 : P(X) \to P(X)$ as follows:

$\gamma_1(A) = O_1$, for all $A \supset O_1$ and $A \cap O_2 = \emptyset$.
$\gamma_1(A) = O_3$, for all $A \supset O_2$ and $A \cap O_1 = \emptyset$.
$\gamma_1(A) = \bigcup_{O_i \subset A, i \in \ell} \gamma_1(O_i)$, where $\ell \subset L = \{1, 2\}$.
$\gamma_1(A) = \emptyset$, for all $A \subset X$ and A does not contain any set $O_j, j \in \{1, 2\}$.
$\gamma_2(A) = O_1$, for all $A \supset O_1$ and A does not contain O_2.
$\gamma_2(A) = O_4$, for all $A \supset O_2$ and A does not contain O_1.
$\gamma_2(A) = \bigcup_{O_i \subset A, i \in \ell} \gamma_2(O_i)$, where $\ell \subset L = \{1, 2\}$.
$\gamma_2(A) = \emptyset$, for all $A \subset X$ and A does not contain any set $O_j, j \in \{1, 2\}$.
$\gamma_3(A) = O_1$, for all $A \supset O_1$.
$\gamma_3(A) = \emptyset$, for all $A \subset X$ and A does not contain O_1.

Therefore,
$\gamma_1(O_1) = \gamma_2(O_1) = \gamma_3(O_1) = O_1$, and $\gamma_i(A) \not\supset A$, for all $A \subset X, A \notin \{\emptyset, O_1\}, i \in \{1, 2, 3\}$.

which implies that $g_{\gamma_1} = g_{\gamma_2} = g_{\gamma_3} = \{\emptyset, O_1\}$.

Hence, $g_{\gamma_1}, g_{\gamma_2}$ and g_{γ_3} are different monotonic maps, defining the same generalized topology. At the same time, we have the following:

$(\gamma_1 \cup \gamma_2)(O_1) = \gamma_1(O_1) \cup \gamma_2(O_1) = O_1$.
$(\gamma_1 \cup \gamma_2)(O_2) = \gamma_1(O_2) \cup \gamma_2(O_2) = O_3 \cup O_4 = O_2$.
$(\gamma_1 \cup \gamma_3)(O_1) = \gamma_1(O_1) \cup \gamma_3(O_1) = O_1$.
$(\gamma_1 \cup \gamma_3)(A) \not\supset A$, for all $A \subset X$ and $A \notin \{\emptyset, O_1\}$.

Therefore,

$$g_{\gamma_1 \cup \gamma_2} = G\{O_1, O_2\} = \{\emptyset, O_1, O_2, O_1 \cup O_2\}, \quad g_{\gamma_1 \cup \gamma_3} = \{\emptyset, O_1\}.$$

Then,

$$G\{g_{\gamma_1}, g_{\gamma_2}\} = g_{\gamma_1} \cup g_{\gamma_2} = g_{\gamma_1 \cup \gamma_3} = \{\emptyset, O_1\} \subset g_{\gamma_1 \cup \gamma_2}.$$

And

$$g_{\gamma_1} \cup g_{\gamma_2} \neq g_{\gamma_1 \cup \gamma_2}, \quad g_{\gamma_1} \cup g_{\gamma_3} = g_{\gamma_1 \cup \gamma_3}.$$

Obtainment of generalized topologies generated by special monotonic maps

Let g_δ be a given generalized topology on X, which is generated by the monotonic map δ, whose generalized interior and generalized closure are denoted by int_δ and cl_δ respectively. It is known that the monotonic maps $int_\delta \circ int_\delta = int_\delta$ and $cl_\delta \circ cl_\delta = cl_\delta$. But the composition of the monotonic functions $int_\delta \circ cl_\delta$ and $cl_\delta \circ int_\delta$ have different behaviors.

$\sigma-$generalized topological space (X, g_σ)

Theorem 32 *Let g_δ be a given generalized topology on X, which is generated by the monotonic map δ. The non-empty elements of the generalized topology g_σ which is defined by the map $\sigma = cl_\delta \circ int_\delta$ consists of all subsets $A \subset X$, having non-empty $int_\delta(A)$ and (each $\delta-$open subset $O \in g_\delta$ intersects $int_\delta(A)$) or (each $\delta-$open subset $O \in g_\delta$, which does not intersect $int_\delta(A)$, does not intersect A also).*

Proof Let $A \subset X$. Then:

1 If $int_\delta(A) = \emptyset$, then $\sigma(A) = \emptyset$.
2 If $int_\delta(A) \neq \emptyset$, then

$$\begin{aligned}
\sigma(A) =& cl_\delta(int_\delta(A) = \bigcap_{O\in g_\delta} \{X - O \quad : \quad int_\delta(A) \subset X - O\} \\
=& \bigcap_{O\in g_\delta} \{X - O \quad : \quad int_\delta(A) \cap O = \emptyset\} \\
=& X - \bigcup_{O\in g_\delta} \{O \quad : \quad int_\delta(A) \cap O = \emptyset\} \\
=& \begin{cases} X & : \quad \bigcup_{O\in g_\delta, int_\delta \cap O=\emptyset} O = \emptyset \\ \bigcap_{O\in g_\delta, int_\delta \cap O=\emptyset}(X - O) & : \quad \bigcup_{O\in g_\delta, int_\delta \cap O=\emptyset} O \neq \emptyset \end{cases}
\end{aligned}$$

