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# A characterization of high transitivity for groups acting on trees

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**Abstract:** We establish a sharp sufficient condition for groups acting on trees to be highly transitive when the action on the tree is minimal of general type. This gives new examples of highly transitive groups, including icc non-solvable Baumslag-Solitar groups, thus answering a question of Hull and Osin.

**Key words and phrases:** high transitivity, groups acting on trees, HNN extensions, amalgamated products, Baumslag-Solitar groups, Baire category theorem.

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# 1 Introduction

Given a countably infinite group $\Gamma$, one is naturally led to the study of its transitive actions, or equivalently of the homogeneous spaces $\Gamma/\Lambda$ where $\Lambda$ is a subgroup of $\Gamma$. A basic invariant for such an action is the **transitivity degree**, namely the supremum of the $n \in \mathbb{N}$ such that for any two $n$-tuples of distinct points, the first can be taken to the second by a group element. Note that the transitivity degree of an action can be infinite, as is witnessed by the natural action of the group of finitely supported permutations of a countably infinite set. One can then lift the transitivity degree to a group invariant $\mathrm{td}(\Gamma)$ defined as the supremum of the transitivity degrees of the *faithful* $\Gamma$-actions. The most transitive groups are the **highly transitive** groups, namely those which admit a faithful action whose transitivity degree is infinite. Note that such groups automatically have infinite transitivity degree. As noted by Hull and Osin in [HO16], it is actually unknown whether there is a countable group $\Gamma$ with infinite transitivity degree, but which fails to be highly transitive.

## 1.1 Some highly transitive groups

Let us now give a brief overview of groups which are known to be highly transitive. First, the group of finitely supported permutations of a countably infinite set is highly transitive. Other examples of locally finite highly transitive groups are provided by the forward orbit stabilizers of minimal $\mathbb{Z}$-actions on the Cantor space, such as the group of dyadic permutations, and by the Hall group.

For finitely generated amenable groups, one can upgrade the group $S_f(\mathbb{Z})$ of finitely supported permutations of $\mathbb{Z}$ to the 2-generated group $S_f(\mathbb{Z}) \rtimes \mathbb{Z}$ of permutations which are translations except on a finite set. Other natural examples are provided by derived groups of topological full groups of minimal $\mathbb{Z}$-subshifts acting on an orbit (the fact that they are finitely generated is due to Matui [Mat06], while their amenability is a celebrated result of Juschenko and Monod [JM13]).

In the non-amenable realm, the first explicit examples of highly transitive groups are free groups $\mathbb{F}_n$ for $2 \leq n \leq +\infty$, as was shown in 1976 by McDonough [McD77] (see also the work of Dixon in [Dix90]). The case of a general free product has been studied Glass and McCleary in [GM91] and later settled by Gunhouse [Gun92] and independently by Hickin [Hic92].

In the last few years, many new examples of highly transitive groups have been discovered such as surface groups [Kit12], $\mathrm{Out}(\mathbb{F}_n)$ for $n \geq 4$ [GG13], and non-elementary hyperbolic groups with trivial finite radical [Cha12]. A vast generalization of these results was then found by Hull and Osin.

**Theorem** ([HO16, Theorem 1.2])**.** *Every countable acylindrically hyperbolic group admits a highly transitive action with finite kernel. In particular, every countable acylindrically hyperbolic group with trivial finite radical is highly transitive.*

Let us recall that a group is called **acylindrically hyperbolic** if it admits a non-elementary acylindrical action on a hyperbolic space. For equivalent definitions, and for more background on acylindrically hyperbolic groups, we refer the reader to [Osi16] or [Osi19].

On the other hand, examples which are not entirely covered by Hull and Osin's result come from groups acting on trees as in the work of the first, third and fourth authors [FMS15]. Other examples are provided by a recent result of Gelander, Glasner and Soĭfer, which states that any center free unbounded and non-virtually solvable countable subgroups of $\mathrm{SL}_2(k)$ is highly transitive, where $k$ is a local field [GGS20].

Our main result is an optimal generalization of the aforementioned result of the first, third and fourth authors.

**Theorem A.** *Let $\Gamma \curvearrowright \mathcal{T}$ be a minimal action of general type of a countable group $\Gamma$ on a tree $\mathcal{T}$. If the action on the boundary $\Gamma \curvearrowright \partial\mathcal{T}$ is topologically free, then $\Gamma$ admits a highly transitive and highly faithful action; in particular, $\Gamma$ is highly transitive.*

The above **minimality** assumption means that there are no nontrivial invariant subtrees, while the **topological freeness** assumption means that no half-tree can be pointwise fixed by a non-trivial group element (in particular, the action is faithful). An action on a tree is of **general type** when there are two transverse hyperbolic elements (see Section 2.3). All these hypotheses are necessary in Theorem A: for topological freeness this is discussed in the next section, while for the type of the action and the minimality this is discussed in section 9.

Finally, **high faithfulness** is a natural strengthening of faithfulness introduced in [FMS15], which states that the intersection of the supports of finitely many nontrivial group elements is always infinite (see Section 2.1 for equivalent definitions). Let us remark that the group of finitely supported permutations does not admit highly transitive highly faithful actions [FLMM22, Remark 8.23], and that the natural highly transitive action of a topological full group is never highly faithful. It would be interesting to understand whether the highly transitive actions of acylindrically hyperbolic groups with trivial finite radical built by Hull and Osin are highly faithful.

## 1.2 Obstructions to high transitivity

Let us now move on to obstructions to high transitivity, which will lead us to a reformulation of our main theorem as a series of equivalences thanks to the work of Hull and Osin [HO16] and of Le Boudec and Matte Bon [LBMB22].

First, one can use the fact that the group of permutations of a countably infinite set is topologically simple for the product of the discrete topology, and that high transitivity can be reformulated as arising as a dense subgroup of this group. This yields the well-known fact that in a highly transitive group, the centralizer of every non-trivial group element is core-free (see Corollary 9.4). In particular, highly transitive groups cannot be solvable or contain nontrivial commuting normal subgroups, and they must be icc (all their non-trivial conjugacy classes are infinite).

In another direction, Hull and Osin have shown that given a highly transitive faithful action of a group $\Gamma$, the following are equivalent:

(1) There is a non-trivial group element with finite support;

(2) The alternating group over an infinite countable set embeds into $\Gamma$;

(3) The group $\Gamma$ satisfies a *mixed identity* .

In particular, any simple highly transitive group which is not the alternating group over an infinite countable set must be MIF (mixed identity free). Moreover, the fact that the highly transitive actions of the groups we consider in Theorem A are highly faithful yields that those groups are MIF. We refer the reader to [HO16, Sec. 5] for the definition of mixed identities, and the proof of the above-mentioned result.

Finally, there are some groups for which one can actually classify sufficiently transitive actions, and show that none of them are highly transitive. The first and only examples have been uncovered by Le Boudec and Matte Bon , who proved the following remarkable result.

**Theorem** ([LBMB22])**.** *Suppose a group $\Gamma$ admits a faithful minimal action of general type on a tree $\mathcal{T}$ which is not topologically free on the boundary. Then every faithful $\Gamma$-action of transitivity degree at least 3 is conjugate to the restriction to one orbit of the $\Gamma$-action on the boundary of $\mathcal{T}$ (whose transitivity degree is at most 3), and the group is not MIF.*

They also proved a similar statement for groups acting on the circle, and provided examples of groups, coming from [LB16, LB17], satisfying the above assumptions. Combining their result with ours, we obtain a large class of groups for which high transitivity is completely understood:

**Theorem B.** *Let $\Gamma \curvearrowright \mathcal{T}$ be a faithful minimal action of general type on a tree $\mathcal{T}$. The following are equivalent*

*(1)* $\mathrm{td}(\Gamma) \geq 4$*;*

*(2)* $\Gamma$ *is highly transitive;*

*(3)* $\Gamma$ *is MIF;*

*(4)* $\Gamma \curvearrowright \partial\mathcal{T}$ *is topologically free.*

Note that the topological freeness of the action on the boundary $\partial\mathcal{T}$ (item ((4))) is a strengthening of the global assumption that the $\Gamma$ action on the tree $\mathcal{T}$ is faithful.

In relation to the above quoted question by Hull and Osin, let us note that Theorem B yields the equivalence between high transitivity and infinite transitivity degree for countable groups admitting a faithful and minimal action of general type on a tree.

### 1.3 The cases of amalgams and HNN extensions

In order to prove Theorem A, we use Bass-Serre theory and reduce the proof to the case of an HNN extension or an amalgamated free product.

Let us first describe the case of an HNN extension $\Gamma = \mathrm{HNN}(H,\Sigma,\vartheta)$. Let $\mathcal{T}$ be the Bass-Serre tree of $\Gamma$ (see section 2.5). Then it is easy to check that the action $\Gamma \curvearrowright \mathcal{T}$ is minimal of general type if and only if $\Sigma \neq H \neq \vartheta(\Sigma)$.

The HNN extension case of Theorem A that we show in the present paper is the following.

**Theorem C.** *Let $\Gamma$ by an HNN extension $\Gamma = \mathrm{HNN}(H,\Sigma,\vartheta)$ with $\Sigma \neq H \neq \vartheta(\Sigma)$. If the action of $\Gamma$ on the boundary of its Bass-Serre tree is topologically free, then $\Gamma$ admits a highly transitive and highly faithful action; in particular, $\Gamma$ is highly transitive.*

Examples of HNN extensions which are not acylindrically hyperbolic and which do not satisfy the hypothesis of [FMS15] are Baumslag-Solitar groups. A direct application of Theorem C allows us to answer a question raised by Hull and Osin in [HO16, Question 6.3]: what is the transitivity degree of the non-solvable icc Baumslag-Solitar groups? Given $m,n \in \mathbb{Z}^*$, recall that the Baumslag-Solitar group with parameter $m,n$ is:

$$\mathrm{BS}(m,n) := \langle a,b \,:\, ab^m a^{-1} = b^n \rangle.$$

It is not solvable if and only if $|n| \neq 1$ and $|m| \neq 1$, and icc if and only if $|n| \neq |m|$. As noted by Hull and Osin, if a Baumslag-Solitar group is either solvable or not icc, then its transitivity degree is equal to 1. We prove the following in Section 8.1.1, and provide more new examples to which Theorem C applies in Sections 8.2.1 and 8.2.2.

**Corollary D.** *All the non-solvable icc Baumslag-Solitar groups are highly transitive. In particular, the group $\mathrm{BS}(2,3)$ is highly transitive.*

Then, let us describe the case of an amalgamated free product (or amalgam for short) $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$, where $\Sigma$ is a common subgroup of $\Gamma_1$ and $\Gamma_2$. Such an amalgam is said to be **non-trivial** if $\Gamma_1 \neq \Sigma \neq \Gamma_2$, and **non-degenerate** if moreover $[\Gamma_1 : \Sigma] \geq 3$ or $[\Gamma_2 : \Sigma] \geq 3$.

Let $\mathcal{T}$ be the Bass-Serre tree of the amalgam $\Gamma$ (see Section 2.6). It easy to see that the action $\Gamma \curvearrowright \mathcal{T}$ always minimal, is of general type if and only if the amalgam is non-degenerate, and is faithful if and only if $\Sigma$ is core-free in $\Gamma$. Let us now state our result in the case of an amalgam.

**Theorem E.** *Consider a non-degenerate amalgam $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ and its Bass-Serre tree $\mathcal{T}$. If the induced $\Gamma$-action on $\partial\mathcal{T}$ is topologically free, then $\Gamma$ admits a highly transitive and highly faithful action; in particular, $\Gamma$ is highly transitive.*

Notice that, in the context of the theorem, if the induced $\Gamma$-action on $\partial\mathcal{T}$ is topologically free, then obviously the action on $\mathcal{T}$ is faithful, hence $\Sigma$ is core-free in $\Gamma$. Sections 8.1.2 and 8.2.3 provides new highly transitive examples obtained via Theorem E.

## 1.4 Comparison with former results

Here is a corollary of our result which does not mention the action on the boundary, where given a subtree $\mathcal{T}'$ of $\mathcal{T}$, we denote by $\Gamma_{\mathcal{T}'}$ the pointwise stabilizer of $\mathcal{T}'$ in $\Gamma$.

**Corollary F.** *Let $\Gamma \curvearrowright \mathcal{T}$ be an action of a countable group $\Gamma$ on a tree $\mathcal{T}$, which is faithful, minimal, and of general type. If there exist a bounded subtree $\mathcal{B}$ and a vertex $u$ in $\mathcal{B}$ such that $\Gamma_{\mathcal{B}}$ is core-free in $\Gamma_u$, then, $\Gamma$ admits an action on a countable set which is both highly transitive and highly faithful. In particular, $\Gamma$ is highly transitive.*

This corollary encompasses all the previously known results of high transitivity for groups with a minimal action of general type on a tree, which fall in two categories. The first examples are the acylindrically hyperbolic ones, for which one can use the following result by Minasyan and Osin, combined with the high transitivity for acylindrically hyperbolic groups result of Hull and Osin. In its statement, we denote by $[u,v]$ the geodesic between $u$ and $v$.

**Theorem.** *[MO15, Theorem 2.1] Let $\Gamma$ be a group acting minimally on a tree $\mathcal{T}$. Suppose that $\Gamma$ is not virtually cyclic, $\Gamma$ does not fix any point in $\partial\mathcal{T}$, and there exist vertices $u, v$ of $\mathcal{T}$ such that $\Gamma_{[u,v]}$ is finite. Then $\Gamma$ is acylindrically hyperbolic.*

We will check in Proposition 7.4, that all groups satisfying the hypotheses of the above theorem, and having a trivial finite radical, also satisfy the hypotheses of Corollary F.

Furthermore, as Hull and Osin noticed [HO16, Corollary 5.12], there are groups acting on trees which are non-acylindrically hyperbolic, but highly transitive thanks to the following result by the first, third, and fourth authors. In the terminology of the present article, the assertion "$\Gamma_e$ is highly core-free in $\Gamma_v$" means that the action $\Gamma_v \curvearrowright \Gamma_v/\Gamma_e$ is highly faithful.

**Theorem.** *[FMS15, Theorem 4.1] Let a countable group $\Gamma$ act without inversion on a tree $\mathcal{T}$, and let $R \subset E(\mathcal{T})$ be a set of representatives of the edges of the quotient graph $\Gamma\backslash\mathcal{T}$. Then $\Gamma$ is highly transitive, provided $\Gamma_v$ is infinite and $\Gamma_e$ is **highly core-free** in $\Gamma_v$, for **every couple** $(e,v)$ where $e \in R$ and $v$ is one of its endpoints.*

The fact that the groups satisfying the hypotheses of the above theorem also satisfy those of Corollary F is checked in Proposition 7.5.

There are examples of groups acting on trees which are highly transitive thanks to Corollary F, but to which the previously known results do not apply. We also check that the icc non-solvable Baumslag-Solitar groups provide examples of HNN extensions for which Theorem A applies while Corollary F does not. All these examples can be found in Section 8.

## 1.5 About the proofs

In order to prove high transitivity for a general class of groups without constructing an explicit highly transitive action, two approaches can be tried. The first is by working in the space of subgroups of $\Gamma$, and proceeds by inductively building a subgroup $\Lambda \leq \Gamma$ such that the associated homogeneous space $\Gamma/\Lambda$ is highly transitive. To our knowledge, this approach made its first appearance in a paper of Hickin [Hic88], and was then made more explicit in [Hic92]. It was notably used by Chaynikov when proving that hyperbolic groups with trivial finite radical are highly transitive, and also by Hull and Osin in their aforementioned result.

The second approach, pioneered by Dixon, goes by fixing an infinite countable set $X$, considering a well-chosen Polish space of group actions on $X$, and showing that in there, the space of faithful highly transitive actions is a countable intersection of dense open sets, hence not empty by the Baire category theorem. In this work, we follow this second approach, using the same space of actions as the one considered in [FMS15], but with a much finer construction in order to show density.

As explained before, the proof of the general result goes through the HNN and the amalgam cases. The two proofs are actually very similar, so for this introduction we only explain in more details what goes on for HNN extensions.

Given a non-degenerate HNN extension $\Gamma = \mathrm{HNN}(H, \Sigma, \vartheta)$, the idea is to start with a free $H$-action on a set $X$ with infinitely many orbits, and then to turn it into a highly transitive faithful $\Gamma$-action via a generic permutation. To be more precise, the Polish space under consideration is the set of all permutations which intertwine the $\Sigma$ and the $\vartheta(\Sigma)$-actions, thus yielding a natural $\Gamma$-action. The result is then that there is a dense $G_\delta$ of such permutations which induce a highly transitive faithful $\Gamma$-action.

For this to work, the notion of high core freeness was handy in [FMS15]: it allows one to "push" the situation by a group element in $H$ so as to get to a place where both $\Sigma$ and $\vartheta(\Sigma)$ act in a more controllable way. Let us note that this approach was generalized in [FLMM22] to show that all the groups considered in [FMS15] actually have a faithful homogeneous action onto *any* bounded $S$-Urysohn space.

Here, we use a different approach, similar to the one due to the third and fourth named authors when they re-discovered the characterization of free products of finite groups which are highly transitive [MS13]. The main difficulty is that the group element that we use to "push" things out does not belong to $H$, in particular it can contain a number of powers of the permutation at hand.

In order to solve this, we first modify the permutation so as to make sure such a push is possible. The modification is actually very natural. Informally speaking, there are two steps:

1. "erasing the permutation" outside a suitable finite set of $\Sigma$-orbits and $\vartheta(\Sigma)$-orbits, which leaves us with a partial bijection;

2. make a "free globalization" of this partial bijection, which is obtained by gluing partial bijections inducing portions of the $\Gamma$-action by right translations on itself.

This results in a new $\Gamma$-action satisfying a very natural universal property, which we state by introducing the notion of *pre-action* of an HNN extension, see Theorem 3.18 (and Theorem 5.17 for its amalgam counterpart). The construction in step (2) allows us to use the topological freeness of the left action on the boundary in order to find the further modification of the permutation which yields high transitivity, following an approach close to the proof of [MS13, Theorem 3.3]. We do not know if our approach can be generalized so as to obtain faithful homogeneous action onto bounded $S$-Urysohn space.

## 1.6 Organization of the paper

Section 2 is a preliminary section in which we introduce our notations and definitions concerning group actions, graphs, amalgams and HNN extensions. Section 3 contains the main technical tools to prove Theorem C: the notion of a pre-action of an HNN extension, its Bass-Serre graph and its free globalization. In Section 4 we prove Theorem C. Section 5 contains the main technical tools to prove Theorem E: the notion of a pre-action of an amalgam, its Bass-Serre graph and its free globalization. In Section 6 we prove Theorem E while in Section 7 we prove Theorem A, Theorem B and Corollary F. Section 8 is dedicated to concrete examples where our results apply. Finally, in Section 9 we show that the minimality assumption in Theorem A is needed, and we discuss other types of actions on trees.

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## 2 Preliminaries

The notation $A \Subset B$ means that $B$ is a set and $A$ is a finite subset of $B$.

### 2.1 Group actions

Throughout the article, we will use the symbol $X$ to denote an infinite countable set. Then, $S(X)$ denotes the Polish group of bijections of $X$. Unless specified otherwise, groups will act on $X$ on the *right*. One of our motivations for doing so is that we will associate paths to words in our groups, so it will be much easier to read both in the same order (see for instance Section 3.3).

So given two permutations $\sigma, \tau \in S(X)$ and $x \in X$, the image of $x$ by $\sigma$ is denoted $x\sigma$, and the product $\sigma\tau$ is the permutation obtained by applying $\sigma$ first and then $\tau$. This way, $S(X)$ acts on $X$ on the right and any right $G$-action $X \curvearrowleft^{\alpha} G$ is equivalent to a morphism of groups $\alpha : G \to S(X)$. The image of an element $g \in G$ by $\alpha$ will be denoted by $\alpha(g)$ or $g^{\alpha}$, or just $g$ if there is no possible confusion. Similarly, the image of a subgroup $H$ of $G$ by $\alpha$ will be denoted by $\alpha(H)$ or $H^{\alpha}$, or just $H$.

Notice however that actions on other kinds of spaces, especially on Bass-Serre trees, will be on the left.

**Definition 2.1.** An action $X \curvearrowleft G$ is **highly transitive** if, for any $k \in \mathbb{N}^*$ and any $k$-tuples $(x_1, \ldots, x_k)$, $(y_1, \ldots, y_k) \in X^k$, each with pairwise distinct coordinates, there exists $\gamma \in G$ such that $x_i\gamma = y_i$ for all $i = 1 \ldots, k$.

**Lemma 2.2.** *An action $X \curvearrowleft G$ is highly transitive if and only if, for every $k \in \mathbb{N}^*$, and every $x_1, \ldots, x_k, y_1, \ldots, y_k$ all pairwise distinct, we can find $g \in G$ such that $x_i g = y_i$ for $i = 1, \ldots, k$.*

*Proof.* Take $x_1, \ldots, x_k$ pairwise distinct, and $z_1, \ldots, z_k$ pairwise distinct, we need to find $\gamma$ such that $x_i\gamma = z_i$ for $i = 1, \ldots, k$. Since $X$ is infinite, we find $y_1, \ldots, y_k$ pairwise distinct and distinct from all the $x_i$'s and $z_i$'s, then by our assumption there are both $g$ and $h$ such that for all $i = 1, \ldots, k$ we have $x_i g = y_i$ and $y_i h = z_i$, so the element $\gamma = gh$ is the element we seek. □

Given a bijection $\sigma \in S(X)$, its **support** is the set $\operatorname{supp}\sigma = \{x \in X : x\sigma \neq x\}$. Recall that an action $X \curvearrowleft G$ is **faithful** if for every $g \in G \setminus \{1\}$ the support of $g$ is not empty.

**Definition 2.3.** An action $X \curvearrowleft G$ is called **strongly faithful** if, for any finite subset $F \subseteq G \setminus \{1\}$, the intersection of the supports of the elements of $F$ is not empty. It is called **highly faithful** if for every finite subset $F \subseteq G \setminus \{1\}$, the intersection of the supports of the elements of $F$ is infinite.

Given a strongly faithful action $X \curvearrowleft G$, and a finite subset $F \subseteq G$, it is easy to see that there exists $x \in X$ such that the translates $xg$, for $g \in F$, are pairwise distinct. Indeed, any element $x \in \bigcap_{g,h \in F} \operatorname{supp}(gh^{-1})$ will do.

Let us check that our definition of high faithfulness coincides with the one given in [FMS15].

**Lemma 2.4.** *An action $X \curvearrowleft G$ is highly faithful if and only if for every $n \in \mathbb{N}$, if $F$ is a finite subset of $X$ and $X_1, \ldots, X_n$ are subsets of $X$ such that $X = F \cup X_1 \cup \cdots \cup X_n$, then there is some $k \in \{1, \ldots, n\}$ such that for every $g \in G \setminus \{1\}$, there is $x \in X_k$ such that $x \cdot g \neq x$.*

*Proof.* We prove the lemma by the contrapositive in both directions.

Suppose that for a fixed $n \in \mathbb{N}$, we can find a decomposition $X = F \cup X_1 \cup \cdots \cup X_n$ such that for all $k \in \{1, \ldots, n\}$, there is $g_k \in G$ whose support is disjoint from $X_k$. Then in particular, the intersection of the supports of the $g_k$'s is contained in $F$, hence finite, contradicting high faithfulness.

Conversely, suppose that we found $g_1, \ldots, g_n \in G$ whose supports have finite intersection. Then the sets $F = \bigcap_{k=1}^{n} \operatorname{supp} g_k$ and $X_k = X \setminus \operatorname{supp} g_k$ satisfy that for all $k \in \{1, \ldots, n\}$, there is some $g \in G$ (namely $g_k$) such that for all $x \in X_k$, $x \cdot g = x$. □

Of course, we have the implications:

$$\text{free} \;\Rightarrow\; \text{highly faithful} \;\Rightarrow\; \text{strongly faithful} \;\Rightarrow\; \text{faithful}$$

Let us now see that strong faithfulness and high faithfulness coincide in many cases.

**Proposition 2.5.** *Given an action $X \curvearrowleft G$ of a nontrivial group $G$, the following assertions are equivalent:*

*(1) the action is strongly faithful, but not highly faithful;*

*(2) there are finite orbits in $X$ on which $G$ acts freely (in particular, $G$ has to be finite), but only finitely many of them.*

*Proof.* Define the *free part* $X_f$ of our action as the union of the orbits in $X$ on which $G$ acts freely. Note that we can write $X_f$ as

$$X_f = \bigcap_{g \in G \setminus \{1\}} \operatorname{supp}(g).$$

Assume first that (2) holds. In this case, $X_f$ is a non-empty finite union of finite orbits, hence a non-empty finite set. Therefore, the action is strongly faithful, since $X_f$ is non-empty, and it is not highly faithful, since $G$ and $X_f$ are finite.

Assume now that (1) holds. Since the action is not highly faithful, there exists $F_0 \Subset G \setminus \{1\}$ such that $\bigcap_{g \in F_0} \operatorname{supp}(g)$ is finite. Thus, the family $(Y_F)_{F_0 \subseteq F \Subset G \setminus \{1\}}$ given by

$$Y_F = \bigcap_{g \in F} \operatorname{supp}(g)$$

is a decreasing family of finite sets, which are all non-empty since the action is strongly faithful. Now, we have

$$X_f = \bigcap_{g \in G \setminus \{1\}} \operatorname{supp}(g) = \bigcap_{F_0 \subseteq F \Subset G \setminus \{1\}} Y_F\,,$$

hence $X_f$ is finite and non-empty. Consequently, there are finite orbits in $X$ on which $G$ acts freely, and their number is finite. □

**Corollary 2.6.** *In case $G$ is infinite, an action $X \curvearrowleft G$ is highly faithful if and only if it is strongly faithful.*

Let us end this section by remarking a reformulation of strong faithfulness which we won't use.

**Remark 2.7.** A transitive action is strongly faithful if and only if the stabilizer of every (or equivalently, some) point is not a *confined* subgroup of the acting group (see Section 1.5 from [Mat18] for a discussion of the notion of confined subgroup).

## 2.2 Graphs

First, let us recall the definition of a non-simple graph.

**Definition 2.8.** A **graph** $\mathcal{G}$ is given by a **vertex set** $V(\mathcal{G})$, an **edge set** $E(\mathcal{G})$, a fixed-point-free involution $\bar{\cdot} : E(\mathcal{G}) \to E(\mathcal{G})$ called the **antipode map**, a **source map** $s : E(\mathcal{G}) \to V(\mathcal{G})$ and a **range map** $r : E(\mathcal{G}) \to V(\mathcal{G})$ subject to the condition:

$$\text{for all } e \in E(\mathcal{G}), \quad s(\bar{e}) = r(e).$$

The graph $\mathcal{G}$ is **oriented** if a partition $E(\mathcal{G}) = E(\mathcal{G})^+ \sqcup E(\mathcal{G})^-$ such that $E(\mathcal{G})^- = \overline{E(\mathcal{G})^+}$ is given. In this case, the edges in $E(\mathcal{G})^+$ are called **positive edges** and the edges in $E(\mathcal{G})^-$ are called **negative edges**.

Recall that a **path** $\omega$ in a graph $\mathcal{G}$ is a finite sequence of edges $\omega = (e_1, \dots, e_n)$, such that, for all $1 \leq k \leq n-1$, $r(e_k) = s(e_k+1)$. We call $s(e_1)$ the **source** of $\omega$ and $r(e_n)$ the **range** of $\omega$. We also say that $\omega$ is a path from $s(\omega) := s(e_1)$ to $r(\omega) := r(e_n)$. The **inverse path** of $\omega$ is defined by $\overline{\omega} := (\overline{e_n}, \dots, \overline{e_1})$. The integer $n$ is called the **length** of $\omega$ and denoted by $\ell(\omega)$. Similarly, an **infinite path**, also called a **ray**, is a sequence of edges $\omega = (e_k)_{k \geq 1}$ such that $r(e_k) = s(e_{k+1})$ for all $k \geq 1$ and the vertex $s(\omega) := s(e_1)$ is called the **source** of $\omega$.

Given a path $\omega = (e_k)_{1 \leq k \leq n}$, respectively an infinite path $\omega = (e_k)_{k \geq 1}$, in $\mathcal{G}$, we use the notation $\omega(n) := r(e_n)$, for $n \geq 1$ and $\omega(0) = s(e_1) = s(\omega)$. A couple $(e_k, e_{k+1})$ such that $e_{k+1} = \overline{e_k}$, if there is one, is called a **backtracking** in $\omega$. If $\omega$ has no backtracking, we also say that it is a **reduced path**. One says that $\omega$ is **geodesic** in $\mathcal{G}$ if, for all $i, j$, the distance in $\mathcal{G}$ between $\omega(i)$ and $\omega(j)$ is exactly $|j-i|$. Obviously, all geodesic paths are reduced.

A **cycle** in $\mathcal{G}$ is a reduced path $c$ of length at least 1 such that $s(c) = r(c)$, that is, a reduced path $c = (e_1, \dots, e_n)$ such that $n \geq 1$ and $c(n) = c(0)$. Such a cycle is **elementary** if moreover the vertices $c(k)$, for $0 \leq k \leq n-1$, are pairwise distinct. Every cycle contains an elementary cycle.

When $\mathcal{G}$ is oriented, a path $\omega = (e_k)_{1 \leq k \leq n}$, respectively an infinite path $\omega = (e_k)_{k \geq 1}$, in $\mathcal{G}$ is called **positively oriented** if $e_k \in \mathrm{E}(\mathcal{G})^+$ for all $k$ and **negatively oriented** if $e_k \in \mathrm{E}(\mathcal{G})^-$ for all $k$; it is called **oriented** if it is either positively oriented or negatively oriented.

**Definition 2.9.** A **morphism of graphs** $f : \mathcal{G} \to \mathcal{G}'$ is a couple of maps $V(\mathcal{G}) \to V(\mathcal{G}')$ and $E(\mathcal{G}) \to E(\mathcal{G}')$, which will both be denoted by $f$ for sake of simplicity, such that $f(\bar{e}) = \overline{f(e)}$, $f(s(e)) = s(f(e))$, and $f(r(e)) = r(f(e))$ for all edges $e$ in $\mathcal{G}$.

The **star** at a vertex $v$ is the set $\mathrm{st}(v)$ of edges whose source is $v$, and its cardinality is called the **degree** of $v$. A morphism of graphs $f : \mathcal{G} \to \mathcal{G}'$ is **locally injective** if, for all $v \in V(\mathcal{G})$, the restriction of $f$ to the star of $v$ is injective. Note that a locally injective morphism from a connected graph to a tree is injective.

**Definition 2.10.** Given a graph $\mathcal{G}$, and a set of edges $E \subseteq E(\mathcal{G})$, we associate to this subset the **induced subgraph** as the graph $\mathcal{H}$ such that $V(\mathcal{H}) = s(E) \cup r(E)$, $E(\mathcal{H}) = E \cup \bar{E}$, and the structure maps of $\mathcal{H}$ are the restrictions of those of $\mathcal{G}$.

**Definition 2.11.** Given an edge $e$, its associated **half-graph** is the subgraph induced by the set of edges $f$ such that there is a reduced path starting by $e$, not using $\bar{e}$, and whose last edge is equal to $f$.

**Remark 2.12.** Suppose $\mathcal{G}$ is connected and $e$ is an edge, let $\mathcal{H}_e$ and $\mathcal{H}_{\bar{e}}$ be the half-graphs associated to $e$ and $\bar{e}$. Then, one has $V(\mathcal{G}) = V(\mathcal{H}_e) \cup V(\mathcal{H}_{\bar{e}})$ and $E(\mathcal{G}) = E(\mathcal{H}_e) \cup E(\mathcal{H}_{\bar{e}})$. Moreover, denoting by $\mathcal{G}_0$ the graph obtained by deleting the edges $e, \bar{e}$ in $\mathcal{G}$:

1. if $\mathcal{G}_0$ remains connected, then one has $\mathcal{H}_e = \mathcal{G} = \mathcal{H}_{\bar{e}}$;

2. if not, then $\mathcal{H}_e$ and $\mathcal{H}_{\bar{e}}$ are obtained by adding the edges $e, \bar{e}$ to the respective connected components of $r(e)$ and $s(e)$ in $\mathcal{G}_0$.

## 2.3 Trees and their automorphisms

For a more detailed account of what follows, we refer the reader to [HP11]. A **forest** is a graph with no cycle and a **tree** is a connected forest. In a forest, we recall that any reduced (finite or infinite) path is geodesic. Moreover, any two vertices in a tree are connected by a unique reduced path.

There is a well-known classification (see e.g. [Ser80]) of the automorphisms of a tree $\mathcal{T}$: if $g$ is such an automorphism, then:

- either $g$ is **elliptic**, which means that $g$ fixes some vertex of $\mathcal{T}$,
- or $g$ is an **inversion**, which means that $g$ sends some edge $e$ onto its antipode $\bar{e}$,
- or $g$ is **hyperbolic**, which means that $g$ acts by a (non-trivial) translation on a bi-infinite geodesic path, called its **axis**.

**Definition 2.13.** The **boundary** $\partial\mathcal{T}$ (or set of **ends**) of a tree $\mathcal{T}$ is the set of geodesic rays quotiented by the equivalence relation which identifies two geodesic rays whose ranges differ by a finite set.

Note that since we are working in a tree, if we fix a vertex $o$ then the set of geodesic rays starting at $o$ is in bijection with the boundary of the tree through the quotient map.

The boundary is equipped with the topology whose basic open sets $U_{\mathcal{H}}$ are given by fixing a half-tree $\mathcal{H}$, and letting $U_{\mathcal{H}}$ be the set of equivalence classes of geodesic rays whose range is contained in $\mathcal{H}$ (so given a geodesic ray $\omega$, its class belongs to $U_{\mathcal{H}}$ if and only if some terminal subpath of $\omega$ is contained in $\mathcal{H}$).

Any $g \in \mathrm{Aut}(\mathcal{T})$ induces a homeomorphism of $\partial\mathcal{T}$, which yields a group homomorphism

$$\mathrm{Aut}(\mathcal{T}) \longrightarrow \mathrm{Homeo}(\partial\mathcal{T}).$$

In case $g$ is hyperbolic, $g$ fixes exactly two points in $\partial\mathcal{T}$, which are the endpoints of its axis. Let us also recall that every action on a tree $\Gamma \curvearrowright \mathcal{T}$ satisfies exactly one of the following:

- it is **elliptic**, which means that (the image of) $\Gamma$ stabilizes some vertex, or some pair of antipodal edges;
- it is **parabolic**, or **horocyclic**, that is, $\Gamma$ contains no hyperbolic elements, without being elliptic itself;
- it is **lineal**, that is $\Gamma$ contains hyperbolic elements, all of them sharing the same axis;
- it is **quasi-parabolic**, or **focal**, that is $\Gamma$ contains hyperbolic elements with different axes, but all hyperbolic elements of $\Gamma$ share a common fixed point in $\partial\mathcal{T}$;
- it is **of general type**, which means that $\Gamma$ contains two hyperbolic elements with no common fixed point in $\partial\mathcal{T}$ (such hyperbolic elements are called **transverse**).

In the parabolic case, it can be shown that every element of $\Gamma$ is elliptic, and that $\Gamma$ stabilizes a unique point in $\partial\mathcal{T}$. In the quasi-parabolic case, it can be shown that the common fixed point in $\partial\mathcal{T}$ of the hyperbolic elements is unique and fixed by $\Gamma$. If $\Gamma \curvearrowright \mathcal{T}$ is of general type, it is easy to produce infinitely many pairwise transverse hyperbolic elements. An action on a tree is called **minimal** if there is no invariant subtree. Since we will be interested in *faithful* minimal actions of infinite countable groups on trees, elliptic actions won't occur. Moreover every vertex will have degree at least 2, since otherwise we could trim off all vertices of degree 1 and get a proper invariant subtree.

Let us recall that the action $\Gamma \curvearrowright \partial\mathcal{T}$ by homeomorphisms is **topologically free** if the trivial element is the only element in $\Gamma$ which fixes a non-empty open subset of $\partial\mathcal{T}$ pointwise. We will rather use the following concrete characterization.

**Proposition 2.14.** *Let $\mathcal{T}$ be a tree with at least three ends. Given a faithful minimal action of an infinite group $\Gamma$ on $\mathcal{T}$, the following are equivalent:*

*(i) the induced action $\Gamma \curvearrowright \partial\mathcal{T}$ is topologically free;*

*(ii) no element of $\Gamma\setminus\{1\}$ can fix pointwise a half-tree in $\mathcal{T}$.*

*Proof.* The implication from (i) to (ii) is clear since half-trees do define open subsets for the topology of $\partial\mathcal{T}$. Conversely, assume (ii). To prove (i), let us fix $\gamma\in\Gamma\setminus\{1\}$ and show that $\gamma$ does not fix any basic open set $U_{\mathcal{H}}$ pointwise.

Notice that an inversion does not fix any point in $\partial\mathcal{T}$. Moreover a hyperbolic automorphism $h$ has exactly two fixed points $\xi^{\pm}$ and, taking a third end $\eta\in\partial\mathcal{T}$, one has $\eta h^{\pm n}\to\xi^{\pm}$ as $n\to+\infty$, so that $\{\xi\in\partial\mathcal{T} : \xi h=\xi\}$ has empty interior. Hence, the only case to check is when $\gamma$ is elliptic. Notice also that, by minimality of the action, every vertex has degree at least 2 in $\mathcal{T}$, so that every half-tree is covered by the geodesic rays in it.

Consider any basic open set $U_{\mathcal{H}}$, given by a half-tree $\mathcal{H}$. We claim that $\mathcal{H}$ contains a geodesic ray which does not meet the subtree $\mathrm{Fix}(\gamma)$ of $\gamma$-fixed points. Indeed, take any geodesic ray $r$ in $\mathcal{H}$. If $r$ meets $\mathrm{Fix}(\gamma)$ at some vertex $v$, then consider the edge $e$ in $r$ whose source is $v$, and the half-tree $\mathcal{H}_e$ it defines. By (ii), there is a vertex $w$ in $\mathcal{H}_e$ which $\gamma$ does not fix. Extend the geodesic from $v$ to $w$ to a geodesic ray in $\mathcal{H}_e$. The tail of this ray from $w$ does not meet $\mathrm{Fix}(\gamma)$, since the latter is a subtree containig $v$. The claim is proved.

Now, since $\gamma$ is elliptic, our ray which does not meet $\mathrm{Fix}(\gamma)$ is moved by $\gamma$ onto a disjoint geodesic ray in $\mathcal{T}$. This corresponds to a point $\xi\in U_{\mathcal{H}}$ such that $\xi\gamma\neq\xi$. □

Note that topological freeness of the action on the boundary is called *slenderness* by de la Harpe and Préaux [HP11]. Although we won't use it, let us mention that for a minimal action of general type, the topological freeness of the action on the boundary is also equivalent to the action on the tree itself being strongly faithful (see [BIO20, Prop. 3.8] for this and other characterizations).

## 2.4 Treeing edges

Let us now turn to the link between half-graphs, seen in Section 2.2, and trees.

**Definition 2.15.** An edge in a graph $\mathcal{G}$ is a **treeing edge** when its associated half-graph is a tree, in which case we also call the latter its **half-tree**.

Here is an easy characterization of treeing edges that will prove useful.

**Lemma 2.16.** *Let $\mathcal{G}$ be a graph, let $e$ be an edge. Then the following are equivalent:*

*(i) the edge $e$ is a treeing edge;*

*(ii) the map which takes a reduced path starting by $e$ to its range is injective;*

*(iii) there is no reduced path from $s(e)$ to $s(e)$ starting by the edge $e$.*

*Proof.* First note that (i) implies (ii) since when $e$ is a treeing edge, all the reduced paths starting by $e$ must belong to its half-tree, and hence have distinct ranges.

We then show that (ii) implies (iii) by the contrapositive. If (iii) does not hold, let $c$ be a reduced path starting by the edge $e$ from $s(e)$ to $s(e)$. Then $c$ and the reduction of $cc$ have the same range, so (ii) does not hold.

Finally we show that (iii) implies (i) by the contrapositive. If $e$ is not a treeing edge, consider the following two cases:

- In the half-graph of $e$, the vertex $s(e)$ has degree at least two. We then fix some $e' \neq e$ such that $s(e') = s(e)$. If $r(e') = r(e)$ then the reduced path $e\overline{e'}$ witnesses that (iii) does not hold.

  Otherwise by the definition of the half-graph we find a reduced path $\omega$ starting by $e$ whose last edge is either $e'$ or $\overline{e'}$. If the last edge is $\overline{e'}$, then $\omega$ witnesses that (iii) does not hold. If the last edge of $\omega$ is $e'$, then write $\omega = \omega' e'$ and note that $\omega'$ witnesses that (iii) does not hold. So in any case, (iii) does not hold.

- In the half-graph of $e$, the vertex $s(e)$ has degree 1. Then since $e$ is not a treeing edge, we find a non-empty reduced path $\omega$ starting and ending at $r(e)$, and using neither $e$ nor $\bar{e}$. Then $e\omega\bar{e}$ witnesses that (iii) does not hold.

This finishes the proof of the equivalences. □

Note that if a reduced path uses a treeing edge at some point, then from that point on it only uses treeing edges. Moreover, we have the following result.

**Lemma 2.17.** *Let $\mathcal{G}$ be a connected graph admitting a treeing edge, and let $\omega$ be a reduced path in $\mathcal{G}$. Then $\omega$ can be extended to a reduced path $\omega'$ whose last edge is a treeing edge.*

*Proof.* Let $e$ be the last edge of $\omega$. If $e$ is a treeing edge, we can take $\omega' = \omega$. If not, by the previous lemma there is a reduced path of the form $ec$ from $s(e)$ to $s(e)$. Let $e'$ be a treeing edge, and denote by $C$ the set of vertices visited by the reduced path $c$.

We then claim that $s(e')$ is strictly closer to $C$ than $r(e')$. Indeed, otherwise, if we fix a geodesic $\eta$ from $r(e')$ to $C$, the geodesic $\eta$ cannot start by $\bar{e'}$, and there exists a cycle $\kappa$ based at $r(\eta)$ and whose vertices belong to $C$. Then the reduced path $e'\eta\kappa\bar{\eta}\bar{e'}$ witnesses that $e'$ does not satisfy condition (iii) from the previous lemma, so $e'$ is not a treeing edge, a contradiction.

Now let $\xi$ be a geodesic from $C$ to $s(e')$, by the previous claim we know that $\xi e'$ is still a reduced path. Let $c'$ be the initial segment of $c$ which connects $r(e)$ to the source of $\xi$, then $\omega' = \omega c' \xi e'$ is the desired extension of $\omega$. □

## 2.5 HNN extensions

Let $H$ be a group, and let $\vartheta : \Sigma \to \vartheta(\Sigma)$ be an isomorphism between subgroups of $H$. The **HNN extension** associated to these data is the group defined by the following presentation

$$\mathrm{HNN}(H,\Sigma,\vartheta) := \left\langle H, t \,|\, t^{-1}\sigma t = \vartheta(\sigma) \text{ for all } \sigma \in \Sigma \right\rangle , [1]$$

[1]This notation means that $\mathrm{HNN}(H,\Sigma,\vartheta)$ is the quotient of the free product $H * \langle t \rangle$ by its smallest normal subgroup containing all elements $t^{-1}\sigma t\vartheta(\sigma)^{-1}$ where $\sigma \in \Sigma$.

where $t$ is an extra generator, called the stable letter, not belonging to $H$. We refer the reader to [Ser80, Chap. 1, Prop. 5] for the fact that the HNN extension defined above does contain $H$ as a natural subgroup. Note that the defining relation $t^{-1}\sigma t = \vartheta(\sigma)$ is different from the one chosen in [FMS15, FLMM22]. This change is coherent with our choice to let groups act on the right on sets.

We will denote this HNN extension by $\Gamma$. Recall that it is called **ascending** if one of the subgroups $\Sigma, \vartheta(\Sigma)$ is equal to $H$.

Let us fix a set of representatives $C^+$ of left $\Sigma$-cosets in $H$, and a set of representatives $C^-$ of left $\vartheta(\Sigma)$-cosets in $H$, which both contain 1, so that we have

$$H = \bigsqcup_{c\in C^+} c\Sigma = \Sigma \sqcup \bigsqcup_{c\in C^+\setminus\{1\}} c\Sigma \quad \text{and} \quad H = \bigsqcup_{c\in C^-} c\vartheta(\Sigma) = \vartheta(\Sigma) \sqcup \bigsqcup_{c\in C^-\setminus\{1\}} c\vartheta(\Sigma).$$

It is well-known, see e.g. [LS01], that every element $\gamma \in \Gamma$ admits a unique **normal form**

$$\gamma = c_1 t^{\varepsilon_1} \cdots c_n t^{\varepsilon_n} h_{n+1},$$

where $n \geq 0$, $\varepsilon_i = \pm 1$ for $1 \leq i \leq n$, $\varepsilon_i = +1$ implies $c_i \in C^+$, $\varepsilon_i = -1$ implies $c_i \in C^-$, $h_{n+1} \in H$, and there is no subword of the form $t^{\varepsilon} 1 t^{-\varepsilon}$. Note that the case $n = 0$ corresponds to elements in $H$.

The **Bass-Serre tree** of the HNN extension $\Gamma$ is the oriented graph $\mathcal{T}$ defined by

$$V(\mathcal{T}) = \Gamma/H; \quad E(\mathcal{T})^+ = \Gamma/\Sigma; \quad E(\mathcal{T})^- = \Gamma/\vartheta(\Sigma);$$

where the structural maps are given by the following formulas

$$\overline{\gamma\Sigma} = \gamma t \vartheta(\Sigma); \quad s(\gamma\Sigma) = \gamma H; \quad r(\gamma\Sigma) = \gamma t H;$$
$$\overline{\gamma\vartheta(\Sigma)} = \gamma t^{-1}\Sigma; \quad s(\gamma\vartheta(\Sigma)) = \gamma H; \quad r(\gamma\vartheta(\Sigma)) = \gamma t^{-1} H.$$

This graph is naturally endowed with a left $\Gamma$-action by graph automorphisms (respecting the orientation), and classical Bass-Serre theory [Ser80] ensures it is a tree. The action is always minimal since it is transitive on the vertices. Let us now recall what kind of action $\Gamma \curvearrowright \mathcal{T}$ is, depending on the inclusions $\Sigma \subseteq H$ and $\vartheta(\Sigma) \subseteq H$. Note that the stable letter $t$ always induces a hyperbolic automorphism.

- If $\Sigma = H = \vartheta(\Sigma)$, then the Bass-Serre tree is a bi-infinite line (each vertex has degree 2), hence, the action is lineal.
- If $\Sigma = H$ and $\vartheta(\Sigma) \neq H$, then there is exactly one positive edge and several negative edges in the star at each vertex. Hence, to each vertex $v$, one can associate a reduced infinite path $\omega_v^+$ starting at $v$ by taking the unique positive edge at each vertex. Given two vertices $u$ and $v$, the paths $\omega_u^+$ and $\omega_v^+$ share a common terminal subpath. Indeed, this is obvious if $u, v$ are linked by an edge, and then, denoting $v_0, \ldots, v_n$ the vertices on the geodesic between $u$ and $v$, all the paths $\omega_{v_i}^+$ share a common terminal subpath.

  Now, let $\xi \in \partial\mathcal{T}$ be the common endpoint of all paths $\omega_v^+$. Given any hyperbolic element $g \in \Gamma$, and any vertex $v$ in $\mathcal{T}$, we have $g \cdot \omega_v^+ = \omega_{gv}^+$ since $\Gamma$ preserves the orientation, whence $g\xi = \xi$. Therfore, all hyperbolic elements of $\Gamma$ fix $\xi$.

  On the other hand, it is easy to see that $t$ and $c^{-1}tc$ don't have the same axis. Hence, the action is quasi-parabolic.
- Similarly, if $\Sigma \neq H$ and $\vartheta(\Sigma) = H$, then the action is quasi-parabolic.
- If the HNN extension is non-ascending, then taking $h$ in $C^+ \setminus \{1\}$ and $g \in C^- \setminus \{1\}$, it is fairly easy to see that $gt$ and $th$ are transverse hyperbolic elements. Hence, the action is of general type.

## 2.6 Group amalgams

Let $\iota_1 : \Sigma \to \Gamma_1$ and $\iota_2 : \Sigma \to \Gamma_2$ be injective morphisms of countable groups. We will denote by $\Sigma_j$ the image of $\iota_j$, and by $\vartheta : \Sigma_1 \to \Sigma_2$ the isomorphism sending $\iota_1(\sigma)$ to $\iota_2(\sigma)$ for all $\sigma \in \Sigma$. The **free product with amalgamation** (or **amalgam** for short) associated to these data is

$$\Gamma_1 *_\Sigma \Gamma_2 := \langle \Gamma_1, \Gamma_2 \,|\, \iota_1(\sigma) = \iota_2(\sigma) \text{ for all } \sigma \in \Sigma \rangle = \langle \Gamma_1, \Gamma_2 \,|\, \sigma = \vartheta(\sigma) \text{ for all } \sigma \in \Sigma_1 \rangle \,.$$ [2]

We will denote the amalgam $\Gamma_1 *_\Sigma \Gamma_2$ by $\Gamma$. We will still denote by $\Sigma, \Gamma_1, \Gamma_2$ the images of these groups in the amalgam $\Gamma$ when there is no risk of confusion. In $\Gamma$, one has $\Gamma_1 \cap \Gamma_2 = \Sigma$. Recall that such an amalgam is said to be **non-trivial** if $\Gamma_j \neq \Sigma_j$ for $j = 1, 2$, and **non-degenerate** if moreover $[\Gamma_1 : \Sigma_1] \geq 3$ or $[\Gamma_2 : \Sigma_2] \geq 3$.

Let us fix sets of representatives $C_j$ of left $\Sigma_j$-cosets in $\Gamma_j$, for $j = 1, 2$, which both contain 1, so that we have

$$\Gamma_1 = \bigsqcup_{c \in C_1} c\Sigma_1 = \Sigma_1 \sqcup \bigsqcup_{c \in C_1 \setminus \{1\}} c\Sigma_1 \quad \text{and} \quad \Gamma_2 = \bigsqcup_{c \in C_2} c\Sigma_2 = \Sigma_2 \sqcup \bigsqcup_{c \in C_2 \setminus \{1\}} c\Sigma_2 \,.$$

Notice that the intersection of the images of $C_1$ and $C_2$ in $\Gamma$ is just $\{1\}$. It is well-known, see e.g. [Ser80], that any element $\gamma \in \Gamma \setminus \Sigma$ admits a unique **normal form**

$$\gamma = c_1 \cdots c_n \sigma$$

where $n \in \mathbb{N}$, $c_1, \dots, c_n$ lie alternatively in $C_1 \setminus \{1\}$ and $C_2 \setminus \{1\}$, and $\sigma \in \Sigma$.

The **Bass-Serre tree** of the amalgam $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is the oriented graph $\mathcal{T}$ defined by

$$V(\mathcal{T}) = \Gamma/\Gamma_1 \sqcup \Gamma/\Gamma_2 \,; \quad E(\mathcal{T})^+ = \Gamma/\Sigma \,; \quad s(\gamma\Sigma) = \gamma\Gamma_1 \,; \quad r(\gamma\Sigma) = \gamma\Gamma_2$$

(the set of negative edges $E(\mathcal{T})^-$ just being $\overline{\Gamma/\Sigma} := \{\bar{e} : e \in \Gamma/\Sigma\}$, which is another copy of $\Gamma/\Sigma$). Again, this graph is naturally endowed with a left $\Gamma$-action by graph automorphisms (respecting the orientation), and classical Bass-Serre theory [Ser80] ensures it is a tree. The action is always minimal since $\Gamma$ acts transitively on the set of positive edges. Let us now recall what kind of action $\Gamma \curvearrowright \mathcal{T}$ is, depending on the inclusions $\Sigma_j \subseteq \Gamma_j$.

- If $\Sigma_1 = \Gamma_1$, then $\Gamma = \Gamma_2$, and the vertex $\Gamma_2$ of $\mathcal{T}$ is fixed. Hence the action is elliptic. Similarly, if $\Sigma_2 = \Gamma_2$, then the action is elliptic.

- If the amalgam is non-trivial, and $[\Gamma_1 : \Sigma_1] = 2 = [\Gamma_2 : \Sigma_2]$, then the Bass-Serre tree is a bi-infinite line (each vertex has degree 2) and for any $\gamma_j \in \Gamma_j - \Sigma_j$, $j = 1, 2$, the element $\gamma_1\gamma_2$ is hyperbolic. Hence, the action is lineal.

- If the amalgam is non-degenerate, then the action is of general type. Indeed, assuming $[\Gamma_1 : \Sigma] \geq 3$, and taking $g_1 \neq g_2$ in $C_1 \setminus \{1\}$ and $h \in C_2 \setminus \{1\}$, it is fairly easy to see that $g_1 h$ and $g_2 h$ are transverse hyperbolic elements. The case $[\Gamma_2, \Sigma] \geq 3$ is similar.

[2] More precisely, $\Gamma_1 *_\Sigma \Gamma_2$ is the quotient of the free product $\Gamma_1 * \Gamma_2$ by its smallest normal subgroup containing all elements $\iota_1(\sigma)\iota_2(\sigma)^{-1}$ where $\sigma \in \Sigma$.

### 2.7 Partial actions

The *pre-actions* that we will define below are tightly linked with the notion of *partial action*. Although the latter do not play an essential role in our construction, we will see that every pre-action yields a natural partial action, so we feel these are worth mentioning. For more details on partial actions, we refer the reader to [KL04].

Given a set $X$, we denote by $\mathcal{I}(X)$ the set of all partial bijections of $X$, which we think of as subsets of $X\times X$ whose vertical and horizontal fibers all have cardinality at most 1. We have a natural composition law on subsets of $X\times X$ given by: for all $A,B\subseteq X\times X$,

$$AB=\{(x,z)\colon \exists y\in X,(x,y)\in A \text{ and } (y,z)\in B\},$$

and this restricts to a composition law on $\mathcal{I}(X)$. The inclusion provides us with a natural partial order on $\mathcal{I}(X)$. The projection on the first coordinate of a partial bijection $\tau$ is its **domain** $\operatorname{dom}\tau$, and the projection on the second coordinate is its **range** $\operatorname{rng}\tau$. Finally, we define the inversion map by $\sigma^{-1}=\{(y,x):(x,y)\in\sigma\}$.

**Definition 2.18.** A (right) **partial action** of a group $\Gamma$ on a set $X$ is a map $\pi:\Gamma\to\mathcal{I}(X)$ such that for all $g,h\in\Gamma$

(1) $\pi(1_\Gamma)=\mathrm{id}_X$;

(2) $\pi(g)\pi(h)\subseteq\pi(gh)$;

(3) $\pi(g)^{-1}=\pi(g^{-1})$.

The main example of a partial action is provided by the restriction of an action to a subset. Conversely, every partial action is the restriction of a global action, and there is a *universal* such global action provided by the following result.

**Theorem 2.19** (see [KL04, Theorem 3.4])**.** *Given a partial action of a countable group $\Gamma$ on a set $X$, there is a $\Gamma$-action on a larger set $\tilde{X}$ such that whenever $Y\curvearrowleft\Gamma$ is a $\Gamma$-action on a set $Y$ which contains $X$, there is a unique $\Gamma$-equivariant map $f:\tilde{X}\to Y$ which restricts to the identity on $X$.*

The action $\tilde{X}\curvearrowleft\Gamma$ from the previous theorem is called the **universal globalization** of the partial $\Gamma$-action on $X$. It is tacit in the theorem that the $\Gamma$-actions on sets containing $X$ extend the initial partial $\Gamma$-action on $X$.

**Definition 2.20.** A partial action $X\curvearrowleft^{\pi}\Gamma$ is called **strongly faithful** if for every $F\Subset\Gamma\setminus\{1\}$, there is $x\in X$ such that for all $g\in F$, we have $x\pi(g)\neq x$ (in particular $x\in\bigcap_{g\in F}\operatorname{dom}\pi(g)$).

**Example 2.21.** The partial action of the free group on two generator $\mathbb{F}_2$ on the set of reduced words which begin by $a$ is strongly faithful.

## 3 Free globalizations for pre-actions of HNN extensions

For this section, as in Section 2.5, let us fix an HNN extension $\Gamma=\mathrm{HNN}(H,\Sigma,\vartheta)$. Let us also fix a set of representatives $C^+$ of left $\Sigma$-cosets in $H$, and a set of representatives $C^-$ of left $\vartheta(\Sigma)$-cosets in $H$, which both contain 1, so that normal forms of elements of $\Gamma$ are well-defined. Let us also denote by $\Gamma^+$, respectively $\Gamma^-$, the set of elements whose normal form leftmost's letter is $t$, respectively $t^{-1}$. Note that $\Gamma^+$ is invariant by left $\Sigma$-multiplication, while $\Gamma^-$ is invariant by left $\vartheta(\Sigma)$-multiplication. We then have $\Gamma=H\sqcup C^+\Gamma^+\sqcup C^-\Gamma^-$.

### 3.1 Pre-actions of HNN extensions and their Bass-Serre graph

Given an action on an infinite countable set $X \curvearrowleft^{\pi} H$, and a bijection $\tau : X \to X$ such that $\sigma^{\pi}\tau = \tau\vartheta(\sigma)^{\pi}$ for all $\sigma \in \Sigma$, there exists a unique action $X \curvearrowleft^{\pi_\tau} \Gamma$ such that $h^{\pi_\tau} = h^{\pi}$ for all $h \in H$, and $t^{\pi_\tau} = \tau$. If the action $\pi$ is free, we obtain an example of the following situation.

**Definition 3.1.** A **pre-action** of the HNN extension $\Gamma$ is a couple $(X,\tau)$ where $X$ is an infinite countable set endowed with a free action $X \curvearrowleft^{\pi} H$, and

$$\tau : \mathrm{dom}(\tau) \to \mathrm{rng}(\tau)$$

is a partial bijection where $\mathrm{dom}(\tau), \mathrm{rng}(\tau) \subseteq X$, and $\sigma^{\pi}\tau = \tau\vartheta(\sigma)^{\pi}$ for all $\sigma \in \Sigma$.

The relations $\sigma^{\pi}\tau = \tau\vartheta(\sigma)^{\pi}$ in Definition 3.1 are equalities between partial bijections. In particular $\sigma^{\pi}\tau$ and $\tau\vartheta(\sigma)^{\pi}$ must have the same domain and the same range. As a consequence, for any pre-action $(X,\tau)$, the domain of $\tau$ is necessarily $\Sigma$-invariant, its range is necessarily $\vartheta(\Sigma)$-invariant, and $\tau$ sends $\Sigma$-orbits onto $\vartheta(\Sigma)$-orbits.

A pre-action $(X,\tau)$ is called **global** if $\tau$ is a genuine permutation of $X$. In this case there is an associated action $X \curvearrowleft^{\pi_\tau} \Gamma$ as above. We will often identify global pre-actions and $\Gamma$-actions.

**Example 3.2.** If $X \curvearrowleft^{\pi} \Gamma$ is an action, where $H$ is acting freely, then denoting by $X \curvearrowleft^{\pi_H} H$ its restriction, one obtains a global pre-action $(X, t^{\pi})$, where $X$ is endowed with $\pi_H$. The action $X \curvearrowleft^{\pi_\tau} \Gamma$ coincides with $X \curvearrowleft^{\pi} \Gamma$ in this case. In particular, the right translation action $\Gamma \curvearrowleft \Gamma$ gives rise to a pre-action $(\Gamma, t^{\rho})$, where $t^{\rho} : \gamma \mapsto \gamma t$, called the **translation pre-action**.

The above notion of pre-action is close to the notion of a partial action developed in [KL04] as we will see. As seen before, actions of $\Gamma$ (such that $H$ acts freely) correspond to pre-actions with a global bijection. Another source of examples of pre-actions is the following.

**Definition 3.3.** Given a pre-action $(X,\tau)$, and an infinite $H$-invariant subset $Y \subseteq X$, the **restriction** of $(X,\tau)$ to $Y$ is the pre-action $(Y,\tau')$, where $Y$ is endowed with the restriction of $\pi$, and the partial bijection is $\tau' = \tau_{\restriction Y \cap Y\tau^{-1}}$. An **extension** of $(X,\tau)$ is a pre-action $(\tilde{X},\tilde{\tau})$ whose restriction to $X$ is $(X,\tau)$.

**Example 3.4.** The sets $\Gamma^+$ and $\Gamma^-$ are $H$-invariant (by right multiplications), thus so are $T^+ := \Gamma^+ \sqcup H$ and $T^- := \Gamma^- \sqcup H$. The translation pre-action $(\Gamma, t^{\rho})$ admits the restrictions $(T^+, \tau_+)$, and $(T^-, \tau_-)$, which we call the **positive translation pre-action** and the **negative translation pre-action** respectively.

Let us compute the domain and range of the partial bijection $\tau_+$ corresponding to the positive translation pre-action. Let $x \in T^+$. If $x$ belongs to $\Gamma^+$, then so does $xt$, and so $\tau_+(x)$ is defined. But if $x$ belongs to $H$ instead, then $xt \notin H$, and $xt \in \Gamma^+$ if and only if its lefttmost letter is $t$, which happens if and only if $x \in \Sigma$. Reciprocally, one has $yt^{-1} \in T^+$ for every $y \in \Gamma^+$, and $yt^{-1} \notin T^+$ for every $y \in H$. We conclude that the domain of $\tau_+$ is $\Sigma \sqcup \Gamma^+$, while its range is $\Gamma^+$.

The same computation can be made for the partial bijection $\tau_-$ associated to the negative translation pre-action on $T^-$: the domain of $\tau_-$ is equal to $\Gamma^-$, and its range is equal to $\Gamma^- \sqcup \vartheta(\Sigma)$.

Let us now associate a graph to any $\Gamma$-pre-action $(X,\tau)$ as follows. Informally speaking, we start with a graph whose vertices are of two kinds: the $\Sigma$-orbits in $X$, and the $\vartheta(\Sigma)$-orbits in $X$. Then we put an edge from $x\Sigma$ to $y\vartheta(\Sigma)$ when $(x\Sigma)\tau = y\vartheta(\Sigma)$, and finally we identify all the $\Sigma$-orbits and all $\vartheta(\Sigma)$-orbits that are in a same $H$-orbit. We may, and will, identify $H$ with its image in $S(X)$ by the action $X \curvearrowleft^{\pi} H$, since the action $\pi$ is free, hence faithful. Consequently, we don't write superscripts $\pi$ from now, as soon as there is no risk of confusion.

**Definition 3.5.** The **Bass-Serre graph** of $(X,\tau)$ is the oriented graph $\mathcal{G}_\tau$ defined by

$$V(\mathcal{G}_\tau) = X/H\,, \quad E(\mathcal{G}_\tau)^+ = \mathrm{dom}(\tau)/\Sigma\,, \quad E(\mathcal{G}_\tau)^- = \mathrm{rng}(\tau)/\vartheta(\Sigma)\,,$$

where the structural maps are given by the following formulas

$$\overline{x\Sigma} = x\tau\vartheta(\Sigma)\,; \quad s(x\Sigma) = xH\,; \quad r(x\Sigma) = x\tau H\,;$$

$$\overline{y\vartheta(\Sigma)} = y\tau^{-1}\Sigma; \quad s(y\vartheta(\Sigma)) = yH\,; \quad r(y\vartheta(\Sigma)) = y\tau^{-1}H\,.$$

The Bass-Serre graph will also be denoted by $\mathbf{BS}(X,\tau)$.

**Example 3.6.** (1) The Bass-Serre graph of the translation pre-action $(\Gamma, t^\rho)$ is the classical Bass-Serre tree $\mathcal{T}$ of $\Gamma$.

(2) The Bass-Serre graph of the positive translation pre-action $(T^+,\tau_+)$ is the half-tree of the edge $\Sigma$ in $\mathcal{T}$.

(3) The Bass-Serre graph of the negative translation pre-action $(T^-,\tau_-)$ is the half-tree of the edge $\vartheta(\Sigma)$ in $\mathcal{T}$.

(4) The Bass-Serre graph of a global $\Gamma$-pre-action is actually a forest if and only if the associated $\Gamma$-action is free.

Example (1) is obvious. Examples (2) and (3), if not obvious yet, will become clear after Remark 3.12. Example (4) will be seen in Remark 3.17.

Now, let us link the star at a vertex in a Bass-Serre graph $\mathbf{BS}(X,\tau)$ to small normal forms in $\Gamma$. Given a vertex in an oriented graph, let us denote by $\mathrm{st}^+(v)$, respectively $\mathrm{st}^-(v)$, the set of positive, respectively negative, edges whose source is $v$, so that we have a partition $\mathrm{st}(v) = \mathrm{st}^+(v) \sqcup \mathrm{st}^-(v)$ of the star at $v$. Given a point $x \in X$, there are natural (maybe sometimes empty) maps

$$\begin{array}{rccc} e_x^+ : & \{ct : c \in C^+, xc \in \mathrm{dom}(\tau)\} & \to & \mathrm{st}^+(xH) \\ & ct & \mapsto & xc\Sigma \\ e_x^- : & \{ct^{-1} : c \in C^-, xc \in \mathrm{rng}(\tau)\} & \to & \mathrm{st}^-(xH) \\ & ct^{-1} & \mapsto & xc\vartheta(\Sigma) \end{array}$$

and we notice that $e_x^+(ct)$ goes from $xH$ to $xc\tau H$, while $e_x^-(ct^{-1})$ goes from $xH$ to $xc\tau^{-1}H$.

These maps are surjective, since the orbits $xc\Sigma$ for $c \in C^+$, respectively the orbits $xc\vartheta(\Sigma)$ for $c \in C^-$, cover $xH$. Since the action $X \curvearrowleft^\pi H$ is free, we have $xH = \bigsqcup_{c\in C^+} xc\Sigma$ and $xH = \bigsqcup_{c\in C^-} xc\vartheta(\Sigma)$, so that $e_x^+, e_x^-$ are in fact bijective. Then, by merging $e_x^+$ and $e_x^-$, we get a bijection

$$e_x : \{ct : c \in C^+, xc \in \mathrm{dom}(\tau)\} \sqcup \{ct^{-1} : c \in C^-, xc \in \mathrm{rng}(\tau)\} \to \mathrm{st}(xH)\,.$$

## 3.2 Morphisms and functoriality of Bass-Serre graphs

We shall now see that there is a functor, that we will call the Bass-Serre functor, from the category of $\Gamma$-pre-actions to the category of graphs, which extends Definition 3.5. Let us start by turning $\Gamma$-pre-actions into a category.

**Definition 3.7.** A **morphism of pre-actions** from $(X,\tau)$ to $(X',\tau')$ is a $H$-equivariant map $\varphi : X \to X'$, such that for all $x \in \mathrm{dom}\,\tau$, $\varphi(x\tau) = \varphi(x)\tau'$.

Note that in particular, $\varphi$ maps $\mathrm{dom}(\tau)$ into $\mathrm{dom}(\tau')$, and $\mathrm{rng}(\tau)$ into $\mathrm{rng}(\tau')$. Now, given a morphism of pre-actions $\varphi : (X,\tau) \to (X',\tau')$, and denoting by $\mathcal{G}_\tau$ and $\mathcal{G}_{\tau'}$ the corresponding Bass-Serre graphs, let us define a map $V(\mathcal{G}_\tau) \to V(\mathcal{G}_{\tau'})$ by

$$xH \mapsto \varphi(x)H\,, \text{ for } x \in X\,,$$

and a map $E(\mathcal{G}_\tau) \to E(\mathcal{G}_{\tau'})$ by

$$x\Sigma \mapsto \varphi(x)\Sigma,\ \text{for } x \in \mathrm{dom}(\tau) \quad \text{and} \quad y\vartheta(\Sigma) \mapsto \varphi(y)\vartheta(\Sigma)\,,\ \text{for } y \in \mathrm{rng}(\tau)\,.$$

It is routine to check that these maps define a morphism of graphs, that we denote by $\mathcal{G}_\varphi$. For instance, the image of $x\Sigma$ is $\varphi(x)\Sigma$, the image of $\overline{x\Sigma} = x\tau\vartheta(\Sigma)$ is $\varphi(x\tau)\vartheta(\Sigma) = \varphi(x)\tau'\vartheta(\Sigma)$, and one has $\overline{\varphi(x)\Sigma} = \varphi(x)\tau'\vartheta(\Sigma)$ in $\mathcal{G}_{\tau'}$ as expected.

**Lemma 3.8.** *The assignments $(X,\tau) \mapsto \mathcal{G}_\tau$ and $\varphi \mapsto \mathcal{G}_\varphi$ define a functor from the category of $\Gamma$-pre-actions to the category of graphs.*

We will denote this functor by **BS** and call it the **Bass-Serre functor** of $\Gamma$. The morphism $\mathcal{G}_\varphi$ will also be denoted by $\mathbf{BS}(\varphi)$.

*Proof.* First, given the identity morphism on a pre-action $(X,\tau)$ it is obvious that the associated morphism of graphs is the identity on $\mathcal{G}_\tau$.

Now, take two morphisms of pre-actions $\varphi : (X,\tau) \to (X',\tau')$ and $\psi : (X',\tau') \to (X'',\tau'')$. It is also clear that the composition of $\mathcal{G}_\varphi$ followed by $\mathcal{G}_\psi$, and the morphism $\mathcal{G}_{\psi\circ\varphi}$ are both given by the map $V(\mathcal{G}_\tau) \to V(\mathcal{G}_{\tau''})$ by

$$xH \mapsto \psi\circ\varphi(x)H\,, \text{ for } x \in X\,,$$

and the map $E(\mathcal{G}_\tau) \to E(\mathcal{G}_{\tau''})$ by

$$x\Sigma \mapsto \psi\circ\varphi(x)\Sigma,\ \text{for } x \in \mathrm{dom}(\tau) \quad \text{and} \quad y\vartheta(\Sigma) \mapsto \psi\circ\varphi(y)\vartheta(\Sigma)\,,\ \text{for } y \in \mathrm{rng}(\tau)\,.$$

This completes the proof. □

To conclude this section, let us notice a consequence of freeness of the $H$-actions in the definition of $\Gamma$-pre-actions.