Therefore, the set $A \subset X$ is $\sigma-$open subset (the class of all $g-$semi-open sets) if $\sigma(A) \supset A$. This statement is valid in the following cases:

(a) For every A, for which $int_\delta(A) \neq \emptyset$ and $\bigcup_{O\in g_\delta, int_\delta \cap O=\emptyset} O = \emptyset$, which means that each $\delta-$open subset O intersects $int_\delta(A)$.
(b) For every A, for which $int_\delta(A) \neq \emptyset$ and $\bigcup_{O\in g_\delta, int_\delta \cap O=\emptyset} O \neq \emptyset$, and

$$A \subset \bigcap_{O\in g_\delta, int_\delta \cap O=\emptyset} (X - O) = X - \bigcap_{O\in g_\delta, int_\delta \cap O=\emptyset} O.$$

Therefore,

$$A \cap \bigcap_{O\in g_\delta, int_\delta \cap O=\emptyset} O = \emptyset.$$

This means that each $\delta-$open subset O, which does not intersect $int_\delta(A)$, does not intersect also with A.

□

Remark 33 The set X is $\sigma-$open, if X is $\delta-$open or there exists a subset $A \subset X$ with non-empty $\delta-$interior, and intersects each $\delta-$open subset $O \in g_\delta$.

Notation 34 Throughout the rest of our study, the three special generalized topological spaces $(X, g_{i\delta}), i \in \{1, 2, 3\}$, will be defined as follows:

1 Let $(X, g_{1\delta})$ be a $1\delta-$generalized topological space on the non-empty set X, where $g_{1\delta}$ is generated by the non-empty subsets O_1, O_2, O_3,satisfying the following conditions:

$O_1 \cap O_2 = \{x_1\}, O_2 \cap O_3 = \{x_2\}$ and $O_3 \cap O_1 = \{x_3\}$. Moreover, $x_1 \notin O_3$, $x_2 \notin O_3$ and $x_3 \notin O_2$.

2 Let $(X, g_{2\delta})$ be a $2\delta-$generalized topological space on the non-empty set X, where $g_{2\delta}$ is generated by the disjoint non-empty subsets $\{O_1, O_2\}$.

3 Let $(X, g_{3\delta})$ be a $3\delta-$generalized topological space on non-empty set X, where $g_{3\delta}$ is generated by the non-empty subsets O_1, O_2, O_3, O_4, satisfying the below: $O_1 \cap O_2 = \{x_1\}$, $O_2 \cap O_3 = \{x_2\}, O_3 \cap O_4 = \{x_3\}$ and $O_4 \cap O_1 = \{x_4\}$. Moreover, $O_1 \cap O_3 = \emptyset$ and $O_2 \cap O_4 = \emptyset$.

$\sigma-$generalized topological spaces which are defined by special generalized topological spaces $(X, g_{i\delta}), i \in \{1, 2, 3\}$

By using Theorem 32, the $\sigma-$generalized topological spaces can be constructed as follows:

1 The $\sigma-$generalized topology g_σ on $g_{1\delta}$ is defined as follows:

Let $A \neq \emptyset$ be $\sigma-$open subset, then $int_{1\delta}(A) \neq \emptyset$, and it contains at least one $1\delta-$open subset $O_{i_0} \in g_{1\delta}$. Consequently, $int_{1\delta}(A)$ intersects all the elements of $g_{1\delta}$. Therefore, the family of $\sigma-$open subset of $g_{1\delta}$ consists of each subset of X, containing at least one of the non-empty elements of $g_{1\delta}$.

It is clear that X is $\sigma-$open set, but X is $1\delta-$open only if $X = O_1 \cup O_2 \cup O_3$.

2 The $\sigma-$generalized topology g_σ on $g_{2\delta}$ is defined as follows:
Let $A \neq \emptyset$ be $\sigma-$open subset, then $int_{2\delta}(A) \neq \emptyset$, and it contains one element of $\{O_1, O_2, O_1 \cup O_2\}$. Therefore, the family of $\sigma-$open subset of $g_{2\delta}$ consists of the following subfamilies:

(i) Any subset of X, containing $O_1 \cup O_2$.
(ii) Any subset of X, containing O_1 and does not intersect O_2.
(iii) Any subset of X, containing O_2 and does not intersect O_1.

It is clear that X is $\sigma-$open set, but X is $2\delta-$open only, if $X = O_1 \cup O_2$.

3 The $\sigma-$generalized topology g_σ on $g_{3\delta}$ is defined as follows:
Let $A \neq \emptyset$ be $\sigma-$open subset, then $int_{3\delta}(A) \neq \emptyset$, and it contains at least one element of $\{O_1, O_2, O_3, O_4\}$. Therefore, the family of $\sigma-$open subset of $g_{3\delta}$ consists of the following subfamilies:

(i) A is $\sigma-$open subset if $int_{3\delta}(A)$ contains at least two elements of $\{O_1, O_2, O_3, O_4\}$, since $int_{3\delta}(A)$ intersects all the elements of $g_{3\delta}$.
(ii) If $int_{3\delta}(A)$ contains O_1 only (or O_3 only), then A is $\sigma-$open subset if it contains O_1 and does not intersect O_3 (or if it contains O_3 and does not intersect O_1).
(iii) If $int_{3\delta}(A)$ contains O_2 only (or O_4 only), then A is $\sigma-$open subset if it contains O_2 and does not intersect O_4 (or if it contains O_4 and does not intersect O_2).

It is clear that X is $\sigma-$open set, but X is $3\delta-$open only, if $X = O_1 \cup O_2 \cup O_3 \cup O_4$.

$\alpha-$generalized topological space (X, g_α)

Theorem 35 *Let g_δ be a given generalized topology on X, which is generated by the monotonic map δ. The non-empty elements of the generalized topology g_α on X, which is defined by the map $\alpha = int_\delta \circ cl_\delta \circ int_\delta$ consists of all subsets $A \subset X$, satisfying the following conditions*:

1 *If $X \notin g_\delta$ and $O \cap int_\delta(A) \neq \emptyset$, for all $O \in g_\delta$. Then, $A \in g_\alpha$ if*:

$$\alpha(A) = \bigcup_{O \in g_\delta} O \supset A \supset int_\delta(A).$$

2 *If $X \in g_\delta$ and $O \cap int_\delta(A) \neq \emptyset$, for all $O \in g_\delta$. Then, $A \in g_\alpha$ if*:

$$\alpha(A) = X \supset A \supset int_\delta(A).$$

This means that $X \in g_\delta \Rightarrow X \in g_\sigma \cap g_\alpha$.

3 *If $int_\delta(A) \neq \emptyset$, and*

$$\bigcup_{O \in g_\delta, O \cap int_\delta(A) \neq \emptyset, \exists U \in g_\delta, U \cap int_\delta(A) = \emptyset, U \cap O \neq \emptyset} O = \emptyset.$$

Then, $A \in g_\alpha$ if:

$$\alpha(A) = \bigcup_{O \in g_\delta, O \cap int_\delta(A) \neq \emptyset} O \supset A \supset int_\delta(A).$$

4 *If $int_\delta(A) \neq \emptyset$, $\bigcup_{O \in g_\delta, O \cap int_\delta(A) \neq \emptyset} O \supset A$,*

$$\bigcup_{O \in g_\delta, O \cap int_\delta(A) \neq \emptyset, \exists U \in g_\delta, U \cap int_\delta(A) = \emptyset, U \cap O \neq \emptyset} O \neq \emptyset$$

and

$$A \cap \bigcup_{O \in g_\delta, O \cap int_\delta(A) \neq \emptyset, \exists U \in g_\delta, U \cap int_\delta(A) = \emptyset, U \cap O \neq \emptyset} O = \emptyset.$$

Then, $A \in g_\alpha$ if:

$$\alpha(A) = \bigcup_{O \in g_\delta, O \cap int_\delta(A) \neq \emptyset, U \cap O = \emptyset \forall U \in g_\delta, U \cap int_\delta(A) = \emptyset} O \supset A \supset int_\delta(A).$$

Proof

(*a*) Let $A \subset X$ such that $int_\delta(A) = \emptyset$, then $\alpha(A) = \emptyset$. Hence, $\emptyset \in g_\alpha$.
(*b*) Let $A \subset X$ such that $int_\delta(A) \neq \emptyset$, then:

$$\alpha(A) = int_\delta(\sigma(A)) = int_\delta \begin{cases} X & : \bigcup_{O \in g_\delta, int_\delta \cap O = \emptyset} O = \emptyset \\ \bigcap_{O \in g_\delta, int_\delta \cap O = \emptyset} (X - O) & : \bigcup_{O \in g_\delta, int_\delta \cap O = \emptyset} O \neq \emptyset \end{cases}$$

$$= \begin{cases} \bigcup_{O \in g_\delta, int_\delta \cap O \neq \emptyset} O & : X \notin g_\delta, \bigcup_{O \in g_\delta, int_\delta \cap O = \emptyset} O = \emptyset \\ X & : X \in g_\delta, \bigcup_{O \in g_\delta, int_\delta \cap O = \emptyset} O = \emptyset \\ \bigcup_{O \in g_\delta, O \cap int_\delta(A) \neq \emptyset, U \cap O = \emptyset \forall U \in g_\delta, U \cap int_\delta(A) = \emptyset} O & : \bigcup_{O \in g_\delta, int_\delta \cap O = \emptyset} O \neq \emptyset \end{cases}$$

The subset A is $\alpha-$open subset, if $\alpha(A) \supset A$. This statement is valid in the following cases:

1 For every $A \subset X$, for which $int_\delta(A) \neq \emptyset$, $X \notin g_\delta, \bigcup_{O \in g_\delta, int_\delta \cap O = \emptyset} O = \emptyset$ (i.e. each $\delta-$open subset O intersects $int_\delta(A)$), and

$$\alpha(A) = \bigcup_{O \in g_\delta} O \supset A \supset int_\delta(A).$$

(Therefore, X is not $\alpha-$open set if it is not $\delta-$open set).