**Lemma 3.9.** *Every morphism of the form $\mathbf{BS}(\varphi)$ is locally injective. More precisely, its restriction to the star at a vertex $xH$, is the composition $e_{\varphi(x)} \circ e_x^{-1}$, which is an injection into the star at $\varphi(x)H$.*

*Proof.* Consider a morphism of pre-actions $\varphi : (X,\tau) \to (X',\tau')$, and give names to the actions involved: $X \curvearrowleft^{\pi} H$, and $X' \curvearrowleft^{\pi'} H$. Let us also recall from Section 3.1 that the maps $e_x$ and $e_{\varphi(x)}$ are bijective, since these actions are free. Now, given $x \in X$ and $e \in \mathrm{st}(xH)$ in $\mathcal{G}_\tau$, one must have $e = e_x(ct^\varepsilon)$, that is:

- either $e = xc^\pi\Sigma$ for a unique $c \in C^+$ satisfying $xc^\pi \in \mathrm{dom}(\tau)$,
- or $e = xc^\pi\vartheta(\Sigma)$ for a unique $c \in C^-$ satisfying $xc^\pi \in \mathrm{rng}(\tau)$.

Then, in the graph $\mathcal{G}_{\tau'}$, one has:

- in the first case, $\varphi(x)c^{\pi'} = \varphi(xc^\pi) \in \mathrm{dom}(\tau')$, so that
$$\mathcal{G}_\varphi(e) = \varphi(xc^\pi)\Sigma = \varphi(x)c^{\pi'}\Sigma = e_{\varphi(x)}(ct) = e_{\varphi(x)} \circ e_x^{-1}(e)\,.$$
- in the second case, $\varphi(x)c^{\pi'} = \varphi(xc^\pi) \in \mathrm{rng}(\tau')$, so that
$$\mathcal{G}_\varphi(e) = \varphi(xc^\pi)\vartheta(\Sigma) = \varphi(x)c^{\pi'}\vartheta(\Sigma) = e_{\varphi(x)}(ct^{-1}) = e_{\varphi(x)} \circ e_x^{-1}(e)\,.$$

In other words, the restriction of $\mathcal{G}_\varphi$ to the star at a vertex $xH$ is the composition $e_{\varphi(x)} \circ e_x^{-1}$. Furthermore, the latter map is an injection into the star at $\varphi(x)H$. □

### 3.3 Paths in Bass-Serre graphs of global pre-actions

Let us turn to the case of a global pre-action $(X,\tau)$. In this case, the bijections $e_x$ defined at the end of Section 3.1 become just:

$$e_x : \{ct : c\in C^+\}\sqcup\{ct^{-1} : c\in C^-\}\longrightarrow \mathrm{st}^+(xH)\sqcup\mathrm{st}^-(xH) = \mathrm{st}(xH)\,.$$

Given a point $x\in X$, and an element $\gamma\in\Gamma\setminus H$ with normal form $c_1t^{\varepsilon_1}\cdots c_nt^{\varepsilon_n}h_{n+1}$, where $n\geq 1$, we associate a sequence $(x_0,x_1,\dots,x_n)$ in $X$ by setting $x_0 = x$ and $x_i = x_{i-1}c_i\tau^{\varepsilon_i}$ for $1\leq i\leq n$, and notice that $x_nh_{n+1} = x\gamma^{\pi_\tau}$. Then we associate a sequence $(e_1,\dots,e_n)$ of edges in the Bass-Serre graph using the bijections $e_x$: for $i = 1,\dots,n$, we set

$$e_i = e_{x_{i-1}}(c_it^{\varepsilon_i})\,.$$

Notice that, for any $i = 1,\dots,n-1$, one has $r(e_i) = x_{i-1}c_i\tau^{\varepsilon_i}H = x_iH = s(e_{i+1})$. Hence $(e_1,\dots,e_n)$ is a path, that we denote by $\mathsf{path}_x(\gamma)$.

**Remark 3.10.** The vertices $(v_i)_{i=0}^n$ visited by $\mathsf{path}_x(\gamma)$ are given by $v_i = x_iH$, where:

$$x_i = xc_1\tau^{\varepsilon_1}\cdots c_i\tau^{\varepsilon_i}.$$

Moreover, defining $\Sigma_1 = \Sigma$ and $\Sigma_{-1} = \vartheta(\Sigma)$ one has $e_1 = xc_1\Sigma_{\varepsilon_1}$ and, for all $2\leq k\leq n$,

$$e_k = xc_1t^{\varepsilon_1}\dots c_{k-1}t^{\varepsilon_{k-1}}c_k\Sigma_{\varepsilon_k}.$$

Now, for $1\leq i\leq n$, let us remark the equivalence

$$e_{i+1} = \bar{e}_i \;\Leftrightarrow\; (\varepsilon_{i+1} = -\varepsilon_i \text{ and } c_{i+1} = 1)\,.$$

Indeed, in case $\varepsilon_i = 1$, one has $e_i = x_{i-1}c_i\Sigma$, therefore

$$e_{i+1} = \bar{e}_i \;\Leftrightarrow\; e_{x_i}(c_{i+1}t^{\varepsilon_{i+1}}) = x_{i-1}c_i\tau\cdot\vartheta(\Sigma) = x_i\vartheta(\Sigma) \;\Leftrightarrow\; (\varepsilon_{i+1} = -1 \text{ and } c_{i+1} = 1)$$

and the case $\varepsilon_i = -1$ is similar. As we started with a normal form of $\gamma$, we obtain that $\mathsf{path}_x(\gamma)$ is a reduced path. Moreover, given a reduced path $(e'_1,\dots,e'_n)$ starting at $xH$, one has $\mathsf{path}_x(\gamma) = (e'_1,\dots,e'_n)$ if and only if

$$\text{for all } i = 1,\dots,n,\qquad e_{x_{i-1}}(c_it^{\varepsilon_i}) = e'_i\,.$$

Since the maps $e_x$ are bijective, there is exactly one element $\gamma\in\Gamma\setminus H$, with normal form $c_1t^{\varepsilon_1}c_2t^{\varepsilon_2}\cdots c_nt^{\varepsilon_n}$ such that $\mathsf{path}_x(\gamma) = (e'_1,\dots,e'_n)$.

**Remark 3.11.** For any $x\in X$, the map $\mathsf{path}_x$, from $\Gamma\setminus H$ to the set of reduced paths starting at the vertex $xH$, is surjective. Its restriction to the set of elements with normal form $c_1t^{\varepsilon_1}c_2t^{\varepsilon_2}\cdots c_nt^{\varepsilon_n}$ is bijective.

Let us say that $\gamma\in\Gamma\setminus H$ is a **path-type element** if its normal form has the form $t^{\varepsilon_1}c_2t^{\varepsilon_2}\cdots c_nt^{\varepsilon_n}$ where $n\geq 1$, that is, if $c_1 = 1$ and $h_{n+1} = 1$. It is said to be **positive** if $\varepsilon_1 = 1$, and **negative** if $\varepsilon_1 = -1$. When $\gamma$ is a positive, respectively negative, path-type element, the first edge of $\mathsf{path}_x(\gamma)$ is $x\Sigma$, respectively $x\vartheta(\Sigma)$. If $n\leq k$, we also say that an element $\tilde{\gamma}$ with normal form $t^{\varepsilon_1}c_2t^{\varepsilon_2}\cdots c_kt^{\varepsilon_k}$ is a **path-type extension** of $\gamma = t^{\varepsilon_1}c_2t^{\varepsilon_2}\cdots c_nt^{\varepsilon_n}$. In this case, $\mathsf{path}_x(\tilde{\gamma})$ extends $\mathsf{path}_x(\gamma)$.

**Remark 3.12.** The map $\mathsf{path}_x$ induces bijections:

- between the subset of positive path-type elements in $\Gamma$, and the set of reduced paths in $\mathbf{BS}(X,\tau)$ whose first edge is $x\Sigma$;
- between the subset of negative path-type elements in $\Gamma$, and the set of reduced paths in $\mathbf{BS}(X,\tau)$ whose first edge is $x\vartheta(\Sigma)$.

Hence, if $x\Sigma$ (respectively $x\vartheta(\Sigma)$) is a treeing edge then, the images of the first (respectively the second) bijection cover exactly the half-tree of $\Sigma$ (respectively the half-graph of $\vartheta(\Sigma)$) in $\mathbf{BS}(X,\tau)$.

Let us end this section by linking paths in Bass-Serre trees and Bass-Serre graphs so as to understand which edges are treeing edges in the Bass-Serre graph.

**Remark 3.13.** Consider a global pre-action $(X,\tau)$, and a basepoint $x\in X$. There exists a unique morphism of pre-actions

$$\varphi:(\Gamma,t^{\rho})\to(X,\tau)$$

from the translation pre-action, such that $\varphi(1)=x$. In fact, $\varphi$ is the orbital map $\gamma\mapsto x\gamma^{\pi_\tau}$ of the associated $\Gamma$-action. By restriction, one obtains morphisms

$$\varphi_+:(T^+,\tau_+)\to(X,\tau)$$
$$\varphi_-:(T^-,\tau_-)\to(X,\tau)$$

from the positive and negative translation pre-actions.

**Lemma 3.14.** *In the context of the above remark, the Bass-Serre morphism $\mathbf{BS}(\varphi)$, from the Bass-Serre tree $\mathcal{T}$ to the Bass-Serre graph $\mathcal{G}_\tau$, sends $\mathsf{path}_{1_\Gamma}^{\mathcal{T}}(\gamma)$ onto $\mathsf{path}_x^{\mathcal{G}_\tau}(\gamma)$.*

*Proof.* Let us consider $\gamma\in\Gamma\setminus H$, and write its normal form: $\gamma=c_1t^{\varepsilon_1}\cdots c_nt^{\varepsilon_n}h_{n+1}$. Let us denote by $(e_1,\dots,e_n)$ the edges of $\mathsf{path}_{1_\Gamma}^{\mathcal{T}}(\gamma)$, and by $(e'_1,\dots,e'_n)$ the edges of $\mathsf{path}_x^{\mathcal{G}_\tau}(\gamma)$. The auxiliary sequences in $\Gamma$ and $X$ used in the construction of the paths will be denoted by $(\gamma_0,\dots,\gamma_n)$ and $(x_0,\dots,x_n)$ respectively.

An easy induction shows that $x_i=\varphi(\gamma_i)$ for all $i=0,\dots,n$. Then, we notice that the source of $e_{\gamma_{i-1}}(c_it^{\varepsilon_i})$ is $\gamma_{i-1}H$ for all $i=1,\dots,n$. Thus, using Lemma 3.9, we get

$$\mathcal{G}_\varphi(e_i)=e_{\varphi(\gamma_{i-1})}\circ e_{\gamma_{i-1}}^{-1}\left(e_{\gamma_{i-1}}(c_it^{\varepsilon_i})\right)=e_{x_{i-1}}(c_it^{\varepsilon_i})=e'_i$$

for all $i=1,\dots,n$. □

Therefore, if $x\Sigma$ is a treeing edge then, the image of $\mathbf{BS}(\varphi_+)$ is the half-tree of $x\Sigma$, while if $x\vartheta(\Sigma)$ is a treeing edge then, the image of $\mathbf{BS}(\varphi_-)$ is the half-tree of $x\vartheta(\Sigma)$.

**Proposition 3.15.** *Consider a global pre-action $(X,\tau)$, and a basepoint $x\in X$. The following are equivalent:*

*(i) the morphism of pre-actions $\varphi_+:(T^+,\tau_+)\to(X,\tau)$ of Remark 3.13 is injective;*

*(ii) the morphism of graphs $\mathbf{BS}(\varphi_+)$ is injective;*

*(iii) the edge $x\Sigma$ in the Bass-Serre graph $\mathbf{BS}(X,\tau)$ is a treeing edge.*

*Proof.* For all $\gamma \in T^+$, recall that $\varphi_+(\gamma) = x\gamma^{\pi_\tau}$, so that $\mathbf{BS}(\varphi_+)$ sends vertices $\gamma H$ to $x\gamma^{\pi_\tau}H$. At the level of positive edges, it sends $\gamma\Sigma$ to $x\gamma^{\pi_\tau}\Sigma$. Fixing $\gamma$, we get $\varphi_+(\gamma h) = x\gamma^{\pi_\tau}h^\pi$ for $h \in H$; since $X \curvearrowleft^\pi H$ is free, $\varphi_+$ realizes a bijection between $\gamma H$ and $x\gamma^{\pi_\tau}H$, and also a bijection between $\gamma\Sigma$ and $x\gamma^{\pi_\tau}\Sigma$. Consequently, $\varphi_+$ is injective if an only if $\gamma H \mapsto x\gamma^{\pi_\tau}H$ and $\gamma\Sigma \mapsto x\gamma^{\pi_\tau}\Sigma$ are both injective. This proves that (i) and (ii) are equivalent. Note that (iii) implies (ii) is obvious since, when $x\Sigma$ is a treeing edge $\mathbf{BS}(\varphi_+)$ is locally injective from the half-tree of $\Sigma$ to the half-tree of $x\Sigma$, hence $\mathbf{BS}(\varphi_+)$ is injective. Finally assume (ii) and let $\omega$ be a reduced path starting by the edge $x\Sigma$. By Remark 3.12 there exists a positive path type element $\gamma \in \Gamma^+$ such that $\omega = \mathsf{path}_x(\gamma)$. By Lemma 3.14, $\omega$ is the image by $\mathbf{BS}(\varphi_+)$ of $\mathsf{path}_1^{\mathcal{T}}(\gamma)$. Since $\mathbf{BS}(\varphi_+)$ is supposed to by injective and since the last vertex of $\mathsf{path}_1^{\mathcal{T}}(\gamma)$ is not $H$, we deduce that the last vertex of $\omega$ is not $xH$. Hence, $x\Sigma$ is a treeing edge by Lemma 2.16. □

By a very similar argument, wet get also the following result.

**Proposition 3.16.** *Consider a global pre-action $(X,\tau)$, and a basepoint $x \in X$. The following are equivalent:*

*(i) the morphism of pre-actions $\varphi_- : (T^-,\tau_-) \to (X,\tau)$ of Remark 3.13 is injective;*

*(ii) the morphism of graphs $\mathbf{BS}(\varphi_-)$ is injective;*

*(iii) the edge $x\vartheta(\Sigma)$ in the Bass-Serre graph $\mathbf{BS}(X,\tau)$ is a treeing edge.*

**Remark 3.17.** Putting the two previous propositions together, one can show that given a $\Gamma$-action where $H$ is acting freely, the Bass-Serre graph of the associated pre-action is a forest if and only if the $\Gamma$-action is free.

### 3.4 The free globalization of a pre-action of an HNN extension

Say that a pre-action is **transitive** when its Bass-Serre graph is connected. Note that a global pre-action $(X,\tau)$ is transitive if and only if the associated $\Gamma$-action is transitive. We will show that every transitive pre-action has a canonical extension to a transitive action, which is **as free as possible**. The construction is better described in terms of Bass-Serre graph: we are going to attach as many treeing edges as possible to it.

**Theorem 3.18.** *Every transitive $\Gamma$-pre-action $(X,\tau)$ on a non-empty set $X$ admits a transitive and global extension $(\tilde{X},\tilde{\tau})$ which satisfies the following universal property: given any transitive and global extension $(Y,\tau')$ of $(X,\tau)$, there is a unique morphism of pre-actions $\varphi : (\tilde{X},\tilde{\tau}) \to (Y,\tau')$ such that*

$$\varphi_{\restriction X} = \mathrm{id}_X .$$

*Moreover, all the edges from the Bass-Serre graph $\mathbf{BS}(X,\tau)$ to its complement in $\mathbf{BS}(\tilde{X},\tilde{\tau})$ are treeing edges.*

In terms of $\Gamma$-actions, the extension $(\tilde{X},\tilde{\tau})$ of the theorem corresponds to an action $\tilde{X} \curvearrowleft \Gamma$ such that, given any action $Y \curvearrowleft^\alpha \Gamma$ satisfying $X \subseteq Y$ as $H$-sets and $yt^\alpha = y\tau$ for all $y \in \mathrm{dom}(\tau)$, there exists a unique $\Gamma$-equivariant map $\varphi : \tilde{X} \to Y$ extending $\mathrm{id}_X$.

*Proof.* We will obtain the Bass-Serre graph of the pre-action $(\tilde{X},\tilde{\tau})$ by adding only treeing edges to the Bass-Serre graph of the pre-action.

First we enumerate the $\Sigma$-orbits which do not belong to the domain of $\tau$ as $(x_i\Sigma)_{i\in I}$. Then, we take disjoint copies of $(T^+,\tau_+)$, for $i \in I$, also disjoint from $X$, which we denote as $(T_i^+,\tau_i)$. Similarly, we enumerate the $\vartheta(\Sigma)$-orbits which do not belong to the range of $\tau$ as $(y_j\vartheta(\Sigma))_{j\in J}$. We then take disjoint copies of $(T^-,\tau_-)$, for $j\in J$, also disjoint from $X$, which we denote as $(T_j^-,\tau_j)$. Now, our extension $(\tilde{X},\tilde{\tau})$ is obtained as follows. We set

$$\tilde{X}=\left(X\sqcup\bigsqcup_{i\in I}T_i^+\sqcup\bigsqcup_{j\in J}T_j^-\right)\Big/_{\sim}$$

where $\sim$ identifies the element $x_ih\in X$ with $h\in H\subset T_i^+$, for each $i\in I$ and $h\in H$, and the element $x_jh\in X$ with $h\in H\subset T_j^-$, for each $j\in J$ and $h\in H$. Since the identifications just glue some orbits pointwise and respect the $H$-actions, $\tilde{X}$ is endowed with a free $H$-action. Then, we set

$$\tilde{\tau}=\tau\sqcup\bigsqcup_{i\in I}\tau_i\sqcup\bigsqcup_{j\in J}\tau_j\,,$$

which is possible since the domain of $\tau_i$, for $i\in I$, intersects other components in $\tilde{X}$ only in the orbit $x_i\Sigma$, the range of $\tau_i$, for $i\in I$, does not intersect other components in $\tilde{X}$, and the situation is analogue for $\tau_j$ with $j\in J$. We have got a pre-action $(\tilde{X},\tilde{\tau})$.

This pre-action is transitive. Indeed, all pre-actions $(X,\tau)$, $(T_i^+,\tau_i)$ and $(T_j^-,\tau_j)$ are, and the identifications make connections between all these components in the Bass-Serre graph $\mathbf{BS}(\tilde{X},\tilde{\tau})$.

The pre-action is also global. Indeed, every $\Sigma$-orbit, respectively $\vartheta(\Sigma)$-orbit, in $T_i^+$, which is not in the domain, respectively the range, of $\tau_i$, has been identified with an orbit in $X$, and the situation is similar for $T_j^-$. We conclude by noting that all $\Sigma$-orbits and $\vartheta(\Sigma)$-orbits in $X$ are now in the domain and in the range of $\tilde{\tau}$.

Moreover, the (oriented) edges from the Bass-Serre graph $\mathbf{BS}(X,\tau)$ to its complement in $\mathbf{BS}(\tilde{X},\tilde{\tau})$ are exactly the edges $x_i\Sigma$ for $i\in I$, and the edges $x_j\vartheta(\Sigma)$ for $j\in J$. For each $i\in I$, the morphism of pre-actions $\varphi_+:(T^+,\tau_+)\to(\tilde{X},\tilde{\tau})$ of Remark 3.13, with basepoint $x_i\in\tilde{X}$, is injective since it realizes an isomorphism onto $(T_i^+,\tau_i)$, hence $x_i\Sigma$ is a treeing edge by Proposition 3.15. One proves similarly that the edges $x_j\vartheta(\Sigma)$ are treeing edges using Proposition 3.16.

It now remains to prove the universal property. To do so, take any transitive and global extension $(Y,\tau')$ of $(X,\tau)$. The unique morphism of pre-actions $\varphi$ from $(\tilde{X},\tilde{\tau})$ to $(Y,\tau')$ such that $\varphi_{\restriction X}=\mathrm{id}_X$ is obtained by taking the union of the morphisms $\varphi_i:(T_i^+,\tau_i)\to(Y,\tau')$ and $\varphi_j:(T_j^-,\tau_j)\to(Y,\tau')$, coming from Remark 3.13 with respect to basepoints $x_i$ or $x_j$, with $\mathrm{id}_X$ (all these morphisms are unique). □

It is straightforward to deduce from the universal property above that the global pre-action we just built is unique up to isomorphism. We thus call it **the free globalization** of the pre-action $(X,\tau)$.

**Example 3.19.** The free globalizations of the positive and negative translation pre-action are equal to the right $\Gamma$-action on itself by translation. Indeed, this is true of any transitive pre-action obtained as a restriction of the (global) $\Gamma$-pre-action on itself by right translation, since the latter is universal among transitive $\Gamma$-actions.

Let us furthermore observe that we can always build this pre-action on a fixed set $\bar{X}$ containing $X$, provided it contains infinitely many free $H$-orbits.

**Theorem 3.20.** *Let $\bar{X}$ be a countable set equipped with a free $H$-action, suppose $X\subseteq\bar{X}$ is $H$-invariant and $\bar{X}\setminus X$ contains infinitely many $H$-orbits. Suppose further that $\tau$ is a partial bijection on $X$ such that $(X,\tau)$ is a transitive pre-action of $\Gamma$. Then there is a permutation $\bar{\tau}$ of $\bar{X}$ such that $(\bar{X},\bar{\tau})$ is (isomorphic to) the free globalization of $(X,\tau)$.*

*Proof.* Let $(\tilde{X}, \tilde{\tau})$ be the free globalization of $(X, \tau)$. The fact that $\tilde{X} \setminus X$ contains infinitely many $H$-orbits and is countable implies that there exist a $H$-equivariant bijection $\varphi : \tilde{X} \to \bar{X}$ whose restriction to $X$ is the identity. Then, one can push forward the permutation $\tilde{\tau}$, to obtain a permutation $\bar{\tau} : \bar{X} \to \bar{X}$ defined by

$$\varphi(x)\bar{\tau} := \varphi(x\tilde{\tau}) \quad \text{for all } x \in \tilde{X},$$

which extends $\tau$. Now, $\varphi$ is an isomorphism of pre-actions between $(\tilde{X}, \tilde{\tau})$ and $(\bar{X}, \bar{\tau})$. □

## 3.5 Connection with partial actions and strong faithfulness

**Definition 3.21.** Given a transitive pre-action of the HNN extension $\Gamma$ on $(X, \tau)$, let us denote by $(\tilde{X}, \tilde{\tau})$ its free globalization. The **partial action associated** to $(X, \tau)$ is the restriction to $X$ of the action $\tilde{X} \curvearrowleft^{\pi_{\tilde{\tau}}} \Gamma$. We denote it by $\alpha_\tau$.

In order to have shorter statements in what follows, we will also call the action $\tilde{X} \curvearrowleft^{\pi_{\tilde{\tau}}} \Gamma$ "free globalization" of $(X, \tau)$.

**Remark 3.22.** One could also construct the partial action directly as follows. Let us denote by $\pi$ the $H$-action on $X$. Given $\gamma \in \Gamma$ with normal form $c_1 t^{\varepsilon_1} \cdots c_n t^{\varepsilon_n} h_{n+1}$, define the partial bijection $\gamma^{\alpha_\tau}$ by

$$\gamma^{\alpha_\tau} := c_1^\pi \tau^{\varepsilon_1} \cdots c_n^\pi \tau^{\varepsilon_n} h_{n+1}^\pi,$$

where we compose partial bijections as described in Section 2.7. The relation $\gamma_1^{\alpha_\tau} \gamma_2^{\alpha_\tau} \subseteq (\gamma_1\gamma_2)^{\alpha_\tau}$ follows from the fact that in order to obtain the normal form of $\gamma_1\gamma_2$ from the concatenation of the normal forms of $\gamma_1$ and $\gamma_2$, one only needs to iterate the following three types of operations:

(1) replacing a subword $h_1 \sigma t h_2$ by $h_1 t \vartheta(\sigma) h_2$;

(2) replacing a subword $h_1 \vartheta(\sigma) t^{-1} h_2$ by $t^{-1} \sigma h_2$;

(3) deleting the occurrences of $tt^{-1}$ or $t^{-1}t$.

By the definition of a pre-action, types (1) and (2) do not affect the partial bijection that we get in the end, while type (3) can only produce extensions (note that $\tau\tau^{-1}$ and $\tau^{-1}\tau$ are *restrictions* of the identity on $X$).

We can now connect the *free* globalization that we constructed to the *universal* globalization of Kellendonk-Lawson that was presented in Theorem 2.19.

**Proposition 3.23.** *The free globalization of a transitive pre-action $(X, \tau)$ is equal to the universal globalization of the partial action $\alpha_\tau$.*

*Proof.* There is a unique $\Gamma$-equivariant map $g$ from the universal globalization $Z$ constructed by Kellendonk-Lawson to the free globalization $\tilde{X}$ because of its universal property. Moreover, in the free globalization, we have that $H$ acts freely, so it follows that $H$ is also acting freely on the universal globalization.

It is then straightforward to check that since the pre-action $\tau$ is transitive, the associated partial action $\alpha_\tau$ is transitive, i.e. for every $x, y \in X$ there is $\gamma \in \Gamma$ such that $y = x\gamma^{\alpha_\tau}$. Since the $\Gamma$-closure of $X$ inside $Z$ satisfies the same universal property as $Z$, we conclude that the universal globalization is transitive.

We can thus apply the universal property of the free globalization from Theorem 3.18 so as to obtain a unique $\Gamma$-equivariant map $f : \tilde{X} \to Z$ which restricts to the identity on $X$. Recalling that $g : Z \to \tilde{X}$ is the unique $\Gamma$-equivariant map provided by Kellendonk and Lawson's theorem, we conclude by uniqueness that both $g \circ f$ and $f \circ g$ are identity maps, which concludes the proof. □

An important property of the partial action associated to a pre-action $(X,\tau)$ is that it is *contained* in the partial action associated to any of the globalizations of $(X,\tau)$, in the following sense.

**Definition 3.24.** Let $X \curvearrowleft^{\alpha} \Gamma$ and $X \curvearrowleft^{\beta} \Gamma$ be two partial actions. We say that $\alpha$ is **contained** in $\beta$ when for every $\gamma\in\Gamma$, we have $\alpha(\gamma)\subseteq\beta(\gamma)$.

**Proposition 3.25.** *Let $(X,\tau)$ be a transitive $\Gamma$-pre-action, let $X \curvearrowleft^{\alpha_\tau} \Gamma$ be the associated partial action. Then for every global extension $(Y,\sigma)$ of the pre-action, we have that the restriction to $X$ of the action $Y \curvearrowleft^{\pi_\sigma} \Gamma$ contains $\alpha_\tau$.*

*Proof.* Suppose $(Y,\sigma)$ is a global extension of the pre-action $(X,\tau)$. Since $(X,\tau)$ is transitive, up to shrinking $(Y,\sigma)$ we may assume it it transitive. We can now apply Theorem 3.18: the universal property gives a $\Gamma$-equivariant map $\rho:\tilde{X}\to Y$ with respect to the actions $\tilde{\pi}:=\pi_{\tilde{\tau}}$ and $\pi_\sigma$. Then, for every $x\in\tilde{X}$, and every $\gamma\in\Gamma$, we have $\rho(x\gamma^{\tilde{\pi}})=\rho(x)\gamma^{\pi_\sigma}$. In particular, for every $x\in X$ such that $x\gamma^{\pi_\sigma}\in X$, we obtain $x\gamma^{\alpha_\tau}=x\gamma^{\tilde{\pi}}=x\gamma^{\pi_\sigma}$, which yields directly the desired result. □

Let us now show that the free globalization of any non-global transitive pre-action is highly faithful. We start with a lemma.

**Lemma 3.26.** *The partial actions associated to the positive and negative translation pre-action are strongly faithful.*

*Proof.* By Example 3.19, the partial action associated to the positive translation pre-action is the restriction to $T^+=\Gamma^+\sqcup H$ of the action $\Gamma\curvearrowleft\Gamma$ by right translations. Thus, it suffices to show that for every $F\Subset\Gamma\setminus\{1\}$, there exists $x\in\Gamma^+\sqcup H$ such that for all $\gamma\in F$, we have $x\gamma\in\Gamma^+\sqcup H$ and $x\gamma\neq x$. The latter assertion is always true, and the former holds if we take $x\in\Gamma^+$ whose normal form is longer than the normal form of all the elements of $F$. We conclude that the partial action associated to the positive translation pre-action is strongly faithful.

A similar argument shows that the partial action associated to the negative translation pre-action is strongly faithful. □

**Proposition 3.27.** *The free globalization of any non-global transitive pre-action is highly faithful.*

*Proof.* Let $\pi$ be the action associated to the free globalization of a non-global transitive pre-action $(X,\tau)$. By Corollary 2.6, it suffices to prove that $\pi$ is strongly faithful. But since the pre-action $(X,\tau)$ is not global, its free globalization contains a copy of either the positive or the negative translation pre-action. The partial actions of the latter being strongly faithful by the previous lemma, we conclude using Proposition 3.25 that $\pi$ itself is strongly faithful because it contains a strongly faithful partial action. □

**Remark 3.28.** More generally, it follows from Propositions 3.15 and 3.16 that every $\Gamma$-action whose Bass-Serre graph contains a treeing edge must be highly faithful.

## 4 High transitivity for HNN extensions

As in Section 3, we fix an HNN extension $\Gamma=\mathrm{HNN}(H,\Sigma,\vartheta)$, a set of representatives $C^+$ of left $\Sigma$-cosets in $H$, and a set of representatives $C^-$ of left $\vartheta(\Sigma)$-cosets in $H$, which both contain 1, so that normal forms of elements of $\Gamma$ are well-defined. From now on, **we assume that the HNN extension $\Gamma$ is non-ascending** and that **the $\Gamma$-action on the boundary of its Bass-Serre tree is topologically free**, since these assumptions will become essential. We still denote by $\Gamma^+$, respectively $\Gamma^-$, the set of elements whose normal form first letter is $t$, respectively $t^{-1}$, so that we have $\Gamma=H\sqcup C^+\Gamma^+\sqcup C^-\Gamma^-$.

### 4.1 Using the free globalization towards high transitivity

This section is devoted to a key result which we will use towards proving high transitivity for HNN extensions. It will allow us to extend any given transitive pre-action which is not an action to one which sends one fixed tuple to another fixed tuple.

**Proposition 4.1.** *Let $\bar{X}$ be a countable set, let $\bar{X} \curvearrowleft^{\pi} H$ be a free action, with infinitely many $H$-orbits, let $X$ be a finite union of $H$-orbits in $\bar{X}$, and suppose that $(X,\tau)$ is a transitive non-global pre-action. For any pairwise distinct points $x_1,\dots,x_k,y_1,\dots,y_k \in \bar{X}$, there exists a transitive and global extension $(\bar{X},\tilde{\tau})$ of $(X,\tau)$ such that:*

*(1) the action $\bar{X} \curvearrowleft^{\pi_{\tilde{\tau}}} \Gamma$ is (transitive and) highly faithful;*

*(2) there is an element $\gamma \in \Gamma$ satisfying $x_i\gamma^{\pi_{\tilde{\tau}}} = y_i$ for all $i$.*

*Proof.* The set $\{x_1,\dots,x_k,y_1,\dots,y_k\}$ will be denoted by $F$. First, by Theorem 3.20, we find a permutation $\bar{\tau} \in S(\bar{X})$ such that $(\bar{X},\bar{\tau})$ is the free globalization of $(X,\tau)$. Given $x \in \bar{X}$, and a path-type element $\gamma$, we will denote by $\mathcal{H}_x(\gamma)$ the half-graph of the last edge of $\mathsf{path}_x(\gamma)$ in the Bass-Serre graph of $(\bar{X},\bar{\tau})$.

**Claim.** *There exists a path-type element $\gamma$ in $\Gamma \setminus H$ such that for every $x \in F$, the last edge of $\mathsf{path}_x(\gamma)$ is a treeing edge (that is, $\mathcal{H}_x(\gamma)$ is a tree).*

*Proof of the claim.* Using the correspondence between path-type elements and reduced paths established in Section 3.3, it follows from Lemma 2.17 that for every $x \in \bar{X}$, and every path-type element $\gamma$, there is a path-type extension $\gamma'$ of $\gamma$ such that the last edge of $\mathsf{path}_x(\gamma')$ is a treeing edge. Now, it suffices to start with any path-type element $\gamma_0$, to extend it to a path-type element $\gamma_1$ such that the last edge of $\mathsf{path}_{x_1}(\gamma_1)$ is a treeing edge, then to extend $\gamma_1$ to a path-type element $\gamma_2$ such that the last edge of $\mathsf{path}_{y_1}(\gamma_2)$ is a treeing edge (note that $\mathsf{path}_{x_1}(\gamma_2)$ also ends with a treeing edge since it extends $\mathsf{path}_{x_1}(\gamma_1)$), ..., and to iterate this extension procedure until we reach an element $\gamma_{2k}$ such that all last edges of $\mathsf{path}_x(\gamma_{2k})$, for $x \in F$, are treeing edges. $\square_{\text{claim}}$

Let us fix some element $c \in C^+ \setminus \{1\}$ (here we use that $\Sigma \neq H$). Then, any path-type element $\gamma$ admits $\gamma ct$ as a path-type extension.

**Claim.** *There exists a path-type element $\gamma$ in $\Gamma \setminus H$ such that for every $x \in F$, the last edge of $\mathsf{path}_x(\gamma)$ is a treeing edge, and the half-trees $\mathcal{H}_x(\gamma)$, for $x \in F$, are pairwise disjoint subgraphs, and disjoint from* $\mathbf{BS}(X,\tau)$.

*Proof of the claim.* We start with a path-type element $\gamma$ such that for every $x \in F$, the last edge of $\mathsf{path}_x(\gamma)$ is a treeing edge. Since $X$ is a finite union of $H$-orbits, $\mathbf{BS}(X,\tau)$ has finitely many vertices. Hence, by extending further the path-type element $\gamma$, we may and will assume that, for every $x \in F$, the half-tree $\mathcal{H}_x(\gamma)$ does not intersect $\mathbf{BS}(X,\tau)$.