2 Let $X \in g_\delta$, then A is $\alpha-$open subset for all $A \subset X$, since $\alpha(A) = X \supset A \supset int_\delta(A)$. Therefore, X is $\alpha-$open set if it is $\delta-$open set.

3 For every $A \subset X$, for which $int_\delta(A) \neq \emptyset$, and

$$\bigcup_{O \in g_\delta, O \cap int_\delta(A) \neq \emptyset, \exists U \in g_\delta, U \cap int_\delta(A) = \emptyset, U \cap O \neq \emptyset} O = \emptyset.$$

(i.e. each $\delta-$open subset O, intersecting $int_\delta(A)$ does not intersect any $\delta-$open subset U, for which $U \cap int_\delta(A) = \emptyset$).
Then, $A \in g_\alpha$ if:

$$\alpha(A) = \bigcup_{O \in g_\delta, O \cap int_\delta(A) \neq \emptyset} O \supset A \supset int_\delta(A).$$

4 For every $A \subset X$, for which $int_\delta(A) \neq \emptyset$, and

$$\bigcup_{O\in g_\delta, O\cap int_\delta(A)\neq\emptyset, \exists U\in g_\delta, U\cap int_\delta(A)=\emptyset, U\cap O\neq\emptyset} O \neq \emptyset.$$

(i.e. each $\delta-$open subset O, intersecting $int_\delta(A)$, intersects some $\delta-$open subset U, for which $U \cap int_\delta(A) = \emptyset$).
Then, $A \in g_\alpha$ if $int_\delta(A) \neq \emptyset$ and

$$\alpha(A) = \bigcup_{O\in g_\delta, O\cap int_\delta(A)\neq\emptyset, U\cap O=\emptyset \forall U\in g_\delta, U\cap int_\delta(A)=\emptyset} O \supset A \supset int_\delta(A).$$

□

Remark 36
The map $\alpha = int_\delta \circ cl_\delta \circ int_\delta$ is called controlled by the generalized topology g_δ. It can be denoted by α_δ.

$\alpha-$generalized topological spaces which are defined by special generalized topological spaces $(X, g_{i\delta}), i \in \{1, 2, 3\}$
By using Theorem 35, the $\alpha-$generalized topological spaces can be constructed as follows:

1 The $\alpha-$generalized topology g_α on $g_{1\delta}$ is defined as follows:

Let $A \neq \emptyset$ be $\alpha-$open subset, then $int_{1\delta}(A) \neq \emptyset$, then it contains at least one $1\delta-$open subset $O_{i_0} \in g_{1\delta}$. Consequently, $int_{1\delta}(A)$ intersects all the elements of $g_{1\delta}$. Therefore, the family of $\alpha-$open subset of $g_{1\delta}$ consists of:

(i) Each subset A of X, containing at least one of the non-empty elements of $g_{1\delta}$ (if X is $1\delta-$open set).
(ii) Each subset A, containing at least one $1\delta-$open subset and $A \subset O_1 \cup O_2 \cup O_3$ (if X is not $1\delta-$open set).

2 The $\alpha-$generalized topology g_α on $g_{2\delta}$ is defined as follows:
Let $A \neq \emptyset$ be $\sigma-$open subset, then $int_{2\delta}(A) \neq \emptyset$, then it contains one element of $\{O_1, O_2, O_1 \cup O_2\}$. Therefore, the family of $\alpha-$open subsets of $g_{2\delta}$ consists of the following subfamilies:

(i) $A = O_1 \cup O_2$, if X is not $1\delta-$open set.
(ii) Any subset A of X, containing $O_1 \cup O_2$, if X is $2\delta-$open set.
(iii) The subset A of X, if $A = O_1$ or $A = O_2$.

Therefore, $g_\alpha = g_{2\delta}$, if X is not $2\delta-$open set. This result is true for every g_δ generated by a family of disjoint subsets, when X is not $\delta-$open set.

3 The $\alpha-$generalized topology g_α on $g_{3\delta}$ is defined as follows:

Let $A \neq \emptyset$ be $\alpha-$open subset, then $int_{3\delta}(A) \neq \emptyset$, then it contains at least one element of $\{O_1, O_2, O_3, O_4\}$. Therefore, the family of $\alpha-$open subset of $g_{3\delta}$ consists of the following subfamilies:

(i) The subset A of X, if $int_{3\delta}(A)$ contains at least two elements of $\{O_1, O_2, O_3, O_4\}$, since $int_{3\delta}(A)$ intersects all the elements of $g_{3\delta}$ if: X is $3\delta-$open set, or $A \subset O_1 \cup O_2 \cup O_3 \cup O_4$ and X is not $3\delta-$open set.
(ii) The subset A of X, if $A = O_1$ or $A = O_2$ or $A = O_3$ or $A = O_4$.

$\pi-$generalized topological space (X, g_π)

Notation 37 Let g_δ be a given generalized topology on X, which is generated by the monotonic map δ. Each subset $A \subset X$ divides the elements of the generalized topology g_δ into two classes:

$$\triangle_A = \{O \in g_\delta : A \cap O \neq \emptyset\}, \quad \nabla_A = \{U \in g_\delta : A \cap U = \emptyset\}.$$

For each $O \in \triangle_A$, we define

$$U_O = \bigcup_{U \in \nabla_A, U \cap O \neq \emptyset} U.$$

And

$$\varepsilon_A = \left\{ x \in A : x \notin \bigcup_{O \in g_\delta} O \right\}.$$

It is clear that the family $\{\triangle_A, \nabla_A\}$, for all $A \subset X$ forms a partition for the $\delta-$generalized topology g_δ on X. Moreover, $(\triangle_A = \emptyset \Rightarrow \nabla_A = g_\delta)$ and $(\nabla_A = \emptyset \Rightarrow \triangle_A = g_\delta)$.

Theorem 38 *Let g_δ be a given generalized topology on X, which is generated by the monotonic map δ. The non-empty elements of the generalized topology g_π on X (the family of all $\pi-$preopen sets), which is defined by the monotonic map $\pi = int_\delta \circ cl_\delta$ consists of all non-empty subsets $A \subset X$, satisfying the following conditions*:

(i) $\nabla_A = \emptyset$ *and X is $\delta-$open set.*
(ii) *If $\nabla_A = \emptyset$, and X is not $\delta-$open set and $A \subset \bigcup_{O \in g_\delta} O$.*
(iii) *If $\nabla_A \neq \emptyset, \triangle_A \neq \emptyset$, and $A \subset \bigcup_{O \in \triangle_A, O \cap U = \emptyset; U \in \nabla_A} O$.*

Proof Consider the action of the map π on the subset A of X:
Let $A = \emptyset$, then $\pi(A) = \emptyset$. Hence, $\emptyset \in g_\pi$.

Let $A \neq \emptyset$, then we get

$$\pi(A) = int_\delta(cl_\delta(A)) = int_\delta \begin{cases} \bigcap_{U\in g_\delta}(X-U) & : \quad \triangle_A = \emptyset. \\ \bigcap_{U\in\nabla_A}(X-U) & : \quad \triangle_A \neq \emptyset, \nabla_A \neq \emptyset. \\ X & : \quad \nabla_A = \emptyset, . \end{cases}$$

$$= \begin{cases} \emptyset & : \quad \triangle_A = \emptyset \quad or \quad \triangle_A \neq \emptyset, \nabla_A \neq \emptyset, U_O \neq \emptyset; O \in \triangle_A. \\ \bigcup_{U_O=\emptyset, O\in g_\delta} O & : \quad \triangle_A \neq \emptyset, \nabla_A \neq \emptyset, \exists O \in \triangle_A, U_O = \emptyset. \\ X & : \quad \nabla_A = \emptyset, X \in g_\delta. \\ \bigcup_{O\in g_\delta} O & : \quad \nabla_A = \emptyset, X \notin g_\delta. \end{cases}$$

The nonempty subset A is $\pi-$open subset, if $\pi(A) \supset (A)$. Therefore, the subset A is $\pi-$open subset in the following cases:

(i) If $\nabla_A = \emptyset$ and $X \in g_\delta$. Then, A is $\pi-$open subset, since $\pi(A) = X \supset A$.
(ii) If $\nabla_A = \emptyset$, and X is not $\delta-$open set. Then, A is $\pi-$open subset if $A \subset \bigcup_{O\in g_\delta} O$.
(iii) If $\triangle_A \neq \emptyset$ and $\nabla_A \neq \emptyset$, then the nonempty subset A is $\pi-$open subset, if:

$$\pi(A) = \bigcup_{U_O=\emptyset, O\in g_\delta} O = \bigcup_{O\in\triangle_A, O\cap U=\emptyset; U\in\nabla_A} O \supset A.$$

□

$\pi-$generalized topological spaces which are defined by special generalized topological spaces $(X, g_{i\delta}), i \in \{1, 2, 3\}$

By using Theorem 38, the $\pi-$generalized topological spaces can be constructed as follows:

1 The $\pi-$generalized topology g_π on $g_{1\delta}$ is defined as follows:

A is $\pi-$open subset:

(i) If A intersects each of the subsets $\{O_1, O_2, O_3\}$ and X is $1\delta-$open set.
(ii) If A intersects each of the subsets $\{O_1, O_2, O_3\}$ and X is not $1\delta-$open set, then $A \subset O_1 \cup O_2 \cup O_3$.

2 The $\pi-$generalized topology g_π on $g_{2\delta}$ is defined as follows:
A is $\pi-$open subset in the following cases:

(i) $A \cap O_1 \neq \emptyset, A \cap O_2 = \emptyset$ and $\pi(A) = O_1 \supset A$.
(ii) $A \cap O_2 \neq \emptyset, A \cap O_1 = \emptyset$ and $\pi(A) = O_2 \supset A$.
(iii) $A \cap O_1 \neq \emptyset, A \cap O_2 \neq \emptyset$ and $\pi(A) = O_1 \cup O_2 \supset A$.