We now notice that, given $x,y \in F$, if the half-trees $\mathcal{H}_x(\gamma)$ and $\mathcal{H}_y(\gamma)$ are disjoint, then so are the half-trees $\mathcal{H}_x(\gamma')$ and $\mathcal{H}_y(\gamma')$ for every path-type extension $\gamma'$ of $\gamma$. Hence, it suffices to prove that, for any $x,y \in F$, with $x \neq y$ and such that $\mathcal{H}_x(\gamma)$ and $\mathcal{H}_y(\gamma)$ intersect, there exists a path-type extension $\gamma'$ of $\gamma$ such that $\mathcal{H}_x(\gamma')$ and $\mathcal{H}_y(\gamma')$ are disjoint. Indeed, an easy induction gives then an extension $\gamma^{(n)}$ such that the half-trees $\mathcal{H}_x(\gamma^{(n)})$, for $x \in F$, are pairwise disjoint.

Take now $x,y \in F$ with $x \neq y$ and such that $\mathcal{H}_x(\gamma)$ and $\mathcal{H}_y(\gamma)$ intersect. These half-trees have to be nested. Indeed, if they aren't, $\mathcal{H}_x(\gamma)$ contains the antipode of the last edge of $\mathsf{path}_y(\gamma)$, hence contains $\mathbf{BS}(X,\tau)$, which is impossible. Without loss of generality, we assume $\mathcal{H}_x(\gamma) \subseteq \mathcal{H}_y(\gamma)$. We now distinguish two cases.

- If $\mathcal{H}_x(\gamma) \subsetneq \mathcal{H}_y(\gamma)$, there is a path-type extension $\gamma''$ of $\gamma$, with $\gamma'' \neq \gamma$, such that $\mathsf{path}_x(\gamma)$ and $\mathsf{path}_y(\gamma'')$ have the same last edge. Since the HNN extension $\Gamma$ is non-ascending, there is another path-type extension $\gamma'$ of $\gamma$ with the same length as $\gamma''$. Now, $\mathsf{path}_y(\gamma')$ and $\mathsf{path}_y(\gamma'')$ are distinct reduced paths extending $\mathsf{path}_y(\gamma)$, hence $\mathcal{H}_y(\gamma')$ and $\mathcal{H}_y(\gamma'') = \mathcal{H}_x(\gamma)$ have to be disjoint. Since $\mathcal{H}_x(\gamma') \subset \mathcal{H}_x(\gamma)$, the half-trees $\mathcal{H}_y(\gamma')$ and $\mathcal{H}_x(\gamma')$ are disjoint.

- If $\mathcal{H}_x(\gamma) = \mathcal{H}_y(\gamma)$, then $\mathsf{path}_x(\gamma)$ and $\mathsf{path}_y(\gamma)$ have the same terminal edge. Let us assume that $\mathsf{path}_x(\gamma ct)$ and $\mathsf{path}_y(\gamma ct)$ have the same terminal edge, since otherwise we are done with $\gamma' = \gamma ct$. This edge is $e := x'c\Sigma = y'c\Sigma$, where $x' = x\gamma^{\pi_{\bar\tau}}$ and $y' = y\gamma^{\pi_{\bar\tau}}$. Consequently, one has $y'c = x'c\sigma$ for some $\sigma \in \Sigma \setminus \{1\}$. Consider now the morphism from the translation pre-action

$$\varphi : (\Gamma, t^{\rho}) \to (\bar X, \bar\tau), \qquad \text{with basepoint } x'c$$

coming from Remark 3.13, and note that the left translation $\psi_\sigma : \gamma^* \mapsto \sigma\gamma^*$ defines an automorphism of $(\Gamma, t^{\rho})$. Using Lemma 3.14, one sees that, for any $\gamma^* \in \Gamma^+$:

  - $\mathbf{BS}(\psi_\sigma)$ maps $\mathsf{path}_1^{\mathcal{T}}(\gamma^*)$ onto $\mathsf{path}_\sigma^{\mathcal{T}}(\gamma^*)$ in the Bass-Serre tree $\mathcal{T}$, and these paths both start by the edge $\Sigma$ hence, they are both in the half-tree of $\Sigma$.
  - $\mathbf{BS}(\varphi)$ maps $\mathsf{path}_1^{\mathcal{T}}(\gamma^*)$ onto $\mathsf{path}_{x'c}(\gamma^*)$; and $\mathsf{path}_\sigma^{\mathcal{T}}(\gamma^*)$ onto $\mathsf{path}_{y'c}(\gamma^*)$.

  Since the left $\Gamma$-action on the boundary $\partial\mathcal{T}$ of its Bass-Serre tree is topologically free, the identity is the only element of $\Gamma$ fixing the half-tree of $\Sigma$ in $\mathcal{T}$ pointwise. Hence, there exists a path $\omega$ in this half-tree whose first edge is $\Sigma$, and such that $\omega$ and $\sigma\cdot\omega$ have distinct ranges. Then, by Remark 3.12, there exists a path-type element $\gamma^+ \in \Gamma^+$ such that $\mathsf{path}_1^{\mathcal{T}}(\gamma^+) = \omega$. We have $\mathsf{path}_\sigma^{\mathcal{T}}(\gamma^+) = \mathbf{BS}(\psi_\sigma)(\omega) = \sigma\cdot\omega$, so that $\mathsf{path}_1^{\mathcal{T}}(\gamma^+)$ and $\mathsf{path}_\sigma^{\mathcal{T}}(\gamma^+)$ have distinct ranges.

  Since the edge $e$ is a treeing edge, the restriction $\varphi_+$ of $\varphi$ to the positive translation pre-action $(T^+, \tau_+)$ is injective by Proposition 3.15, and so is $\mathbf{BS}(\varphi_+)$. Consequently, $\mathsf{path}_{x'c}(\gamma^+)$ and $\mathsf{path}_{y'c}(\gamma^+)$ diverge at some point in the half-tree of $e$ in $\mathbf{BS}(\bar X, \bar\tau)$.

  Finally, for any path-type element $\gamma^*$ in $\Gamma^+$, by construction, $\mathsf{path}_x(\gamma c\gamma^*)$ is the concatenation of $\mathsf{path}_x(\gamma)$ and $\mathsf{path}_{x'c}(\gamma^*)$, and $\mathsf{path}_y(\gamma c\gamma^*)$ is the concatenation of $\mathsf{path}_y(\gamma)$ and $\mathsf{path}_{y'c}(\gamma^*)$. Hence $\mathsf{path}_x(\gamma c\gamma^+)$ and $\mathsf{path}_y(\gamma c\gamma^+)$ diverge at some point in the half-tree of $e$. Setting $\gamma' = \gamma c\gamma^+$, we obtain that $\mathcal{H}_x(\gamma')$ and $\mathcal{H}_y(\gamma')$ are disjoint.

We are done in both cases. $\square_{\text{claim}}$

We then modify the bijection $\bar\tau$ to get the pre-action $(\bar X, \tilde\tau)$ we are looking for. First, given an element $\gamma$ as in the previous claim, we consider, for each $z \in F$, the morphism of pre-actions from the negative translation pre-action coming from Remark 3.13:

$$\psi_z : (T^-, \tau_-) \to (\bar X, \bar\tau) \quad \text{with basepoint } z'c\,,$$

where $z' := z\gamma^{\pi_{\bar\tau}}$. Note that the image of this morphism corresponds to the half-graph opposite to the half-tree $\mathcal{H}_z(\gamma ct)$. Then, we define $X' = \bigcap_{z\in F} \operatorname{rng}(\psi_z) \subset \bar X$, and take the restriction $(X', \tau')$ of $(\bar X, \bar\tau)$. Informally speaking, we erase $\bar\tau$ on the $\Sigma$-orbits corresponding to edges in the half-trees $\mathcal{H}_z(\gamma ct)$ for $z \in F$. Note that this leaves infinitely many $H$-orbits in $\bar X$ outside $\operatorname{dom}(\tau')$ and $\operatorname{rng}(\tau')$, and the pre-action $(X', \tau')$ is transitive. Now, we extend $\tau'$. Pick some orbits $z_1H, \ldots, z_kH$ in $\bar X \setminus (\operatorname{dom}(\tau') \cup \operatorname{rng}(\tau'))$, add them to $X'$, take some $c^- \in C^- \setminus \{1\}$ (here we use that $\vartheta(\Sigma) \neq H$), and set

$$x_i'c\sigma\tau' := z_i\vartheta(\sigma) \quad \text{and} \quad y_i'c\sigma\tau' := z_ic^-\vartheta(\sigma)$$

for $i = 1,\dots,k$ and $\sigma \in \Sigma$. This is possible since the $\vartheta(\Sigma)$-orbits at points $z_i$ and $z_i c^-$ are pairwise disjoint (we use again the freeness of the $H$-action), and since the $\Sigma$-orbits at $x_i' c$ and $y_i' c$ were not initially in the domain of $\tau'$. Note that, after this extension, the pre-action $(X', \tau')$ is still transitive. Then we apply Theorem 3.20 to get an extension $\tilde{\tau} : \bar{X} \to \bar{X}$ of $\tau'$ such that $(\bar{X}, \tilde{\tau})$ is the free globalization of $(X', \tau')$. A computation shows then that $x_i(\gamma c t c^-(\gamma c t)^{-1})^{\pi_{\tilde{\tau}}} = y_i$ for all $i = 1,\dots,k$. Finally, the action $\pi_{\tilde{\tau}}$ is highly faithful by Proposition 3.27. □

## 4.2 High transitivity for HNN extensions

From now on, we fix a free action $X \curvearrowleft^{\pi} H$ with infinitely many orbits. We then consider the space $\mathbf{A}$ of $\Gamma$-actions on $X$ which extend $\pi$, which can be written as

$$\mathbf{A} = \{\tau \in S(X) : \sigma^{\pi}\tau = \tau\vartheta(\sigma)^{\pi} \text{ for all } \sigma \in \Sigma\}.$$

In other words, $\mathbf{A}$ is the set of permutations $\tau$ of $X$ such that $(X, \tau)$ is a global pre-action of $\Gamma$. The set $\mathbf{A}$ is clearly a closed subset of $S(X)$ for the topology of pointwise convergence, hence a Polish space. Recall that the action associated to a permutation $\tau \in \mathbf{A}$ is denoted by $\pi_\tau$.

**Definition 4.2.** Let us set

$$\begin{aligned} \mathbf{TA} &= \{\tau \in \mathbf{A} \colon \pi_\tau \text{ is transitive}\}; \\ \mathbf{HFA} &= \{\tau \in \mathbf{A} \colon \pi_\tau \text{ is highly faithful}\}; \\ \mathbf{HTA} &= \{\tau \in \mathbf{A} \colon \pi_\tau \text{ is highly transitive}\}. \end{aligned}$$

The subset $\mathbf{TA}$ is not closed for the topology of pointwise convergence. However, we have the following result.

**Lemma 4.3.** *The set* $\mathbf{TA}$ *is* $G_\delta$ *in* $\mathbf{A}$*, hence it is a Polish space. Moreover, it is non-empty.*

*Proof.* Since $X \curvearrowleft^{\pi} H$ has infinitely many orbits, there is an $H$-equivariant bijection $\varphi : \Gamma \to X$. It then suffices to push-forward the translation pre-action by $\varphi$ to get an element of $\mathbf{TA}$ (its Bass-Serre graph will be isomorphic to the classical Bass-Serre tree and $\pi_\tau$ will be conjugated to the translation action $\Gamma \curvearrowleft \Gamma$).

To show that $\mathbf{TA}$ is a $G_\delta$ subset, we write $\mathbf{TA} = \bigcap_{x,y\in X} O_{x,y}$, where for $x, y \in X$,

$$O_{x,y} = \{\tau \in \mathbf{A} : \text{ there exists } \gamma \in \Gamma \text{ such that } x\gamma^{\pi_\tau} = y\}.$$

The latter sets are clearly open in $\mathbf{A}$ for all $x, y \in X$, thus finishing the proof. □

We now show that our HNN extension $\Gamma$ has a highly transitive highly faithful action, thus proving Theorem C.

**Theorem 4.4.** *The set* $\mathbf{HTA} \cap \mathbf{HFA}$ *is dense* $G_\delta$ *in* $\mathbf{TA}$*. In particular,* $\Gamma$ *admits actions which are both highly transitive and highly faithful.*

*Proof.* For $k \geq 1$ and $x_1,\dots x_k, y_1,\dots,y_k \in X$ pairwise distinct, consider the open subsets

$$V_{x_1,\dots,x_k,y_1,\dots,y_k} = \{\tau \in \mathbf{TA} \colon \exists \gamma \in \Gamma,\, x_i\gamma^{\pi_\tau} = y_i \text{ for all } 1 \leq i \leq k\}$$

Their intersection is the set $\mathbf{HTA}$ by Lemma 2.2, so $\mathbf{HTA}$ is $G_\delta$ in $\mathbf{TA}$. Similarly, we can write the set of strongly faithful actions as the intersection over all finite subsets $F \Subset \Gamma \setminus \{1\}$ of the open sets

$$W_F = \{\tau \in \mathbf{TA} \colon \exists x \in X\,,\, xf^{\pi_\tau} \neq x \text{ for all } f \in F\}$$

Now, since strong faithfulness is equivalent to high faithfulness by Corollary 2.6, the set **HFA** is $G_\delta$ in **TA**.

To conclude, it suffices to show that each set $(V_{x_1,\dots,x_k,y_1,\dots,y_k}) \cap \mathbf{HFA}$ is dense in **TA**, since this immediately implies that each open set $(V_{x_1,\dots,x_k,y_1,\dots,y_k}) \cap W_F$ is dense in **TA**. To do this, let $\tau \in \mathbf{TA}$ and let $F'$ be a finite subset of $X$. Consider a finite connected subgraph $\mathcal{G}$ of $\mathbf{BS}(X,\tau)$ containing the edges $z\Sigma$ for $z \in F'$, and denote by $\tau_0$ the restriction of $\tau$ to the union of the $\Sigma$-orbits in $X$ corresponding to the edges of $\mathcal{G}$.

Then, apply Proposition 4.1 to the transitive pre-action $(\mathrm{dom}(\tau_0)\cdot H \cup \mathrm{rng}(\tau_0)\cdot H, \tau_0)$, whose Bass-Serre graph is $\mathcal{G}$, to get an extension $\tau'$ such that $\tau' \in V_{x_1,\dots,x_k,y_1,\dots,y_k} \cap \mathbf{HFA}$. Moreover, since $F' \subset \mathrm{dom}(\tau_0)$, it follows that $\tau$ and $\tau'$ coincide on $F'$. □

**Remark 4.5.** We give below a direct proof that $\mathbf{HFA} \cap \mathbf{HTA}$ is dense in **TA**, without relying on Baire's Theorem.

*Proof.* Start with an element $\tau_0 \in \mathbf{TA}$, consider the transitive and global pre-action $(X,\tau_0)$, and fix a finite subset $F_0 \Subset X$. What we have to prove is that there exists $\tau' \in \mathbf{HFA} \cap \mathbf{HTA}$ such that the restrictions of $\tau_0$ and $\tau'$ on $F_0$ coincide.

Let us now take an enumeration $(g_n)_{n\geq 0}$ of $\Gamma \setminus \{1\}$, and an enumeration $(k_n,\bar{x}_n,\bar{y}_n)_{n\geq 0}$ of the set of triples $(k,\bar{x},\bar{y})$, where $k$ is a positive integer and $\bar{x} = (x_1,\dots,x_k), \bar{y} = (y_1,\dots,y_k)$ are $k$-tuples of elements of $X$ such that $x_1,\dots,x_k,y_1,\dots,y_k$ are pairwise distinct. Starting with $\tau_0$ and $F_0$, we construct inductively a sequence $(\tau_n)_{n\geq 0}$ in **TA**, and an increasing sequence $(F_n)_{n\geq 0}$ of finite subsets of $X$, as follows.

1. Starting with $F_n \Subset X$ and $\tau_n \in \mathbf{TA}$, we set $F$ to be the union of $F_n$ and the coordinates of $\bar{x}_n$ and $\bar{y}_n$, and consider the smallest connected subgraph $\mathcal{G}$ of $\mathbf{BS}(X,\tau_n)$ which contains all edges $z\Sigma$ and $z\vartheta(\Sigma)$ for $z \in F$. (Notice that $\mathbf{BS}(X,\tau_n)$ is connected, since $\tau_n \in \mathbf{TA}$).

2. We take the restriction $\tau$ of $\tau_n$ on the union of the $\Sigma$-orbits in $X$ corresponding to edges in $\mathcal{G}$, and get a transitive pre-action $(\mathrm{dom}(\tau)\cdot H \cup \mathrm{rng}(\tau)\cdot H, \tau)$, whose Bass-Serre graph is $\mathcal{G}$, and such that $\tau^{\pm 1}$ coincides with $\tau_n^{\pm 1}$ on $F$.

3. By Proposition 4.1, we get an extension $\tau_{n+1}$ of $\tau$, which lies in $\mathbf{TA} \cap \mathbf{HFA}$, and an element $\gamma_n \in \Gamma$ such that $\bar{x}_n\gamma_n^{\pi_{\tau_{n+1}}} = \bar{y}_n$. Moreover, $\tau_{n+1}^{\pm 1}$ coincides with $\tau_n^{\pm 1}$ on $F_n$ by construction. Let also $v_n$ be an element of $X$ such that $v_n g_k^{\pi_{\tau_{n+1}}} \neq v_n$ for all $k = 0,\dots,n$ (which exists since $\pi_{\tau_{n+1}}$ is highly faithful).

4. We take a finite subset $F_{n+1} \Subset X$ which contains $F$, and all elements $z \in X$ such that $z\Sigma$, or its antipode, is in $\mathsf{path}_{v_n}(g_k)$ for some $k \leq n$, or in $\mathsf{path}_u(\gamma_n)$ for some coordinate $u$ of $\bar{x}_n$. Now, for any $\tau^*$ coinciding with $\tau_{n+1}$ on $F_{n+1}$, one has $\bar{x}_n\gamma_n^{\pi_{\tau^*}} = \bar{y}_n$ and $v_n g_k^{\pi_{\tau^*}} \neq v_n$ for all $k = 0,\dots,n$.

Theses sequences satisfy $\bar{x}_m\gamma_m^{\pi_{\tau_n}} = \bar{y}_m$ and $v_m g_k^{\pi_{\tau_n}} \neq v_m$ for all $0 \leq k \leq m < n$. Moreover, the subsets $F_n$ exhaust $X$, and $\tau_n^{\pm 1}$ coincides with $\tau_m^{\pm 1}$ on $F_m$ for all $n > m$. Consequently the sequence $(\tau_n)$ converges to a bijection $\tau'$ and the action $\pi_{\tau'}$ is highly transitive by Lemma 2.2. Notice finally that $\pi_{\tau'}$ is also strongly faithful, since it satisfies $v_m g_k^{\pi_{\tau'}} \neq v_m$ for all $k \leq m$. Thus, $\pi_{\tau'}$ is highly faithful by Corollary 2.6. □

# 5 Free globalization for pre-actions of amalgams

We now turn to the case of amalgams, where the notion of pre-action is a bit less intuitive than for HNN extensions since it will involve two sets, reflecting the fact that the corresponding graph of groups has two vertices. The results and proofs are very similar to those we have proved for HNN extensions in Sections 3 and 4, but for the convenience of the reader we will give full proofs.

For this section, as in Section 2.6, let us fix an amalgam $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$, and sets of representatives $C_j$ of left $\Sigma_j$-cosets in $\Gamma_j$ such that $1 \in C_j$, for $j = 1,2$, so that normal forms of elements of $\Gamma$ are well-defined. Let us also denote by $N_{C_j}$ the set of elements of $\Gamma$ whose normal form begins (from the left) with an element of $C_j \setminus \{1\}$, for $j = 1,2$, so that we have $\Gamma = \Sigma \sqcup N_{C_1} \sqcup N_{C_2}$.

## 5.1 Actions and pre-actions of amalgams, and Bass-Serre graphs

Given two actions on infinite countable sets $X_1 \curvearrowleft^{\pi_1} \Gamma_1$ and $X_2 \curvearrowleft^{\pi_2} \Gamma_2$, and a bijection $\tau : X_1 \to X_2$ such that $\sigma^{\pi_1}\tau = \tau\vartheta(\sigma)^{\pi_2}$ for all $\sigma \in \Sigma_1$, there exists a unique action $X_1 \curvearrowleft^{\pi_{1,\tau}} \Gamma$ such that $g^{\pi_{1,\tau}} = g^{\pi_1}$ for all $g \in \Gamma_1$, and $h^{\pi_{1,\tau}} = \tau h^{\pi_2}\tau^{-1}$ for all $h \in \Gamma_2$. Similarly, there exists a unique action $X_2 \curvearrowleft^{\pi_{2,\tau}} \Gamma$ such that $h^{\pi_{2,\tau}} = h^{\pi_2}$ for all $h \in \Gamma_2$, and $g^{\pi_{2,\tau}} = \tau^{-1}g^{\pi_1}\tau$ for all $g \in \Gamma_1$. Of course, these actions are conjugate: one has $\gamma^{\pi_{2,\tau}} = \tau\gamma^{\pi_{1,\tau}}\tau^{-1}$ for every $\gamma \in \Gamma$. Turning back to the general case, if the actions $\pi_1, \pi_2$ are free, we obtain an example of the following situation.

**Definition 5.1.** A **pre-action** of the amalgam $\Gamma$ is a triple $(X_1, X_2, \tau)$ where $X_1, X_2$ are infinite countable sets endowed with free actions $X_1 \curvearrowleft^{\pi_1} \Gamma_1$ and $X_2 \curvearrowleft^{\pi_2} \Gamma_2$, and

$$\tau : \mathrm{dom}(\tau) \to \mathrm{rng}(\tau)$$

is a partial bijection such that $\mathrm{dom}(\tau) \subseteq X_1$, $\mathrm{rng}(\tau) \subseteq X_2$, and $\sigma^{\pi_1}\tau = \tau\vartheta(\sigma)^{\pi_2}$ for all $\sigma \in \Sigma_1$.

The relations $\sigma^{\pi_1}\tau = \tau\vartheta(\sigma)^{\pi_2}$ are equalities between partial bijections. In particular $\sigma^{\pi_1}\tau$ and $\tau\vartheta(\sigma)^{\pi_2}$ must have the same domain and the same range. As a consequence, for any pre-action $(X_1, X_2, \tau)$, the domain of $\tau$ is necessarily $\Sigma_1$-invariant, its range is necessarily $\Sigma_2$-invariant, and $\tau$ sends $\Sigma_1$-orbits onto $\Sigma_2$-orbits.

A pre-action $(X_1, X_2, \tau)$ is called **global** if $\tau$ is a global bijection between $X_1$ and $X_2$. In this case there are associated actions $X_1 \curvearrowleft^{\pi_{1,\tau}} \Gamma$ and $X_2 \curvearrowleft^{\pi_{2,\tau}} \Gamma$ as above.

**Example 5.2.** If $X \curvearrowleft^{\pi} \Gamma$ is an action, where $\Gamma_1$ and $\Gamma_2$ are acting freely, then denoting by $X \curvearrowleft^{\pi_1} \Gamma_1$ and $X \curvearrowleft^{\pi_2} \Gamma_2$ its restrictions, one obtains a global pre-action $(X, X, \mathrm{id}_X)$, where the first copy of $X$ is endowed with $\pi_1$ and the second with $\pi_2$. The actions $X_1 \curvearrowleft^{\pi_{1,\tau}} \Gamma$ and $X_2 \curvearrowleft^{\pi_{2,\tau}} \Gamma$ both coincide with $X \curvearrowleft^{\pi} \Gamma$ in this case. In particular, the right translation action $\Gamma \curvearrowleft \Gamma$ gives rise to a pre-action $(\Gamma, \Gamma, \mathrm{id})$, called the (right) **translation pre-action**.

As seen before, actions of $\Gamma$ (such that the factors act freely) correspond to pre-actions with a global bijection. Another source of examples of pre-actions is the following.

**Definition 5.3.** Given a pre-action $(X_1, X_2, \tau)$, and infinite $\Gamma_j$-invariant subsets $Y_j \subseteq X_j$, the **restriction** of $(X_1, X_2, \tau)$ to $(Y_1, Y_2)$ is the pre-action $(Y_1, Y_2, \tau')$, where $Y_j$ is endowed with the restrictions of $\pi_j$, and the partial bijection is $\tau' = \tau_{\restriction Y_1 \cap Y_2\tau^{-1}}$. An **extension** of $(X_1, X_2, \tau)$ is a pre-action $(\tilde{X}_1, \tilde{X}_2, \tilde{\tau})$ whose restriction to $(X_1, X_2)$ is $(X_1, X_2, \tau)$.

In the following important example of restrictions, for $j \in \{1,2\}$ we denote by $N_{C_j}$ the set of elements of $\Gamma$ whose normal form begins with an element of $C_j$, so that we have $\Gamma = \Sigma \sqcup N_{C_1} \sqcup N_{C_2}$.

**Example 5.4.** The set $N_{C_2}$ is $\Gamma_1$-invariant by right multiplication, and $N_{C_1}$ is $\Gamma_2$-invariant by right multiplication. Thus, by taking complements, $\Sigma \sqcup N_{C_1}$ is $\Gamma_1$-invariant, and $\Sigma \sqcup N_{C_2}$ is $\Gamma_2$-invariant. The translation pre-action $(\Gamma, \Gamma, \mathrm{id})$ admits the restrictions $(\Gamma_1 \sqcup N_{C_2}, \Gamma_2 \cup N_{C_2}, \tau_+)$, and $(\Gamma_1 \cup N_{C_1}, \Gamma_2 \sqcup N_{C_1}, \tau_-)$, which we call the **positive translation pre-action** and the **negative translation pre-action** respectively. Notice that $\tau_+ = \mathrm{id}_{\restriction \Sigma \sqcup N_{C_2}}$ and $\tau_- = \mathrm{id}_{\restriction \Sigma \sqcup N_{C_1}}$.

Let us now associate a graph to any $\Gamma$-pre-action $(X_1, X_2, \tau)$ as follows. Informally speaking, we start with a graph whose vertices are of two kinds: the $\Sigma_1$-orbits in $X_1$, and the $\Sigma_2$-orbits in $X_2$. Then we put an edge from $x\Sigma_1$ to $y\Sigma_2$ when $(x\Sigma_1)\tau = y\Sigma_2$, and finally we identify all the $\Sigma_j$-orbits that are in a same $\Gamma_j$-orbit, for $j = 1,2$. We may, and will, identify the groups $\Gamma_j, \Sigma_j$ with their images by $\pi_j$, since the actions $\pi_1, \pi_2$ are free, hence faithful. Consequently, we don't write superscripts $\pi_1, \pi_2$ from now, as soon as there is no risk of confusion.

**Definition 5.5.** The **Bass-Serre graph** of $(X_1, X_2, \tau)$ is the oriented graph $\mathcal{G}_\tau$ defined by

$$V(\mathcal{G}_\tau) = X_1/\Gamma_1 \sqcup X_2/\Gamma_2\,, \quad E(\mathcal{G}_\tau)^+ = \mathrm{dom}(\tau)/\Sigma_1\,, \quad E(\mathcal{G}_\tau)^- = \mathrm{rng}(\tau)/\Sigma_2\,,$$

where the structural maps are given by the following formulas, for $x \in \mathrm{dom}(\tau)$ and $y \in \mathrm{rng}(\tau)$,

$$\overline{x\Sigma_1} = x\tau\Sigma_2\,; \quad s(x\Sigma_1) = x\Gamma_1\,; \quad r(x\Sigma_1) = x\tau\Gamma_2\,;$$

$$\overline{y\Sigma_2} = y\tau^{-1}\Sigma_1\,; \quad s(y\Sigma_2) = y\Gamma_2\,; \quad r(y\Sigma_2) = y\tau^{-1}\Gamma_1\,.$$

The Bass-Serre graph will also be denoted by $\mathbf{BS}(X_1, X_2, \tau)$.

**Example 5.6.** (1) The Bass-Serre graph of the translation pre-action $(\Gamma, \Gamma, \mathrm{id})$ is the classical Bass-Serre tree $\mathcal{T}$ of $\Gamma$.

(2) The Bass-Serre graph of the positive translation pre-action $(\Gamma_1 \sqcup N_{C_2}, \Gamma_2 \cup N_{C_2}, \tau_+)$ is the half-tree of the edge $\Sigma_1$ in $\mathcal{T}$.

(3) The Bass-Serre graph of the negative translation pre-action $(\Gamma_1 \cup N_{C_1}, \Gamma_2 \sqcup N_{C_1}, \tau_-)$ is the half-tree of the edge $\Sigma_2$ in $\mathcal{T}$.

Given points $x \in X_1, y \in X_2$, there are natural (maybe sometimes empty) maps from some coset representatives to the stars at $x\Gamma_1$ and $y\Gamma_2$:

$$\begin{array}{rccc} e_{1,x}: & \{c \in C_1 : xc \in \mathrm{dom}(\tau)\} & \longrightarrow & \mathrm{st}(x\Gamma_1) \\ & c & \longmapsto & xc\Sigma_1 \\ e_{2,y}: & \{c \in C_2 : yc \in \mathrm{rng}(\tau)\} & \longrightarrow & \mathrm{st}(y\Gamma_2) \\ & c & \longmapsto & yc\Sigma_2 \end{array}$$

These maps are are surjective, since, for $j = 1,2$, the orbits $xc\Sigma_j$, for $c \in C_j$, cover $x\Gamma_j$. Moreover, since the actions $X_j \curvearrowleft^{\pi_j} \Gamma_j$ are free, we have $x\Gamma_j = \bigsqcup_{c \in C_j} xc\Sigma_j$, so that these maps are in fact bijective.

If $x \in X_1 \cap X_2$, then by merging $e_{1,x}$ and $e_{2,x}$, we get a bijection

$$e_x : \{c \in C_1 : xc \in \mathrm{dom}(\tau)\} \sqcup \{c \in C_2 : xc \in \mathrm{rng}(\tau)\} \to \mathrm{st}(x\Gamma_1) \sqcup \mathrm{st}(x\Gamma_2)\,.$$

We also set $e_x = e_{1,x}$ when $x \in X_1 \setminus X_2$, and $e_x = e_{2,x}$ when $x \in X_2 \setminus X_1$.

## 5.2 Morphisms and functoriality of Bass-Serre graphs

We shall now see that there is a functor, that we will call the Bass-Serre functor, from the category of $\Gamma$-pre-actions to the category of graphs, which extends Definition 5.5. Let us start by turning $\Gamma$-pre-actions into a category.

**Definition 5.7.** A **morphism of pre-actions** from $(X_1, X_2, \tau)$ to $(X_1', X_2', \tau')$ is a couple $(\varphi_1, \varphi_2)$, where $\varphi_j : X_j \to X_j'$ is a $\Gamma_j$-equivariant map for $j = 1,2$, and for all $x \in \operatorname{dom}\tau$, $\varphi_2(x\tau) = \varphi_1(x)\tau'$.

Again, we have in particular that $\varphi_1$ maps $\operatorname{dom}(\tau)$ into $\operatorname{dom}(\tau')$ and $\varphi_2$ maps $\operatorname{rng}(\tau)$ into $\operatorname{rng}(\tau')$.

Now, given a morphism of pre-actions $(\varphi_1, \varphi_2) : (X_1, X_2, \tau) \to (X_1', X_2', \tau')$, and denoting by $\mathcal{G}_\tau$ and $\mathcal{G}_{\tau'}$ the corresponding Bass-Serre graphs, let us define a map $V(\mathcal{G}_\tau) \to V(\mathcal{G}_{\tau'})$ by

$$x\Gamma_1 \mapsto \varphi_1(x)\Gamma_1\,,\ \text{for } x \in X_1 \quad \text{and} \quad y\Gamma_2 \mapsto \varphi_2(y)\Gamma_2\,,\ \text{for } y \in X_2\,,$$

and a map $E(\mathcal{G}_\tau) \to E(\mathcal{G}_{\tau'})$ by

$$x\Sigma_1 \mapsto \varphi_1(x)\Sigma_1\,,\ \text{for } x \in \operatorname{dom}(\tau) \quad \text{and} \quad y\Sigma_2 \mapsto \varphi_2(y)\Sigma_2\,,\ \text{for } y \in \operatorname{rng}(\tau)\,.$$

It is routine to check that these maps define a morphism of graphs, that we denote by $\mathcal{G}_{(\varphi_1,\varphi_2)}$. For instance, the image of $x\Sigma_1$ is $\varphi_1(x)\Sigma_1$, the image of $\overline{x\Sigma_1} = x\tau\Sigma_2$ is $\varphi_2(x\tau)\Sigma_2 = \varphi_1(x)\tau'\Sigma_2$, and one has $\overline{\varphi_1(x)\Sigma_1} = \varphi_1(x)\tau'\Sigma_2$ in $\mathcal{G}_{\tau'}$.

**Lemma 5.8.** *The assignments $(X_1, X_2, \tau) \mapsto \mathcal{G}_\tau$ and $(\varphi_1, \varphi_2) \mapsto \mathcal{G}_{(\varphi_1,\varphi_2)}$ define a functor from the category of $\Gamma$-pre-actions to the category of graphs.*

We will denote this functor by **BS** and call it the **Bass-Serre functor** of $\Gamma$. The morphism $\mathcal{G}_{(\varphi_1,\varphi_2)}$ will also be denoted by $\mathbf{BS}(\varphi_1, \varphi_2)$.

*Proof.* First, given the identity morphism on a pre-action $(X_1, X_2, \tau)$ it is obvious that the associated morphism of graphs id the identity on $\mathcal{G}_\tau$.

Now, let us consider two morphisms of pre-actions $(\varphi_1, \varphi_2) : (X_1, X_2, \tau) \to (X_1', X_2', \tau')$ and $(\psi_1, \psi_2) : (X_1', X_2', \tau') \to (X_1'', X_2'', \tau'')$. It is also clear that the composition of $\mathcal{G}_{(\varphi_1,\varphi_2)}$ followed by $\mathcal{G}_{(\psi_1,\psi_2)}$, and the morphism $\mathcal{G}_{(\psi_1\circ\varphi_1,\psi_2\circ\varphi_2)}$ are both given by the map $V(\mathcal{G}_\tau) \to V(\mathcal{G}_{\tau'})$ by

$$x\Gamma_1 \mapsto \psi_1 \circ \varphi_1(x)\Gamma_1\,,\ \text{for } x \in X_1 \quad \text{and} \quad y\Gamma_2 \mapsto \psi_2 \circ \varphi_2(y)\Gamma_2\,,\ \text{for } y \in X_2\,,$$

and the map $E(\mathcal{G}_\tau) \to E(\mathcal{G}_{\tau'})$ by

$$x\Sigma_1 \mapsto \psi_1 \circ \varphi_1(x)\Sigma_1\,,\ \text{for } x \in \operatorname{dom}(\tau) \quad \text{and} \quad y\Sigma_2 \mapsto \psi_2 \circ \varphi_2(y)\Sigma_2\,,\ \text{for } y \in \operatorname{rng}(\tau)\,.$$

This completes the proof. □

Let us notice a consequence of freeness of the $\Gamma_j$-actions in the definition of $\Gamma$-pre-actions, analogous to Lemma 3.9.