3 The $\pi-$generalized topology g_π on $g_{3\delta}$ is defined as follows:

A is $\pi-$open subset in the following cases:

(i) If it is included in O_1 and intersects O_2, O_4.
(ii) If it is included in O_2 and intersects O_1, O_3.
(iii) If it is included in O_3 and intersects O_2, O_4.
(iv) If it is included in O_4 and intersects O_3, O_1.
(v) If it is included in $O_1 \cup O_2 \cup O_3 \cup O_4$ and intersects all the elements $\{O_1, O_2, O_3, O_4\}$.

Remark 39 Consider the following case: If $\nabla_A \neq \emptyset$ and $U_O \neq \emptyset$, for some $O \in \triangle_{1A} \subset \triangle_A$. It follows that A is not $\pi-$open subset. Since if $U_O \neq \emptyset$, for some $O \in \triangle_{1A} \subset \triangle_A$, then the points of A in O are not included in $\pi(A) = \bigcup_{U_O=\emptyset, O\in\triangle_A} O$. Then, A is not included in $\pi(A)$ and is not $\pi-$open subset.

$\beta-$generalized topological space (X, g_β)

Theorem 40 *Let g_δ be a given generalized topology on X, which is generated by the monotonic map δ. The non-empty elements of the generalized topology g_β on X, which is defined by the monotonic map $\beta = cl_\delta \circ int_\delta \circ cl_\delta$ consists of all non-empty subsets $A \subset X$, satisfying the following conditions*:

1 *If for some $O_0 \in g_\delta$, A intersects O_0 and A intersects each $O \in g_\delta$, intersecting O_0. Moreover, $A \subset \bigcup_{U\cap A\neq\emptyset, U\in g_\delta}(X - U)$.*
2 *If A intersects every $O \in g_\delta$.*

Proof Consider the action of the map β on the subset A of X:
Let $A = \emptyset$, then $\beta(A) = \emptyset$. Hence, $\emptyset \in g_\beta$.

Let $A \neq \emptyset$, then we get:

$$\beta(A) = cl_\delta(\pi(A)) = cl_\delta \begin{cases} \emptyset \quad : \quad \triangle_A = \emptyset \quad or \quad \triangle_A \neq \emptyset, \nabla_A \neq \emptyset, U_O \neq \emptyset;\ O \in \triangle_A. \\ \bigcap_{U\in\nabla_A}(X - O) \quad : \quad \triangle_A \neq \emptyset, \nabla_A \neq \emptyset, \exists O \in \triangle_A, U_O = \emptyset. \\ X \quad : \quad \nabla_A = \emptyset, X \in g_\delta. \\ \bigcup_{O\in g_\delta} O \quad : \quad \nabla_A = \emptyset, X \notin g_\delta. \end{cases}$$

$$= \begin{cases} \emptyset \quad : \quad \triangle_A = \emptyset \quad or \quad \triangle_A \neq \emptyset, \nabla_A \neq \emptyset, U_O \neq \emptyset;\ O \in \triangle_A. \\ \bigcup_{U_O=\emptyset, O\in g_\delta} O \quad : \quad \triangle_A \neq \emptyset, \nabla_A \neq \emptyset, \exists O \in \triangle_A, U_O = \emptyset. \\ X \quad : \quad \nabla_A = \emptyset. \end{cases}$$

The nonempty subset A is $\beta-$open subset, if $\beta(A) \supset (A)$. Therefore, the subset A is $\beta-$open subset in the following two cases:

1 If for some $O_0 \in g_\delta$, A intersects O_0 and A intersects each $O \in g_\delta$, intersecting O_0. Moreover, $A \subset \bigcup_{U \cap A \neq \emptyset, U \in g_\delta} (X - U)$.
2 If A intersects every $O \in g_\delta$.

□

Remark 41 The nonempty subset A is $\beta-$open subset, if A intersects every $O \in g_\delta$. It follows that X is $\beta-$open set.

$\beta-$*generalized topological spaces which are defined by special generalized topological spaces* $(X, g_{i\delta}), i \in \{1, 2, 3\}$

By using Theorem 40, the $\beta-$generalized topological spaces can be constructed as follows:

1 The $\beta-$generalized topology g_β on $g_{1\delta}$ is defined as follows:

A is $\beta-$open subset:

(i) If A intersects each of the subsets $\{O_1, O_2, O_3\}$ and X is $1\delta-$open set.
(ii) If A intersects each of the subsets $\{O_1, O_2, O_3\}$ and X is not $1\delta-$open set, then $A \subset O_1 \cup O_2 \cup O_3$.

2 The $\beta-$generalized topology g_β on $g_{2\delta}$ is defined as follows:
A is $\beta-$open subset in the following cases:

(i) $A \cap O_1 \neq \emptyset, A \cap O_2 = \emptyset$. Therefore, each subset of O_1 is $\beta-$open subset.
(ii) $A \cap O_2 \neq \emptyset, A \cap O_1 = \emptyset$. Therefore, each subset of O_2 is $\beta-$open subset.
(iii) A intersects $O_1 \cup O_2$.