**Lemma 5.9.** *Every morphism of the form $\mathbf{BS}(\varphi_1, \varphi_2) = \mathcal{G}_{(\varphi_1,\varphi_2)}$ is locally injective. More precisely, the restriction of $\mathbf{BS}(\varphi_1, \varphi_2)$ to the star at a vertex $x\Gamma_1$, respectively $y\Gamma_2$, is the composition $e_{1,\varphi_1(x)} \circ e_{1,x}^{-1}$, respectively $e_{2,\varphi_2(y)} \circ e_{2,y}^{-1}$, which is an injection into the star at $\varphi_1(x)\Gamma_1$, respectively $\varphi_2(y)\Gamma_2$.*

*Proof.* Consider a morphism of pre-actions $(\varphi_1,\varphi_2):(X_1,X_2,\tau)\to(X_1',X_2',\tau')$, and give names to the $\Gamma_1$-actions involved: $X_1 \curvearrowleft^{\pi_1} \Gamma_1$, and $X_1' \curvearrowleft^{\pi_1'} \Gamma_1$. Let us also recall from Section 5.1 that maps of the form $e_{1,x}$ and $e_{2,y}$ are bijective, since the $\Gamma_j$-actions are free. Now, given $x\in X_1$ and $e\in \mathrm{st}(x\Gamma_1)$ in $\mathcal{G}_\tau$, one has $e=e_{1,x}(c)=xc^{\pi_1}\Sigma_1$ for a unique $c\in C_1$ satisfying $xc^{\pi_1}\in\mathrm{dom}(\tau)$. Then, one has $\varphi(x)c^{\pi_1'}=\varphi(xc^{\pi_1})\in\mathrm{dom}(\tau')$, so that in $\mathcal{G}_{\tau'}$:

$$\mathcal{G}_{(\varphi_1,\varphi_2)}(e)=\varphi_1(xc^{\pi_1})\Sigma_1=\varphi_1(x)c^{\pi_1'}\Sigma_1=e_{1,x\varphi_1}(c)\,.$$

In other words, the restriction of $\mathcal{G}_{(\varphi_1,\varphi_2)}$ to the star at $x\Gamma_1$ is the composition $e_{1,\varphi_1(x)}\circ e_{1,x}^{-1}$. Furthermore, this map is an injection into the star at $\varphi_1(x)\Gamma_1$.

Similarly, one can prove that the restriction of $\mathcal{G}_{(\varphi_1,\varphi_2)}$ to the star at a vertex $y\Gamma_2$ is the composition $e_{2,\varphi_2(y)}\circ e_{2,y}^{-1}$, which is an injection into the star at $\varphi_2(y)\Gamma_2$. □

### 5.3 Paths in Bass-Serre graphs of global pre-actions

Let us turn to the case of a global pre-action $(X_1,X_2,\tau)$. In this case, the bijections $e_{1,x}$ and $e_{2,y}$, defined at the end of Section 5.1, become just

$$e_{1,x}:C_1\longrightarrow \mathrm{st}(x\Gamma_1)\quad\text{and}\quad e_{2,y}:C_2\longrightarrow \mathrm{st}(y\Gamma_2)\,.$$

Given a point $x\in X_1$ and an element $\gamma\in N_{C_2}$ with normal form $\gamma=c_1\cdots c_n\sigma$ where $n\geq 1$ and $c_1\in C_2\setminus\{1\}$, we associate a sequence $(x_0,x_1,\dots,x_{n+1})$ in $X_1\cup X_2$ and a sequence $(e_0,e_1,...,e_n)$ of edges in the Bass-Serre graph as follows. We set $x_0=x$, $c_0=1\in C_1$, and then inductively for $i=0,\dots,n$:

- for $i$ such that $c_i\in C_1$, set $e_i=e_{1,x_i}(c_i)$, and $x_{i+1}=x_ic_i\tau$;
- for $i$ such that $c_i\in C_2$, set $e_i=e_{2,x_i}(c_i)$, and $x_{i+1}=x_ic_i\tau^{-1}$.

Notice that, for any $i=0,\dots,n-1$, if $c_i\in C_1$ (or equivalently if $i$ is even), one has $r(e_i)=x_ic_i\tau\Gamma_2=x_{i+1}\Gamma_2=s(e_{i+1})$ , and similarly if $c_i\in C_2$ we have $r(e_i)=s(e_{i+1})$. Hence $(e_0,\dots,e_n)$ is a path, that we denote by $\mathtt{path}_{1,x}(\gamma)$. Note that this path begins by the edge $e_0=x\Sigma_1$.

Let us check that $\mathtt{path}_{1,x}(\gamma)$ is a reduced path. For $0\leq i\leq n-1$ and $c_i\in C_1$, we have

$$e_{i+1}=\bar{e}_i\Leftrightarrow x_{i+1}c_{i+1}\Sigma_2=\overline{x_ic_i\Sigma_1}\Leftrightarrow x_ic_i\tau c_{i+1}\Sigma_2=x_ic_i\tau\Sigma_2\Leftrightarrow c_{i+1}=1$$

since $X_2\curvearrowleft^{\pi_2}\Gamma_2$ is free. Since $c_1\cdots c_n\sigma$ is the normal form of $\gamma$, we cannot have $c_{i+1}=1$, so $\mathtt{path}_{1,x}(\gamma)$ is reduced.

Finally, since the maps $e_{1,x}$ and $e_{2,x}$ are bijective, given a reduced path $(e_0,\dots,e_n)$ beginning by $x\Sigma_1$, there is exactly one normal form $c_1\cdots c_n$ with $c_1\in C_2\setminus\{1\}$ such that $\mathtt{path}_{1,x}(c_1\cdots c_n)=(e_0,\dots,e_n)$. The following remark is now clear.

**Remark 5.10.** For any $x\in X_1$, the map $\mathtt{path}_{1,x}$ is a surjection from $N_{C_2}$ to the set of reduced paths starting by the edge $x\Sigma_1$. It becomes a bijection if we restrict it to the subset of elements $\gamma\in N_{C_2}$ whose normal form is $c_1\cdots c_n$ with $c_1\in C_2\setminus\{1\}$. So if $x\Sigma_1$ is a treeing edge, then the images $\mathtt{path}_{1,x}(\gamma)$, for $\gamma\in N_{C_2}$, cover exactly the half-tree of $x\Sigma_1$ in $\mathcal{G}_\tau$.

We now give a definition of path-type elements which is analogous to the one for HNN extensions, except that we only want to consider paths which end in $X_1$. An element $\gamma\in N_{C_2}$ with normal form $\gamma=c_1\cdots c_n$ such that $c_1\in C_2\setminus\{1\}$ and $n\geq 1$ is *odd* will be called a **path type element** of $N_{C_2}$. Note that the corresponding path then has *even* length. If $\gamma' = c_1 \cdots c_k$, for some $k \leq n$, is a path type element in $N_{C_2}$, then $\gamma$ is called a **path type extension** of $\gamma'$.

Similarly, given a point $x \in X_2$, we can associate to every element $\gamma \in N_{C_1}$ with normal form $\gamma = c_1 \cdots c_n \sigma$, where $n \geq 1$ and $c_1 \in C_1 \setminus \{1\}$, a reduced path $\mathsf{path}_{2,x}(\gamma) := (e_0, e_1, ..., e_n)$, as follows. We set $x_0 = x$, $c_0 = 1 \in C_2$, and then inductively for $i = 0, \dots, n$:

- for $i$ such that $c_i \in C_1$, set $e_i = e_{1,x_i}(c_i)$, and $x_{i+1} = x_i c_i \tau$;
- for $i$ such that $c_i \in C_2$, set $e_i = e_{2,x_i}(c_i)$, and $x_{i+1} = x_i c_i \tau^{-1}$.

This defines a surjective map $\mathsf{path}_{2,x}$ from $N_{C_1}$ to the set of reduced paths starting by the edge $x\Sigma_2$. Hence, if $x\Sigma_2$ is a treeing edge then, the images $\mathsf{path}_{2,x}(\gamma)$, for $\gamma \in N_{C_1}$, cover exactly the half-tree of $x\Sigma_2$ in $\mathcal{G}_\tau$. Moreover, the map $\mathsf{path}_{2,x}$ becomes a bijection if we restrict it to the set of elements of with normal form $\gamma = c_1 \cdots c_n$ where $n \geq 1$ and $c_1 \in C_1 \setminus \{1\}$, and such elements will be called **path type elements** of $N_{C_1}$ when moreover $n$ is odd. As before, there is a notion of path type extension for path type elements in $N_{C_1}$.

**Remark 5.11.** Let $i, j \in \{1,2\}$ with $i \neq j$ and $x \in X_j$, $\gamma \in N_{C_i}$ with normal form $c_1 \cdots c_n \sigma$.

1. By construction, $\mathsf{path}_{j,x}(\gamma) = \mathsf{path}_{j,x}(c_1 \cdots c_n)$ and, for every $1 \leq k \leq n$, the path $\mathsf{path}_{j,x}(c_1 \cdots c_n)$ is an extension of $\mathsf{path}_{j,x}(c_1 \cdots c_k)$.
2. The source of $\mathsf{path}_{j,x}(\gamma)$ is $s(x\Sigma_j) = x\Gamma_j$.
3. If $\gamma = c_1 \cdots c_n$ is a path type element, then the range of $\mathsf{path}_{j,x}(\gamma)$ is $x\gamma^{\pi_{j,\tau}}\Gamma_j$.
4. If for some $1 \leq k \leq n$ the last edge of $\mathsf{path}_{j,x}(c_1 \cdots c_k)$ is a treeing edge, then for all $k \leq l \leq n$ the last edge of $\mathsf{path}_{j,x}(c_1 \cdots c_l)$ is also a treeing edge.

Let us end this section by establishing a link between paths in Bass-Serre trees and Bass-Serre graphs.

**Remark 5.12.** Consider a global pre-action $(X_1, X_2, \tau)$, and basepoints $x_1 \in X_1$ and $x_2 \in X_2$ such that $x_2 = x_1 \tau$. There exists a unique morphism of pre-actions

$$(\varphi_1, \varphi_2) : (\Gamma, \Gamma, \mathrm{id}) \to (X_1, X_2, \tau)$$

from the translation pre-action, such that $\varphi_j(1) = x_j$ for $j = 1, 2$. It satisfies $\varphi_j(\gamma) = x_j \gamma^{\pi_{j,\tau}}$ for all $j = 1, 2$ and $\gamma \in \Gamma$. By restriction, one obtains morphisms

$$(\varphi_{1,+}, \varphi_{2,+}) : (\Gamma_1 \sqcup N_{C_2}, \Gamma_2 \cup N_{C_2}, \tau_+) \to (X_1, X_2, \tau)$$
$$(\varphi_{1,-}, \varphi_{2,-}) : (\Gamma_1 \cup N_{C_1}, \Gamma_2 \sqcup N_{C_1}, \tau_-) \to (X_1, X_2, \tau)$$

from the positive and negative translation pre-actions.

**Lemma 5.13.** *In the context of the above remark, the Bass-Serre morphism* $\mathbf{BS}(\varphi_1, \varphi_2)$*, from the Bass-Serre tree* $\mathcal{T}$ *to the Bass-Serre graph* $\mathcal{G}_\tau$*, sends* $\mathsf{path}^{\mathcal{T}}_{j,1_\Gamma}(\gamma)$ *onto* $\mathsf{path}^{\mathcal{G}_\tau}_{j,x_j}(\gamma)$ *for all* $j \in \{1,2\}$ *and all* $\gamma \in N_{C_i}$ *with* $i \in \{1,2\} \setminus \{j\}$.

*Proof.* We make the proof in the case $j = 1$ only; the case $j = 2$ is similar.

Let us consider $\gamma \in N_{C_2}$, and write its normal form: $\gamma = c_1 \cdots c_n \sigma$. Let us denote by $(e_0, e_1, \dots, e_n)$ the edges of $\mathrm{path}^{\mathcal{T}}_{1,1_\Gamma}(\gamma)$, and by $(e'_0, e'_1, \dots, e'_n)$ the edges of $\mathrm{path}^{\mathcal{G}_\tau}_{1,x_1}(\gamma)$. The auxiliary sequences in $\Gamma$ and $X$ used in the construction of the paths will be denoted by $(\gamma_0, \dots, \gamma_{n+1})$ and $(x_0, \dots, x_{n+1})$ respectively.

An easy induction shows that $x_i = \varphi_1(\gamma_i)$ when $i$ even, and $x_i = \varphi_2(\gamma_i)$ when $i$ is odd. Then, we notice that the source of $e_i = e_{1,\gamma_i}(c_i)$ is $\gamma_i\Gamma_1$ when $i$ even, and the source of $e_i = e_{2,\gamma_i}(c_i)$ is $\gamma_i\Gamma_2$ when $i$ odd. Thus, using Lemma 5.9, we get

$$\mathbf{BS}(\varphi_1, \varphi_2)(e_i) = e_{1,\varphi_1(\gamma_i)} \circ e^{-1}_{1,\gamma_i}\big(e_{1,\gamma_i}(c_i)\big) = e_{1,x_i}(c_i) = e'_i$$

when $i$ is even, and

$$\mathbf{BS}(\varphi_1, \varphi_2)(e_i) = e_{2,\varphi_2(\gamma_i)} \circ e^{-1}_{2,\gamma_i}\big(e_{2,\gamma_i}(c_i)\big) = e_{2,x_i}(c_i) = e'_i$$

when $i$ is odd. □

Therefore, if $x_1\Sigma_1$ is a treeing edge then, the image of $\mathbf{BS}(\varphi_{1,+}, \varphi_{2,+})$ is the half-tree of $x_1\Sigma_1$ while, if $x_2\Sigma_2$ is a treeing edge, the image of $\mathbf{BS}(\varphi_{1,-}, \varphi_{2,-})$ is the half-tree of $x_2\Sigma_2$.

## 5.4 The free globalization of a pre-action of an amalgam

First, let us notice that, for any $\sigma \in \Sigma$, there is an automorphism of pre-actions induced by left translation by $\sigma$

$$(\gamma \mapsto \sigma\gamma, \gamma \mapsto \sigma\gamma)$$

for each of the following pre-actions:

- the translation pre-action $(\Gamma, \Gamma, \mathrm{id})$;
- the positive translation pre-action $(\Gamma_1 \sqcup N_{C_2}, \Gamma_2 \cup N_{C_2}, \tau_+)$;
- the negative translation pre-action $(\Gamma_1 \cup N_{C_1}, \Gamma_2 \sqcup N_{C_1}, \tau_-)$.

Indeed, all sets $\Gamma, \Gamma_1, \Gamma_2, \Sigma, N_{C_1}, N_{C_2}$ are invariant by left translation by $\sigma$, hence the domains and range of $\tau_+$ and $\tau_-$ are invariant by left translation by $\sigma$. Then checking we have morphisms of pre-actions is a straightforward computation, and invertibility is obvious.

**Proposition 5.14.** *Consider a global pre-action $(X_1, X_2, \tau)$, and basepoints $x_1 \in X_1$ and $x_2 = x_1\tau \in X_2$. The following are equivalent:*

*(i) the morphism of pre-actions $(\varphi_{1,+}, \varphi_{2,+}) : (\Gamma_1 \sqcup N_{C_2}, \Gamma_2 \cup N_{C_2}, \tau_+) \to (X_1, X_2, \tau)$ of Remark 5.12 is injective;*

*(ii) the morphism of graphs $\mathbf{BS}(\varphi_{1,+}, \varphi_{2,+})$ is injective;*

*(iii) the edge $x_1\Sigma_1$ in the Bass-Serre graph $\mathbf{BS}(X_1, X_2, \tau)$ is a treeing edge.*

*Proof.* For all $\gamma \in \Gamma_j \cup N_{C_2}$, recall that $\varphi_{j,+}(\gamma) = x_j\gamma^{\pi_{j,\tau}}$, so that $\mathbf{BS}(\varphi_{1,+}, \varphi_{2,+})$ sends vertices $\gamma\Gamma_j$ to $x_j\gamma^{\pi_{j,\tau}}\Gamma_j$, and edges $\gamma\Sigma_j$ to $x_j\gamma^{\pi_{j,\tau}}\Sigma_j$. Fixing $\gamma$, we get $\varphi_{j,+}(\gamma g) = x_j\gamma^{\pi_{j,\tau}}g^{\pi_j}$ for $g \in \Gamma_j$; since $X_j \curvearrowleft^{\pi_j} \Gamma_j$ is free, $\varphi_{j,+}$ realizes a bijection between $\gamma\Gamma_j$ and $x_j\gamma^{\pi_{j,\tau}}\Gamma_j$, and also a bijection between $\gamma\Sigma_j$ and $x_j\gamma^{\pi_{j,\tau}}\Sigma_j$. Consequently, $\varphi_{j,+}$ is injective if an only if $\gamma\Gamma_j \mapsto x_j\gamma^{\pi_{j,\tau}}\Gamma_j$ and $\gamma\Sigma_j \mapsto x_j\gamma^{\pi_{j,\tau}}\Sigma_j$ are both injective. This proves that (i) and (ii) are equivalent.

The implication (iii) $\implies$ (ii) follows from the fact that when $x_1\Sigma_1$ is a treeing edge $\mathbf{BS}(\varphi_{1,+},\varphi_{2,+})$ is locally injective from the half-tree of $\Sigma_1$ to the half-tree of $x_1\Sigma_1$, hence $\mathbf{BS}(\varphi_{1,+},\varphi_{2,+})$ is injective.

Finally assume (ii) and let $\omega$ be a reduced path starting by the edge $x_1\Sigma_1$. By Remark 5.10 there exists $\gamma \in N_{C_2}$ such that $\omega = \mathsf{path}_{1,x_1}(\gamma)$. By Lemma 5.13, $\omega$ is the image by $\mathbf{BS}(\varphi_{1,+},\varphi_{2,+})$ of $\mathsf{path}^{\mathcal{T}}_{1,1_\Gamma}(\gamma)$. Since $\mathbf{BS}(\varphi_{1,+},\varphi_{2,+})$ is supposed to be injective and since the last vertex of $\mathsf{path}^{\mathcal{T}}_{1,1_\Gamma}(\gamma)$ is not $\Gamma_1$, we deduce that the last vertex of $\omega$ is not $x_1\Gamma_1$. Hence, $x_1\Sigma_1$ is a treeing edge by Lemma 2.16. □

By a very similar argument, we get also the following result.

**Proposition 5.15.** *Consider a global pre-action $(X_1,X_2,\tau)$, and basepoints $x_1 \in X_1$ and $x_2 = x_1\tau \in X_2$. The following are equivalent:*

*(i) the morphism of pre-actions $(\varphi_{1,-},\varphi_{2,-}) : (\Gamma_1 \cup N_{C_1}, \Gamma_2 \sqcup N_{C_1}, \tau_-) \to (X_1,X_2,\tau)$ of Remark 5.12 is injective;*

*(ii) the morphism of graphs $\mathbf{BS}(\varphi_{1,-},\varphi_{2,-})$ is injective;*

*(iii) the edge $x_2\Sigma_2$ in the Bass-Serre graph $\mathbf{BS}(X_1,X_2,\tau)$ is a treeing edge.*

**Remark 5.16.** Putting the two previous propositions together, one can show that, given a global pre-action of $\Gamma$, its Bass-Serre graph is a forest if and only if the action $X_1 \curvearrowleft^{\pi_{1,\tau}} \Gamma$ (or equivalently $X_2 \curvearrowleft^{\pi_{2,\tau}} \Gamma$) is free.

Say that a pre-action is **transitive** when its Bass-Serre graph is connected. Note that a global pre-action $(X_1,X_2,\tau)$ is transitive if and only if the action $X_1 \curvearrowleft^{\pi_{1,\tau}} \Gamma$, or equivalently $X_2 \curvearrowleft^{\pi_{2,\tau}} \Gamma$, is a transitive action. We will show that every transitive pre-action has a canonical extension to a transitive action, which is **as free as possible**. The construction is again better described in terms of the Bass-Serre graph: we are going to attach as many treeing edges as possible to it.

**Theorem 5.17.** *Every transitive $\Gamma$-pre-action $(X_1,X_2,\tau)$ admits a transitive and global extension $(\tilde{X}_1,\tilde{X}_2,\tilde{\tau})$ which satisfies the following universal property: given any transitive and global extension $(Y_1,Y_2,\tau')$ of $(X_1,X_2,\tau)$, there exists a unique morphism of pre-actions $(\varphi_1,\varphi_2) : (\tilde{X}_1,\tilde{X}_2,\tilde{\tau}) \to (Y_1,Y_2,\tau')$ such that*

$$(\varphi_{1\restriction X_1},\varphi_{2\restriction X_2}) = (\mathrm{id}_{X_1},\mathrm{id}_{X_2}).$$

*Moreover, all the (oriented) edges from the Bass-Serre graph $\mathbf{BS}(X_1,X_2,\tau)$ to its complement in $\mathbf{BS}(\tilde{X}_1,\tilde{X}_2,\tilde{\tau})$ are treeing edges.*

*Proof.* We will obtain the Bass-Serre graph of this action by adding only treeing edges to the Bass-Serre graph of the pre-action. First enumerate the $\Sigma_1$-orbits which do not belong to the domain of $\tau$ as $(x_i\Sigma_1)_{i\in I_+}$, and the $\Sigma_2$-orbits which do not belong to the range of $\tau$ as $(x_i\Sigma_2)_{i\in I_-}$, with disjoint index sets $I_+,I_-$. Then, we take copies $(Y_{1,i},Y_{2,i},\tau_i)$, of the positive translation pre-action $(\Gamma_1 \sqcup N_{C_2},\Gamma_2 \cup N_{C_2},\tau_+)$, for $i \in I_+$, and copies $(Y_{1,i},Y_{2,i},\tau_i)$ , of the negative translation pre-action $(\Gamma_1 \cup N_{C_1},\Gamma_2 \sqcup N_{C_1},\tau_-)$, for $i \in I_-$, which are pairwise disjoint (by this, we mean $Y_{1,i} \cup Y_{2,i}$ is disjoint from $Y_{1,i'} \cup Y_{2,i'}$ whenever $i \neq i'$), and disjoint from the original pre-action $(X_1,X_2,\tau)$. We set then

$$\tilde{X}_1 = \left( X_1 \sqcup \bigsqcup_{i\in I_+} Y_{1,i} \sqcup \bigsqcup_{i\in I_-} Y_{1,i} \right) \Big/ _{\sim_1} \quad \text{and} \quad \tilde{X}_2 = \left( X_2 \sqcup \bigsqcup_{i\in I_+} Y_{2,i} \sqcup \bigsqcup_{i\in I_-} Y_{2,i} \right) \Big/ _{\sim_2}$$

where $\sim_1$ identifies the element $x_i g \in X_1$ with $g \in \Gamma_1 \subset Y_{1,i}$, for each $i \in I_+$ and $g \in \Gamma_1$, and $\sim_2$ identifies the element $x_i h \in X_2$ with $h \in \Gamma_2 \subset Y_{2,i}$, for each $i \in I_-$ and $h \in \Gamma_2$. Since the identifications just glue some orbits pointwise and respect the $\Gamma_j$-actions, $\tilde{X}_1$ is endowed with a free $\Gamma_1$-action, and $\tilde{X}_2$ is endowed with a free $\Gamma_2$-action. Now, we set

$$\tilde{\tau} = \tau \sqcup \bigsqcup_{i \in I_+} \tau_i \sqcup \bigsqcup_{i \in I_-} \tau_i \,,$$

which is possible since the domain of $\tau_i$, for $i \in I_+$, intersects other components in $\tilde{X}_1$ only in the orbit $x_i \Sigma_1$, the range of $\tau_i$, for $i \in I_+$, does not intersect other components in $\tilde{X}_2$, and the situation is analogue for $\tau_i$ with $i \in I_-$. We have got a pre-action $(\tilde{X}_1, \tilde{X}_2, \tilde{\tau})$.

This pre-action is transitive, since all pre-actions $(X_1, X_2, \tau)$ and $(Y_{1,i}, Y_{2,i}, \tau_i)$ are, and the identifications make connections between each $(Y_{1,i}, Y_{2,i}, \tau_i)$ and $(X_1, X_2, \tau)$ in the Bass-Serre graph. It is also global, since every $\Sigma_1$-orbit in $Y_{1,i}$, respectively $\Sigma_2$-orbit in $Y_{2,i}$, which is not in the domain, respectively the range, of $\tau_i$ has been identified with an orbit in $X_1$, respectively $X_2$, and every $\Sigma_1$-orbit in $X_1$, respectively $\Sigma_2$-orbit in $X_2$, is now in the domain, respectively the range, of $\tilde{\tau}$.

Moreover, the (oriented) edges from the Bass-Serre graph $\mathbf{BS}(X_1, X_2, \tau)$ to its complement in $\mathbf{BS}(\tilde{X}_1, \tilde{X}_2, \tilde{\tau})$ are exactly the edges $x_i \Sigma_1$ for $i \in I_+$, and the edges $x_i \Sigma_2$ for $i \in I_-$. For each $i \in I_+$, the morphism of pre-actions $(\varphi_{1,+}, \varphi_{2,+}) : (\Gamma_1 \sqcup N_{C_2}, \Gamma_2 \cup N_{C_2}, \tau_+) \to (\tilde{X}_1, \tilde{X}_2, \tau)$ of Remark 5.12, with basepoints $x_i \in \tilde{X}_1$ and $x_i \tilde{\tau} \in \tilde{X}_2$, is injective since it realizes an isomorphism onto $(Y_{1,i}, Y_{2,i}, \tau_i)$, hence $x_i \Sigma_1$ is a treeing edge by Proposition 5.14. One proves similarly that the edges $x_i \Sigma_2$ are treeing edges using Proposition 5.15.

It now remains to prove the universal property. To do so, take any transitive and global extension $(Y_1, Y_2, \tau')$ of $(X_1, X_2, \tau)$. Then, the unique morphism of pre-actions $(\varphi_1, \varphi_2)$ from $(\tilde{X}_1, \tilde{X}_2, \tilde{\tau})$ to $(Y_1, Y_2, \tau')$ such that $(\varphi_{1\restriction X_1}, \varphi_{2\restriction X_2}) = (\mathrm{id}_{X_1}, \mathrm{id}_{X_2})$ is obtained by taking the union of $(\mathrm{id}_{X_1}, \mathrm{id}_{X_2})$ with the morphisms $(\varphi_{1,i}, \varphi_{2,i})$ from $(Y_{1,i}, Y_{2,i}, \tau_i)$ to $(Y_1, Y_2, \tau')$ coming from Remark 5.12 with respect to basepoints $x_i$ and $x_i \tilde{\tau}^{\pm 1}$, which are unique. □

It is straightforward to deduce from the universal property above that the action we just built is unique up to isomorphism. We thus call it **the free globalization** of the pre-action $(X_1, X_2, \tau)$. The interested reader can establish a connection with the notion of partial action, as we did in section 3.5 for HNN extensions. For the sake of brevity, we just observe the following useful analogue of Proposition 3.27.

**Remark 5.18.** In the context of Theorem 5.17, if the pre-action $(X_1, X_2, \tau)$ is not global, then the conjugate actions $\pi_{1,\tilde{\tau}}$ and $\pi_{2,\tilde{\tau}}$ induced by the free globalization $(\tilde{X}_1, \tilde{X}_2, \tilde{\tau})$ are highly faithful. Indeed, by Corollary 2.6, it suffices to prove that $\pi_{1,\tilde{\tau}}$ is strongly faithful. Notice that $(\tilde{X}_1, \tilde{X}_2, \tilde{\tau})$ contains a copy of the positive (or of the negative) translation pre-action, which correspond to a half-tree in $\mathbf{BS}(\tilde{X}_1, \tilde{X}_2, \tilde{\tau})$. Now note that the positive translation pre-action is strongly faithful, meaning that given $F \Subset \Gamma$, we can find $x \in \Gamma_1 \cup N_{C_1}$ such that for all $f \in F$, we have $xf \neq x$ and $xf \in \Gamma_1 \cup N_{C_1}$ (indeed it suffices to take $x \in N_{C_1}$ with a sufficiently long normal form). Similarly, the negative translation is strongly faithful. It follows that the free globalization is strongly faithful, hence highly faithful as wanted.

Let us furthermore observe that we can always build the free globalization on a fixed couple of sets $(\bar{X}_1, \bar{X}_2)$ with $\bar{X}_j$ containing $X_i$, provided $\bar{X}_j$ contains infinitely many free $\Gamma_j$-orbits.

**Theorem 5.19.** *Let $\bar{X}_j$ be a countable set equipped with a free $\Gamma_j$-action for $j = 1, 2$. Suppose $X_j \subseteq \bar{X}_j$ is $\Gamma_j$-invariant, and $\bar{X}_j \setminus X_j$ contains infinitely many $\Gamma_j$-orbits. Suppose further that we have a pre-action*

$(X_1,X_2,\tau)$. *Then there is a bijection* $\bar{\tau}:\bar{X}_1\to\bar{X}_2$ *which extends* $\tau$ *such that* $(\bar{X}_1,\bar{X}_2,\bar{\tau})$ *is (isomorphic to) the free globalization of* $(X_1,X_2,\tau)$.

*Proof.* Let $(\tilde{X}_1,\tilde{X}_2,\tilde{\tau})$ be the free globalization of $(X_1,X_2,\tau)$. The fact that $\tilde{X}_j\setminus X_j$ contains infinitely many $\Gamma_j$-orbits and is countable implies that there exist $\Gamma_j$-equivariant bijections $\varphi_j:\tilde{X}_j\to\bar{X}_j$ whose restrictions to $X_j$ are the identities. Then, one can push forward the bijection $\tilde{\tau}$, to obtain a bijection $\bar{\tau}:\bar{X}_1\to\bar{X}_2$ defined by

$$x\varphi_1\bar{\tau}:=x\tilde{\tau}\varphi_2 \quad \text{for all } x\in\tilde{X}_1\,,$$

which extends $\tau$. Now, $(\varphi_1,\varphi_2)$ is an isomorphism of pre-actions between $(\tilde{X}_1,\tilde{X}_2,\tilde{\tau})$ and $(\bar{X}_1,\bar{X}_2,\bar{\tau})$. □

# 6 High transitivity for amalgams

As in Section 5, we fix an amalgam $\Gamma=\Gamma_1*_\Sigma\Gamma_2$, and sets of representatives $C_j$ of left $\Sigma_j$-cosets in $\Gamma_j$ such that $1\in C_j$, for $j=1,2$, so that normal forms of elements of $\Gamma$ are well-defined. We still denote by $N_{C_j}$ the set of elements of $\Gamma$ whose normal form begins with an element of $C_j\setminus\{1\}$, for $j=1,2$, so that we have $\Gamma=\Sigma\sqcup N_{C_1}\sqcup N_{C_2}$.

Non-degeneracy and topological freeness become now essential. Hence, **we assume from now on that our amalgam $\Gamma$ is non-degenerate** and that **the $\Gamma$-action on the boundary of its Bass-Serre tree is topologically free**.

## 6.1 Using the free globalization towards high transitivity

This section is devoted to a key proposition which will allow us to extend any given transitive pre-action which is not global to a global one such that the associated $\Gamma$-action sends one fixed tuple to another fixed tuple.

**Proposition 6.1.** *Suppose* $(X_1,X_2,\tau)$ *is a transitive non-global pre-action, that* $X_j$ *is a finite union of orbits of a free action* $\bar{X}_j\curvearrowleft\Gamma_j$*, where* $\bar{X}_j$ *is countable, and that the complement* $\bar{X}_j\setminus X_j$ *contains infinitely many* $\Gamma_j$*-orbits. Let* $x_1,\dots,x_k,y_1,\dots,y_k\in\bar{X}_1$ *be pairwise distinct points. Then* $(X_1,X_2,\tau)$ *can be extended to a transitive and global pre-action* $(\bar{X}_1,\bar{X}_2,\bar{\tau})$ *so that there is an element* $\gamma\in\Gamma$ *such that* $x_i\gamma^{\pi_{1,\bar{\tau}}}=y_i$*, and the action* $\pi_{1,\bar{\tau}}$ *is highly faithful.*

Notice that the choice to work in $\bar{X}_1$ is arbitrary. We could prove a similar statement for the $\Gamma_2$-action on $\bar{X}_2$.

*Proof.* We will denote the set $\{x_1,\dots,x_k,y_1,\dots,y_k\}$ by $F$. First, by Theorem 5.19, we find a bijection $\bar{\tau}:\bar{X}_1\to\bar{X}_2$ such that $(\bar{X}_1,\bar{X}_2,\bar{\tau})$ is the free globalization of $(X_1,X_2,\tau)$.

**Claim.** *There exists a path-type element* $\gamma$ *in* $N_{C_2}$ *such that for every* $x\in F$*, the last edge of* $\mathsf{path}_{1,x}(\gamma)$ *is a treeing edge.*

*Proof of the claim.* Recall the correspondence established in Section 5.3 between path-type elements and reduced paths of even length. Since $\mathbf{BS}(\bar{X}_1,\bar{X}_2,\bar{\tau})$ is connected and has treeing edges, it follows from Lemma 2.17 that for every $x\in\bar{X}_1$, and every path-type element $\gamma\in N_{C_2}$, there is a path-type extension $\gamma'$ of $\gamma$ such that the last edge of $\mathsf{path}_{1,x}(\gamma')$ is a treeing edge. Now, it suffices to start with any path-type element $\gamma_0\in N_{C_2}$, to extend it to a path-type element $\gamma_1$ such that the last edge of $\mathsf{path}_{1,x_1}(\gamma_1)$ is a treeing edge, then to extend $\gamma_1$ to a path-type element $\gamma_2$ such that the last edge of $\mathsf{path}_{1,y_1}(\gamma_2)$ is a treeing edge, ..., and iterate this extension procedure until we reach an element $\gamma_{2k} \in N_{C_2}$ such that all last edges of $\mathsf{path}_{1,x}(\gamma_{2k})$, for all $x \in F$, are treeing edges (by Remark 5.11 (4)). $\square_{\text{claim}}$

Given $x \in \bar{X}_1$, and a path-type element $\gamma$ in $N_{C_2}$, we will denote by $\mathcal{H}_x(\gamma)$ the half-graph of the last edge of $\mathsf{path}_{1,x}(\gamma)$.

**Claim.** *There exists a path-type element $\gamma$ in $N_{C_2}$ such that for every $x \in F$, the last edge of* $\mathsf{path}_{1,x}(\gamma)$ *is a treeing edge, and the half-trees $\mathcal{H}_x(\gamma)$, for $x \in F$, are pairwise disjoint subgraphs, and all disjoint from* $\mathbf{BS}(X_1, X_2, \tau)$.

*Proof of the claim.* We start with a path-type element $\gamma$ in $N_{C_2}$ such that for every $x \in F$, the last edge of $\mathsf{path}_{1,x}(\gamma)$ is a treeing edge. Since $X_j$ is a finite union of $\Gamma_j$-orbits for $j = 1, 2$, the Bass-Serre graph $\mathbf{BS}(X_1, X_2, \tau)$ has finitely many vertices. Hence, by extending further the path-type element $\gamma$, we can assume that for every $x \in F$, the half-tree $\mathcal{H}_x(\gamma)$ does not intersect $\mathbf{BS}(X_1, X_2, \tau)$.

Notice that, given $x, y \in F$, if the half-trees $\mathcal{H}_x(\gamma)$ and $\mathcal{H}_y(\gamma)$ are disjoint, then so are the half-trees $\mathcal{H}_x(\gamma')$ and $\mathcal{H}_y(\gamma')$ for every path-type extension $\gamma'$ of $\gamma$, since $\mathcal{H}_x(\gamma') \subseteq \mathcal{H}_x(\gamma)$ and $\mathcal{H}_y(\gamma') \subseteq \mathcal{H}_y(\gamma)$. Hence, it suffices to prove that, for any $x, y \in F$ with $x \neq y$ and such that $\mathcal{H}_x(\gamma)$ and $\mathcal{H}_y(\gamma)$ intersect, there exists a path-type extension $\gamma'$ of $\gamma$ such that $\mathcal{H}_x(\gamma')$ and $\mathcal{H}_y(\gamma')$ are disjoint. Indeed, an easy induction gives then an extension $\gamma^{(n)}$ such that the half-trees $\mathcal{H}_x(\gamma^{(n)})$, for $x \in F$, are pairwise disjoint.

Take now $x, y \in F$ with $x \neq y$ and such that $\mathcal{H}_x(\gamma)$ and $\mathcal{H}_y(\gamma)$ intersect. These half-trees have to be nested. Indeed, if they are not, $\mathcal{H}_x(\gamma)$ contains the antipode of the last edge of $\mathsf{path}_y(\gamma)$, hence contains $\mathbf{BS}(X_1, X_2, \tau)$, which is impossible. Without loss of generality, we assume $\mathcal{H}_x(\gamma) \subseteq \mathcal{H}_y(\gamma)$. We now distinguish two cases.

- If $\mathcal{H}_x(\gamma) \subsetneq \mathcal{H}_y(\gamma)$, there is a path type extension $\gamma''$ of $\gamma$ such that $\mathsf{path}_{1,x}(\gamma)$ and $\mathsf{path}_{1,y}(\gamma'')$ have the same last edge. We have the product of normal forms

$$\gamma'' = \gamma \cdot (c_1 \cdots c_n),$$

where $n \geq 2$ is even. Since the amalgam $\Gamma$ is non-degenerate, we can obtain another normal form $\gamma' = \gamma \cdot (c'_1 \cdots c'_n)$ by replacing a letter $c_i$ in the factor $\Gamma_j$ such that $[\Gamma_j : \Sigma_j] \geq 3$ by another letter $c'_i$ in $C_j \setminus \{1\}$. This change has the effect that $\mathsf{path}_{1,y}(\gamma')$ and $\mathsf{path}_{1,y}(\gamma'')$ are distinct reduced paths (which are both extensions of the $\mathsf{path}_{1,y}(\gamma)$). Hence, since $\mathcal{H}_y(\gamma)$ is a tree, the sub-trees $\mathcal{H}_y(\gamma'')$ and $\mathcal{H}_y(\gamma')$ must be disjoint. Since $\mathcal{H}_x(\gamma) = \mathcal{H}_y(\gamma'')$ we are done.

- If $\mathcal{H}_x(\gamma) = \mathcal{H}_y(\gamma)$, then $\mathsf{path}_{1,x}(\gamma)$ and $\mathsf{path}_{1,y}(\gamma)$ have the same terminal edge, which is

$$e := \overline{x'\Sigma_1} = \overline{y'\Sigma_1}, \quad \text{where} \quad x' = x\gamma^{\pi_{1,\bar{\tau}}} \text{ and } y' = y\gamma^{\pi_{1,\bar{\tau}}}.$$

Consequently, one has $y' = x'\sigma^{\pi_{1,\bar{\tau}}}$ for some $\sigma \in \Sigma$. Note that, since $x \neq y$, one has $\sigma \neq 1$ and consider the morphism of pre-actions from the negative translation pre-action $(\varphi_{1,-}, \varphi_{2,-})$ : $(\Gamma_1 \cup N_{C_1}, \Gamma_2 \sqcup N_{C_1}, \tau_-) \to (\bar{X}_1, \bar{X}_2, \bar{\tau})$ coming from Remark 5.12, with basepoints $x_1 = x'$ and $x_2 = x'\bar{\tau}$. Since $e$ is a treeing edge, this morphism is injective by Proposition 5.15. The half-tree $\mathcal{H}_x(\gamma) = \mathcal{H}_y(\gamma)$ is thus isomorphic, via $\mathbf{BS}(\varphi_{1,-}, \varphi_{2,-})$, to the half-tree $\mathcal{H}$ of $\Sigma_2$ in the Bass-Serre tree $\mathcal{T}$.

Note that the left translation by $\sigma$ (i.e. $\gamma^* \mapsto \sigma\gamma^*$) defines an automorphism of the negative translation pre-action, which we write as $(\sigma_1, \sigma_2)$. The morphism of graphs $\mathbf{BS}(\sigma_1, \sigma_2)$ maps $\mathsf{path}^{\mathcal{T}}_{2,1_\Gamma}(\gamma^*)$ to $\mathsf{path}^{\mathcal{T}}_{2,\sigma}(\gamma^*)$ in $\mathcal{H}$ by Lemma 5.13 (note that these paths both have $\Sigma_2$ as first edge). Since the left $\Gamma$-action on the boundary $\partial\mathcal{T}$ of its Bass-Serre tree is topologically free, the left $\sigma$ action does not fix the half-tree $\mathcal{H}$ pointwise. Hence there exists an element $\gamma^* \in \Sigma \sqcup N_{C_1}$ such that $\mathsf{path}^{\mathcal{T}}_{2,1}(\gamma^*)$ and $\mathsf{path}^{\mathcal{T}}_{2,\sigma}(\gamma^*)$ have distinct ranges. Moreover, up to extending $\gamma^*$, we can further assume that $\gamma^*$ is a path type element of $N_{C_1}$ so that the element $\gamma' := \gamma\gamma^*$ is a path-type element of $N_{C_2}$.

Now, the images of $\mathsf{path}^{\mathcal{T}}_{2,1}(\gamma^*)$ and $\mathsf{path}^{\mathcal{T}}_{2,\sigma}(\gamma^*)$ by $\mathbf{BS}(\varphi_{1,-},\varphi_{2,-})$ are $\mathsf{path}_{2,x'\bar{\tau}}(\gamma^*)$ and $\mathsf{path}_{2,y'\bar{\tau}}(\gamma^*)$ by Lemma 5.13, and these paths diverge in the half-tree $\mathcal{H}_x(\gamma) = \mathcal{H}_y(\gamma)$. Note finally that the starting edges of these paths are both equal to $e$; this implies that $\mathsf{path}_{1,x}(\gamma')$ and $\mathsf{path}_{1,y}(\gamma')$ don't have the same range. Hence $\mathcal{H}_x(\gamma')$ and $\mathcal{H}_y(\gamma')$ are disjoint.

We are done in both cases. $\square_{\text{claim}}$

We then modify the bijection $\bar{\tau}$ to get the pre-action $(\bar{X}_1, \bar{X}_2, \tilde{\tau})$ we are looking for. First, given an element $\gamma$ as in the previous claim, we consider for each $z \in F$ the morphism of pre-actions from the positive translation pre-action

$$(\psi_{1,z}, \psi_{2,z}) : (\Gamma_1 \sqcup N_{C_2}, \Gamma_2 \cup N_{C_2}, \tau_+) \to (\bar{X}_1, \bar{X}_2, \bar{\tau})$$

coming from Remark 5.12, with basepoints $z' := z\gamma^{\pi_{1,\bar{\tau}}} \in \bar{X}_1$ and $z'\bar{\tau} \in \bar{X}_2$. Note that the image of this morphism corresponds to the half-graph opposite to the half-tree $\mathcal{H}_z(\gamma)$. Then, we define $X'_j = \bigcap_{z\in F} \mathrm{rng}(\psi_{j,z}) \subset \bar{X}_j$, and consider the restriction $(X'_1, X'_2, \tau')$ of $(\bar{X}_1, \bar{X}_2, \bar{\tau})$. Informally speaking, we erase $\bar{\tau}$ on the $\Sigma_1$-orbits corresponding to edges in the half-trees $\mathcal{H}_z(\gamma)$ for $z \in F$. Note that this leaves infinitely many $\Gamma_1$-orbits in $\bar{X}_1$ outside $\mathrm{dom}(\tau')$, respectively infinitely many $\Gamma_2$-orbits in $\bar{X}_2$ outside $\mathrm{rng}(\tau')$, and the pre-action $(X'_1, X'_2, \tau')$ is transitive. Notice also that, for any $z \in F$, we have $z'\Gamma_1 \cap \mathrm{dom}(\tau') = z'\Sigma_1$ (in other words, the only edge in the star at $z'$ which belongs to $\mathbf{BS}(X'_1, X'_2, \tau')$ is $z'\Sigma_1$). In particular, given any $c_1 \in C_1 \setminus \{1\}$, the orbits $x'_i c_1 \Sigma_1$ and $y'_i c_1 \Sigma_1$, for $1 \le i \le k$, are not in $\mathrm{dom}(\tau')$.

We now extend $\tau'$. Pick some orbits $z_1\Gamma_2, \ldots, z_k\Gamma_2$ in $\bar{X}_2 \setminus \mathrm{rng}(\tau')$, add them to $X'_2$, take $c_j$ in $C_j \setminus \{1\}$ for $j = 1,2$, and set $x'_i c_1 \sigma \tau' := z_i \vartheta(\sigma)$ and $y'_i c_1 \sigma \tau' := z_i c_2 \vartheta(\sigma)$ for $i = 1, \ldots, k$ and $\sigma \in \Sigma_1$. This is possible since the $\Sigma_2$-orbits of the points $z_i$ and $z_i c_2$ are pairwise disjoint (we use again the freeness of the $\Gamma_2$-action), and since the $\Sigma_1$-orbits at $x'_i c_1$, $y'_i c_1$ for $1 \le i \le k$ are pairwise disjoint and were not initially in the domain of $\tau'$. Note that, after this extension, $(X'_1, X'_2, \tau')$ is still transitive. Then we apply Theorem 5.19 to get an extension $\tilde{\tau} : \bar{X}_1 \to \bar{X}_2$ of $\tau'$ such that $(\bar{X}_1, \bar{X}_2, \tilde{\tau})$ is the free globalization of $(X'_1, X'_2, \tau')$. A computation shows then that $x_i(\gamma c_1 c_2 c_1^{-1} \gamma^{-1})^{\pi_{1,\tilde{\tau}}} = y_i$ for all $i = 1, \ldots, k$. Finally, the action $\pi_{1,\tilde{\tau}}$ is highly faithful by Remark 5.18. $\square$

## 6.2 Highly transitive actions of amalgams

From now on, we fix free actions $X_1 \curvearrowleft^{\pi_1} \Gamma_1$ and $X_2 \curvearrowleft^{\pi_2} \Gamma_2$ with infinitely many orbits. We endow the set of bijections from $X_1$ onto $X_2$ with the topology of pointwise convergence, which is a Polish topology. We then set

$$\mathbf{PA} = \{\tau : X_1 \to X_2 \text{ bijective} : x\sigma\tau = x\tau\vartheta(\sigma) \text{ for all } \sigma \in \Sigma_1\}.$$

In other words, $\mathbf{PA}$ is the set of bijections $\tau : X_1 \to X_2$ such that $(X_1, X_2, \tau)$ is a (global) pre-action of $\Gamma$. This is clearly a closed subset for the topology of pointwise convergence, hence a Polish space. Recall that every $\tau \in \mathbf{PA}$ induces an action $X_j \curvearrowleft^{\pi_{j,\tau}} \Gamma$ for $j = 1,2$. We will focus on the action $\pi_{1,\tau}$, which we will abbreviate by $\pi_\tau$.

**Definition 6.2.** Let us set

$$\begin{aligned}\mathbf{TA} &= \{\tau \in \mathbf{PA} \colon \pi_\tau \text{ is transitive}\}\ ;\\ \mathbf{HFA} &= \{\tau \in \mathbf{PA} \colon \pi_\tau \text{ is highly faithful }\}\ ;\\ \mathbf{HTA} &= \{\tau \in \mathbf{PA} \colon \pi_\tau \text{ is highly transitive }\}\ .\end{aligned}$$

As in the HNN case, the subset **TA** isn't closed for the topology of pointwise convergence, but we have the following result.

**Lemma 6.3.** *The set* **TA** *is* $G_\delta$ *in* **PA**, *hence a Polish space. Moreover,* $\mathbf{TA} \neq \emptyset$.

*Proof.* Since $X_1 \curvearrowleft^{\pi_1} \Gamma_1$ and $X_2 \curvearrowleft^{\pi_2} \Gamma_2$ have infinitely many orbits, there are $\Gamma_j$-equivariant bijections $\varphi_j : \Gamma \to X_j$ for $j = 1,2$. It then suffices to push-forward the translation pre-action by $(\varphi_1, \varphi_2)$ to get an element of **TA** (its Bass-Serre graph will be isomorphic to the classical Bass-Serre tree and $\pi_\tau$ will be conjugated to the translation action $\Gamma \curvearrowleft \Gamma$). Hence, **TA** is non-empty. To show that **TA** is $G_\delta$ in **PA**, it suffices to write $\mathbf{TA} = \bigcap_{x,x' \in X_1} O_{x,y}$, where for $x, y \in X_1$, $O_{x,y} = \{\tau \in \mathbf{PA} :$ there exists $\gamma \in \Gamma$ such that $x\gamma^{\pi_\tau} = y\}$. Since $O_{x,y}$ is obviously open in **PA** for all $x, y \in X_1$, this shows that **TA** is a $G_\delta$ subset of **PA**. ☐

Here comes the theorem proving that our amalgam $\Gamma$ admits a highly transitive highly faithful action, thus proving Theorem E.

**Theorem 6.4.** *The set* $\mathbf{HTA} \cap \mathbf{HFA}$ *is dense* $G_\delta$ *in* **TA**. *In particular,* $\Gamma$ *admits actions which are both highly transitive and highly faithful.*

*Proof.* For $k \geq 1$ and $x_1, \dots x_k, y_1, \dots, y_k \in X_1$ pairwise distinct, the sets

$$V_{x_1,\dots,x_k,y_1,\dots,y_k} = \{\tau \in \mathbf{TA} : \exists \gamma \in \Gamma, x_i\gamma^{\pi_\tau} = y_i \text{ for all } 1 \leq i \leq k\}$$

are obviously open in **TA**. Similarly, for finite subsets $F$ of $\Gamma \setminus \{1\}$, the sets

$$W_F = \{\tau \in \mathbf{TA} : \exists x \in X_1\,, x f^{\pi_\tau} \neq x \text{ for all } f \in F\}$$

are also obviously open in **TA**. Now, using Lemma 2.2, and since every strongly faithful action of $\Gamma$ is highly faithful by Corollary 2.6, we have

$$\mathbf{HTA} \cap \mathbf{HFA} = \bigcap_{\substack{F \Subset \Gamma\setminus\{1\}, k\geq 1, x_1,\dots x_k, y_1,\dots,y_k \in X_1 \\ \text{pairwise distinct}}} (V_{x_1,\dots,x_k,y_1,\dots,y_k} \cap W_F)\,.$$

To conclude, it suffices to show that each set $(V_{x_1,\dots,x_k,y_1,\dots,y_k}) \cap \mathbf{HFA}$ is dense in **TA**, since this immediately implies that each open set $(V_{x_1,\dots,x_k,y_1,\dots,y_k}) \cap W_F$ is dense in **TA**. To do this, let $\tau \in \mathbf{TA}$ and let $F$ be a finite subset of $X_1$. Fix a finite connected subgraph $\mathcal{G}$ of $\mathbf{BS}(X_1, X_2, \tau)$ containing the edges $z\Sigma_1$ for $z \in F$, and denote by $\tau_0$ the restriction of $\tau$ to the union of the $\Sigma_1$-orbits in $X_1$ corresponding to the edges of $\mathcal{G}$. Then apply Proposition 6.1 to the transitive pre-action $(\mathrm{dom}(\tau_0) \cdot \Gamma_1, \mathrm{rng}(\tau_0) \cdot \Gamma_2, \tau_0)$, whose Bass-Serre graph is $\mathcal{G}$, to get an extension $\tau'$ such that $\tau' \in V_{x_1,\dots,x_k,y_1,\dots,y_k} \cap \mathbf{HFA}$. Moreover, since $F \subset \mathrm{dom}(\tau_0)$, it follows that $\tau$ and $\tau'$ coincide on $F$. ☐

**Remark 6.5.** As in Remark 4.5, one can give a direct proof of the previous theorem without relying on Baire's theorem.

# 7 Highly transitive actions of groups acting on trees

## 7.1 Proofs of Theorem A and B

Let us begin with a few preliminaries.

Suppose we are given an action of a countable group $G$ on a tree $\mathcal{T}$, and a proper subtree $\mathcal{T}'$ such that $\mathcal{T}'$ and $g\mathcal{T}'$, where $g \in G$, are either equal or disjoint subtrees.

We can then can form a "quotient" tree $\bar{\mathcal{T}}$ by shrinking each subtree $g\mathcal{T}'$ to a single vertex, that we will denote by $(g\mathcal{T}')$. The tree $\bar{\mathcal{T}}$ is naturally endowed with a $G$-action and the "quotient map" $q : \mathcal{T} \to \bar{\mathcal{T}}$ is $G$-equivariant. The image by $q$ of a path in $\mathcal{T}$ is a path which is obtained by shrinking each subpath contained in a subtree $g\mathcal{T}'$ to the vertex $(g\mathcal{T}')$ in $\bar{\mathcal{T}}$. In case of a geodesic ray, its image by $q$ is either a geodesic ray in $\bar{\mathcal{T}}$, or a geodesic which ends at a vertex $(g\mathcal{T}')$. Hence $q$ induces a map $\partial q : \partial\mathcal{T} \to V(\bar{\mathcal{T}}) \cup \partial\bar{\mathcal{T}}$.

**Remark 7.1.** The restriction of $\partial q$ to $(\partial q)^{-1}(\partial\bar{\mathcal{T}})$ is injective.

*Proof.* Given $\xi, \xi' \in \partial\mathcal{T}$ such that $\partial q(\xi)$ and $\partial q(\xi')$ lie in $\partial\bar{\mathcal{T}}$, consider geodesic rays $\omega, \omega'$ in $\mathcal{T}$ tending to $\xi, \xi'$. Each ray contains all edges of its image under $q$. Hence $\partial q(\xi) = \partial q(\xi')$ implies that $\omega$ and $\omega'$ have infinitely many common edges, and therefore $\xi = \xi'$. □

One can also notice, although we do not need this fact below, that $\partial q$ is continuous at each point $\xi \in (\partial q)^{-1}(\partial\bar{\mathcal{T}})$. In case $\mathcal{T}'$ is bounded, one has in fact $\partial q : \partial\mathcal{T} \to \partial\bar{\mathcal{T}}$, and $\partial q$ is continuous and injective.

**Lemma 7.2.** *In the context above, assume that $G \curvearrowright \mathcal{T}$ is a minimal action. Then:*

*(1) if $G \curvearrowright \mathcal{T}$ is of general type, then so is $G \curvearrowright \bar{\mathcal{T}}$;*

*(2) if $G \curvearrowright \partial\mathcal{T}$ is topologically free, then so is $G \curvearrowright \partial\bar{\mathcal{T}}$.*

*Proof.* Assume first $G \curvearrowright \mathcal{T}$ is of general type, in order to prove (1). The hypotheses on $\mathcal{T}'$ guarantee the existence of an edge $e$ in $\mathcal{T}$ which lies outside all translates $g\mathcal{T}'$. By minimality of $G \curvearrowright \mathcal{T}$, there exists a hyperbolic element $h \in G$ whose axis in $\mathcal{T}$ contains $e$. Pick $g_1, g_2 \in G$ which induce transverse hyperbolic automorphisms of $\mathcal{T}$. For $n$ sufficiently large, $h_1 = g_1^n h g_1^{-n}$ and $h_2 = g_2^n h g_2^{-n}$ induce transverse hyperbolic automorphisms of $\mathcal{T}$. Moreover, their axes do contain edges in the orbit of $e$, so that their images by $q$ lie in $\partial\bar{\mathcal{T}}$. Hence, by Remark 7.1, $h_1$ and $h_2$ induce transverse hyperbolic automorphisms of $\bar{\mathcal{T}}$. This proves that $G \curvearrowright \bar{\mathcal{T}}$ is of general type.

Assume now that $G \curvearrowright \partial\mathcal{T}$ topologically free, in order to prove (2). Then assume that $g \in G$ fixes a half-tree $\mathcal{H}'$ in $\bar{\mathcal{T}}$, corresponding to some edge $e$ in $\mathcal{T}'$, pointwise. Notice that $e$ is an edge of $\mathcal{T}$ that $q$ does not shrink, and denote by $\mathcal{H}$ its half-tree in $\mathcal{T}$. One has $q(\mathcal{H}) = \mathcal{H}'$. By minimality of $G \curvearrowright \mathcal{T}$, the edges of the orbit $G \cdot e$ which lie in $\mathcal{H}$ do generate $\mathcal{H}$. Since they also lie in $\mathcal{H}'$, they are fixed pointwise by $g$, therefore the half-tree $\mathcal{H}$ itself is fixed pointwise by $g$. Since $G \curvearrowright \partial\mathcal{T}$ is topologically free, $g$ has to be the trivial element, and this proves that $G \curvearrowright \partial\bar{\mathcal{T}}$ is topologically free. □

We now recall the statement of Theorem A before proving it.

**Theorem.** *Let $\Gamma \curvearrowright \mathcal{T}$ be a minimal action of general type of a countable group $\Gamma$ on a tree $\mathcal{T}$. If the action on the boundary $\Gamma \curvearrowright \partial\mathcal{T}$ is topologically free, then $\Gamma$ admits a highly transitive and highly faithful action; in particular, $\Gamma$ is highly transitive.*

*Proof of Theorem A.* Let us consider an edge $e$ in $\mathcal{T}$. The complement of the orbits $\Gamma \cdot e$ and $\Gamma \cdot \bar{e}$ in $E(\mathcal{T})$ is either empty, or generates a disjoint union of subtrees of $\mathcal{T}$. Since each of these subtrees contains some endpoint of some translate of $e$, there are at most two orbits of subtrees. Hence by applying Lemma 7.2 zero, one, or two times, one gets a tree $\bar{\mathcal{T}}$ endowed with a $\Gamma$-action which is still of general type, and such that $\Gamma \curvearrowright \partial\bar{\mathcal{T}}$ is topologically free. Moreover, the action on $E(\bar{\mathcal{T}})$ is transitive. Now, the quotient $\Gamma\backslash\bar{\mathcal{T}}$ is either a segment or a loop, the fundamental group of the corresponding graph of groups, which is either an amalgam or an HNN extension, is isomorphic to $\Gamma$, and $\bar{\mathcal{T}}$ is the associated Bass-Serre tree. Applying Theorem E or Theorem C, we finally get that $\Gamma$ admits a highly transitive and highly faithful action. □

We can now also prove Theorem B, but let us first recall its statement.

**Theorem.** *Let $\Gamma \curvearrowright \mathcal{T}$ be a faithful minimal action of general type of a countable group $\Gamma$ on a tree $\mathcal{T}$. The following are equivalent*

*(1)* $\mathrm{td}(\Gamma) \geq 4$*;*

*(2)* $\Gamma$ *is highly transitive;*

*(3)* $\Gamma$ *is MIF;*

*(4)* $\Gamma \curvearrowright \partial\mathcal{T}$ *is topologically free.*

*Proof of Theorem B.* The implication (2) $\implies$ (1) is clear. The implication (1) $\implies$ (4) is Le Boudec and Matte Bon's main result [LBMB22, Thm 1.4]. The implication (4) $\implies$ (2) is a consequence of Theorem A. So (1), (2) and (4) are all equivalent.

To prove that these three statements are also equivalent to (3), note that Theorem A shows moreover that, under the assumption (4), the group $\Gamma$ admits a highly transitive *highly faithful* action. So for such an action, all its elements have infinite support, which by [HO16, Corollary 5.8] implies that $\Gamma$ is MIF. The implication (4) $\implies$ (3) thus holds. Finally, the implication (3) $\implies$ (4) follows from [LBMB22, Proposition 3.7] □

## 7.2 Corollary F and its implication of former results

We now turn to the proof of a lemma which directly implies Corollary F via Theorem A, and then we check that Corollary F applies to all groups acting on trees which can be proven to be highly transitive by previous results quoted in the introduction.

Recall that, given a subtree $\mathcal{U}$ of $\mathcal{T}$, we denote by $G_{\mathcal{U}}$ the pointwise stabilizer of $\mathcal{U}$ in $G$. The following lemma is a generalization of Prop. 19 (iv) and Prop. 20 (iv) from [HP11].

**Lemma 7.3.** *Let $G \curvearrowright \mathcal{T}$ be a faithful and minimal action such that $G$ contains a hyperbolic element $h$. If there exist a bounded subtree $\mathcal{B}$ and a vertex $u$ in $\mathcal{B}$ such that $G_{\mathcal{B}}$ is core-free in $G_u$, then the induced action $G \curvearrowright \partial\mathcal{T}$ is topologically free.*

*Proof.* Let $\mathcal{B}'$ be the union of the translates $g\mathcal{B}$ for $g \in G_u$. This is a subtree, since all $g\mathcal{B}$ contain $u$, which is $G_u$-invariant and contained in the ball of radius $\mathrm{diam}(\mathcal{B})$ centered at $u$. Let $g_0$ be an element of $G$ fixing a half-tree $\mathcal{H}$ pointwise. Up to conjugating by a suitable power of $h$, we may and will assume that $\mathcal{H}$ contains $\mathcal{B}'$, so that $g_0$ is in $G_{\mathcal{B}'}$. Now, as $G_{\mathcal{B}}$ is core-free in $G_u$, we have $G_{\mathcal{B}'} = \bigcap_{g \in G_u} G_{g\mathcal{B}} = \bigcap_{g \in G_u} g G_{\mathcal{B}} g^{-1} = \{1\}$. Thus, we get $g_0 = 1$, which proves that $G \curvearrowright \partial\mathcal{T}$ is topologically free by Corollary 2.14. □

We now prove that Corollary F applies to all groups acting on trees which are highly transitive via the combination of the results of Minasyan-Osin [MO15] and Hull-Osin [HO16].

**Proposition 7.4.** *Let $\Gamma$ be a countable group acting minimally on a tree $\mathcal{T}$. Suppose that*

*(i) $\Gamma$ is not virtually cyclic,*

*(ii) $\Gamma$ does not fix any point of $\partial\mathcal{T}$,*

*(iii) there exist vertices $u,v$ of $\mathcal{T}$ such that the stabilizer $\Gamma_{[u,v]}$ is finite;*

*(iv) the finite radical of $\Gamma$ is trivial.*

*Then the action $\Gamma \curvearrowright \mathcal{T}$ is faithful, of general type, and there exists a bounded subtree $\mathcal{B}$ such that $\Gamma_{\mathcal{B}}$ is trivial. In particular, $\Gamma_{\mathcal{B}}$ is core-free in $\Gamma_u$ for every vertex $u$ in $\mathcal{B}$.*

*Proof.* Starting with a finite stabilizer $\Gamma_{[u,v]}$ given by (iii), one observes that $\bigcap_{\gamma\in\Gamma}\gamma\Gamma_{[u,v]}\gamma^{-1}$ is contained in the finite radical, hence trivial by (iv). In particular, $\Gamma \curvearrowright \mathcal{T}$ is faithful.

The action $\Gamma \curvearrowright \mathcal{T}$ cannot be elliptic. Indeed, if it were, then $\mathcal{T}$ would be a singleton, $\Gamma$ would be finite by (iii), and this contradicts (i). Furthermore, this action cannot be lineal, because of (i), faithfulness and minimality, nor parabolic, nor quasi-parabolic, because of (ii). Hence, $\Gamma \curvearrowright \mathcal{T}$ is of general type.

Finally, there is a finite subset $F \Subset \Gamma$, containing 1, such that $\bigcap_{f\in F} f\Gamma_{[u,v]}f^{-1}$ is already trivial, hence $\bigcap_{f\in F}\Gamma_{f\cdot[u,v]}$ is trivial. Now, we are done by considering the smallest subtree $\mathcal{B}$ of $\mathcal{T}$ containing the geodesics $f\cdot[u,v]$, for $f \in F$. □

Second, we prove that Corollary F applies to all groups which are highly transitive thanks to [FMS15]. This is a straightforward consequence of the following result.

**Proposition 7.5.** *Let a countable group $\Gamma$ act without inversion on a tree $\mathcal{T}$, and consider a set $R \subset E(\mathcal{T})$ of representatives of the edges of the quotient graph $\Gamma\backslash\mathcal{T}$. Assume that $\Gamma_v$ is infinite and $\Gamma_e$ is highly core-free in $\Gamma_v$, for every couple $(e,v)$ where $e \in R$ and $v$ is one of its endpoints. Then, there exists a subtree $\mathcal{T}'$ of $\mathcal{T}$ such that:*

*(1) the action $\Gamma \curvearrowright \mathcal{T}'$ is faithful, of general type, and minimal;*

*(2) there exist a bounded subtree $\mathcal{B}$ of $\mathcal{T}'$ and $u \in V(\mathcal{B})$ such that $\Gamma_{\mathcal{B}}$ is core-free in $\Gamma_u$.*

*Proof.* First, assume that $\gamma \in \Gamma$ fixes $\mathcal{T}$ pointwise. Then, one has $\gamma \in \Gamma_v$ for some endpoint $v$ of an edge $e \in R$. As $\Gamma_e$ is highly core-free in $\Gamma_v$, the $\Gamma_v$-action on the orbit $\Gamma_v\cdot e$, which is conjugate to $\Gamma_v \curvearrowright \Gamma_v/\Gamma_e$, is highly faithful. Hence, we get $\gamma = 1$, so that $\Gamma \curvearrowright \mathcal{T}$ is faithful.

Second, let us consider any edge $e \in R$, and set $v = s(e)$, $w = r(e)$. By high core-freeness, the indexes $[\Gamma_v : \Gamma_e]$ and $[\Gamma_w : \Gamma_e]$ are both infinite. Thus, there exist elements $g_1, g_2 \in \Gamma_w$ and $h_1, h_2 \in \Gamma_v$ such that the edges $e, g_1^{-1}e, g_2^{-1}e, h_1e, h_2e$ are pairwise distinct. Notice that $(g_1^{-1}e, \bar{e}, h_1e)$ and $(g_2^{-1}e, \bar{e}, h_2e)$ are oriented paths. For $j = 1,2$, the element $h_jg_j$ is hyperbolic and its axis contains $(g_j^{-1}e, \bar{e}, h_je)$. We have got transverse hyperbolic elements $h_1g_1, h_2g_2$, which proves that $\Gamma \curvearrowright \mathcal{T}$ is of general type.

In fact, $\Gamma_v$ is infinite and $\Gamma_e$ is highly core-free in $\Gamma_v$, for every couple $(e,v)$ where $e$ is any edge of $\mathcal{T}$ and $v$ is one of its endpoints. Notice this property passes to the smallest subtree $\mathcal{T}'$ of $\mathcal{T}$ containing the axes of all hyperbolic elements in $\Gamma$. This subtree is $\Gamma$-invariant, and the action $\Gamma \curvearrowright \mathcal{T}'$ is still faithful and of general type. Of course, it is also minimal, and Assertion (2) is trivially satisfied when $\mathcal{B}$ is any segment (that is, any subtree with exactly two vertices). □

Finally, we state two natural consequences of Corollary F when considering the natural actions of HNN extensions (resp. amalgams) on their Bass-Serre tree.

**Corollary 7.6.** *Consider a non-ascending HNN extension* $\Gamma = \mathrm{HNN}(H,\Sigma,\vartheta)$*. If one of the subgroups* $\Sigma, \vartheta(\Sigma)$ *is core-free in* $H$*, then* $\Gamma$ *admits a highly transitive and highly faithful action; in particular,* $\Gamma$ *is highly transitive.*

*Proof.* Apply Corollary F to the tree induced by the edge $\Sigma$ and the vertex $H$, or to the tree induced by the edge $\vartheta(\Sigma)$ and the vertex $H$. □

**Corollary 7.7.** *Consider a non-degenerate amalgam* $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$*. If* $\Sigma$ *is core-free in one factor* $\Gamma_j$*, then then* $\Gamma$ *admits a highly transitive and highly faithful action; in particular,* $\Gamma$ *is highly transitive.*

*Proof.* Apply Corollary F to the tree induced by the edge $\Sigma$ and the vertex $\Gamma_j$. □

# 8 Examples and applications

As mentioned in the Introduction, it is worth giving examples of groups which are highly transitive thanks to Theorem A, or to its consequences, but for which previous results from [MO15, HO16, FMS15, GGS20] do not apply. In particular, we will prove that some groups are neither acylindrically hyperbolic nor linear. To that end, we will use the following well-known results.

**Proposition 8.1.** *[Osi16, Corollary 1.5] If a group* $\Gamma$ *is acylindrically hyperbolic, then so is any s-normal subgroup of* $\Gamma$*.*

Let us recall that a subgroup $\Lambda \leq \Gamma$ is called **s-normal** if for every $\gamma \in \Gamma$, the subgroup $\gamma\Lambda\gamma^{-1} \cap \Lambda$ is infinite. Every infinite normal subgroup is clearly s-normal.

**Proposition 8.2.** *Every acylindrically hyperbolic group contains a non-abelian free subgroup. In particular, every acylindrically hyperbolic group is non-amenable.*

The latter proposition can either be proved by a standard ping-pong argument, or deduced from Theorem 6.8 and Theorem 8.1 in [DGO17], which imply that every acylindrically hyperbolic group is SQ-universal (see also the discussion around Conjecture 9.6 in the same book).

Let us recall that a group is called **linear over a field** $k$ if it is isomorphic to a subgroup of $\mathrm{GL}(V)$, where $V$ is a finite dimensional $k$-vector space. Note that if $k'$ is an extension of $k$ then any group linear over $k$ is linear over $k'$. Hence, if a group is linear over $k$ then it is also linear over the algebraic closure of $k$.

A group $\Gamma$ is called **linear** if there exists a field $k$ such that $\Gamma$ is linear over $k$. It follows from the preceding discussion that $\Gamma$ is linear if and only if there exists an algebraically closed field $k$ such that $\Gamma$ is linear over $k$.

## 8.1 Examples around Baumslag-Solitar groups

### 8.1.1 Baumslag-Solitar groups themselves

Let us recall the definition: for any $m,n \in \mathbb{Z}^*$, the Baumslag-Solitar group with parameters $m,n$ is

$$\mathrm{BS}(m,n) := \langle a,b \,|\, ab^m a^{-1} = b^n \rangle.$$

Hull and Osin asked what the transitivity degree of Baumslag-Solitar groups is [HO16, Question 6.3], and noted that it was actually unknown whether $\mathrm{BS}(2,3)$ is highly transitive or not. We completely answer this question in Proposition 8.8 and Corollary 8.12 below.