3 The $\beta-$generalized topology g_β on $g_{3\delta}$ is defined as follows:
A is $\beta-$open subset in the following cases:

(i) If it intersects three subsets only of $\{O_1, O_2, O_3, O_4\}$, and $A \subset X - O_i$ or $A \cap O_i = \emptyset$, for only one element i in the family $\{1, 2, 3, 4\}$.
(ii) If it intersects all the elements of $g_{3\delta}$.

Properties of the composition binary operation on the monotonic functions $int_\delta, cl_\delta, \sigma, \alpha, \pi, \beta$

Let (X, g_δ) be any $\delta-$generalized topological space, generated by $\delta \in \Gamma_\delta$. Then, for any $A \subset X$, the definitions of the monotonic maps σ, α, π and β, implies the following relations:

1 $\alpha(A) = int_\delta(\sigma(A))$.
2 $\beta(A) = cl_\delta(\pi(A))$.
3 $int_\delta(A) \subset int_\delta(\sigma(A)), \quad cl_\delta(\pi(A)) \subset cl_\delta(A)$.
4 $\sigma(A) = \sigma(int_\delta(A)), \quad \pi(A) = \pi(cl_\delta(A))$.

Theorem 42 *Let (X, g_δ) be any $\delta-$generalized topological space, generated by $\delta \in \Gamma_\delta$. Then, for any $A \subset X$, the following conditions are satisfied*:

1 $\pi(X - A) = X - \sigma(A)$.
2 $\sigma(X - A) = X - \pi(A)$.
3 $\alpha(X - A) = X - \beta(A)$.
4 $\beta(X - A) = X - \alpha(A)$.

Proof The proof is straightforward, using the relations: $int_\delta(X - A) = X - cl_\delta(A)$ and $cl_\delta(X - A) = X - int_\delta(A)$. □

Theorem 43 *Let (X, g_δ) be any $\delta-$generalized topological space, generating by $\delta \in \Gamma_\delta$. Then, for any $A \subset X$, and $\gamma \in \{\sigma, \alpha, \pi, \beta\}$, it follows that $\gamma(A) = \gamma^2(A)$.*

Proof

1
$$\sigma(A) = \bigcap_{V \in g_\delta} \{X - V : int_\delta(A) \cap V = \emptyset\},$$
$$\sigma^2(A) = \bigcap_{V \in g_\delta} \{X - V : int_\delta(\sigma(A)) \cap V = \emptyset\}.$$

Since $int_\delta(A) \subset \sigma(A)$, then $int_\delta(A) \subset int_\delta(\sigma(A))$, which implies that for any $V \in g_\delta$, if $int_\delta(\sigma(A)) \cap V = \emptyset$, then $int_\delta(A) \cap V = \emptyset$. Therefore $\sigma^2(A) \subset \sigma(A)$.

Conversely, let $x \in \sigma(A)$ and $x \notin \sigma^2(A)$, then there exists $V \in g_\delta$ such that $x \in V$ and $int_\delta(\sigma(A)) \cap V = \emptyset$. Since $int_\delta(A) \subset \sigma(A)$, then $int_\delta(A) \cap V = \emptyset$, which contradicts $x \in \sigma(A)$. Hence $x \in \sigma^2(A)$, and $\sigma^2(A) \supset \sigma(A)$, which implies that $\sigma^2(A) = \sigma(A)$.

2 $\pi^2(A) = \pi(\pi(A)) = \pi(X - \sigma(X - A)) = X - \sigma(\sigma(X - A)) = X - \sigma^2(X - A) = X - \sigma(X - A) = \pi(A)$.

3 $\alpha^2(A) = \alpha(\alpha(A)) = \alpha(int_\delta(\sigma(A))) = int_\delta(\sigma(int_\delta(\sigma(A)))) = int_\delta(\sigma(\sigma(A))) = int_\delta(\sigma(A)) = \alpha(A)$.
4 $\beta^2(A) = \beta(\beta(A)) = \beta(cl_\delta(\pi(A))) = cl_\delta(\pi(cl_\delta(\pi(A)))) = cl_\delta(\pi(\pi(A))) = cl_\delta(\pi(A)) = \beta(A)$.

□

Theorem 44 *Let (X, g_δ) be any $\delta-$generalized topological space, generated by $\delta \in \Gamma_\delta$.*

The composition operation $\circ$ on the set of all functions is a binary operation on the family of monotonic maps $\{int_\delta, cl_\delta, \sigma, \alpha, \pi, \beta\}$.

Proof One can construct the following table easily.