Notice that $\mathrm{BS}(m,n)$ is isomorphic to $\mathrm{HNN}(\mathbb{Z},n\mathbb{Z},\vartheta)$, where $\vartheta(nq)=mq$ for all $q\in\mathbb{Z}$, and the isomorphism from $\mathrm{HNN}(\mathbb{Z},n\mathbb{Z},\vartheta)$ to $\mathrm{BS}(m,n)$ is given by $t\mapsto a$ and $q\mapsto b^q$ for $q\in\mathbb{Z}$. We will freely identify $\mathrm{BS}(m,n)$ to this HNN extension below without recalling it explicitly. Hence, $\mathrm{BS}(m,n)$ has a natural action on the Bass-Serre tree of this HNN extension, which we denote by $\mathcal{T}_{m,n}$.

**Remark 8.3.** The following facts are well-known:

- $\mathrm{BS}(m,n)$ is solvable if and only if $|m|=1$ or $|n|=1$;
- $\mathrm{BS}(m,n)$ is icc if and only if $|m|\neq|n|$;
- $\mathrm{BS}(m,n)$ is residually finite if and only if $|m|=1$, $|n|=1$, or $|m|=|n|$; see [Mes72];
- $\mathrm{BS}(m,n)$ is non-linear whenever $|m|\neq 1$, $|n|\neq 1$, and $|m|\neq|n|$; this is a consequence of the former fact and Malcev's theorem [Mal40].

**Remark 8.4.** Hull and Osin observed that $\mathrm{BS}(m,n)$ is never acylindrically hyperbolic (for $m,n\in\mathbb{Z}^*$); this is [Osi16, Example 7.4]. Let us recall the argument: since the cyclic subgroup $\langle b\rangle$ is s-normal, the group $\mathrm{BS}(m,n)$ is not acylindrically hyperbolic by Proposition 8.1.

Let us note the following result for later use.

**Lemma 8.5.** *For all $m,n\in\mathbb{Z}^*$ and $r\geq 1$, the subgroup $\langle b^r\rangle$ is s-normal in $\mathrm{BS}(m,n)$.*

*Proof.* Let $\gamma$ be any element of $\mathrm{BS}(m,n)$, with normal form $b^{s_0}a^{\varepsilon_1}b^{s_1}\cdots a^{\varepsilon_k}b^{s_k}$. It is easy to check that $\gamma b^{rm^kn^k}\gamma^{-1}$ is still a non-trivial power of $b$, say $b^s$. Then, $\langle b^r\rangle\cap\gamma\langle b^r\rangle\gamma^{-1}$ contains $\langle b^{rm^kn^ks}\rangle$, hence is infinite. □

Let us now turn to a crucial lemma before stating our result. It is due to de la Harpe and Préaux [HP11, Lem. 21], but we include a proof for the reader's convenience.

**Lemma 8.6** (de la Harpe-Préaux)**.** *The action $\mathrm{BS}(m,n)\curvearrowright\partial\mathcal{T}_{m,n}$ is topologically free if and only if $|n|\neq|m|$.*

*Proof.* ($\Longleftarrow$) Since $|n|\neq|m|$, either $n\nmid m$ or $m\nmid n$. Assume that $n\nmid m$ and let $d=\gcd(n,m)$ and $n=dn_0$, $m=dm_0$. Since $n\nmid m$ we have $|n_0|\geq 2$. In particular, we have $\Sigma\neq H$, hence there are several positive edges in the star at any vertex in $\mathcal{T}_{m,n}$. Consequently every half-tree in $\mathcal{T}_{m,n}$ contains a half-tree corresponding to a positive edge. Now, let $\gamma\in\mathrm{BS}(m,n)$, suppose $\gamma$ fixes pointwise a half-tree. Since the action of $\mathrm{BS}(m,n)$ is transitive on the positive edges of $\mathcal{T}_{m,n}$, we may assume that the fixed half-tree $\mathcal{H}$ contains the one given by the edge $\Sigma$. For all $k\geq 1$, one has $\mathsf{path}_1(t^k)\subset\mathcal{H}$, hence $\gamma$ fixes $\mathsf{path}_1(t^k)$ pointwise. It follows that $\gamma\in\Sigma\cap t^k\Sigma t^{-k}=n_0^{k+1}d\mathbb{Z}$, for all $k\geq 1$. Hence, $\gamma=1$, since $|n_0|\geq 2$. In the case $m\nmid n$ the proof is similar (we could also deduce this case from the isomorphism $\mathrm{BS}(m,n)\simeq\mathrm{BS}(n,m)$).

($\Longrightarrow$) Suppose that $|n|=|m|$, so that $\Sigma=\vartheta(\Sigma)$ and $\vartheta=\pm\mathrm{id}$. In this case, $\Sigma$ is a non-trivial normal subgroup of $\mathrm{BS}(m,n)$, and $\mathcal{T}_{m,n}$ itself is fixed pointwise by any element of $\Sigma$. □

**Remark 8.7.** The lemma immediately implies that $\langle b\rangle$ is a core-free subgroup of $\mathrm{BS}(m,n)$ whenever $|n|\neq|m|$. Indeed, the action on $\mathrm{BS}(m,n)\curvearrowright\partial\mathcal{T}_{m,n}$ being topologically free, the action $\mathrm{BS}(m,n)\curvearrowright\mathcal{T}_{m,n}$ is faithful. Hence $\langle b\rangle$ is core-free in $\mathrm{BS}(m,n)$, since the conjugates of $\langle b\rangle$ in $\mathrm{BS}(m,n)$ are exactly the vertex stabilizers. See also Lemma 8.13.

Our first new examples of highly transitive groups are given by the following result.

**Proposition 8.8.** *Let $m,n \in \mathbb{Z}^*$. The following are equivalent:*

*(i)* $|m| \neq 1$, $|n| \neq 1$, *and* $|m| \neq |n|$;

*(ii)* $\mathrm{BS}(m,n)$ *admits a highly transitive and highly faithful action;*

*(iii)* $\mathrm{BS}(m,n)$ *is highly transitive;*

*(iv)* $\mathrm{BS}(m,n)$ *is non-solvable and icc.*

*Proof.* The implication (i) $\Longrightarrow$ (ii) is a direct consequence of Theorem C and Lemma 8.6. Then, (ii) $\Longrightarrow$ (iii) is trivial, (iii) $\Longrightarrow$ (iv) results from classical obstructions to high transitivity recalled in the Introduction, and (iv) $\Longrightarrow$ (i) results from Remark 8.3. □

**Remark 8.9.** As reminded in Remark 8.3 and Remark 8.4, the highly transitive groups arising in Proposition 8.8 are non-acylindrically hyperbolic and non-linear. Moreover, edge-stabilizers are not highly core-free in their endpoints stabilizers (they have finite index). Hence, results from [MO15, HO16, FMS15, GGS20] do not apply.

**Remark 8.10.** The following lemma proves that Corollary F cannot apply to the action $\mathrm{BS}(m,n) \curvearrowright \mathcal{T}_{m,n}$. Consequently, Theorem A is stronger than Corollary F. Note that Baumslag-Solitar groups are our only examples which testify to this fact.

**Lemma 8.11.** *Set $\Gamma := \mathrm{BS}(m,n)$. If $\mathcal{B}$ is any bounded subtree of $\mathcal{T}_{m,n}$ and $u$ is any vertex of $\mathcal{B}$, then the pointwise stabilizer $\Gamma_{\mathcal{B}}$ is not core-free in $\Gamma_u$.*

*Proof.* There exists a positive integer $r$ such that $\mathcal{B}$ is contained in the ball $\mathcal{B}(r)$ of radius $r$ at $\langle b \rangle$. Then, every stabilizer $\Gamma_v$, where $v \in V(\mathcal{B}(r))$, is a conjugate subgroup $\gamma\langle b\rangle\gamma^{-1}$, where the normal form of $\gamma$ contains at most $r$ occurrences of $a^{\pm 1}$. Consequently $\gamma^{-1}b^{m^r n^r}\gamma$ is still a power of $b$, so that $b^{m^r n^r}$ lies in $\gamma\langle b\rangle\gamma^{-1} = \Gamma_v$. This proves that $b^{m^r n^r}$ lies in the pointwise stabilizer $\Gamma_{\mathcal{B}(r)}$. Now, for every $\gamma \in \Gamma_u$, one has $\gamma\Gamma_{\mathcal{B}}\gamma^{-1} = \Gamma_{\gamma\mathcal{B}} \supseteq \Gamma_{\mathcal{B}(r)}$, since $\gamma\mathcal{B} \subseteq \mathcal{B}(r)$. Thus all conjugates $\gamma\Gamma_{\mathcal{B}}\gamma^{-1}$ where $\gamma \in \Gamma_u$ contain $b^{m^r n^r}$, so $\Gamma_{\mathcal{B}}$ is not core-free in $\Gamma_u$. □

Let us now complete the answer to Hull and Osin's question. As they noticed in [HO16, Lemma 4.2 and Corollary 4.6], infinite non-icc groups and infinite residually finite solvable groups have transitivity degree 1. Hence we can compute the transitivity degree of all Baumslag-Solitar groups.

**Corollary 8.12.** *Let $m,n \in \mathbb{Z}^*$. The following hold.*

*(1) If* $|n| = 1$ *or* $|m| = 1$ *or* $|n| = |m|$, *then* $\mathrm{td}(\mathrm{BS}(m,n)) = 1$.

*(2) In the other cases,* $\mathrm{BS}(m,n)$ *is highly transitive, so* $\mathrm{td}(\mathrm{BS}(m,n)) = +\infty$.

*Proof.* (1) If $|n| = 1$ or $|m| = 1$, the group $\mathrm{BS}(m,n)$ is infinite, residually finite, and solvable, hence $\mathrm{td}(\mathrm{BS}(m,n)) = 1$. If $|n| = |m|$, the group $\mathrm{BS}(m,n)$ is infinite and non-icc, hence $\mathrm{td}(\mathrm{BS}(m,n)) = 1$.

(2) This follows from Proposition 8.8 directly. □

### 8.1.2 Amalgams with Baumslag-Solitar groups

Let us now turn to examples of highly transitive groups given by amalgams. Let us begin by some more preliminaries.

**Lemma 8.13.** *Let $m,n \in \mathbb{Z}^*$. If $|n| \neq |m|$, the subgroup $\langle b\rangle$ is highly core-free in* $\mathrm{BS}(m,n)$.

Notice this is essentially the same example as the one given in [HO16, Corollary 5.12]. As the action on $\mathrm{BS}(m,n) \curvearrowright \partial\mathcal{T}_{m,n}$ is topologically free, it is a particular case of a general phenomenon described in the following lemma.

**Lemma 8.14.** *Let $\Gamma \curvearrowright \mathcal{T}$ be a minimal action of a countable group $\Gamma$ on a tree $\mathcal{T}$. If the action on the boundary $\Gamma \curvearrowright \partial\mathcal{T}$ is topologically free, then for every vertex $v$ in $\mathcal{T}$, the stabilizer $\Gamma_v$ is highly core-free in $\Gamma$.*

*Proof.* We have to prove that the action $\Gamma \curvearrowright \Gamma/\Gamma_v$ is highly faithful; by Corollary 2.6 it is sufficient to prove it is strongly faithful. Notice the orbit $\Gamma v$ in $\mathcal{T}$ meets every half-tree in $\mathcal{T}$ by minimality of the action $\Gamma \curvearrowright \mathcal{T}$, and that $\Gamma \curvearrowright \Gamma v$ is conjugate to $\Gamma \curvearrowright \Gamma/\Gamma_v$.

Take any non-trivial elements $\gamma_1,\dots,\gamma_k \in \Gamma$. Let us start with any half-tree $\mathcal{H}_0$. Then, since the fixed points of $\gamma_1$ form a subtree, and since $\Gamma \curvearrowright \partial\mathcal{T}$ is topologically free, there exists a half-tree $\mathcal{H}_1 \subseteq \mathcal{H}_0$, all of whose vertices are moved by $\gamma_1$. Then applying the same argument to $\gamma_2$, we get a half-tree $\mathcal{H}_2 \subseteq \mathcal{H}_1$, all of whose vertices are moved by $\gamma_1$ and $\gamma_2$. And so on, and so forth, we finish with a half-tree $\mathcal{H}_k$, all of whose vertices are moved by all elements $\gamma_1,\dots,\gamma_k$. Finally $\mathcal{H}_k$ contains a point of $\Gamma v$, which is moved by all elements $\gamma_1,\dots,\gamma_k$. This proves that $\Gamma \curvearrowright \Gamma v$ is strongly faithful. ☐

We also need a general fact about s-normality in amalgams.

**Lemma 8.15.** *Let us consider an amalgam $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$. If all infinite subgroups $\Sigma' < \Sigma$ are s-normal in both $\Gamma_1$ and $\Gamma_2$, then they are also all s-normal in $\Gamma$.*

*Proof.* Let $\Sigma_0$ be any infinite subgroup of $\Sigma$, and let $\gamma$ be any element of $\Gamma$, that we write as a product $\gamma = \gamma_1\cdots\gamma_n$ of elements of $\Gamma_1$ or $\Gamma_2$. Set $\Sigma_k = \Sigma_{k-1}\cap\gamma_k^{-1}\Sigma_{k-1}\gamma_k$ for $k = 1,\dots,n$. Let us prove by induction that $\Sigma_k$ is infinite, and contained in $\Sigma_0\cap(\gamma_1\cdots\gamma_k)^{-1}\Sigma_0(\gamma_1\cdots\gamma_k)$ for $k = 0,\dots,n$.

For $k = 0$, the group $\Sigma_0$ has been supposed infinite, and it coincides with the intersection $\Sigma_0\cap(\gamma_1\cdots\gamma_k)^{-1}\Sigma_0(\gamma_1\cdots\gamma_k)$ in this case. Then, for $k \geq 1$, the subgroup $\Sigma_{k-1}$ is infinite by induction hypothesis, therefore $\Sigma_{k-1}$ is s-normal in $\Gamma_1$ and $\Gamma_2$. Consequently, the subgroup $\Sigma_k = \Sigma_{k-1}\cap\gamma_k^{-1}\Sigma_{k-1}\gamma_k$ is infinite. Moreover, one has

$$\begin{aligned}\Sigma_k &= \Sigma_{k-1}\cap\gamma_k^{-1}\Sigma_{k-1}\gamma_k\\ &\subseteq \Sigma_0\cap(\gamma_1\cdots\gamma_{k-1})^{-1}\Sigma_0(\gamma_1\cdots\gamma_{k-1})\cap\gamma_k^{-1}\Sigma_0\gamma_k\cap(\gamma_1\cdots\gamma_k)^{-1}\Sigma_0(\gamma_1\cdots\gamma_k)\end{aligned}$$

by induction hypothesis, whence $\Sigma_k \subseteq \Sigma_0\cap(\gamma_1\cdots\gamma_k)^{-1}\Sigma_0(\gamma_1\cdots\gamma_k)$.

Finally, applying the result with $k = n$, we get that $\Sigma_n$ is infinite and $\Sigma_n \subseteq \Sigma_0\cap\gamma^{-1}\Sigma_0\gamma$, which proves that $\Sigma_0$ is s-normal in $\Gamma$, as desired. ☐

Let us finally turn to our examples of highly transitive amalgams.

**Proposition 8.16.** *Let $m,n,k \in \mathbb{Z}^*$, and let $\Lambda$ be a countable group containing a proper infinite cyclic subgroup $\langle c\rangle$. The amalgam $\Gamma = \mathrm{BS}(m,n) *_{\langle b^k=c\rangle} \Lambda$ has the following properties:*

*(1) if $|n| \neq |m|$, then $\Gamma$ admits an action which is both highly transitive and highly faithful;*

*(2) if $|n| \neq |m|$ and $\langle c \rangle$ is s-normal in $\Lambda$, then $\Gamma$ is not acylindrically hyperbolic;*

*(3) if $|m| \neq 1$, $|n| \neq 1$, and $|m| \neq |n|$, then $\Gamma$ is not a linear group.*

*Proof.* (1) The amalgam $\Gamma$ is non-degenerate. By Lemma 8.13, or Remark 8.7, $\langle b^k \rangle$ is a core-free subgroup of $\mathrm{BS}(m,n)$. Then Corollary F, or in this case Corollary 7.7, implies that $\Gamma$ admits an action which is both highly transitive and highly faithful.

(2) By Lemma 8.5, every non-trivial subgroup of $\langle b \rangle$ is s-normal in $\mathrm{BS}(m,n)$. Furthermore, in $\Lambda$, for any $\lambda \in \Lambda$, the intersection $\langle c \rangle \cap \lambda \langle c \rangle \lambda^{-1}$ is infinite cyclic, say generated by $c_\lambda$, since $\langle c \rangle$ is s-normal in $\Lambda$. Then, for every $l \geq 1$, one has $\langle c^l \rangle \cap \lambda \langle c^l \rangle \lambda^{-1} = \langle c_\lambda^l \rangle$, which is infinite. Hence, every non-trivial subgroup of $\langle c \rangle$ is s-normal in $\Lambda$.

Then, Lemma 8.15 implies that every non-trivial subgroup of $\langle b^k \rangle = \langle c \rangle$ is s-normal in the amalgam $\Gamma$. Now, $\langle c \rangle$ is cyclic, so it is not acylindrically hyperbolic by Proposition 8.2, so $\Gamma$ is not acylindrically hyperbolic either by Proposition 8.1.

(3) The group $\Gamma$ contains a copy of $\mathrm{BS}(m,n)$, which is non-linear (see Remark 8.3). Hence $\Gamma$ cannot be a linear group. □

**Remark 8.17.** The previous proposition shows in particular that if $|m| \neq 1$, $|n| \neq 1$, $|m| \neq |n|$, and if one chooses $\Lambda$ such that $\langle c \rangle$ is not highly core-free in $\Lambda$ (e.g. $\Lambda = \mathbb{Z}$ and $c \geq 2$), then results from [MO15, HO16, FMS15, GGS20] do not apply to prove that $\Gamma$ is highly transitive.

**Remark 8.18.** On the other hand, if $|m| \neq 1$, $|n| \neq 1$, $|m| \neq |n|$, and if one chooses $\Lambda$ such that $\langle c \rangle$ is highly core-free in $\Lambda$ (e.g. $\Lambda = \mathrm{BS}(m,n)$ and $c = b^k$), then Corollary B of [FLMM22] shows that $\Gamma$ admits homogeneous actions on bounded Urysohn spaces.

## 8.2 Examples around finitely supported permutations

We now turn to examples constructed from the group of finitely supported permutations on an infinite countable set.

### 8.2.1 Examples of HNN extensions over $S_f(X)$

We denote by $S_f(X)$ the subgroup of $S(X)$ consisting of finitely supported permutations.

The group $S_f(X)$ is known to be not linear but we could not find any elementary proof in the literature and this is why we have chosen to include a complete proof below. We thank Julien Bichon for explaining to us the following argument.

**Lemma 8.19.** *Let $\Gamma$ be a group. If, for any prime number $q$ and any $N \in \mathbb{N}^*$, $\Gamma$ contains a subgroup $G$ with $G \simeq (\mathbb{Z}/q\mathbb{Z})^N$ then $\Gamma$ is not linear.*

*Proof.* Let $k$ be any algebraically closed field and denote by $p$ its characteristic. Let us recall some elementary facts. For $n \in \mathbb{N}^*$, let us denote by $U_n(k) \subset k^*$ the multiplicative subgroup of $n$-th roots of unity. Elements of $U_n(k)$ are exactly the roots of the polynomial $P = X^n - 1 \in k[X]$. Since $P' = nX^{n-1}$ all the roots of $P$ are simple if $p = 0$ or if $p$ is a prime number which does not divide $n$. Hence, if $p = 0$ or $p$ is prime and does not divide $n$ one has $|U_n(k)| = n$. To deduce the Lemma, it suffices to prove the following claim.

**Claim.** *Let $V$ be a finite dimensional vector space over an algebraically closed field $k$. If $\mathrm{GL}(V)$ contains a subgroup $G$ isomorphic to $(\mathbb{Z}/q\mathbb{Z})^N$, where $N \in \mathbb{N}^*$ and $q$ is any prime number with $q \neq \mathrm{char}(k)$ then $N \leq \dim(V)$.*

Note that any element $g \in G$ satisfies $g^q = 1$ hence, the minimal polynomial $\mu_g$ of $g$ divides $X^q - 1$. Since $q \neq \mathrm{char}(k)$, $X^q - 1$ has only simple roots so $\mu_g$ has only simple roots and $g$ is diagonalizable with eigenvalues in $U_q(k)$. Moreover, since $G$ is finite abelian and all its elements are diagonalisable, there exists a basis $\mathcal{B} = (e_1, \dots, e_n)$ of $V$ which simultaneously diagonalises every element of $G$. Let us denote by $\lambda(g) \in U_q(k)^n$ the element $\lambda(g) = (\lambda_1(g), \dots, \lambda_n(g))$, where $g(e_k) = \lambda_k(g)e_k$. This defines an injective map $G \to U_q(k)^n$, $g \mapsto \lambda(g)$. It follows that $|G| = q^N \leq |U_q(k)^n| = q^n$, hence $N \leq n = \dim(V)$. □

**Proposition 8.20.** *The group $S_f(X)$ is not linear.*

*Proof.* Let $N \in \mathbb{N}^*$ and $q$ be a prime number. Since $X$ is infinite, one can choose $q$-cycles $\sigma_1, \dots, \sigma_N$ in $S_f(X)$ with pairwise disjoint supports. It is then easy to check that there is an injective morphism of groups defined by

$$(\mathbb{Z}/q\mathbb{Z})^N \to S_f(X) \quad ; \quad (x_1, \dots, x_n) \mapsto \sigma_1^{x_1} \cdots \sigma_N^{x_N}.$$

Hence, the proof follows from Lemma 8.19. □

For any subset $F \subseteq X$, let us denote by $\Sigma(F)$ the pointwise stabilizer of $F$ in $S_f(X)$, and remark that whenever $F$ is finite, the subgroup $\Sigma(F)$ is infinite, in fact isomorphic to $S_f(X)$ itself. We will abbreviate $\Sigma(\{x\})$ as $\Sigma(x)$. For any $k \geq 1$, let $X^{(k)}$ denote the set of $k$-tuples of pairwise distinct points in $X$.

**Lemma 8.21.** *Let $F$ be a non-empty subset of $X$. The following hold:*

*(1) the stabilizer $\Sigma(F)$ is a core-free subgroup of $S_f(X)$, with infinite index;*

*(2) if $F$ is finite, then $\Sigma(F)$ is not highly core-free in $S_f(X)$;*

*(3) if $F$ is finite, then $\Sigma(F)$ is s-normal in $S_f(X)$.*

*Proof.* (1) Let $x \in F$. Since the action $X \curvearrowleft S_f(X)$ is transitive (even highly transitive) and faithful we have $\bigcap_{g \in S_f(X)} g^{-1}\Sigma(x)g = \bigcap_{g \in S_f(X)} \Sigma(x \cdot g) = \bigcap_{y \in X} \Sigma(y) = \{1\}$, hence $\Sigma(x)$ is core-free in $S_f(X)$. A fortiori, $\Sigma(F)$ is core-free in $S_f(X)$. Let us denote by $\tau_y \in S_f(X)$ the transposition $\tau_y = (x\ y)$ for $y \neq x$. The subgroup $\Sigma(x)$ has infinite index since, for all $y, z \in X \setminus \{x\}$, one has $\tau_y^{-1}\tau_z \in \Sigma(x) \Leftrightarrow y = z$. A fortiori, $\Sigma(F)$ has infinite index in $S_f(X)$.

(2) Let us write $F = \{x_1, \dots, x_k\}$, with $\bar{x} = (x_1, \dots, x_k) \in X^{(k)}$. The action $X^{(k)} \curvearrowleft S_f(X)$ is transitive, since the action $X \curvearrowleft S_f(X)$ is highly transitive, and the stabilizer of $\bar{x}$ is $\Sigma(F)$. Consequently, the action $\Sigma(F)\backslash S_f(X) \curvearrowleft S_f(X)$ is conjugate to $X^{(k)} \curvearrowleft S_f(X)$. Now, $X^{(k)} \curvearrowleft S_f(X)$ is not strongly faithful, since taking $k+1$ permutations with pairwise disjoint and finite supports in $X$, every point in $X^{(k)}$ will be fixed by at least one of them. Consequently, $\Sigma(F)\backslash S_f(X) \curvearrowleft S_f(X)$ is not highly faithful.

(3) For any $g \in S_f(X)$, we have $\Sigma(F) \cap g^{-1}\Sigma(F)g = \Sigma(F \cup F \cdot g)$, and $F \cup F \cdot g$ is still finite, hence $\Sigma(F) \cap g^{-1}\Sigma(F)g$ is infinite. □

**Proposition 8.22.** *Let $Y$ and $Z$ be two distinct infinite proper subsets of $X$, let $\tau : Y \to Z$ be a bijection, and let $\vartheta = \tau_* : S_f(Y) \to S_f(Z)$ be the isomorphism defined by $\vartheta(\sigma) = \tau^{-1}\sigma\tau$. Then, the HNN extension $\Gamma = \mathrm{HNN}(S_f(X), S_f(Y), \vartheta)$ has the following properties:*

*(1) it admits an action which is both highly transitive and highly faithful;*

*(2) it is not linear;*

*(3) if $Y$ and $Z$ are both cofinite, then it is not acylindrically hyperbolic;*

*(4) if $Y$ and $Z$ are both cofinite, then for every bounded subtree $\mathcal{B}$ of its Bass-Serre tree, the pointwise stabilizer $\Gamma_{\mathcal{B}}$ is not highly core-free in a vertex stabilizer $\Gamma_u$.*

*Proof.* (1) Note that $S_f(Y) = \Sigma(X \setminus Y)$. Thus, $S_f(Y)$ is a core-free subgroup of $S_f(X)$ by Lemma 8.21. Hence, Corollary F, or in this case Corollary 7.6, applies.

(2) Follows from Proposition 8.20.

(3) As $Y$ and $Z$ are both cofinite and since the intersection of finitely many cofinite subsets is cofinite, the powers $\tau^n$, defined by composition of partial bijections in $X$ (for $n \in \mathbb{Z}$) all have a cofinite domain and a cofinite range, that we will denote by $Y_n$ and $Z_n$ respectively. Let $U$ be any cofinite subset of $X$. For any $g \in S_f(X)$, one has $S_f(U) \cap g^{-1} S_f(U) g = S_f(U \cap U \cdot g)$, and $U \cap U \cdot g$ is still cofinite. Moreover, for any $n \in \mathbb{Z}$, one has $S_f(U) \cap t^{-n} S_f(U) t^n = S_f(U \cap U\tau^n)$, where $t \in \Gamma$ is the stable letter. Note that $U\tau^n$ is cofinite since the bijection $\tau^n$ realizes a bijection between $U \cap Y_n$ and $U\tau^n$, so that the subset $U\tau^n$ is cofinite in $Z_n$, hence cofinite. Therefore, $U \cap U\tau^n$ is cofinite.

Then, given $\gamma \in \Gamma$, one can write $\gamma = \gamma_1 \cdots \gamma_k$, where each $\gamma_j$ is either a power of the stable letter $t$, or an element of $S_f(X)$, and an easy induction based on previous facts shows that $S_f(Y) \cap \gamma^{-1} S_f(Y) \gamma$ contains $S_f(V)$ for some cofinite set $V$. This proves that $S_f(Y)$ is an s-normal subgroup in $\Gamma$.

Furthermore, $S_f(Y)$ is an amenable group, hence it is not acylindrically hyperbolic by Proposition 8.2. Finally, Proposition 8.1 implies that $\Gamma$ is not acylindrically hyperbolic.

(4) Up to conjugating and to enlarging $\mathcal{B}$, we may and will assume without loss of generality that the stabilizer $\Gamma_u$ is $S_f(X)$, and that $\Gamma_e = S_f(Y)$ for some edge $e$ in $\mathcal{B}$. Since $\Gamma$ acts transitively on the positive edges, there exists $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that

$$\Gamma_{\mathcal{B}} = S_f(Y) \cap \bigcap_{j=1}^{k} \gamma_j^{-1} S_f(Y) \gamma_j .$$

As in the proof of (2), we see that there exist cofinite sets $V_1, \ldots, V_k$ such that the intersection $S_f(Y) \cap \gamma_j^{-1} S_f(Y) \gamma_j$ contains $S_f(V_j)$ for every $j$, hence $\Gamma_{\mathcal{B}}$ contains $S_f(\bigcap_{j=1}^{k} V_j)$, where $\bigcap_{j=1}^{k} V_j$ is cofinite. Now, $S_f(\bigcap_{j=1}^{k} V_j)$ is not highly core-free in $\Gamma_u = S_f(X)$ by Lemma 8.21, hence $\Gamma_{\mathcal{B}}$ is not either. □

**Remark 8.23.** When both $Y$ and $Z$ are cofinite in $X$, this proposition provides more explicit new examples of highly transitive groups, since items (1), (3) and (4) show that the results from [MO15, HO16, FMS15, GGS20] do not apply.

**Remark 8.24.** In the context of Proposition 8.22, notice that $\Gamma$ obviously admits a highly transitive action when $\tau$ can be extended to a permutation $\tilde{\tau} \in S(X)$. Indeed, the $\Gamma$-action defined by $t \mapsto \tilde{\tau}$ and $\sigma \mapsto \sigma$ for $\sigma \in S_f(X)$ is highly transitive since its restriction to $S_f(X)$ already is (in the terminology of Section 3, this action corresponds to the global pre-action $(X, \tilde{\tau})$). Nevertheless, the $\Gamma$-action we obtain factors through the semi-direct product $S_f(X) \rtimes \langle \tilde{\tau} \rangle$, which is amenable while $\Gamma$ is not; hence the $\Gamma$-action is not faithful. Furthermore, such an extension to a permutation $\tilde{\tau}$ is not possible when $X - Y$ and $X - Z$ have different cardinalities.

### 8.2.2 Examples of HNN extensions over $S_f(\mathbb{Z}) \rtimes \mathbb{Z}$

Let us now move to a modification of former examples to get groups which are moreover finitely generated. For these examples, we consider the permutation $s \in S(\mathbb{Z})$ given by $k \cdot s = k + 1$. It is straightforward to check that the subgroup $\langle S_f(\mathbb{Z}), s \rangle < S(\mathbb{Z})$ is finitely generated, and isomorphic to a semi-direct product of the form $S_f(\mathbb{Z}) \rtimes \mathbb{Z}$. As before, for any subset $F \subseteq \mathbb{Z}$, let us denote by $\Sigma(F)$ the pointwise stabilizer of $F$ in $S_f(\mathbb{Z})$. For some purposes, we will need the action

$$\mathbb{Z} \times \mathbb{Z} \curvearrowleft \langle S_f(\mathbb{Z}), s\rangle, \qquad (k,l) \cdot s^n g := (ks^n g, ls^n) = ((k+n)g, l+n)$$

for $g \in S_f(\mathbb{Z})$ and $n \in \mathbb{Z}$. Notice that this action is faithful, as is the action $\mathbb{Z} \curvearrowleft \langle S_f(\mathbb{Z}), s\rangle$. Moreover, given a subset $F \subseteq \mathbb{Z}$, we observe that the pointwise stabilizer in $\langle S_f(\mathbb{Z}), s\rangle$ of the subset $F \times \{0\} \subset \mathbb{Z} \times \mathbb{Z}$ is the subgroup $\Sigma(F) < S_f(\mathbb{Z})$.

**Lemma 8.25.** *Let $F$ be a non-empty subset of $\mathbb{Z}$. The following hold:*

*(1) the stabilizer $\Sigma(F)$ is a core-free subgroup of $\langle S_f(\mathbb{Z}), s\rangle$, with infinite index;*

*(2) if $F$ is finite, then $\Sigma(F)$ is not highly core-free in $\langle S_f(\mathbb{Z}), s\rangle$;*

*Proof.* (1) The group $\Sigma(F)$ is already core-free and has infinite index in $S_f(\mathbb{Z})$ by Lemma 8.21.

(2) Let us write $F = \{x_1, \ldots, x_k\}$, with $x_1, \ldots, x_k$ pairwise distinct, and set

$$\bar{x} = \big((x_1, 0), \ldots, (x_k, 0)\big) \in (\mathbb{Z} \times \mathbb{Z})^{(k)}.$$

Let us denote by $\Omega$ the orbit of $\bar{x}$ under $\langle S_f(\mathbb{Z}), s\rangle$. As the action $\mathbb{Z} \curvearrowleft S_f(\mathbb{Z})$ is highly transitive, $\Omega$ is the union $\bigcup_{n\in\mathbb{Z}} (\mathbb{Z} \times \{n\})^{(k)}$. Furthermore, the stabilizer of $\bar{x}$ is the pointwise stabilizer of $F \times \{0\}$, that is, $\Sigma(F)$. Consequently, the action $\Sigma(F)\backslash\langle S_f(\mathbb{Z}), s\rangle \curvearrowleft \langle S_f(\mathbb{Z}), s\rangle$ is conjugate to $\Omega \curvearrowleft \langle S_f(\mathbb{Z}), s\rangle$. Now, $\Omega \curvearrowleft \langle S_f(\mathbb{Z}), s\rangle$ is not strongly faithful, since taking $k+1$ elements of $S_f(\mathbb{Z})$ with pairwise disjoint supports, every point in $\Omega = \bigcup_{n\in\mathbb{Z}} (\mathbb{Z} \times \{n\})^{(k)}$ will be fixed by at least one of them. Consequently, $\Sigma(F)\backslash\langle S_f(\mathbb{Z}), s\rangle \curvearrowleft \langle S_f(\mathbb{Z}), s\rangle$ is not highly faithful. ☐

Using Lemma 8.25 we can prove the following Proposition exactly as we proved Proposition 8.22.

**Proposition 8.26.** *Let $Y$ and $Z$ be two distinct infinite proper subsets of $\mathbb{Z}$, let $\tau : Y \to Z$ be a bijection, and let $\vartheta = \tau_* : S_f(Y) \to S_f(Z)$ be the isomorphism defined by $\vartheta(\sigma) = \tau^{-1}\sigma\tau$. Then, the HNN extension $\Gamma = \mathrm{HNN}(\langle S_f(\mathbb{Z}), s\rangle, S_f(Y), \vartheta)$ has the following properties:*

*(1) it admits an action which is both highly transitive and highly faithful;*

*(2) it is finitely generated and not linear;*

*(3) if $Y$ and $Z$ are both cofinite in $\mathbb{Z}$, then it is not acylindrically hyperbolic;*

*(4) if $Y$ and $Z$ are both cofinite in $\mathbb{Z}$, then for every bounded subtree $\mathcal{B}$ of its Bass-Serre tree, the pointwise stabilizer $\Gamma_{\mathcal{B}}$ is not highly core-free in a vertex stabilizer $\Gamma_u$.*

**Remark 8.27.** Again, when both $Y$ and $Z$ are cofinite in $\mathbb{Z}$, this proposition provides explicit new examples of groups which are highly transitive.

Note that when the complements of $Y$ and $Z$ have the same cardinality, these groups admit a natural highly transitive action, but it fails to be faithful. Indeed, the subgroup $\langle S_f(\mathbb{Z}), t\rangle < \Gamma$ is isomorphic to the HNN extension $\mathrm{HNN}(S_f(\mathbb{Z}), S_f(Y), \vartheta)$, and the action of this subgroup is not faithful by Remark 8.24.

### 8.2.3 Examples of amalgams

We now switch to the context of amalgams. We will need a refinement of Lemma 8.15.

**Lemma 8.28.** *Let us consider an amalgam $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$. Assume there is a collection $\mathcal{C}$ of infinite subgroups of $\Sigma$ such that, for every $\Sigma' \in \mathcal{C}$ and every $\gamma \in \Gamma_1 \cup \Gamma_2$, the intersection $\Sigma' \cap \gamma^{-1}\Sigma'\gamma$ contains an element of $\mathcal{C}$. Then, for every $\Sigma' \in \mathcal{C}$ and every $\gamma \in \Gamma$, the intersection $\Sigma' \cap \gamma^{-1}\Sigma'\gamma$ contains an element of $\mathcal{C}$. In particular, all $\Sigma' \in \mathcal{C}$ are s-normal subgroups of $\Gamma$.*

*Proof.* Let $\Sigma_0$ be any element of $\mathcal{C}$, and let $\gamma$ be any element of $\Gamma$, that we write as a product $\gamma = \gamma_1 \cdots \gamma_n$ of elements of $\Gamma_1$ or $\Gamma_2$. Let us prove by induction that, for $k = 0, \dots, n$, there exists $\Sigma_k \in \mathcal{C}$ which is contained in $\Sigma_0 \cap (\gamma_1 \cdots \gamma_k)^{-1}\Sigma_0(\gamma_1 \cdots \gamma_k)$.

For $k = 0$, the group $\Sigma_0$ has been chosen in $\mathcal{C}$, and it coincides with $\Sigma_0 \cap (\gamma_1 \cdots \gamma_k)^{-1}\Sigma_0(\gamma_1 \cdots \gamma_k)$ in this case. Then, for $k \geq 1$, the subgroup $\Sigma_{k-1} \cap \gamma_k^{-1}\Sigma_{k-1}\gamma_k$ contains some $\Sigma_k \in \mathcal{C}$. Moreover, one has $\Sigma_{k-1} \subseteq \Sigma_0 \cap (\gamma_1 \cdots \gamma_{k-1})^{-1}\Sigma_0(\gamma_1 \cdots \gamma_{k-1})$ by induction hypothesis, hence

$$\begin{aligned}\Sigma_k &\subseteq \Sigma_{k-1} \cap \gamma_k^{-1}\Sigma_{k-1}\gamma_k \\ &\subseteq \Sigma_0 \cap (\gamma_1 \cdots \gamma_{k-1})^{-1}\Sigma_0(\gamma_1 \cdots \gamma_{k-1}) \cap \gamma_k^{-1}\Sigma_0\gamma_k \cap (\gamma_1 \cdots \gamma_k)^{-1}\Sigma_0(\gamma_1 \cdots \gamma_k)\end{aligned}$$

whence $\Sigma_k \subseteq \Sigma_0 \cap (\gamma_1 \cdots \gamma_k)^{-1}\Sigma_0(\gamma_1 \cdots \gamma_k)$.

Finally, for $k = n$, we get $\Sigma_n \subseteq \Sigma_0 \cap \gamma^{-1}\Sigma_0\gamma$ with $\Sigma_n \in \mathcal{C}$, as desired. □

**Proposition 8.29.** *Let $X, Y, Z$ be infinite countable sets such that $Z$ is proper subset of $X \cap Y$. Then, the amalgam $\Gamma = S_f(X) *_{S_f(Z)} S_f(Y)$ has the following properties:*

*(1) it admits an action which is both highly transitive and highly faithful;*

*(2) it is not a linear group;*

*(3) if $Z$ is cofinite in both $X$ and $Y$, then it is not acylindrically hyperbolic;*

*(4) if $Z$ is cofinite in $X$ (resp. $Y$), then $S_f(Z)$ is not highly core-free in $S_f(X)$ (resp. $S_f(Y)$).*

*Proof.* Lemma 8.21 and Corollary 7.7 imply (1). Proposition 8.20 implies (2) while (4) follows from Lemma 8.21. Let us prove (3). Let $\mathcal{C}$ be the collection of subgroups of the form $S_f(U)$ where $U$ is cofinite in $Z$ (hence cofinite in both $X$ and $Y$). For every $\Sigma' = S_f(U) \in \mathcal{C}$ and every $\gamma \in S_f(X) \cup S_f(Y)$, one can check as in former proofs that the intersection $\Sigma' \cap \gamma^{-1}\Sigma'\gamma$ contains an element of $\mathcal{C}$. Hence, Lemma 8.28 implies that $S_f(Z)$ is s-normal in $\Gamma$. Since $S_f(Z)$ is amenable it is not acylindrically hyperbolic hence $\Gamma$ is not acylindrically hyperbolic. □

Again, one can easily modify the previous examples to get groups which are moreover finitely generated. The proof of the following Proposition is exactly the same as the proof of Proposition 8.29 (by using Lemma 8.25). We can now provide one last new class of highly transitive examples.

**Proposition 8.30.** *Let $Z$ be an infinite proper subset of $\mathbb{Z}$ and consider the amalgam $\Gamma := \langle S_f(\mathbb{Z}), s\rangle *_{S_f(Z)} \langle S_f(\mathbb{Z}), s\rangle$. The following holds.*

*(1) $\Gamma$ admits an action which is both highly transitive and highly faithful;*

*(2) $\Gamma$ is finitely generated and not linear;*

*(3) If $Z$ is cofinite in $\mathbb{Z}$, then $\Gamma$ is not acylindrically hyperbolic and $S_f(Z)$ is not highly core-free in $\langle S_f(\mathbb{Z}), s\rangle$.*

### 8.3 Faithful actions which are non-topologically free on the boundary

Although our main result provides a complete characterization of high transitivity for groups admitting a faithful minimal action of general type on a tree, one may wonder if Corollaries 7.6 and 7.7 can hold in a wider context, namely, if the core-freeness assumption of the edge group in a vertex group can be weakened to core-freeness in the whole group. We will see that it is not the case.

Thanks to the quoted result from [LBMB22], this amounts to finding examples of amalgams and HNN extensions whose action on their Bass-Serre trees are minimal of general type and faithful, but the action on the boundary is not topologically free. By Bass-Serre theory, we essentially need to find faithful edge transitive actions on trees without inversions which are not topologically free on the boundary, but which in the amalgam case have two vertex orbits, while in the HNN case they have only one vertex orbit.

Our examples belong to a class which was explored in depth by Le Boudec [LB16, LB17], generalizing a construction of Bader-Caprace-Gelander-Mozes [BCGM12] which takes its roots in the work of Burger-Mozes [BM00]. Such examples already appeared in Le Boudec and Matte-Bon's work on high transitivity, so our only contribution here is to point out that some of those naturally decompose as amalgams or HNN extensions. We will focus on specific easy examples instead of seeking large generality. For more examples, we refer the reader to Ivanov's recent work [Iva20].

#### 8.3.1 An example of amalgam

Let $\mathcal{T}_d$ be a $d$-regular tree of finite degree $d \geq 3$. As in [BM00], let us fix a coloring on the set of edges $c : E(\mathcal{T}_d) \to \{1, \ldots, d\}$ such that:

- every edge has the same color as its antipode;
- for any vertex $v$, the restriction of $c$ to the star $\mathrm{st}(v)$ is a bijection onto $\{1, \ldots, d\}$.

For any vertex $v$, any automorphism $g \in \mathrm{Aut}(\mathcal{T}_d)$ induces a bijection $g_v : \mathrm{st}(v) \to \mathrm{st}(gv)$, which itself induces a permutation $\sigma(g, v) \in S_d$, where $S_d = \mathrm{Sym}(\{1, \ldots, d\})$. Let $C_d$ be a cyclic subgroup of $S_d$ generated by a $d$-cycle. Consider the group, coming from [LB16],

$$G = G(C_d) = \{g \in \mathrm{Aut}(\mathcal{T}_d) : \sigma(g, v) \in C_d \text{ for all but finitely many vertices}\}.$$

**Remark 8.31.** The group $G(C_d)$ is countable. Indeed, given an automorphism $g \in G(C_d)$ and any edge $e$ such that $\sigma(g, r(e))$ has to be in $C_d$, the permutation $\sigma(g, r(e))$ is determined by the color of $g(e)$. It follows that $g$ is completely determined by its restriction to any finite subtree containing at least one edge and all stars at vertices $v$ such that $\sigma(g, v) \notin C_d$.

In order to forbid inversions, recall there is a natural equivalence relation $\mathcal{R}_{even}$ on $V(\mathcal{T}_d)$ which relates any two vertices at even distance from each other, and that this equivalence relation is preserved by any automorphism of $\mathcal{T}_d$. We then let

$$\Gamma = G^+ = \{g \in G : g \text{ does not exchange the two classes of } \mathcal{R}_{even}\}.$$

It is fairly easy to see that the action $\Gamma \curvearrowright \mathcal{T}_d$ is transitive on undirected edges, hence minimal, of general type, and without inversion. Let us now fix some edge $e_0$ from $v_1$ to $v_2$, and consider the stabilizers $\Gamma_1$, $\Gamma_2$ and $\Sigma$ of $v_1$, $v_2$ and $e_0$ respectively (in $\Gamma$). By Bass-Serre theory, we have the following.

**Remark 8.32.** The morphism $\Gamma_1 *_\Sigma \Gamma_2 \to \Gamma$ given by inclusions is an isomorphism, and $\mathcal{T}_d$ is the Bass-Serre tree of $\Gamma_1 *_\Sigma \Gamma_2$.

The following result summarizes well-known properties of $\Gamma$ showing that the hypothesis that $\Sigma$ is core-fre in $\Gamma_1$ or $\Gamma_2$ cannot be relaxed in Corollary 7.7. We provide a proof for the reader's convenience.

**Proposition 8.33.** *With the above notations:*

*(1) $\Sigma$ is core-free in $\Gamma$, and the amalgam $\Gamma_1 *_\Sigma \Gamma_2$ is non-degenerate;*

*(2) the $\Gamma$-action on $\partial\mathcal{T}_d$ is not topologically free;*

*(3) $\Gamma$ is not highly transitive;*

*(4) $\Gamma$ is icc.*

*Proof.* (1) The $\Gamma$-action on $\mathcal{T}_d$ is faithful (by definition) and of general type, so that $\Sigma$ is core-free in $\Gamma$, and the amalgam $\Gamma_1 *_\Sigma \Gamma_2$ is non-degenerate.

(2) Consider the half-tree $\mathcal{H}$ associated to $e_0$. It suffices to prove that the pointwise stabilizer of $\mathcal{H}$ is a non-trivial group.

To do so, we follow the proof of [LB17, Theorem C]. First, we take a non-trivial permutation $\sigma \in S_d$ which fixes the color of $e_0$ (this exists since $d \geq 3$; notice it lives in $S_d \setminus C_d$). Then, we define a non-trivial automorphism $\gamma \in \Gamma$ fixing $\mathcal{H}$ pointwise as follows.

- The restriction of $\gamma$ to $\mathcal{H}$ is the identity.
- Then, we let $\gamma$ act on $\mathrm{st}(v_1)$ so that $\sigma(g, v_1) = \sigma$ (this is possible since $\sigma$ fixes the color of $e_0$, and will guarantee that $\gamma$ is non-trivial since $\sigma$ is non-trivial).
- Then, we extend the action inductively: given any vertex $w$ outside $\mathcal{H}$, we set $w'$ to be the unique neigbour of $w$ which is closer to $\mathcal{H}$ than $w$, and define the $\gamma$-action on $\mathrm{st}(w)$ in terms of the (previously defined) $\gamma$-action on $\mathrm{st}(w')$. Namely, denoting by $e_w$ the edge from $w$ to $w'$, the $\gamma$-action on $\mathrm{st}(w')$ provides the edge $\gamma e_w$. Then there is a unique element $\sigma_w \in C_d$ sending $c(e_w)$ onto $c(\gamma e_w)$, and we let $\gamma$ act on $\mathrm{st}(w)$ so that $\sigma(g, w) = \sigma_w$ (note that $\sigma(g, w) = \sigma(g, w')$ as soon as $\sigma(g, w')$ was already in $C_d$).

(3) The $\Gamma$-action on $\mathcal{T}_d$ is minimal, by edge-transitivity, and of general type. Consequently, [LBMB22, Corollary 1.5] applies, and the transitivity degree of $\Gamma$ is at most 2.

(4) Let $\gamma_0$ be a non-trivial element of $\Gamma$ and $\xi$ be a point in $\partial\mathcal{T}_d$ which is not fixed by $\gamma_0$. Given edges $e, e'$ with sources in the same class of vertices, such that $e$ is on the geodesic $[\xi, s(e')]$, there exists $\gamma \in \Gamma$ such that $\gamma e = e'$. This $\gamma$ is a hyperbolic element whose axis contains $e$ and $e'$. Moreover, if we choose $e$ close enough to $\xi$, then the repelling point $\xi^-$ of $\gamma$ in $\partial\mathcal{T}_d$ is close enough to $\xi$ so that $\gamma_0\xi^- \neq \xi^-$. Now, since $\xi^-$ is not fixed by $\gamma_0$, the set of fixed points of $\gamma^n\gamma_0\gamma^{-n}$ moves into smaller and smaller neighborhoods (in $\mathcal{T}_d \cup \partial\mathcal{T}_d$) of the attracting point of $\gamma$ as $n \to +\infty$. Thus, the set $\{\gamma^n\gamma_0\gamma^{-n} : n \geq 1\}$ is infinite. □

### 8.3.2 An example of HNN extension

In the previous example, notice that even the subgroup

$$U(\mathrm{id}) = \{g \in \mathrm{Aut}(\mathcal{T}_d) : \sigma(g, v) = \mathrm{id} \text{ for every vertex } v\}$$

includes inversions, so that one cannot easily find a subgroup of $G$ without inversion and acting transitively on $V(\mathcal{T}_d)$. Hence, we slightly modify the construction in order to get an example of HNN extension.

Let $\mathcal{T}_{d,d}$ be a $(d,d)$-biregular tree, where $d \geq 2$ (by this, we mean an oriented tree in which every star $\mathrm{st}(v)$ contains exactly $d$ positive edges and $d$ negative edges). Let us denote $\mathrm{st}(v)^+ = \mathrm{st}(v) \cap E(\mathcal{T}_{d,d})^+$ and $\mathrm{st}(v)^- = \mathrm{st}(v) \cap E(\mathcal{T}_{d,d})^-$, and fix a coloring on the set of edges $c : E(\mathcal{T}_{d,d}) \to \{1,\dots,2d\}$ such that:

- every edge has the same color as its antipode;
- for any vertex $v$, the restriction of $c$ to the star $\mathrm{st}(v)$ is a bijection onto $\{1,\dots,2d\}$;
- for any vertex $v$, the image $c(\mathrm{st}(v)^+)$ is either $\{1,\dots,d\}$ or $\{d+1,\dots,2d\}$.

By $\mathrm{Aut}(\mathcal{T}_{d,d})$, we mean the group of automorphisms of $\mathcal{T}_{d,d}$ preserving the orientation. For any vertex $v$, any automorphism $g \in \mathrm{Aut}(\mathcal{T}_{d,d})$ induces bijections $g_v^{\pm} : \mathrm{st}(v)^{\pm} \to \mathrm{st}(gv)^{\pm}$, which themselves induces a permutation $\sigma(g,v) \in S_{2d}$, where $S_{2d} = \mathrm{Sym}(\{1,\dots,2d\})$, preserving the partition $\{1,\dots,d\} \sqcup \{d+1,\dots,2d\}$. Let $F_d$ be a the subgroup of $S_{2d}$ generated by the commuting elements

$$\begin{aligned}\sigma_1 &= (1 \quad 2 \quad \cdots \quad d)(d+1 \quad d+2 \quad \cdots \quad 2d),\\ \sigma_2 &= (1 \quad d+1)(2 \quad d+2)\cdots(d \quad 2d).\end{aligned}$$

Then, the group,

$$\Gamma = G(F_d) = \{g \in \mathrm{Aut}(\mathcal{T}_{d,d}) : \sigma(g,v) \in F_d \text{ for all but finitely many vertices}\}$$

is countable (this is not hard to prove, using that $F_d$ acts freely on $\{1,\dots,2d\}$).

It is fairly easy to see that the action $\Gamma \curvearrowright \mathcal{T}_{d,d}$ is transitive on positive edges, hence minimal, of general type. It is moreover without inversion since it preserves the orientation. Let us fix some vertex $v$, some positive edges $e_1, e_2$ such that $r(e_1) = v = s(e_2)$, and some automorphism $\tau \in \Gamma$ such that $\tau(e_1) = e_2$. Now, consider the stabilizers $H = \Gamma_v$ and $\Sigma = \Gamma_{e_2}$, and the isomorphism $\vartheta : \Sigma \to \Gamma_{e_1}$ given by $\vartheta(\sigma)(x) = \tau^{-1}\sigma\tau(x)$. By Bass-Serre theory, we have the following.

**Remark 8.34.** The morphism $\mathrm{HNN}(H,\Sigma,\vartheta) \to \Gamma$ given by inclusions is an isomorphism, and $\mathcal{T}_{d,d}$ is the Bass-Serre tree of $\mathrm{HNN}(H,\Sigma,\vartheta)$.

The following result summarizes well-known properties of $\Gamma$ showing that the hypothesis that one of the subgroups $\Sigma, \vartheta(\Sigma)$ is core-free in $H$ cannot be relaxed in Corollary 7.6. We omit the proof, which is similar to the one of Proposition 8.33.

**Proposition 8.35.** *With the above notations:*

1. *(1) $\Sigma$ is core-free in $\Gamma$, and the HNN extension $\mathrm{HNN}(H,\Sigma,\vartheta)$ is non-ascending;*
2. *(2) the $\Gamma$-action on $\partial\mathcal{T}_{d,d}$ is not topologically free;*
3. *(3) $\Gamma$ is not highly transitive;*
4. *(4) $\Gamma$ is icc.*

# 9 Other types of actions and necessity of the minimality assumption

We now discuss various natural extensions of Theorem A by considering other types of actions (recall that group actions on trees are classified in five different types, see Section 2.3). As we will see, non-general type actions which are topologically free on the boundary seem to play no role regarding high transitivity. We will also see that the minimality hypothesis in Theorem A cannot be avoided.

Let us recall that a group action by homeomorphisms on a topological space is called **minimal** when every orbit is dense. Note that given a group action on a tree, the minimality of the action on the boundary implies the minimality of the action on the tree but the converse does not hold: for instance the standard $\mathbb{Z}$-action on itself is minimal but the action on the boundary is not since it has two distinct fixed points.

**Proposition 9.1.** *Every residually finite group admits an elliptic faithful action on a tree with non-empty boundary such that the action on the boundary is both free and minimal.*

*Proof.* Let $\Gamma$ be a residually finite group, let $(\Gamma_n)_{n\geq 0}$ be a decreasing chain of finite index normal subgroups with trivial intersection, where $\Gamma_0 = \Gamma$. Then the disjoint union of the coset spaces $\Gamma/\Gamma_n$ has a natural tree structure where we connect each $\gamma\Gamma_{n+1}$ to $\gamma\Gamma_n$, and the boudary is non-empty. Since the action is transitive on each level of the tree, this action is minimal on the boundary. It is free because the subgroups $\Gamma_n$ are normal and intersect trivially. Finally it is elliptic because the vertex $\Gamma$ is fixed (moreover, the only non-trivial invariant subtrees are balls around $\Gamma$). □

Since there are both highly transitive residually finite groups (such as $\mathbb{F}_2$) and non-highly transitive residually finite groups (such as $\mathbb{Z}$), we see that there is no hope for a classification of the transitivity degree of groups admitting an elliptic faithful action on a tree such that the induced action on the boundary is free and minimal. We can use also this construction in order to show that the minimality assumption for the action on the tree is needed in Theorem A.

**Proposition 9.2.** *Let $\Gamma$ be a non-abelian free group. Then for every residually finite group $\Lambda$, the group $\Gamma \times \Lambda$ admits an action of general type on a tree which is topologically free on the boundary.*

*Proof.* Since $\Gamma$ is free, we have a free $\Gamma$-action on a tree $\mathcal{T}_1$ which is of general type because $\Gamma$ is not abelian. Let $\Lambda \curvearrowright \mathcal{T}_2$ be an action provided by the previous proposition, let $o$ be its unique fixed point. Our new tree $\mathcal{T}$ is obtained by gluing over each vertex of $\mathcal{T}_1$ a copy of $\mathcal{T}_2$ at its origin $o$. To be more precise, the vertex set is $V(\mathcal{T}) = V(\mathcal{T}_1) \times V(\mathcal{T}_2)$, and on the vertex set $\mathcal{T}_1 \times \{o\}$ we put a copy of the edges of $\mathcal{T}_1$, while for each $v \in V(\mathcal{T}_1)$, we put a copy of the edges of $\mathcal{T}_2$ on the vertex set $\{v\} \times \mathcal{T}_2$.

Then the $\Gamma \times \Lambda$ action on $V(\mathcal{T})$ given by $(\gamma, \lambda) \cdot (x_1, x_2) = (\gamma \cdot x_1, \lambda \cdot x_2)$ is an action by automorphisms on our new tree $\mathcal{T}$. Noting that each half-tree contains a copy of a half-tree of $\mathcal{T}_2$, it is not hard to check that this action is moreover topologically free. Moreover, it is of general type since the $\Gamma$-action on $\mathcal{T}_1$ was of general type. □

The previous proposition will allow us to show that in our main result, the hypothesis of minimality of the action on the tree is needed, using the well-known fact that non-trivial product groups cannot be highly transitive. For the convenience of the reader, we provide a proof of the latter fact via the following stronger result.

**Proposition 9.3.** *Let $G$ be a non-abelian topologically simple group. If $\Gamma$ is a dense subgroup of $G$, then the centralizer of every non-trivial element of $\Gamma$ is core-free.*

*Proof.* Let $\gamma \in \Gamma \setminus \{1_\Gamma\}$. If the centralizer of $\Gamma$ is not core-free, then there exists a normal subgroup $N \leq \Gamma$ such that every element of $N$ commutes with $\gamma$. Since $G$ is topologically simple, $N$ is dense in $G$, so by continuity of group multiplication we conclude that every element of $G$ commutes with $\gamma$. In particular, $G$ has a non-trivial center, which contradicts the topological simplicity of $G$ since $G$ is not abelian. □

**Corollary 9.4.** *Let $\Gamma$ be a highly transitive group. Then the centralizer of every non-trivial element of $\Gamma$ must be core-free, and $\Gamma$ cannot be decomposed as a non-trivial direct product.*

*Proof.* If $\Gamma$ is highly transitive, it can be embedded as a dense subgroup of $S(X)$ for some infinite set $X$, and since the latter is topologically simple, the conclusion follows from the previous result. Moreover, if $\Gamma$ could be decomposed as a non-trivial direct product $\Gamma_1 \times \Gamma_2$, then if $\gamma \in \Gamma_1 \setminus \{1_{\Gamma_1}\}$, the element $(\gamma, 1_{\Gamma_2})$ would commute with every element of the non-trivial normal subgroup $\{1_{\Gamma_1}\} \times \Gamma_2$, a contradiction. □

Applying Proposition 9.2 for instance to $\Gamma = \mathbb{F}_2$ and $\Lambda = \mathbb{Z}$, we see that the finitely generated group $\mathbb{F}_2 \times \mathbb{Z}$ admits a (faithful) action of general type on a tree $\mathcal{T}$ which is topologically free on the boundary, although the group $\mathbb{F}_2 \times \mathbb{Z}$ is not highly transitive because it decomposes as a direct product. We see moreover that when restricting to the minimal component of this action, we will loose the faithfulness of the action (in particular the topological freeness), which is why Theorem A cannot be applied.

We now move on to showing that no general classification can be hoped for in the case of parabolic actions (note that minimal parabolic actions only arise for non-finitely generated groups).

**Lemma 9.5.** *Every non-finitely generated group admits a faithful parabolic action on a tree.*

*Proof.* Since $\Gamma$ is countable, it can be written as a countable increasing union of finitely generated subgroups $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$, where $\Gamma_0 = \{1\}$. We now put a tree structure on the vertex set $\bigsqcup_n \Gamma/\Gamma_n$ by connecting each $\gamma\Gamma_n$ to $\gamma\Gamma_{n+1}$. $\Gamma$ acts on this tree by left translation. Since $\Gamma_0 = \{1\}$, this action is faithful. Note that every group element $g \in \Gamma$ is elliptic (with fixed point $\Gamma_n$, where $n \in \mathbb{N}$ is such that $g \in \Gamma_n$). Since $\Gamma$ is not finitely generated, this action has no global fixed point, so we have a parabolic action. □

Note that the $\Gamma$-invariant subtrees of the action constructed above are exactly those of the form $\sqcup_{n \geq m} \Gamma/\Gamma_n$ for some $m \geq 0$. So this action has no minimal globally invariant subtree, as opposed to what happens for finitely generated groups.

**Proposition 9.6.** *Let $\Gamma$ be a non-finitely generated group, let $\Lambda$ be a residually finite infinite group. Then $\Gamma \times \Lambda$ admits a parabolic action on a tree such that the action on the boundary is topologically free.*

*Proof.* As before we use the $\Gamma$-action on a tree $\mathcal{T}_1$ provided by the previous lemma. Let $\Lambda \curvearrowright \mathcal{T}_2$ be an elliptic action provided by Proposition 9.1, whose unique fixed point is denoted by $o \in V(\mathcal{T}_2)$. This time, we glue a copy of $\mathcal{T}_2$ on each terminal vertex (that is, each vertex of the form $\gamma\Gamma_0$) in the tree $\mathcal{T}_1$, thus yielding a $\Gamma \times \Lambda$ action which is easily seen to be parabolic. Moreover, the action on the boundary is topologically free. Note furthermore that the action on the boundary has a unique fixed point, corresponding to the unique element of the boundary of $\mathcal{T}_1$. □

As an example, we can take for $\Gamma$ the group of finitely supported permutations, and for $\Lambda$ the group $\mathbb{Z}$, and we get a non-highly transitive group with a parabolic action which is topologically free on the boundary, and which has no minimal component. We do not know if there is a highly transitive group with a parabolic action which is topologically free on the boundary.

Let us now treat the quasi-parabolic case. First, note that the Baumslag-Solitar groups $\mathrm{BS}(1,n)$ for $n \geq 2$ provide examples of groups admitting an action on a tree which is minimal and quasi-parabolic (since it is an ascending HNN extension, cf. Sec. 2.5) and topologically free on the boundary (by Lemma 8.6), but which are not highly transitive since they are solvable.

**Remark 9.7.** Another example of a non-highly transitive group with a quasi-parabolic minimal action on a tree which is topologically free on the boundary is provided by Thompson's group $F = \langle x_0, x_1, x_2, \ldots \,|\, x_k^{-1} x_n x_k = x_{n+1} \text{ for all } k < n \rangle$. $F$ is not highly transitive since, by [LBMB22, Corollary 5.3], it has transitivity degree at most 2. Let $H$ be the subgroup generated by $\{x_i : i \geq 1\}$ (which is isomorphic to $F$), and $\vartheta$ be the endomorphism which takes $x_i$ to $x_{i+1}$ and observe that $F = \mathrm{HNN}(H, H, \vartheta)$. By Section 2.5 the action of $F$ on the associated Bass-Serre tree is quasi-parabolic and minimal. Moreover, since $x_0^{-k} \vartheta(H) x_0^k = \langle x_n \,|\, n \geq k+2 \rangle$ for all $k \geq 1$, it is not difficult to check that $\vartheta(H)$ is core-free in $H$ hence, the action is also topologically free on the boundary.

Coupled with the previous examples, the following proposition shows that for groups admitting minimal quasi-parabolic actions on a tree, the topological freeness of the action does not play a role in their high transitivity.

**Proposition 9.8.** *The finitely generated group $\Gamma = S_f(\mathbb{Z}) \rtimes \mathbb{Z}$ is highly transitive and finitely generated but admits a minimal quasi-parabolic action on a tree which is topologically free on the boundary.*

*Proof.* We have already observed that $\Gamma$ is highly transitive thanks to its natural action on $\mathbb{Z}$. We will obtain our desired action on a tree by showing that it can be written as an *ascending* HNN extension. Note that $\Gamma$ is a semi-direct product, so it does have a natural HNN extension decomposition, but this decomposition provides a lineal action so we need another one.

Denote by $\tau$ the translation on $\mathbb{Z}$. Let $\vartheta$ be the corresponding inner automorphism of $S(\mathbb{Z})$, i.e. $\vartheta(\gamma) = \tau^{-1}\gamma\tau$. Consider the subgroups $H = \Sigma = S_f(\mathbb{N})$ which we view as subgroups of $S_f(\mathbb{Z}) < \Gamma$. Note that $\vartheta(\Sigma)$ is the stabilizer of 0 in $S_f(\mathbb{N})$ i.e. $\vartheta(\Sigma) = \Sigma(0) < S_f(\mathbb{N})$ with the notations of Section 8.2. We claim that $\Gamma = \mathrm{HNN}(H, \Sigma, \vartheta)$.

First, since $\vartheta(h) = \tau^{-1} h \tau$ for all $h \in H$, we have a quotient map $\pi : \mathrm{HNN}(H, \Sigma, \vartheta) \to \Gamma$ given by $t \mapsto \tau$ and $h \mapsto h$ for every $h \in H$. To show that $\pi$ is injective, we use the fact that the HNN extension is ascending: since $H = \Sigma$, for every $g \in \mathrm{HNN}(H, \Sigma, \vartheta)$, there exists $k \in \mathbb{Z}$ such that $t^{-k} g t^k = h t^n$ with $h \in H$ and $n \in \mathbb{Z}$ (it suffices to take $k$ sufficently large). If $g \neq 1$, one must have $h \neq \mathrm{id}$ or $n \neq 0$, and it is clear that $\pi(g) \neq 1$ in both cases.

So we do have $\Gamma = \mathrm{HNN}(H, \Sigma, \vartheta)$, in particular it is an ascending non-degenerate HNN extension. So as explained in Section 2.5, its action on its Bass-Serre tree is minimal and quasi-parabolic. Finally, $\vartheta(\Sigma) = \Sigma(0)$ is core-free in $H = S_f(\mathbb{N})$ by Lemma 8.21, so by Lemma 7.3 we conclude that the action on the boundary is topologically free as wanted. □

**Remark 9.9.** One can also construct a tree $\mathcal{T}$ directly, with vertex set $V = \bigsqcup_{k \in \mathbb{Z}} S_f(\mathbb{Z})/H_k$, where $H_k := S_f(\mathbb{Z}_{\geq -k})$, and positive edges corresponding to inclusions $\sigma H_k \subseteq \sigma H_{k+1}$. The reader can verify that $\Gamma = S_f(\mathbb{Z}) \rtimes \mathbb{Z}$ acts on $\mathcal{T}$ via

$$(\sigma, n) \cdot \sigma' H_k := \sigma \tau^n (\sigma' H_k) \tau^{-n} = \sigma(\tau^n \sigma' \tau^{-n}) H_{k+n},$$

where $\tau$ is still the translation on $\mathbb{Z}$, and then check "by hand" that this action has all the properties announced in the above proposition.

We finally mention the lineal case. Note that in this case, minimal actions are not interesting with respect to high transitivity since no subgroup of the automorphism group of the biinfinite line is highly transitive.

In the elliptic case, a natural weakening of the minimality assumption was provided by asking that the action on the boundary is minimal. Here however, this is still too strong a condition since any lineal action will have the two ends corresponding to the axis as an invariant set. We thus replace it by topological transitivity (which means the existence of a dense orbit) and observe that in this setup, there seems to be no connection between high transitivity and lineal actions.

**Proposition 9.10.** *Let $\Gamma$ be a residually finite group, then the group $\Gamma \times \mathbb{Z}$ admits a lineal action on a tree which is both topologically free and topologically transitive on the boundary.*

*Proof.* Let $\mathcal{T}$ be a tree equipped with an elliptic $\Gamma$-action which is minimal and free on the boundary as provided by Proposition 9.1, let $o$ be the fixed point. As in the previous constructions, we then glue a copy of $\mathcal{T}$ at the vertex $o$ on top of every element of $\mathbb{Z}$, thus obtaining a tree with a natural $\Gamma \times \mathbb{Z}$-action which is both topologically free and topologically transitive on the boundary: every element of the boundary which does not belong to the two element set $\partial\mathbb{Z}$ has a dense orbit and is fixed by no nontrivial group element. □

The previous proposition provides us many non-highly transitive groups with a lineal action on a tree which is both topologically free and topologically transitive on the boundary.

In the opposite direction, the group $S_f(\mathbb{Z}) \rtimes \mathbb{Z}$ provides us an example of a highly transitive group satisfying the assumptions of the previous proposition, thus showing that lineal actions which are topologically free and topologically transitive on the boundary do not play a role in high transitivity.

**Proposition 9.11.** *The highly transitive group $S_f(\mathbb{Z}) \rtimes \mathbb{Z}$ admits a lineal action on a tree which is both topologically free and topologically transitive on the boundary.*

*Proof.* Let us construct a tree $\mathcal{T}$ as follows. We start with $\mathbb{Z}$, seen as a bi-infinite line. Then, for every $(\sigma, k) \in S_f(\mathbb{Z}) \rtimes \mathbb{Z}$, we consider an infinite ray $\mathcal{R}_{(\sigma,k)}$, and link its origin $o_{(\sigma,k)}$ to the vertex $k \in \mathbb{Z}$ by an edge. The boundary $\partial\mathcal{T}$ consists of points $\xi_\gamma$ for $\gamma \in S_f(\mathbb{Z}) \rtimes \mathbb{Z}$ (the extremities of the rays, which are isolated in $\partial\mathcal{T}$) and two accumulation points $\eta_\pm$.

The group $S_f(\mathbb{Z}) \rtimes \mathbb{Z}$ acts on $\mathcal{T}$ by $(\sigma, k) \cdot k' = k + k'$ and by $\gamma \cdot \mathcal{R}_{\gamma'} = \mathcal{R}_{\gamma\gamma'}$ (it permutes the rays). The induced action on $\partial\mathcal{T}$ fixes $\eta_\pm$ and is transitive free on $\{\xi_\gamma : \gamma \in S_f(\mathbb{Z}) \rtimes \mathbb{Z}\}$. Thus, it is both topologically free and topologically transitive. □

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