$\circ$	int_δ	cl_δ	σ	π	α	β
int_δ	int_δ	π	α	π	α	π
cl_δ	σ	cl_δ	σ	β	σ	β
σ	σ	β	σ	β	σ	β
π	α	π	α	π	α	π
α	α	π	α	π	α	π
β	σ	β	σ	β	σ	β

Therefore, the composition operation $\circ$ is a binary operation on the family of monotonic maps $\{int_\delta, cl_\delta, \sigma, \alpha, \pi, \beta\}$. □

Corollary 45 *Let (X, g_δ) be any $\delta-$generalized topological space, generated by $\delta \in \Gamma_\delta$.*

For any $\gamma \in \{int_\delta, cl_\delta, \sigma, \alpha, \pi, \beta\}$, it follows that:

1 $\beta \circ \gamma = \sigma \circ \gamma$.
2 $\pi \circ \gamma = \alpha \circ \gamma$.
3 $\gamma \circ \sigma = \gamma \circ \alpha$.
4 $\gamma \circ \pi = \gamma \circ \beta$.
5 $int_\delta \circ \gamma \circ int_\delta = \alpha; \gamma \neq int_\delta$.
6 $cl_\delta \circ \gamma \circ cl_\delta = \beta; \gamma \neq cl_\delta$.
7 $\sigma \circ \gamma \circ \sigma = \sigma$.
8 $\pi \circ \gamma \circ \pi = \pi$.
9 $\alpha \circ \gamma \circ \alpha = \alpha$.
10 $\beta \circ \gamma \circ \beta = \beta$.

Proof The proof can be constructed from the above table in Theorem 44. □

The relation between the $\gamma-$generalized topological spaces, where $\gamma \in \{int_\delta, cl_\delta, \sigma, \alpha, \pi, \beta\}$, can be studied in the following theorem.

Theorem 46 *Let (X, g_δ) be any δ—generalized topological space, generated by $\delta \in \Gamma_\delta$, and (X, D_X) be the discrete space, then*:

1 $g_{int_\delta} = g_\delta \subset g_\alpha \subset g_\sigma \subset g_\beta \subset g_{cl_\delta} = D_X$.
2 $g_\alpha \subset g_\pi \subset g_\beta$.

Proof The proof is easy, since for all $A \subset X$, it follows that:

(a) $int_\delta(A) \subset A \subset cl_\delta(A)$.
(b) $int_\delta(A) \subset \alpha(A) \subset \sigma(A) \subset \beta(A) \subset cl_\delta(A)$.
(c) $\alpha(A) \subset \pi(A) \subset \beta(A)$.

□

Conclusion

In this paper, the family of monotonic functions $\Gamma(X)$ have the following properties:

1 The monotonic map $int_\delta \in \Gamma(X)$ is the smallest monotonic map in the equivalence class of all monotonic maps Γ_δ, which is defined by the same generalized topology δ. Moreover, the monotonic map $cl_\delta \in \Gamma(X)$ is the largest monotonic map in the associated equivalence class Γ^δ to the class Γ_δ.
2 Using the invariant systems $\prec G, \mathcal{H}_G \succ$, it is shown that there exists a one-to-one correspondence between the family of G—continuous functions and the family of $f(G)$—continuous functions: $h \leftrightarrow \gamma_{f^{-1}} \circ h \circ \gamma_f$; $h \in \mathcal{H}_G$; for any one-to-one correspondence $f \in X^X$.
3 The family of monotonic maps $\{int_\delta, cl_\delta, \sigma, \alpha, \pi, \beta\}$, for every δ—Csázsár generalized topological space (X, g_δ) is closed under the composition operation and has interesting relations (see article 4).

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References

1. Bai, S.Z., Zuo, Y.P.: On $g-\alpha$-irresolute functions. Acta Math. Hungar. **130**(4), 382–389 (2011)
2. Bayhan, S., Kanibir, A., Reilly, I.L.: On functions between generalized topological spaces. Appl. Gen. Topol. **14**(2), 195–203 (2013)
3. Cao, C., Yan, J., Wang, W., Wang, B.: Some generalized continuous functions on generalized topological spaces. Hacettepe J. Math. Stat. **42**(2), 159–163 (2013)
4. Császár, Á.: Generalized topology and generalized continuity. Acta Math. Hungar. **96**, 351–357 (2002)
5. Császár, Á.: Generalized open sets in generalized topologies. Acta Math. Hungar. **106**, 53–66 (2005)
6. Császár, Á.: Product of generalized topologies. Acta Math. Hungar. **134**, 132–138 (2009)
7. Khayyeri, R., Mohamadian, R.: On base for generalized topological spaces. Int. J. Contemp. Math. Sci. **6**(48), 2377–2383 (2011)
8. Min, W.K.: Some Results on generalized topological spaces and generalized systems. Acta Math. Hungar. **108**, 171–181 (2005)
9. Min, W.K.: Weak continuity on generalized topological spaces. Acta Math. Hungar. **121**, 283–292 (2008)
10. Min, W.K.: Generalized continuous functions defined by genenralized open sets on generalized topological spaces. Acta Math. Hungar. **128**, 299–306 (2010)
11. Min, W.K.: $(\delta,\delta')-$continuity on generalized topological spaces. Acta Math. Hungar. **129**(4), 350–356 (2010)
12. Popa, V., Noiri, T.: On $\beta-$continuous functions. Real Anal. Exchange. 18, 544–548 (1992/1993)
13. Reilly, I.L., Vamanamurthy, M.K.: On α-continuity in topological spaces. Acta Math. Hungar. **45**, 27–32 (1985)
14. Willard, S.: General Topology. Addison-Wesley Publishing Company, Boston (1970)

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